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Trotter limits of Lie algebra representations and coherent states

U. Cattaneo*

Institut de Physique, Université de Neuchâtel, CH-2000 Neuchâtel (Switzerland)

W. Wreszinski

Instituto de Física, Universidade de São Paulo, São Paulo (Brazil)

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Abstract. The notion of a Trotter limit of a net (π_ι) of Lie algebra representations is introduced and a proposition on the existence of such a limit for a net of skew-symmetric representations satisfying specific conditions is proven. This result is then applied to limits of interest in relativistic quantum field theory or quantum optics, in particular for the study of zero-mass representations of the Poincaré group and of the connection between Bloch and Glauber coherent states.

I. Introduction

Since the pioneering papers of Segal [1] and Inönü and Wigner [2], there has been renewed interest in the theory of Lie algebra and Lie group contraction, both mathematical and in view of physical applications. In particular, there have been several important mathematical contributions to the subject as, for instance, those of Saletan [3] and Lévy-Nahas [4]. Moreover, applications have been found in several branches of theoretical physics, such as relativistic quantum field theory (in connection with the study of zero-mass representations of the Poincaré group [5]) and, more recently, quantum optics [6].

Although, from the algebraic point of view, the theory has been developed aiming at great generality and in a mathematical rigorous way in [3] and [4], it has met with some troubles when dealing with the contraction of representations. The latter problem has been ordinarily studied from the point of view of taking suitable limits of matrix elements. Yet the definition of 'limit operators' (which are, in most relevant cases, properly unbounded) by matrix elements is, in general, beset by serious difficulties ([7], §53).

In a previous paper [8], a theory of contraction of Lie algebra representations has been presented which attempts to avoid these difficulties. Given, for each element ι of a directed system J (usually a subset of $\mathbf{R}$ with the induced ordering), a complex Hilbert space $\mathfrak{H}_\iota$ and a representation π_ι on $\mathfrak{H}_\iota$ of a finite-dimensional real Lie algebra $\mathfrak{g}_\iota$ isomorphic to a reference Lie algebra $\mathfrak{g}$ which is 'contracting

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into $\hat{\mathfrak{g}}'$, we have defined and investigated in [8] a representation $\hat{\pi}_J$ of the contracted Lie algebra $\hat{\mathfrak{g}}$ with carrier space given in terms of the net $(\mathfrak{H}_\iota)$. In particular, we have shown the existence of a subrepresentation $\hat{\pi}$ of $\hat{\pi}_J$, naturally defined under conditions which are often realized in practice.

We prove, in Section II of the present paper, that the representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ is essentially unique and can be defined as a Trotter limit of the net (π_ι). In Section III, we apply the results of Section II and of [8] to several examples which are chosen to illustrate the chief points of the theory of contraction of skew-symmetric Lie algebra representations and in view of their interest for applications, especially to the study of zero-mass representations of the Poincaré group and to quantum optics. Regarding the latter, we prove in Section III.3 that the Glauber coherent states (see, e.g., [9], Ch. 3) are limits in Trotter's sense [10] of sequences of the so-called Bloch (or spin, or atomic) coherent states [11, 6], and this settles the treatment of ([6], Section IV) on a firm ground.

We shall use notation and results of [8] throughout. In particular, V will always stand for a *finite-dimensional real vector space* and $\operatorname{alg}(V, \mu)$ will denote the Lie algebra with underlying vector space V and Lie multiplication μ. Every net considered in the present paper will be indexed by a directed system denoted by J.

II. Trotter limits of nets of Lie algebra representations

In preparation for formulating the main result of this section, we begin by recollecting two concepts already used in [8].

Let $\operatorname{alg}(V, \mu)$ be a reference Lie algebra and let (Γ_ι) be a reference net of automorphisms of V. A net $(\operatorname{alg}(V, \mu_\iota))$ of Lie algebras such that, for each $\iota \in J$, the mapping Γ_ι is an isomorphism of $\operatorname{alg}(V, \mu_\iota)$ onto $\operatorname{alg}(V, \mu)$ is said to be contracting into $\operatorname{alg}(V, \hat{\mu})$ with respect to $\operatorname{alg}(V, \mu)$ if

$$\hat{\mu}(g, g') = \lim_\iota \Gamma_\iota^{-1}(\mu(\Gamma_\iota(g), \Gamma_\iota(g')))$$

for all g, g' in V. By abuse of language, the contracted Lie algebra of the net $(\operatorname{alg}(V, \mu_\iota))$, namely, $\operatorname{alg}(V, \hat{\mu})$, is also called a contraction of $\operatorname{alg}(V, \mu)$.

It is not a goal of the present paper to find necessary and sufficient conditions in order that a net of finite-dimensional real Lie algebras be nontrivially contracting. Particular cases have been studied by different authors. The most important types of Lie algebra contractions appearing in the literature are, in decreasing order of generality, the following.

(a) *Lévy-Nahas contractions* [4]

Using the previous notation with J a subset of $\mathbf{R} - \{0\}$ unbounded from above, the reference nets (Γ_t) of automorphisms of V defining Lévy-Nahas contractions are given by

$$\Gamma_t = t^{-n}(\Gamma + t^{-1} \operatorname{Id}_V),$$

where n is a positive integer and Γ is a noninvertible endomorphism of V such that $\Gamma + t^{-1}\,\mathrm{Id}_V$ is an automorphism of V for all $t \in J$. Let $V = V_R \oplus V_N$ be the Fitting decomposition [12] of V relative to Γ, i.e., let the direct summands V_R and V_N be Γ-stable vector subspaces of V such that $\Gamma \mid V_R$ is an injection onto V_R and $\Gamma \mid V_N$ is nilpotent. Then the net $(\mathrm{alg}\,(V, \mu_t))$ defined by $\mathrm{alg}\,(V, \mu)$ and (Γ_t) is nontrivially contracting if and only if $\Gamma^n \circ \nu = 0$, where ν is a mapping of $V \times V$ into V given by

$$\nu(g, g') = \Gamma^2(\mu(g, g')_N) - \Gamma(\mu(\Gamma(g), g')_N + \mu(g, \Gamma(g'))_N) + \mu(\Gamma(g), \Gamma(g'))_N.$$

Here, and in the next few paragraphs, the subscript N (resp. R) denotes orthogonal projection onto V_N (resp. V_R). If the condition $\Gamma^n \circ \nu = 0$ is satisfied, then $\hat{\mu} = (-\Gamma)^{n-1} \circ \nu$ for $n \geq 1$ and

$$\hat{\mu}(g, g') = (\Gamma \mid V_R)^{-1}(\mu(\Gamma(g), \Gamma(g'))_R) - \Gamma(\mu(g, g')_N) + \mu(\Gamma(g), g')_N + \mu(g, \Gamma(g'))_N$$

for all g, g' in V when $n = 0$.

Notice that this result is obtained for every J of the type considered.

(b) *Saletan contractions* [3]

They are the Lévy-Nahas contractions with $n = 0$.

(c) *Inönü-Wigner contractions* [2]

They are the Saletan contractions such that $\Gamma(V_N) = \{0\}$ ([3], Section I. C. 1).

