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Autor(en): **Gruber, G. / Wisskott, B.**

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Surface tension and phase transition for lattice systems. II

C. Gruber[1]) and **B. Wisskott**[2])

(1) Laboratoire de Physique Théorique
Ecole Polytechnique Fédérale de Lausanne
LAUSANNE, Switzerland

(2) Départment de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
LAUSANNE, Switzerland*)

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Abstract. The geometrical criterion previously given for the existence of phase transitions is shown to be valid for any spin $\frac{1}{2}$ ferromagnetic systems. This criterion appears in relation with a non-local observable, the 'surface tension', whose general properties are investigated.

The connection with other criteria, valid however only for $\mathbb{Z}^\nu$-invariant crystal lattices, is discussed.

1. Introduction

In a recent paper [1] (which we shall refer to (*I*) in this article), it was suggested to introduce the 'surface tension' as a possible definition of phase transitions, i.e. there exists a phase transition associated with surface tension τ if there exists some critical temperature T_c such that $\tau = 0$ for $T > T_c$ and $\tau \neq 0$ (or not defined) for $T < T_c$.

The interest of such a definition is that it is expected to coincide with the usual definition whenever the transition is associated with a local order parameter and may give an extension when the transition is associated with non local order parameters. In this connection we can mention a new result [2] which states that the surface tension is always zero when the spontaneous magnetization is zero for ferro-magnetic system with two body forces.

Now, one of the standard techniques to prove the existence of a local order parameter is to use the 'Peierls argument' to show the existence of a spontaneous magnetization m at low temperatures [3, 4]; however, there exists phase transitions associated with local order parameter for which $m = 0$ for all temperature. This case was studied carefully in [4] where it was shown that it is possible to reduce the system to a new system which will exhibit a spontaneous magnetization m^* if the original system has a transition associated with a local parameter. This

*) Present address: Department of Mathematics, Stanford University, STANFORD CA. 94305, USA.

discussion was however restricted to spin $\frac{1}{2}$ systems on $\mathbb{Z}^\nu$ with $\mathbb{Z}^\nu$-invariant interactions.

On the other hand the discussion of the phase transition using the surface tension (*I*) was rather general (arbitrary lattice and general phase space associated with each lattice sites); unfortunately it was necessary to introduce some conditions which correspond in fact to the condition of a spontaneous magnetization (or decomposition property in the terminology of [4]).

To be able to study the more general phase transitions (e.g. associated with a local order parameter other than the magnetization) we introduce in this paper a slightly modified definition of the surface tension. This new definition is exactly the dual definition of the Wilson loop integral introduced in Gauge Theories [5]; it differs from the previous definition in the fact that it involves two limiting procedures: first the volume of the system has to go to infinity, then the non-local observable associated with the dividing surface. (In the previous definition of the surface tension these limits were simultaneously taken). We should remark that in some explicit examples, it is known that both definitions coincide.

To simplify the following discussion we shall restrict ourselves to spin $\frac{1}{2}$ systems; however, some of the results have been extended to general systems [6] using the same group structure as in (*I*)

As in (*I*), we shall first derive general bounds for this new surface tension and prove that it is always zero at high temperature. We then show that for ferromagnetic systems, this surface tension is well defined, non-negative, bounded and decreases as the temperature increases; furthermore, we prove that the criterion given in (*I*) for the existence of phase transitions remains valid for systems which do not have the decomposition property.

The relation between our approach and the method introduced in [4] is expressed in terms of HT–HT duality transformation whose properties are briefly discussed. In the last section we compare our general criterion with the criterion given in [4] for $\mathbb{Z}^\nu$-invariant crystal lattices and prove the equivalence of these criteria for pair potentials; if we introduce the conjecture that the criteria are equivalent for systems with the decomposition property, they will remain equivalent for systems which do not have this decomposition property.

Finally, we shall discuss some examples which lead to the conjecture that a surface tension will always appear between distinct, quasi-periodic, equilibrium states.

