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On the impossibility of a finite propositional lattice for quantum mechanics

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Abstract. We give a simplified proof of the impossibility of defining an orthogonality relation in vector spaces (of dimension ≥ 3) over finite fields.

In a previous issue [1] of this journal, Eckmann and Zabey showed 'that the lattice of propositions of a quantum mechanical system cannot be represented as subspaces of Hilbert Space with coefficients from a finite field' (with exception of the dimension two), employing arguments from the theory of such fields and of quadratic forms. We here intend to show that this follows in a simple way already from the axiom system for lattices of subspaces of Hilbert spaces.

The axioms needed are:

(1) L is a complete lattice with 0 and 1.
(2) Atomicity: Every non-zero element in L majorizes an atom, i.e. a non-zero element $p \in L$ with $0 < x \leq p$ only if $x = p$.
(3) Atomic covering property: If p is an atom, then $x \leq y \leq x \vee p$ only if $y = x$ or $y = x \vee p$.
(4) Atomic bisection property (irreducibility): If p and q are atoms, $p \neq q$, then there exists an atom r with $r \neq p$, $r \neq q$ and $r \leq p \vee q$.
(5) L is orthocomplemented: An orthocomplementation is an involutive mapping $L \ni x \mapsto x' \in L$, with $x \vee x' = 1$, $x \wedge x' = 0$ and $x \leq y$ iff $y' \leq x'$.
(6) Orthomodularity: If $x \leq y$, then $y = x \vee (x' \wedge y)$.

It is known that if L is of dimension ≥ 4 there exist a division ring K with an involutorial anti-automorphism $\lambda \mapsto \lambda^*$ and a vector space E over K with a Hermitian form f such that L is ortho-isomorphic to the lattice $L_E(E)$ of E-closed subspaces of E [2].

We define the dimension of an element $x \in L$ as the minimum number of atoms p_i with $x = \vee_i p_i$. It follows from an elementary combinatorial argument [3] that every two-dimensional element majorizes the same number of atoms.

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We will show that in an orthomodular lattice of dimension ≥ 3 this number cannot be finite. It is sufficient to prove this for three-dimensional lattices, i.e. lattices for which 1 is three-dimensional, since if the dimension is larger, we may pick a three-dimensional element x and consider the lattice of all elements of L majorized by x. In view of the orthomodularity, $y \mapsto x \wedge y'$ is an orthocomplementation on this lattice.

In the following we will consequently assume that L is a three-dimensional lattice satisfying axioms 1–5, where each two-dimensional element majorizes $n + 1$ atoms (as would be the case in a Hilbert space over a field of order n).

It follows from axioms 1–4 that the total number of atoms in L is $N = n^2 + n + 1$ [3].

We define the $N \times N$–matrix $A = (a_{ij})_{1 \leq i, j \leq N}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } e_i \text{ and } e_j \text{ are orthogonal, i.e. if } e_i \leq e_j' \\ 0 & \text{otherwise} \end{cases}$$

A is then a symmetric matrix with zeros in the diagonal, exactly $n + 1$ 1:s in each row, and for any two rows there is exactly one column where both rows have a 1.

Consequently $A^2 = nE + U$ where E is the identity matrix of order N and every element of U is 1.

One easily finds that A^2 has the eigenvector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ with eigenvalue $n + N = (n + 1)^2$ and the orthogonal complement in R^N of this vector is an eigenspace of A with eigenvalue n. We conclude that A, apart from the eigenvalue $n + 1$, has eigenvalues $\varepsilon_1\sqrt{n}, \varepsilon_2\sqrt{n}, \ldots, \varepsilon_{N-1}\sqrt{n}$, where $\varepsilon_i = \pm 1$.

Since the trace of A (i.e. the sum of the diagonal elements) equals the sum of its eigenvalues, we obtain: $n + 1 + m\sqrt{n} = 0$, where $m = \sum_{i=1}^{N+1} \varepsilon_i$ is an integer. Clearly $m \neq 0$, so $\sqrt{n} = -(n + 1)/m$ is a rational number, that is an integer, since the square root of an integer is either irrational or an integer. Setting $\sqrt{n} = k$ gives:

$$m = -\frac{k^2 + 1}{k} = -\left(k + \frac{1}{k}\right), \qquad \frac{1}{k} = -(m + k).$$

But this means that $1/k$ is an integer, and we get our desired contradiction, since axiom 4 implies $n \geq 2$.

We have thus shown that there exist no 'Hilbert lattices' of dimension ≥ 3 with a finite number of atoms under each two-dimensional element, from which the non-existence of Hilbert spaces of dimension ≥ 3 over finite fields follows.

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