The second concept we need is that of a net $(\mathfrak{H}_\iota)$ of Hilbert spaces approximating a Hilbert space $\mathfrak{H}$ ([10], Section 2) with respect to a net (P_ι), where, for each $\iota \in J$, P_ι is a continuous linear mapping of $\mathfrak{H}$ into $\mathfrak{H}_\iota$. This happens when $\|P_\iota\| \leq 1$ for all $\iota \in J$ and $\lim_\iota \|P_\iota \phi\|_\iota = \|\phi\|_{\mathfrak{H}}$ for all $\phi \in \mathfrak{H}$. Then a net $(\phi_\iota) \in \prod_\iota \mathfrak{H}_\iota$ is said to be (P_ι)-convergent to $\phi \in \mathfrak{H}$ (shortly: $(P_\iota) - \lim \phi_\iota = \phi$) if $\lim_\iota \|\phi_\iota - P_\iota \phi\|_\iota = 0$; a net (A_ι) of operators in $(\mathfrak{H}_\iota)$ is said to be (P_ι)-convergent to the operator A in $\mathfrak{H}$ (shortly: $(P_\iota) - \lim A_\iota = A$) if $A\phi = (P_\iota) - \lim A_\iota P_\iota \phi$ for all $\phi \in D(A)$. Notice that, if $\phi = (P_\iota) - \lim \phi_\iota$ and $\phi' = (P_\iota) - \lim \phi_\iota$, then $\phi = \phi'$ because

$$\|\phi - \phi'\|_{\mathfrak{H}} = \lim_\iota \|P_\iota \phi - P_\iota \phi'\|_\iota \leq \lim_\iota \|\phi_\iota - P_\iota \phi\|_\iota + \lim_\iota \|\phi_\iota - P_\iota \phi'\|_\iota = 0.$$

Definition. *Let* $(\mathfrak{g}_\iota) = (\mathrm{alg}\,(V, \mu_\iota))$ *be a net of Lie algebras contracting into* $\hat{\mathfrak{g}} = \mathrm{alg}\,(V, \hat{\mu})$ *and, for each* $\iota \in J$, *let* π_ι *be a representation of* $\mathfrak{g}_\iota$ *on a complex Hilbert space* $\mathfrak{H}_\iota$. *A representation* $\hat{\pi}$ *of* $\hat{\mathfrak{g}}$ *on a complex Hilbert space* $\mathfrak{H}$ *is said to be a* **Trotter limit of the net** (π_ι) *and, alternatively,* (π_ι) *is said to be* **strictly contracting into** $\hat{\pi}$ *if the following conditions are satisfied:*

(a) *For each* $\iota \in J$, *there exists a continuous linear mapping* P_ι *of* $\mathfrak{H}$ *into* $\mathfrak{H}_\iota$ *such that the net* $(\mathfrak{H}_\iota)$, *approximates* $\mathfrak{H}$ *with respect to the net* (P_ι).

(b) $\hat{\pi}(g) = (P_\iota) - \lim \pi_\iota(g)$ *for all* $g \in V$.

If $\hat{\mathfrak{g}}$ *is the contraction of a Lie algebra* $\hat{\mathfrak{g}} = \mathrm{alg}\,(V, \mu)$ *by means of a reference net* (Γ_ι) *of automorphisms of* V *and if, for each* $\iota \in J$, π_ι *is a representation of* $\hat{\mathfrak{g}}$ *on* $\mathfrak{H}_\iota$, *then* $\hat{\pi}$ *is said to be a Trotter limit of* (π_ι) *(and* (π_ι) *is said to be strictly contracting into* $\hat{\pi}$*) when condition (a) is satisfied together with*

(b′) $\hat{\pi}(g) = (P_\iota) - \lim \pi_\iota(\Gamma_\iota(g))$ *for all* $g \in V$.

More precisely, in both cases, $\hat{\pi}$ is said to be the *Trotter limit of* (π_ι) *relative to* (P_ι).

Proposition 1. *Let* $(\mathfrak{g}_\iota) = (\mathrm{alg}\,(V, \mu_\iota))$ *be a net of Lie algebras contracting into* $\hat{\mathfrak{g}} = \mathrm{alg}\,(V, \hat{\mu})$ *with respect to* $\mathfrak{g} = \mathrm{alg}\,(V, \mu)$ *and let* $\mathcal{G}$ *be a basis of* V. *For each* $\iota \in J$, *let* $\mathfrak{H}_\iota$ *be a complex Hilbert space of dimension* Card (S_ι), *where* S_ι *is a subset of* $\mathbf{R}$ *such that* $S_{\iota'} \subseteq S_\iota$ *whenever* $\iota' < \iota$, *let* π_ι *be a skew-symmetric representation of* $\mathfrak{g}_\iota$ *on* $\mathfrak{H}_\iota$, *let* $\{\phi_\iota^{(s)}\}_{s \in S_\iota}$ *be an orthonormal basis of* $\mathfrak{H}_\iota$ *contained in* $\mathrm{D}\,(\pi_\iota)$, *and, with* $S = \bigcup_\iota S_\iota$, *define* $\{\psi_\iota^{(s)}\}_{s \in S}$ *by* $\psi_\iota^{(s)} = \phi_\iota^{(s)}$ *whenever* $s \in S_\iota$ *and* $\psi_\iota^{(s)} = 0$ *otherwise. Suppose that for* $-k \leqslant m \leqslant k$, *where* k *is a fixed positive integer, for each* $s \in S$, *and each* $g \in \mathcal{G}$, *we have a net* $(c_{\iota,s,m}(g))$ *of complex numbers, which are* 0 *whenever* $s \notin S_\iota$ *or* $s + m \notin S_\iota$, *converging in* $\mathbf{C}$ *to* $c_{s,m}(g)$. *If*

$$\pi_\iota(g)\phi_\iota^{(s)} = \sum_{m=-k}^{k} c_{\iota,s,m}(g)\phi_\iota^{(s+m)} \tag{II.1}$$

for all $\iota \in J$, *all* $g \in \mathcal{G}$, *and all* $s \in S_\iota$, *then, up to unitary equivalence, there exists one and only one skew-symmetric representation* $\hat{\pi}$ *of* $\hat{\mathfrak{g}}$ *on a complex Hilbert space* $\mathfrak{H}$ *of dimension* Card (S) *which is the Trotter limit of the net* (π_ι) *relative to a net* (P_ι) *defined by*

$$P_\iota \psi^{(s)} = \psi_\iota^{(s)} \qquad (\iota \in J;\ s \in S), \tag{II.2}$$

where $\mathfrak{S}_J = \{\psi^{(s)}\}_{s \in S}$ *is some orthonormal basis of* $\mathfrak{H}$ *such that* $\mathrm{D}\,(\hat{\pi}) = \mathrm{sp}\,(\mathfrak{S}_J)$. *In addition, we have*

$$\hat{\pi}(g)\psi^{(s)} = \sum_{m=-k}^{k} c_{s,m}(g)\psi^{(s+m)} \tag{II.3}$$

for all $g \in \mathcal{G}$ *and all* $s \in S$.

Proof. To begin with, let us note that if $\hat{\pi}$ is the Trotter limit of (π_ι) relative to the net (P_ι) defined by (II.2), then (II.3) is satisfied because it follows from (II.1) that

$$\lim_\iota \left\| \pi_\iota(g)\psi_\iota^{(s)} - \sum_{m=-k}^{k} c_{s,m}(g)\psi_\iota^{(s+m)} \right\|_\iota \leqslant \sum_{m=-k}^{k} \lim_\iota |c_{\iota,s,m}(g) - c_{s,m}(g)| = 0$$

for all $g \in \mathcal{G}$ and all $s \in S$.