2. Definitions and notations [3]

We recall that a '*Classical spin $\frac{1}{2}$ lattice system*' (or system) is defined by $\{\mathscr{L}, \mathscr{B}, K\}$ where

a) $\mathscr{L}$ is a countable subset of $\mathbb{R}^\nu$, consisting of elements called '*sites*'
b) $\mathscr{B}$ is a subset of $\mathscr{P}_f(\mathscr{L})$ (the set of all finite subsets of $\mathscr{L}$) consisting of elements B called '*bonds*'
c) K is a real function on $\mathscr{B}$, defining the '*interactions*', such that $K_B \neq 0$ for all B in $\mathscr{B}$.

The sets $\mathscr{P}(\mathscr{L}) = \{X; X \subset \mathscr{L}\}$ and $\mathscr{P}(\mathscr{B}) = \{\beta; \beta \subset \mathscr{B}\}$ are commutative groups for the product defined by the symmetric difference. Furthermore $\mathscr{P}(\mathscr{B})$ is a graph for the incidence relation (B_1, B_2) if $B_1 \cap B_2 \neq \phi$. We denote by $|X|$ and $|\beta|$ the cardinality of X and β.

The functions σ_Y, $Y \in \mathscr{P}_f(\mathscr{L})$, and γ, on $\mathscr{P}(\mathscr{L})$ are defined by:

i) $\sigma_Y(X): \mathscr{P}(\mathscr{L}) \to \{-1, +1\} \qquad \sigma_Y(X) = (-1)^{|Y \cap X|}$
ii) $\gamma: \mathscr{P}(\mathscr{L}) \to \mathscr{P}(\mathscr{B}) \qquad \gamma(X) = \{B \in \mathscr{B}; \sigma_B(X) = -1\}$

and we introduce the subgroups of $\mathscr{P}(\mathscr{B})$ and $\mathscr{P}(\mathscr{L})$ given by:

$$\Gamma = \mathscr{I}m\gamma \qquad \Gamma_f = \Gamma \cap \mathscr{P}_f(\mathscr{B}) \qquad \Gamma^{(f)} = \mathscr{I}m(\gamma\,|_{\mathscr{P}_f(\mathscr{L})})$$

$$\mathscr{K} = \left\{\mathscr{X} \in \mathscr{P}(\mathscr{B});\ \prod_{B \in \mathscr{X}} \sigma_B(X) = 1 \quad \forall X \in \mathscr{P}_f(\mathscr{L})\right\}$$

$$\mathscr{K}_f = \mathscr{K} \cap \mathscr{P}_f(\mathscr{B})$$

$$\mathscr{S} = \text{Ker}\ \gamma = \{S \subset \mathscr{L};\ \sigma_B(S) = 1 \quad \forall B \in \mathscr{B}\} \qquad \mathscr{S}_f = \mathscr{S} \cap \mathscr{P}_f(\mathscr{L})$$

Let $\Lambda \in \mathscr{P}_f(\mathscr{L})$ and $Y \subset \mathscr{L}$; the 'Equilibrium states of the finite system Λ with boundary condition Y' are defined by the probability measure $p_\Lambda^{(Y)}$ on $\mathscr{P}(\Lambda)$

$$p_\Lambda^{(Y)} = Z(\Lambda, Y)^{-1} e^{-(1/kT)H_{\Lambda,Y}} \quad {}^{*)}$$

$$\frac{1}{kT} H_{\Lambda,Y}(X) = -\sum_{B \in \mathscr{B}_\Lambda} K_B \sigma_B(Y_{\Lambda^c} X)$$

$$\mathscr{B}_\Lambda = \{B \in \mathscr{B};\ B \cap \Lambda \neq \phi\} \qquad \Lambda^c = \mathscr{L} \setminus \Lambda \qquad Y_{\Lambda^c} = Y \cap \Lambda^c$$

$$Z(\Lambda, Y) = \sum_{X \subset \Lambda} \prod_{B \in \mathscr{B}_\Lambda} e^{K_B \sigma_B(YX)}$$