The existence of $\hat{\pi}$ with the stated properties was proven in ([8], Proposition 4). Conversely, suppose that together with $\hat{\pi}$ we have a skew-symmetric representation $\hat{\pi}'$ of $\hat{\mathfrak{g}}$ on a complex Hilbert space $\mathfrak{H}'$ of dimension Card (S) which is the Trotter limit of (π_ι) relative to a net (P'_ι) defined, in analogy with (P_ι), via an orthonormal basis $\mathfrak{S}'_J = \{\psi'^{(s)}\}_{s \in S}$. Then, by (II.3), the unitary mapping U of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $U\psi^{(s)} = \psi'^{(s)}$ for all $s \in S$ implies $\hat{\pi}(g) = U^{-1}\hat{\pi}'(g)U$ for all $g \in V$. ■

Remark 1. Alternatively, a representation $\hat{\pi}$ which is a suitable Trotter limit of the net (π_ι) can be constructed as follows. Let $\mathfrak{H}$ be $l^2_{\mathbf{C}}(S)$, the Hilbert space of all complex-valued functions f defined in S and satisfying

$$\sum_{s\in S} |f(s)|^2 < \infty$$

with scalar multiplication $(.|.)$ given by

$$(f \mid h) = \sum_{s\in S} f(s)\overline{h(s)}.$$

If $\mathfrak{S}_J = \{\psi^{(s)}\}_{s\in S}$ is the canonical orthonormal basis of $l^2_{\mathbf{C}}(S)$ such that $\psi^{(s)}(s') = \delta_{ss'}$, then P_ι is given by (II.2) and $\hat{\pi}$, defined by (II.3) with $D(\hat{\pi}) = \mathrm{sp}(\mathfrak{S}_J)$, is a representation with the desired properties.

Remark 2. If the relation (II.6) of [8] is satisfied, then the skew-symmetric representation $\hat{\pi}$ of Proposition 1 is integrable ([13], Theorem 1).

Remark 3. Suppose that, instead of the nets $(\mathfrak{g}_\iota)$, (π_ι), we have a Lie algebra $\mathfrak{g} = \mathrm{alg}(V, \mu)$, a contraction $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by means of a reference net (Γ_ι) of automorphisms of V, and, for each $\iota \in J$, a skew-symmetric representation π_ι of $\mathfrak{g}$ on a complex Hilbert space $\mathfrak{H}_\iota$ of dimension $\mathrm{Card}(S_\iota)$. Then the conclusions of Proposition 1 are still correct when $\pi_\iota \circ \Gamma_\iota$ replaces π_ι in (II.1) and all other assumptions are kept unchanged.

Remark 4. By virtue of ([8], Propositions 1 and 4), Proposition 1 (and its reformulation according to Remark 3) is still true if 'skew-symmetric' is replaced everywhere by 'symmetric' and even if the term 'skew-symmetric' is everywhere dropped, provided in this last case the nets $(\pi_\iota(g))$ satisfy Condition (K) of [8] for all $g \in V$.

III. Examples

In this section, we illustrate the previous theory by several examples, for which we shall verify the assumptions of Proposition 1. Hence, in each case, we are given a contracting net $(\mathfrak{g}_\iota) = (\mathrm{alg}(V, \mu_\iota))$ of Lie algebras, each $\mathfrak{g}_\iota$ being isomorphic to a reference Lie algebra $\mathfrak{g} = \mathrm{alg}(V, \mu)$, and a net $(\mathfrak{H}_\iota)$ of complex Hilbert spaces with, for each $\iota \in J$, a skew-symmetric representation π_ι of $\mathfrak{g}_\iota$ on $\mathfrak{H}_\iota$. In the examples treated here, we require that π_ι be, for each $\iota \in J$, an irreducible representation.

We shall denote by $\mathfrak{su}(2)$, $\mathfrak{e}(2)$, $\mathfrak{h}(1)$, $\mathfrak{r}^3$, and $\mathfrak{su}(1, 1)$, respectively, the Lie algebras of the (real) Lie groups $\mathbf{SU}(2)$, $E(2)$ (the Euclidean group of the plane), $H(1)$ (the 3-dimensional Heisenberg group), the additive group of $\mathbf{R}^3$, and $\mathbf{SU}(1, 1)$.

The following two particular notions concerning Lie algebra representations will be used. Given a connected finite-dimensional real Lie group G whose Lie algebra is isomorphic to a Lie algebra $\mathfrak{g}$ and given a skew-symmetric representation π of $\mathfrak{g}$ on a complex Hilbert space $\mathfrak{H}$, we shall say that π is *integrable to* G if there exist a (unique) strongly continuous unitary representation U of G on $\mathfrak{H}$ and an isomorphism θ of $\mathfrak{g}$ onto $\mathrm{Lie}(G)$ such that $\pi(g) \subseteq dU(\theta(g))$ for all $g \in \mathfrak{g}$. Two skew-symmetric representations π and π' of $\mathfrak{g}$ on $\mathfrak{H}$ which are integrable to G by means of strongly continuous unitary representations U and U', respectively, will be said to be *cognate* if $U = U'$.

In what follows, the symbols $\mathfrak{H}$ and $\mathfrak{S}_J$ (with the suitable J) will have the same meaning as in Proposition 1. In particular, $\mathfrak{H}$ and $\mathfrak{S}_J$ can be constructed as in ([8], Proposition 4).

For the sake of clarity, we divide this section, and then again Section III.1, into three parts. In Section III.1.1 (resp. Section III.1.2, resp. Section III.1.3) we take $\mathfrak{g}=\mathfrak{su}(2)$ and a contracted Lie algebra $\hat{\mathfrak{g}}\approx\mathfrak{e}(2)$ (resp. $\hat{\mathfrak{g}}\approx\mathfrak{h}(1)$, resp. $\hat{\mathfrak{g}}\approx\mathfrak{r}^3$) as an example of the case where $\mathfrak{g}$ is the Lie algebra of a compact Lie group, shortly, a compact Lie algebra. In Section III.2, we give an example where $\mathfrak{g}$ is the Lie algebra of a noncompact Lie group (namely, $\mathfrak{g}=\mathfrak{su}(1,1)$) and $\hat{\mathfrak{g}}$ is once more isomorphic to $\mathfrak{e}(2)$.

In each of the quoted examples, we start with nets of integrable faithful irreducible skew-symmetric representations and then we obtain integrable faithful irreducible skew-symmetric representations of $\hat{\mathfrak{g}}$, except for the cases treated in Section III.1.3 and in Remarks 5 and 6 in which the representations obtained are neither irreducible nor faithful.

By ([8], Corollary to Proposition 4), the results when $\hat{\mathfrak{g}}\approx\mathfrak{e}(2)$ may be interpreted as follows [5]. The Lie algebra $\mathfrak{e}(2)$ is isomorphic to the Lie algebra of the stabilizer of a light-like vector; its faithful irreducible skew-symmetric representations integrable to $E(2)$ are Trotter limits of nets of integrable irreducible skew-symmetric representations of the Lie algebra (isomorphic to $\mathfrak{su}(2)$) of the stabilizer of a time-like vector (resp. of the Lie algebra (isomorphic to $\mathfrak{su}(1,1)$) of the stabilizer of a space-like vector).

The case of $\mathfrak{g}=\mathfrak{su}(2)$ and $\hat{\mathfrak{g}}\approx\mathfrak{h}(1)$ is treated in greater detail, and more concretely, in Section III.3 because of the interest it has recently found in quantum optics regarding the connection between Bloch and Glauber coherent states (see, e.g., [6], Section IV). We prove, in particular, that the latter are limits of sequences of the former in Trotter's sense.

III.1. The compact case: $\mathfrak{g}=\mathfrak{su}(2)$

Let $\{g_1, g_2, g_3\}$ be a basis of the reference Lie algebra $\mathfrak{su}(2)=\operatorname{alg}(V,\mu)$ such that

$$\mu(g_1,g_2)=g_3,\qquad \mu(g_2,g_3)=g_1,\qquad \mu(g_3,g_1)=g_2.$$

Throughout Section III.1, the index set will be $J=\frac{1}{2}\mathbf{N}^*$ or $J=\mathbf{N}^*$. For each $l\in\frac{1}{2}\mathbf{N}^*$, the symbol $\mathfrak{H}_l$ will denote the complex Hilbert space of all polynomials of degree $\leq 2l$ in one complex variable which carries the standard irreducible skew-symmetric representation of dimension $2l+1$ of $\mathfrak{su}(2)$ ([14], Ch. III, §2). We choose a canonical orthonormal basis of $\mathfrak{H}_l$ that we shall denote by $\{\phi_l^{(s)}\}_{s\in S_l}$, where $S_l=\{-l,-l+1,\ldots,l\}$. For each $s\in S_l$ and each $m\in\{-1,0,1\}$, we put

$$c_{l,s,m}^{\mathfrak{su}(2)}(g_1)=\begin{cases}\frac{1}{2}i\sqrt{(l+s)(l-s+1)} & \text{if } m=-1\\ 0 & \text{if } m=0\\ \frac{1}{2}i\sqrt{(l-s)(l+s+1)} & \text{if } m=1,\end{cases}\tag{III.1a}$$

$$c_{l,s,m}^{\mathfrak{su}(2)}(g_2)=\begin{cases}-\frac{1}{2}\sqrt{(l+s)(l-s+1)} & \text{if } m=-1\\ 0 & \text{if } m=0\\ \frac{1}{2}\sqrt{(l-s)(l+s+1)} & \text{if } m=1,\end{cases}\tag{III.1b}$$

$$c_{l,s,m}^{\mathfrak{su}(2)}(g_3) = \begin{cases} -is & \text{if } m=0 \\ 0 & \text{otherwise.} \end{cases} \tag{III.1c}$$

III.1.1. $\hat{\mathfrak{g}} \approx \mathfrak{e}(2)$

For each $l \in \mathbf{N}^*$ and each $r \in \mathbf{R}_+^*$ (the set of strictly positive real numbers), let $\mathfrak{g}_l^{(r)} = \text{alg}\,(V, \mu_l^{(r)})$ be the Lie algebra isomorphic to $\mathfrak{su}(2)$ via the automorphism $\Gamma_l^{(r)}$ of V defined by

$$\Gamma_l^{(r)}(g_1) = (r/l)g_1, \qquad \Gamma_l^{(r)}(g_2) = (r/l)g_2, \qquad \Gamma_l^{(r)}(g_3) = g_3; \tag{III.2}$$

thus

$$\mu_l^{(r)}(g_1, g_2) = (r^2/l^2)g_3, \qquad \mu_l^{(r)}(g_2, g_3) = g_1, \qquad \mu_l^{(r)}(g_3, g_1) = g_2. \tag{III.3}$$

The contracted Lie algebra $\hat{\mathfrak{g}} = \text{alg}\,(V, \hat{\mu})$ of the sequence $(\mathfrak{g}_l^{(r)})_{l \in \mathbf{N}^*}$ has a Lie multiplication $\hat{\mu}$ satisfying

$$\hat{\mu}(g_1, g_2) = 0, \qquad \hat{\mu}(g_2, g_3) = g_1, \qquad \hat{\mu}(g_3, g_1) = g_2,$$

hence is isomorphic to $\mathfrak{e}(2)$. The contraction is an Inönü-Wigner one.

For each $l \in \mathbf{N}^*$ and each $r \in \mathbf{R}_+^*$, we define on $\mathfrak{H}_l$ an (integrable) irreducible skew-symmetric representation $\pi_l^{(r)}$ of $\mathfrak{g}_l^{(r)}$ by

$$\pi_l^{(r)}(g_j)\phi_l^{(s)} = \sum_{m=-1}^{1} c_{l,s,m}^{(r)}(g_j)\phi_l^{(s+m)} \qquad (j = 1, 2, 3;\ s \in S_l), \tag{III.4}$$

where

$$c_{l,s,m}^{(r)}(g_j) = (r/l)c_{l,s,m}^{\mathfrak{su}(2)}(g_j) \qquad (j = 1, 2), \tag{III.5a}$$

$$c_{l,s,m}^{(r)}(g_3) = c_{l,s,m}^{\mathfrak{su}(2)}(g_3). \tag{III.5b}$$

Moreover, we put $c_{l,s,m}^{(r)}(g_j) = 0$ $(j = 1, 2, 3;\ -1 \leq m \leq 1)$ for all $s \in \mathbf{Z} - S_l$. It follows from (III.5) that

$$c_{s,m}^{(r)}(g_j) = \lim_{l \to \infty} c_{l,s,m}^{(r)}(g_j) \qquad (j = 1, 2, 3;\ s \in \mathbf{Z};\ -1 \leq m \leq 1)$$

is given by

$$c_{s,m}^{(r)}(g_1) = \begin{cases} \tfrac{1}{2}ir & \text{if } m = -1 \text{ or } m = 1 \\ 0 & \text{if } m = 0, \end{cases} \tag{III.6a}$$

$$c_{s,m}^{(r)}(g_2 = \begin{cases} -\tfrac{1}{2}r & \text{if } m = -1 \\ 0 & \text{if } m = 0 \\ \tfrac{1}{2}r & \text{if } m = 1, \end{cases} \tag{III.6b}$$

$$c_{s,m}^{(r)}(g_3) = \begin{cases} -is & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{III.6c}$$

By virtue of (III.6), the relation

$$|c_{s,m}^{(r)}(g_j)| \leq (r+1)(|s|+1) \qquad (j = 1, 2, 3;\ r \in \mathbf{R}_+^*;\ s \in \mathbf{Z};\ -1 \leq m \leq 1)$$

is satisfied; hence, the sequence $(\pi_l^{(r)})$ is strictly contracting into an integrable skew-symmetric representation $\hat{\pi}^{(r)}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ with $D(\hat{\pi}^{(r)}) = \mathrm{sp}(\mathfrak{S}_{\mathbf{N}^*})$ (Proposition 1 and Remark 2). By (III.4) and (III.6), $\hat{\pi}^{(r)}$ is given explicitly (with $s \in S = \mathbf{Z}$ and $\psi^{(s)} \in \mathfrak{S}_{\mathbf{N}^*}$) by

$$\hat{\pi}^{(r)}(g_1)\psi^{(s)} = \tfrac{1}{2}ir\psi^{(s-1)} + \tfrac{1}{2}ir\psi^{(s+1)}, \tag{III.7a}$$

$$\hat{\pi}^{(r)}(g_2)\psi^{(s)} = -\tfrac{1}{2}r\psi^{(s-1)} + \tfrac{1}{2}r\psi^{(s+1)}, \tag{III.7b}$$

$$\hat{\pi}^{(r)}(g_3)\psi^{(s)} = -is\psi^{(s)} \tag{III.7c}$$

(cf. [14], Ch. IV, §2.3, where $R = ir$).