The surface tension will be introduced by means of the function $\mu_\Lambda^{(+)}$ on $\mathscr{P}(\mathscr{B}_\Lambda)$

$$\mu_\Lambda^{(+)}(\beta) = \left\langle \prod_{B \in \beta} e^{-2K_B \sigma_B} \right\rangle_{(\Lambda,+)} = \sum_{X \subset \Lambda} p_\Lambda^{(+)}(X) \prod_{B \in \beta} e^{-2K_B \sigma_B(X)}$$

which admits the following expansions [3]:

High temperature (HT)-*expansion*

$$\mu_\Lambda^{(+)}(\beta) = \frac{\sum_{\bar\beta \in \mathscr{K}_\Lambda} \sigma_\beta(\bar\beta) \cdot \prod_{B \in \bar\beta} \text{th}\, K_B}{\sum_{\bar\beta \in \mathscr{K}_\Lambda} \prod_{B \in \bar\beta} \text{th}\, K_B}$$

where $\mathscr{K}_\Lambda = \{\kappa \in \mathscr{P}(\mathscr{B}_\Lambda);\ \prod_{B \in \kappa} \sigma_B(X) = 1\ \forall X \subset \Lambda\} \subset \mathscr{P}(\mathscr{B}_\Lambda)$ is the HT group.

Low Temperature (LT)-*expansion*

$$\mu_\Lambda^{(+)}(\beta) = \frac{\sum_{\bar\beta : \beta\bar\beta \in \Gamma^{(\Lambda)}} \prod_{B \in \bar\beta} e^{-2K_B}}{\sum_{\bar\beta \in \Gamma^{(\Lambda)}} \prod_{B \in \bar\beta} e^{-2K_B}}$$

where $\Gamma^{(\Lambda)} = \{\gamma(X);\ X \subset \Lambda\} = \mathscr{I}m(\gamma)_{\mathscr{P}(\Lambda)} \subset \mathscr{P}(\mathscr{B}_\Lambda)$ is the LT group.

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*) For $Y = \phi$ we write $p_\Lambda^{(+)}$.

To discuss thermodynamic limits we shall consider only sequences $\{\Lambda_i\}$ converging to $\mathscr{L}$ in such a manner that $\Lambda_{i+1} \supset \Lambda_i$ and we define the function $\mu^{(+)}$ on $\mathscr{P}_f(\mathscr{B})$ by

$$\mu^{(+)}(\beta) = \lim_{\Lambda_i \to \mathscr{L}} \mu^{(+)}_{\Lambda_i}(\beta)$$

(we know the limit exists at least for appropriate sequences of $\{\Lambda_i\}$).

Notation: We shall denote by the same symbol Λ a domain in $\mathbb{R}^\nu$ and its intersection with $\mathscr{L}$.

3. Surface tension

3.1. *Definition and general properties*

To introduce the surface tension between the states $p^{(+)}$ and $p^{(S)}$, $S \in \mathscr{S}$, $S \neq \phi$, we decompose $\mathscr{L}$ into $\mathscr{L}^u$ and $\mathscr{L}^d$

$$\mathscr{L}^u = \{x \in \mathscr{L};\ x_\nu > 0\}$$

$$\mathscr{L}^d = \mathscr{L} \setminus \mathscr{L}^u$$

and we consider parallelipipeds Λ with sides $(L_1, \ldots, L_{\nu-1}, 2\mathscr{M})$ symmetric with respect to the plane $x_\nu = 0$ (we consider only infinite lattices $\mathscr{L}$ such for all $\Lambda \subset \mathbb{R}^\nu$ finite $|\mathscr{L} \cap \Lambda^c| = \infty$).