We obtain in this way, up to cognateness, representatives of all equivalence classes of faithful irreducible skew-symmetric representations of $\hat{\mathfrak{g}}$ which are integrable to $E(2)$ ([15], Korollar zu Satz 3).

If we replace in the above $J = \mathbf{N}^*$ by $J = \frac{1}{2}\mathbf{N}^* - \mathbf{N}^*$, and keep the rest unchanged, then, for each $r \in \mathbf{R}_+^*$, the net $(\pi_l^{(r)})$ is strictly contracting into a faithful irreducible skew-symmetric representation $\hat{\pi}'^{(r)}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ which is integrable to a 2-sheeted covering group of $E(2)$ but not to $E(2)$.

Remark 5. Even by changing the sequence $(\Gamma_l^{(r)})$ it is not possible to obtain, proceeding as above, the (non-faithful) 1-dimensional skew-symmetric representations of $\hat{\mathfrak{g}}$. On the other hand, if we put in (III.2) $\Gamma_l^{(r)}(g_j) = (r/l^2)g_j$ and in (III.5) $c_{l,s,m}^{(r)}(g_j) = (r/l^2)c_{l,s,m}^{\mathfrak{su}(2)}(g_j)$ for $j = 1, 2$, with the rest remaining unchanged, we get a direct sum of 1-dimensional skew-symmetric representations of $\hat{\mathfrak{g}}$ ([14], Ch. IV, §2.3).

III.1.2. $\hat{\mathfrak{g}} \approx \mathfrak{h}(1)$

In this case, for each $l \in \frac{1}{2}\mathbf{N}^*$, we define $\mathfrak{g}_l = \mathrm{alg}(V, \mu_l)$ via the automorphism Γ_l of V given by

$$\Gamma_l(g_1) = l^{-1/2}g_1, \qquad \Gamma_l(g_2) = l^{-1/2}g_2, \qquad \Gamma_l(g_3) = (1/l)g_3;$$

then

$$\mu_l(g_1, g_2) = g_3, \qquad \mu_l(g_2, g_3) = (1/l)g_1, \qquad \mu_l(g_3, g_1) = (1/l)g_2.$$

The contracted Lie algebra of the net $(\mathfrak{g}_l)$ is thus $\hat{\mathfrak{g}} = \mathrm{alg}(V, \hat{\mu})$, isomorphic to $\mathfrak{h}(1)$, where

$$\hat{\mu}(g_1, g_2) = g_3, \qquad \hat{\mu}(g_2, g_3) = 0, \qquad \hat{\mu}(g_3, g_1) = 0.$$

The contraction is a Lévy-Nahas one, however not a Saletan one ([4], Section II.E).

For each $l \in \frac{1}{2}\mathbf{N}^*$, we define an (integrable) irreducible skew-symmetric representation π_l of $\mathfrak{g}_l$ on $\mathfrak{H}_l$ by

$$\pi_l(g_j)\phi_l'^{(n)} = \sum_{m=-1}^{1} c'_{l,n,m}(g_j)\phi_l'^{(n+m)} \qquad (j = 1, 2, 3;\ n \in N_l), \tag{III.8}$$

where $N_l = \{0, 1, \ldots, 2l\}$, the orthonormal basis $\{\phi_l'^{(n)}\}_{n \in N_l}$ is defined by $\phi_l'^{(n)} = \phi_l^{(n-l)}$, and

$$c'_{l,n,m}(g_j) = l^{-1/2}c_{l,n-l,m}^{\mathfrak{su}(2)}(g_j) \qquad (j = 1, 2), \tag{III.9a}$$

$$c'_{l,n,m}(g_3) = (1/l)c_{l,n-l,m}^{\mathfrak{su}(2)}(g_3). \tag{III.9b}$$

We put $c'_{l,n,m}(g_j)=0$ $(j=1,2,3; -1\leq m\leq 1)$ for all $n\in \mathbf{N}-N_l$; then it follows from (III.9) that

$$c'_{n,m}(g_j)=\lim_{l\to\infty} c'_{l,n,m}(g_j) \qquad (j=1,2,3;\ n\in\mathbf{N};\ -1\leq m\leq 1),$$

with

$$c'_{n,m}(g_1)=\begin{cases} i2^{-1/2}\sqrt{n} & \text{if } m=-1\\ 0 & \text{if } m=0\\ i2^{-1/2}\sqrt{n+1} & \text{if } m=1,\end{cases} \tag{III.10a}$$

$$c'_{n,m}(g_2)=\begin{cases} -2^{-1/2}\sqrt{n} & \text{if } m=-1\\ 0 & \text{if } m=0\\ 2^{-1/2}\sqrt{n+1} & \text{if } m=1,\end{cases} \tag{III.10b}$$

$$c'_{n,m}(g_3)=\begin{cases} i & \text{if } m=0\\ 0 & \text{otherwise.}\end{cases} \tag{III.10c}$$

The relation

$$|c'_{n,m}(g_j)|\leq n+1 \qquad (j=1,2,3;\ n\in\mathbf{N};\ -1\leq m\leq 1) \tag{III.11}$$

being satisfied, the net (π_l) is strictly contracting into an integrable skew-symmetric representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ with $\mathrm{D}(\hat{\pi})=\mathrm{sp}(\mathfrak{S}_{\frac{1}{2}\mathbf{N}^*})$. By virtue of (III.8) and (III.10), $\hat{\pi}$ is given explicitly (with $n\in S=\mathbf{N}$ and $\psi'^{(n)}\in\mathfrak{S}_{\frac{1}{2}\mathbf{N}^*}$) by

$$\hat{\pi}(g_1)\psi'^{(n)}=i2^{-1/2}\sqrt{n}\psi'^{(n-1)}+i2^{-1/2}\sqrt{n+1}\psi'^{(n+1)}, \tag{III.12a}$$

$$\hat{\pi}(g_2)\psi'^{(n)}=-2^{-1/2}\sqrt{n}\psi'^{(n-1)}+2^{-1/2}\sqrt{n+1}\psi'^{(n+1)}, \tag{III.12b}$$

$$\hat{\pi}(g_3)\psi'^{(n)}=i\psi'^{(n)}. \tag{III.12c}$$

If we take $J=\mathbf{N}^*$, we obtain an equivalent result.

By (III.12), $\mathfrak{H}$ is isomorphic to the Fock space over $\mathbf{C}$, the Hilbert sum of a countable family of copies of $\mathbf{C}$, equipped with an irreducible representation with vacuum of the canonical commutation relations for one degree of freedom. Therefore, by von Neumann's uniqueness theorem, the representation $\hat{\pi}$ is a representative of the unique equivalence class, up to a 'phase factor' and cognateness, of nontrivial irreducible skew-symmetric representations of $\hat{\mathfrak{g}}$ which are integrable to $H(1)$.