Definition. The 'Surface tension' $\hat{\tau}^{(+,S)}$ between the states $p^{(+)}$ and $p^{(S)}$ is defined by:

$$\hat{\tau}^{(+,S)} = -\operatorname*{Lim}_{L_i \to \infty} \operatorname*{Lim}_{M \to \infty} \frac{1}{\Pi L_i} \operatorname{Log} \mu^{(+)}(\beta_\Lambda)$$

$$\beta_\Lambda = \gamma(S_d) \cap \mathscr{B}_\Lambda \qquad S_d = S \cap \mathscr{L}^d$$

For a discussion of this definition, we refer to (*I*) where the surface tension was introduced using thermodynamics analogies and where it was given by:

$$\tau^{(+,S)} = -\operatorname*{Lim}_{L_i \to \infty} \operatorname*{Lim}_{M \to \infty} \frac{1}{\Pi L_i} \operatorname{Log} \mu^{(+)}_\Lambda(\beta_\Lambda)$$

(for lack of better name we shall still call $\hat{\tau}^{(+,S)}$ surface tension).

Theorem 1. *If the surface tension $\hat{\tau}^{(+,S)}$ exists, it satisfies the following inequality:*

$$|\hat{\tau}^{(+,S)}| \leq 2\bar{K} C^{(+,S)}$$

$$\bar{K} = \sup_{B \in \mathscr{B}} |K_B|$$

$$C^{(+,S)} = \operatorname*{Lim\,sup}_{\Lambda \to \mathscr{L}} \frac{1}{\Pi L_i} C_\Lambda \qquad C_\Lambda = \min_{\gamma \in \Gamma^{(f)}} |\beta_\Lambda \cdot \gamma|$$

Proof. Let $\Lambda \in \mathcal{P}_f(\mathcal{L})$, $\beta_\Lambda = \gamma(S_d) \cap \mathcal{B}_\Lambda$ and $\underline{\Lambda} \in \mathcal{P}_f(\mathcal{L})$ such that $\underline{\Lambda} \supset \Lambda$. From the H.T. expansion it follows immediately that

$$\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda) = \mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda \gamma) \qquad \forall \gamma \in \Gamma^{(\Delta)}$$

Therefore

$$e^{-2\bar{K}|\beta_\Lambda \cdot \gamma|} \leq \mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda \gamma) \leq e^{+2\bar{K}|\beta_\Lambda \cdot \gamma|}$$

implies

$$|\mathrm{Log}\, \mu^{(+)}(\beta_\Lambda \gamma)| \leq 2\bar{K} \min_{\gamma \in \Gamma^{(f)}} |\beta_\Lambda \cdot \gamma|$$

which concludes the proof.

Remarks

1) The constant $C^{(+,S)}$ is a geometrical constant independent of temperature.
2) It may happen that $C^{(+,S)} = 0$ (see example), i.e. $\hat{\tau}^{(+,S)} = 0$ for all temperatures. This result may indicate that the state $p^{(S)}$ is identical with the state $p^{(+)}$. Furthermore, if $C^{(+,S)} = 0$ for all S in $\mathcal{S}$ there will be no phase transition associated with surface tension.
   In the rest of this section we consider '*ferromagnetic systems*', i.e. systems such that $K_B > 0$ for all B in $\mathcal{B}$, with *finite range interaction*, i.e. diam $B \leq R < \infty$ for all B. Furthermore, the system has '*finite density of sites*' if there exists some $\delta > 0$ such that $|x - x'| \geq \delta$ for all x, x' in $\mathcal{L}$.

Theorem 2. *Let $\{\mathcal{L}, \mathcal{B}, \underline{K}\}$ be any ferromagnetic system with finite density, finite range interactions and $\bar{K} < \infty$. For any S in $\mathcal{S}$ the surface tension $\hat{\tau}^{(+,S)}$ exists, is non-negative, bounded above, and is a decreasing function of the temperature.*

Proof. For ferromagnetic systems we have the following 'Griffith's inequality'

$$\mu^{(+)}(\beta_1\beta_2) - \mu^{(+)}(\beta_1)\mu^{(+)}(\beta_2) \geq 0$$

Indeed, using successively the HT and LT-expansions, we have:

$$\begin{aligned}
\mu^{(+)}(\beta_1\beta_2) - \mu^{(+)}(\beta_1)\mu^{(+)}(\beta_2) &= A^2 \sum_{\bar{\beta} \in \mathcal{K}_\Lambda} \sum_{\beta' \in \mathcal{K}_\Lambda} \\
&\quad \times [\sigma_{\beta_1\beta_2}(\bar{\beta}) - \sigma_{\beta_1}(\bar{\beta})\sigma_{\beta_2}(\beta')] \prod_{B \in \bar{\beta}} \mathrm{th}\, K_B \prod_{B \in \beta'} \mathrm{th}\, K_B \\
&= A^2 \sum_{\beta' \in \mathcal{K}_\Lambda} \sum_{\bar{\beta} \in \mathcal{K}_\Lambda} [1 - \sigma_{\beta_2}(\bar{\beta}\beta')]\sigma_{\beta_1\beta_2}(\bar{\beta}) \\
&\quad \times \prod_{B \in \bar{\beta}\beta'} \mathrm{th}\, K_B \prod_{B \in \bar{\beta}} (\mathrm{th}\, K_B)^{[1+\sigma_B(\beta')]1/2} \\
&= B^2 \sum_{\beta'' \in \mathcal{K}_\Lambda} [1 - \sigma_{\beta_2}(\beta'')] \prod_{B \in \beta''} \mathrm{th}\, K_B \\
&\quad \times \left[\sum_{\bar{\beta} : \beta_1\beta_2\bar{\beta} \in \Gamma_\Lambda} \prod_{B \in \bar{\beta}} e^{-2K''_B} \right] \geq 0
\end{aligned}$$

with

$$K''_B = K_B[1-\sigma_B(\beta'')]\tfrac{1}{2}$$

The existence of the limit

$$\lim_{L_i\to\infty}\lim_{M\to\infty}\frac{1}{\Pi L_i}\operatorname{Log}\mu^{(+)}(\beta_\Lambda)$$

is then established using the conditions on the systems and the subadditivity arguments of ref. [7].

Using the HT-expansion we see that $\mu_\Lambda^{(+)}(\beta)\leqslant 1$ for ferromagnetic systems and thus $\hat{\tau}^{(+,S)}\geqslant 0$. Finally

$$\mu^{(+)}(\beta_\Lambda)=\left\langle \prod_{\substack{B\in\mathscr{B}_\Lambda\\ \sigma_B(S_d)=-1}} e^{-2K_B\sigma_B}\right\rangle_{(+)} \geqslant e^{-2\bar{K}C_\Lambda'}$$

where C'_Λ is the number of bonds which intersect simultaneously $\mathscr{L}^u$ and $\mathscr{L}^d$.

For finite range interactions and finite density of sites we thus have:

$$0\leqslant\hat{\tau}^{(+,S)}\leqslant 2\bar{K}C'$$

$$C'=\sup_\Lambda\frac{c'_\Lambda}{\Pi L_i}=\text{maximum density of bonds crossing the plane } x_\nu=0.$$

Finally, using the same technique as above (see also the lemma Sec. 3.3.), we have the following Griffith inequality:

$$\mu^{(+)}_{(\Lambda;K^{(2)})}(\beta)\geqslant\mu^{(+)}_{(\Lambda;K^{(1)})}(\beta)\quad\text{if}\quad K_B^{(1)}\geqslant K_B^{(2)}>0\ \forall B,$$

which concludes the proof.

3.2. *Surface tension at high temperature*

Theorem 3. *Let $\{\underline{\mathscr{L}},\mathscr{B},\mathscr{K}\}$ be any system with finite denisty of sites, finite range interactions and $\bar{K}<\infty$. For any $S\in\mathscr{S}$, there exists a temperature T_0 such that*

$$\hat{\tau}^{(+,S)}=0\qquad\text{for all } T\geqslant T_0.$$

Proof. Let Λ be a parallelipiped with sides $(L_1,\ldots,L_{\nu-1},2M)$, $\beta_\Lambda=\gamma(S_d)\cap\mathscr{B}_\Lambda$, and $\underline{\Lambda}\in\mathscr{P}_f(\mathscr{L})$ such that $\underline{\Lambda}\supset\Lambda$.