III.1.3. $\hat{\mathfrak{g}}\approx\mathfrak{r}^3$

For each $l\in\frac{1}{2}\mathbf{N}^*$, the Lie algebra $\mathfrak{g}_l=\mathrm{alg}(V,\mu_l)$ is defined by means of the automorphism Γ_l of V given by $\Gamma_l(g_j)=(1/l)g_j$ $(j=1,2,3)$, so that

$$\mu_l(g_1,g_2)=(1/l)g_3, \qquad \mu_l(g_2,g_3)=(1/l)g_1, \qquad \mu_l(g_3,g_1)=(1/l)g_2.$$

Hence, the contracted Lie algebra $\hat{\mathfrak{g}}$ of the net $(\mathfrak{g}_l)$ is Abelian and is isomorphic to $\mathfrak{r}^3$. We define π_l as in Section III.1.2, but with coefficients $c'_{l,n,m}(g_j)$ related to $c^{\mathfrak{su}(2)}_{l,s,m}(g_j)$ by

$$c'_{l,n,m}(g_j)=(1/l)c^{\mathfrak{su}(2)}_{l,n-l,m}(g_j) \qquad (j=1,2,3;\ n\in N_l;\ -1\leq m\leq 1) \tag{III.13}$$

and equal zero if $n \in \mathbf{N} - N_l$. It follows from (III.13) that the assumptions of Proposition 1 and (III.11) are satisfied, and thus the net (π_l) is strictly contracting into an integrable non-faithful reducible skew-symmetric representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ with $\mathrm{D}(\hat{\pi}) = \mathrm{sp}(\mathfrak{S}_{\frac{1}{2}\mathbf{N}^*})$ defined by

$$\hat{\pi}(g_j) = 0 \qquad (j = 1, 2),$$

$$\hat{\pi}(g_3) = i\mathrm{Id}_{\mathfrak{H}}.$$

III.2. The noncompact case: $\mathfrak{g} = \mathfrak{su}(1, 1)$ *and* $\hat{\mathfrak{g}} \approx \mathfrak{e}(2)$

In Section III.2, $\{g_1, g_2, g_3\}$ will denote a basis of the reference Lie algebra $\mathfrak{su}(1, 1) = \mathrm{alg}(V, \mu)$ such that

$$\mu(g_1, g_2) = -g_3, \qquad \mu(g_2, g_3) = g_1, \qquad \mu(g_3, g_1) = g_2.$$

The irreducible strongly continuous unitary representations of $\mathbf{SU}(1, 1)$ that we shall consider are those of the principal series, labeled by a real number $l \neq 0$ and by a parameter ε taking the values 0 and $\frac{1}{2}$ ([14], Ch. VI, §2.7, where $\rho = l$). The index set will be $J = \mathbf{R} - \{0\}$ in both cases $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$.

For each $l \in J$ and each $r \in \mathbf{R}_+^*$, we define $\mathfrak{g}_l^{(r)} = \mathrm{alg}(V, \mu_l^{(r)})$ via the automorphism $\Gamma_l^{(r)}$ of V given by (III.2); then $\mu_l^{(r)}$ is given by (III.3) with a minus sign in the right-hand side of the first relation. The contracted Lie algebra $\hat{\mathfrak{g}} = \mathrm{alg}(V, \hat{\mu})$ of the net $(\mathfrak{g}_l^{(r)})$ identifies with the Lie algebra $\hat{\mathfrak{g}}$ of Section III.1.1. The contraction is an Inönü-Wigner one.

For each $l \in J$, let $\mathfrak{H}_l$ be the Hilbert space $L^2_{\mathbf{C}}(\mathbf{U}(1))$ of equivalence classes of Lebesgue square-integrable complex-valued functions on the circle group $\mathbf{U}(1)$ and let $\{\phi^{(s)}\}_{s \in \mathbf{Z}}$ be the basis of $L^2_{\mathbf{C}}(\mathbf{U}(1))$ defined, up to equivalence, by

$$\phi^{(s)}(\theta) = (2\pi)^{-1/2} \exp(-is\theta),$$

so that the set S_l of Proposition 1 is a copy of $\mathbf{Z}$. Then we define an integrable irreducible skew-symmetric representation $\pi_l^{(\varepsilon,r)}$ of $\mathfrak{g}_l^{(r)}$ on $L^2_{\mathbf{C}}(\mathbf{U}(1))$ ([14], Ch. VI, §2.3), with $\varepsilon \in \{0, \frac{1}{2}\}$ and $r \in \mathbf{R}_+^*$, by

$$\pi_l^{(\varepsilon,r)}(g_1)\phi^{(s)} = (r/2l)(s - \tfrac{1}{2} + il + \varepsilon)\phi^{(s-1)} - (r/2l)(s + \tfrac{1}{2} - il + \varepsilon)\phi^{(s+1)}, \tag{III.14a}$$

$$\pi_l^{(\varepsilon,r)}(g_2)\phi^{(s)} = -(r/2il)(s - \tfrac{1}{2} + il + \varepsilon)\phi^{(s-1)} - (r/2il)(s + \tfrac{1}{2} - il + \varepsilon)\phi^{(s+1)}, \tag{III.14b}$$

$$\pi_l^{(\varepsilon,r)}(g_3)\phi^{(s)} = -i(s + \varepsilon)\phi^{(s)}. \tag{III.14c}$$

It follows from (III.14) that the assumptions of Proposition 1 and Remark 2 are satisfied, so that we obtain an integrable skew-symmetric representation $\hat{\pi}^{(\varepsilon,r)}$ of $\hat{\mathfrak{g}}$ on $\alpha(L^2_{\mathbf{C}}(\mathbf{U}(1)))$ with $\mathrm{D}(\hat{\pi}^{(\varepsilon,r)}) = \mathrm{sp}(\mathfrak{S}_J)$, where α is the mapping of ([8], Remark 1). The representation $\hat{\pi}^{(\varepsilon,r)}$ has the same form as the representation $\hat{\pi}^{(r)}$ of Section III.1.1 if one replaces the coefficient s in (III.7c) by $s + \varepsilon$ and $\psi^{(s)}$ by $\alpha(\phi^{(s)})$; thus $\hat{\pi}^{(0,r)}$ is equivalent to $\hat{\pi}^{(r)}$ while $\hat{\pi}^{(1/2,r)}$ is equivalent to $\hat{\pi}'^{(r)}$.

Remark 6. Proceeding as in Remark 5, it is possible to obtain a direct sum of 1-dimensional skew-symmetric representations of $\hat{\mathfrak{g}}$.

Remark 7. The results of Section III.2 may also be obtained by using a suitable countable index set J' (e.g., $J' = \mathbf{N}^*$) instead of J.

III.3. Bloch and Glauber coherent states

In this section, we treat the example of Section III.1.2 in greater detail and from another point of view which emphasizes the notion of a Trotter limit. The directed system will be $J=\mathbf{N}^*$.