To establish this theorem, we make use of the HT-expansion of $\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda)$; we note first that any graph $\bar{\beta}$ in $\mathscr{K}_\Lambda$ can be uniquely decomposed as union of connected graphs which are also in $\mathscr{K}_{\underline{\Lambda}}$.

As usual, we introduce the space $\mathscr{K}_{\underline{\Lambda}}=\{\xi=(\beta_1,\ldots,\beta_q);\ q\in\mathbb{N},\ \beta_i\subset\mathscr{B}_{\underline{\Lambda}}$ connected graph$\}$ and the function $G:\chi_{\underline{\Lambda}}\to\{0,1\}$ defined by:

$$G(\{\beta_1,\ldots,\beta_q\})=\prod_{i\neq j}g(\beta_i,\beta_j)$$

$$g(\beta_i,\beta_j)=\begin{cases}0 & \text{if } \beta_i \text{ is connected to } \beta_j\\ 1 & \text{otherwise}\end{cases}$$

to obtain:

$$\sum_{\bar\beta\in\mathcal{K}_{\underline{\Delta}}} \sigma_{\beta_\Lambda}(\bar\beta) \prod_{B\in\bar\beta} \text{th } K_B = \sum_{\xi\in\mathcal{K}_{\underline{\Delta}}} G(\xi) \prod_{\beta\in\xi} [\phi(\beta)\sigma_{\beta_\Lambda}(\beta)]$$

$$= \exp\left[\sum_{\xi\in\mathcal{K}_{\underline{\Delta}}} G_T(\xi) \prod_{B\in\xi} [\phi(\beta)\sigma_{\beta_\Lambda}(\beta)]\right]$$

where $G_T(\xi)$ is the truncated G-function [8], and

$$\phi(\beta) = \begin{cases} \prod_{B\in\beta} \text{th } K_B & \text{if } \beta\in\mathcal{K}_{\underline{\Delta}} \\ 0 & \text{otherwise} \end{cases}$$

We thus have:

$$\text{Log } \mu_{\underline{\Delta}}^{(+)}(\beta_{\underline{\Delta}}) = \sum_{\xi\in\mathcal{K}_{\underline{\Delta}}} G_T(\xi)\left[\prod_{\beta\in\xi} \sigma_{\beta_\Lambda}(\beta) - 1\right] \prod_{\beta\in\xi} \phi(\beta)$$

Let us take $\underline{\Delta}$ such that B in $\mathcal{B}_\Lambda$ implies $B\subset\underline{\Delta}$; then $\beta\in\mathcal{K}_{\underline{\Delta}}\cap\mathcal{P}(\mathcal{B}_\Lambda)$ implies $\beta\in\mathcal{K}_f$ and

$$\sigma_\beta(\beta_\Lambda) = \sigma_\beta(\gamma(S_d)) = 1$$

which yields:

$$\text{Log } \mu_{\underline{\Delta}}^{(+)}(\beta_\Lambda) = \sum_{\substack{\xi\in\mathcal{K}_\Lambda \\ [\xi]\cap\beta_\Lambda\neq\phi \\ [\xi]\cap\partial\Lambda\neq\phi}} G_T(\xi)\left[\prod_{\beta\in\xi} \sigma_{\beta_\Lambda}(\beta) - 1\right] \prod_{\beta\in\xi} \phi(\beta)$$

where $[\xi]\cap\beta_\Lambda\neq\phi$ means there exists β in ξ such that $\beta\cap\beta_\Lambda\neq\phi$
$[\xi]\cap\partial\Lambda\neq\phi$ means there exists β in ξ such that $\beta\notin\mathcal{P}(\mathcal{B}_\Lambda)$