Let $\{|n\rangle\}_{n\in\mathbf{N}}$ be the standard orthonormal basis of the Fock space $\mathfrak{F}$ over $\mathbf{C}$ consisting of eigenvectors of the number operator a^*a, where a and a^* denote, respectively, the annihilation and creation operators satisfying

$$a\,|n\rangle=\sqrt{n}\,|n-1\rangle \qquad (n>0), \qquad a\,|\,0\rangle=0,$$
$$a^*\,|\,n\rangle=\sqrt{n+1}\,|\,n+1\rangle.$$

We can identify $\mathfrak{F}$ with the Hilbert space $\mathfrak{H}$ of Section III.1.2 by putting $|n\rangle=\psi'^{(n)}$ and

$$a=-2^{-1/2}(i\hat{\pi}(g_1)+\hat{\pi}(g_2)), \tag{III.15a}$$
$$a^*=-2^{-1/2}(i\hat{\pi}(g_1)-\hat{\pi}(g_2)). \tag{III.15b}$$

For each $N\in\mathbf{N}^*$, we consider the 2^N-dimensional complex Hilbert space $(\mathbf{C}^2)^{\otimes N}$ which can be interpreted, for instance, as the space of states of an assembly of N 2-level atoms. In what follows, we shall indulge in the usual abuse of language of calling 'states' the vectors of a space of states instead of reserving this name for the rays describing the pure states of the physical system considered. An orthonormal basis of $(\mathbf{C}^2)^{\otimes N}$ is given by all vectors of the form $|\phi^{\varepsilon_1}\rangle\otimes|\phi^{\varepsilon_2}\rangle\otimes\cdots\otimes|\phi^{\varepsilon_N}\rangle$, where $\varepsilon_j\in\{+,-\}$ for $1\leq j\leq N$ and

$$|\phi^+\rangle=\begin{pmatrix}1\\0\end{pmatrix}, \qquad |\phi^-\rangle=\begin{pmatrix}0\\1\end{pmatrix};$$

we set

$$|\phi^-\rangle\otimes|\phi^-\rangle\otimes\cdots\otimes|\phi^-\rangle=|0\rangle_N.$$

In addition, we define operators $S_N^{(k)}$ $(k=1,2,3)$ in $(\mathbf{C}^2)^{\otimes N}$ (the 'total spin' operators) by

$$S_N^{(k)}=\sum_{j=1}^{N}\mathbb{1}\otimes\cdots\otimes\tfrac{1}{2}\sigma_j^{(k)}\otimes\cdots\otimes\mathbb{1},$$

where $\sigma_j^{(k)}$ is a Pauli matrix (acting in the j-th factor), and we define vectors $|n\rangle_N$ $(n\in\mathbf{N})$ of $(\mathbf{C}^2)^{\otimes N}$ by

$$|n\rangle_N=\begin{cases}((N-n)!\,N^n/n!\,N!)^{1/2}(N^{-1/2}S_N^+)^n\,|0\rangle_N & \text{if } n\leq N\\ 0 & \text{if } n>N,\end{cases} \tag{III.16}$$

with $S_N^{\pm}=S_N^{(1)}\pm iS_N^{(2)}$.

The above definitions are motivated in part by the interpretation of the operators $N^{-1}S_N^{(k)}$ $(k=1,2,3)$ as 'intensive observables' and of the operators $N^{-1/2}S_N^{\pm}$ as 'fluctuation operators' (of $N^{-1}S_N^{\pm}$ around 0) [16] and in part by 'spin wave theory' [17].

It follows from (III.16) that, for each $N\in\mathbf{N}^*$,

$$_N\langle m \mid n\rangle_N = \begin{cases}\delta_{mn} & \text{if } m\leq N \text{ and } n\leq N\\ 0 & \text{otherwise.}\end{cases} \tag{III.17}$$

Let $\mathfrak{H}_N$ be the vector subspace of $(\mathbf{C}^2)^{\otimes N}$ generated by $\{|n\rangle_N\}_{n\in\mathbf{N}}$; then we have a continuous linear mapping P_N of $\mathfrak{F}$ onto $\mathfrak{H}_N$ defined in $\{|n\rangle\}_{n\in\mathbf{N}}$ by $P_N\,|n\rangle = |n\rangle_N$ and extended to $\mathfrak{F}$ by linearity and continuity. We shall denote by $S_N^{(k)}$ ($k = 1, 2, 3$), S_N^+, S_N^- also the restrictions to $\mathfrak{H}_N$ of these operators. The Hilbert space $\mathfrak{H}_N$ is $(N+1)$-dimensional and carries an irreducible skew-symmetric representation π_N of $\mathfrak{su}(2)$ given, in the basis $\{g_1, g_2, g_3\}$ of Section III.1, by

$$\pi_N(g_1) = \tfrac{1}{2}i(S_N^+ + S_N^-),$$
$$\pi_N(g_2) = \tfrac{1}{2}(S_N^+ - S_N^-),$$
$$\pi_N(g_3) = -iS_N^{(3)}.$$

Considered as a subspace of the space of states $(\mathbf{C}^2)^{\otimes N}$ of a system of N 2-level atoms, $\mathfrak{H}_N$ consists of all states with maximal 'total spin', namely $\frac{1}{2}N$. By virtue of (III.17), we have $\|P_N\| = 1$ for all $N\in\mathbf{N}^*$ and, for each element $\sum_{n=0}^{\infty}\gamma_n\,|n\rangle$ of $\mathfrak{F}$,

$$\lim_{N\to\infty}\left\|P_N\sum_{n=0}^{\infty}\gamma_n\,|n\rangle\right\|_N = \lim_{N\to\infty}\left(\sum_{n=0}^{N}|\gamma_n|^2\right)^{1/2} = \left\|\sum_{n=0}^{\infty}\gamma_n\,|n\rangle\right\|_{\mathfrak{F}}.$$

Thus the sequence $(\mathfrak{H}_N)$ approximates $\mathfrak{F}$ with respect to the sequence (P_N); obviously, $|n\rangle = (P_N)-\lim |n\rangle_N$ for all $n\in\mathbf{N}$.

Proposition 2. (i)

$$a = (P_N)-\lim N^{-1/2}S_N^-, \tag{III.18a}$$
$$a^* = (P_N)-\lim N^{-1/2}S_N^+, \tag{III.18b}$$
$$Id_{\mathfrak{F}} = (P_N)-\lim(-2N^{-1}S_N^{(3)}). \tag{III.18c}$$

(ii) *For each $z\in\mathbf{C}$, let $|z\rangle$ be the element of $\mathfrak{F}$ defined by*

$$|z\rangle = \exp(-\tfrac{1}{2}|z|^2)\exp(za^*)|0\rangle \tag{III.19}$$

and, for each $z\in\mathbf{C}$ and each $N\in\mathbf{N}^*$, let $|\Omega_z\rangle_N$ be the element of $\mathfrak{H}_N$ defined by

$$|\Omega_z\rangle_N = (1+|z|^2/N)^{-\frac{1}{2}N}\exp(zN^{-1/2}S_N^+)\,|0\rangle_N; \tag{III.20}$$

then $|z\rangle = (P_N)-\lim|\Omega_z\rangle_N$.

Proof. (i) By reason of (III.16), we have, for each $N\in\mathbf{N}^*$,

$$N^{-1/2}S_N^-\,|n\rangle_N = (N/(N-n+1))^{1/2}\sqrt{n}\,|n-1\rangle_N + \mathcal{O}_1(N^{-1}) \qquad (0<n\leq N),$$
$$N^{-1/2}S_N^+\,|n\rangle_N = ((N-n)/N)^{1/2}\sqrt{n+1}\,|n+1\rangle_N \qquad (n\in\mathbf{N}),$$
$$-2N^{-1}S_N^{(3)}\,|n\rangle_N = |n\rangle_N + \mathcal{O}_2(N^{-1}) \qquad (n\in\mathbf{N}),$$

where $\mathcal{O}_j(N^{-1})$ ($j = 1, 2$) stands for an element of $\mathfrak{H}_N$ such that the set $\{N\,\|\mathcal{O}_j(N^{-1})\|_N\}$ is uniformly bounded in N.

(ii) It follows from (III.19), (III.20), and (III.16) that

$$\lim_{N\to\infty} \|\,|\Omega_z\rangle_N - P_N\,|z\rangle\|_N$$

$$= \lim_{N\to\infty} \left\| \sum_{n=0}^{N} \left((1+|z|^2/N)^{-\frac{1}{2}N} \binom{N}{n}^{1/2} N^{-\frac{1}{2}n} z^n - \exp(-\tfrac{1}{2}|z|^2) n!^{-1/2} z^n \right) |n\rangle_N \right\|_N$$

$$= \left(\lim_{N\to\infty} \sum_{n=0}^{N} \left((1+|z|^2/N)^{-N} \binom{N}{n} N^{-n} |z|^{2n} + \exp(-|z|^2) n!^{-1} |z|^{2n} \right.\right.$$

$$\left.\left. -2(1+|z|^2/N)^{-\frac{1}{2}N} \exp(-\tfrac{1}{2}|z|^2) \binom{N}{n}^{1/2} N^{-\frac{1}{2}n} n!^{-1/2} |z|^{2n} \right) \right)^{1/2}$$

$$= \left(2 - 2\exp(-|z|^2) \lim_{N\to\infty} \sum_{n=0}^{N} \binom{N}{n}^{1/2} N^{-\frac{1}{2}n} n!^{-1/2} |z|^{2n} \right)^{1/2} \leqslant 0$$

because

$$\binom{N}{n}^{1/2} N^{-\frac{1}{2}n} n!^{-1/2} \geqslant \binom{N}{n} N^{-n}$$

for all $N \in \mathbf{N}^*$ and all positive integers $n \leqslant N$. ■

The vectors $|z\rangle$ of $\mathfrak{F}$ are the *Glauber coherent states* (for one degree of freedom) so important in quantum optics, and the vectors $|\Omega_z\rangle_N$ of $\mathfrak{H}_N$ are the *Bloch coherent states of radius* $\frac{1}{2}N$ with the 'north pole' of the Bloch sphere of radius $\frac{1}{2}N$ [9] removed.

Remark 8. For each $N \in \mathbf{N}^*$, the Hilbert space $\mathfrak{H}_N$ can be identified with the Hilbert space $\mathfrak{H}_{\frac{1}{2}N}$ of Section III.1.2 by putting $|n\rangle_N = \psi'^{(n)}_{\frac{1}{2}N}$, where $\psi'^{(n)}_{\frac{1}{2}N} = \phi'^{(n)}_{\frac{1}{2}N}$ if $n \leqslant N$ and $\psi'^{(n)}_{\frac{1}{2}N} = 0$ if $n > N$. Moreover, the representation π_N can be identified with the representation $\pi_{\frac{1}{2}N} \circ \Gamma^{-1}_{\frac{1}{2}N}$ of that section and so $\hat{\pi}$ becomes a Trotter limit of the net (π_N) (Remark 3). The contraction parameter l of Section III.1.2 may then be interpreted as a 'fluctuation parameter'; assertion (i) of Proposition 2 follows from Proposition 1 and from (III.15).

Remark 9. In order to show the connection between Bloch and Glauber coherent states, another contraction was considered in [6] and [18], namely, a contraction of the Lie algebra $\mathfrak{u}(2)$ of $\mathbf{U}(2)$ into a Lie algebra isomorphic to that of the harmonic oscillator group for one degree of freedom.

Let $\{g_j\}_{0 \leqslant j \leqslant 3}$ be a basis of $\mathfrak{u}(2)$ such that $\{g_1, g_2, g_3\}$ is the basis of $\mathfrak{su}(2)$ introduced in Section. III.1 and g_0 is in the center of $\mathfrak{u}(2)$. The reference net $(\Gamma_l)_{l \in \frac{1}{2}\mathbf{N}^*}$ of automorphisms of V defining the contraction studied in [6] and [18] can be given by

$$\Gamma_l(g_1) = l^{-1/2} g_1, \qquad \Gamma_l(g_2) = l^{-1/2} g_2, \qquad \Gamma_l(g_3) = g_3 - l g_0, \qquad \Gamma_l(g_0) = g_0.$$

The representation π_l of $\mathfrak{g}_l$ on $\mathfrak{H}_l$ ($l \in \frac{1}{2}\mathbf{N}^*$) is defined by (III.8) for $j = 1, 2$ and by

$$\pi_l(g_3)\phi'^{(n)}_l = -in\phi'^{(n)}_l, \qquad \pi_l(g_0)\phi'^{(n)}_l = i\phi'^{(n)}_l;$$

Proposition 1 can be applied and we obtain an integrable irreducible skew-symmetric representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ (with $\mathrm{D}(\hat{\pi}) = \mathrm{sp}(\mathfrak{S}_{\frac{1}{2}\mathbf{N}^*})$) given by (III.12a), (III.12b),

$$\hat{\pi}(g_3)\psi'^{(n)} = -in\psi'^{(n)},$$
$$\hat{\pi}(g_0)\psi'^{(n)} = i\psi'^{(n)}.$$

With the identifications of the present section, we have $a^*a = i\hat{\pi}(g_3)$; then assertion (i) of Proposition 2, with (III.18c) replaced by

$$a^*a = (P_N) - \lim \left(S_N^{(3)} + \tfrac{1}{2}N\, \mathrm{Id}_{\mathfrak{H}_N}\right),$$

follows from Proposition 1 while assertion (ii) is left unchanged.

To make contact with the notation of the literature, we remark that we have $|0\rangle_N = |\tfrac{1}{2}N, -\tfrac{1}{2}N\rangle$ in standard angular momentum notation. The elements $|n\rangle$ of $\mathfrak{F}$ are sometimes called the Fock states and the elements $|n\rangle_N$ of $\mathfrak{H}_N$ the Dicke states of radius $\tfrac{1}{2}N$. The coherent state $|\Omega_z\rangle_N$ (also denoted by $|\theta, \phi\rangle_N$) is the vector obtained by applying to the ground state $|0\rangle_N$ (which is identified with the 'south pole' of the Bloch sphere of radius $\tfrac{1}{2}N$ [9]) the operator that represents the rotation taking the direction of the south pole into the direction (θ, ϕ) such that

$$N^{-1/2}z = \mathrm{tg}\,(\tfrac{1}{2}\theta)\exp(-i\phi) \qquad (0 \le \theta < \pi;\ 0 \le \phi < 2\pi).$$

We see that the contraction consists here in letting the radius of the Bloch sphere tend to infinity in such a way that 'small rotations' on the sphere go over into translations in the tangent plane at the south pole. This plane corresponds to the phase space of the 1-dimensional harmonic oscillator ([9], Appendix A.2).

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