and

$$|\text{Log } \mu_{\underline{\Delta}}^{(+)}(\beta_\Lambda)| \leq 2 \sum_{\substack{\xi\in\mathcal{K}_{\underline{\Delta}} \\ [\xi]\cap\beta_\Lambda\neq\phi \\ [\xi]\cap\partial\Lambda\neq\phi}} |G_T(\xi)|\,(\text{th } \bar K)^{|\xi|}$$

$$|\xi| = \sum_{\beta\in\xi} |\beta|$$

$$|\text{Log } \mu_{\underline{\Delta}}^{(+)}(\beta_\Lambda)| \leq 2\left\{\sum_{d=0}^{M'} \sum_{\substack{x\in\partial\Lambda \\ |x_\omega|\in[d\delta,(d+1)\delta[}} \sum_{\substack{\xi\in\mathcal{K}_{\underline{\Delta}} \\ [\xi]\ni x \\ \text{diam}[\xi]>d'}} |G_T(\xi)|\,(\text{th } \bar K)^{|\xi|}\right\}$$

$$M' = \frac{1}{\delta}(M+R) \qquad d' = d\delta$$

where $[\xi]\ni x$ means there exists $\beta\in\xi$ and $B\in\beta$ such that $B\ni x$.

$$\text{diam}[\xi] = \text{diam}\left(\bigcup_{B\in\xi} B\right); \partial\Lambda = \{x\in\Lambda^c;\, \gamma(x)\cap\mathcal{P}(\mathcal{B}_\Lambda)\neq\phi\}$$

Using Lemma 2 of (*I*) we know that there exist $z_0 \in]0, 1]$ such that

$$\sup_{x\in\mathcal{L}} \sum_{\substack{\xi\in\mathcal{K}_\Lambda \\ [\xi]\ni x \\ \operatorname{diam}[\xi]>d}} z_0^{|\xi|}\,|G_T(\xi)| \leqslant (C_1 + C_2 d^\nu) e^{-C_3\sqrt{d}}$$

Furthermore, since $th\bar{K} \to 0$ as $T\to\infty$, there exists some T_0 such that $th\bar{K} \leqslant z_0$; therefore if $T \geqslant T_0$

$$\begin{aligned}
|\operatorname{Log} \mu_\Lambda^{(+)}(\beta_\Lambda)| &\leqslant 2\left\{\sum_{d=0}^{M'} \sum_{\substack{x\in\partial\Lambda \\ |x_\omega|\in[d\delta,(d+1)\delta[}} (C_1 + C_2 d'^\nu) e^{-C_3\sqrt{d'}}\right\} \\
&\leqslant 2\delta^{-\omega} R \prod_{i=1}^{\nu-1} L_i \left\{4 \sum_{j=1}^{\nu-1} \frac{1}{L_j} \sum_{d=0}^{\infty} (C_1 + C_2\delta^\nu d^\nu) e^{-C_3\sqrt{\delta d}} \right. \\
&\qquad \left. + 2(C_1 + C_2(M+R)^\nu) e^{-C_3\sqrt{M+R}}\right\} \\
&= \prod_{i=1}^{\nu-1} L_i f(L_i, M)
\end{aligned}$$

where $f(L_i, M)$ is a bounded function which tends to zero as $\{L_i\}$ and M tend to ∞; therefore

$$|\hat{\tau}^{(+,S)}| \leqslant \operatorname*{Lim}_{L_i\to\infty} \operatorname*{Lim}_{M\to\infty} f(L_i, M) = 0$$

3.3. *Surface tension at low temperature*

To establish the low temperature properties we need the following definitions:

Let $\{\kappa^{(j)}\}_{j\in J} \subset \mathcal{K}_f$ be a family of generators for $\mathcal{K}_f$, i.e.

$$\sigma_\beta(\kappa^{(j)} = 1)\ \forall j \in J \Leftrightarrow \beta \in \Gamma$$

The 'dual graph' structure on $\mathcal{P}(\mathcal{B})$ is defined by the incidence relation '$[B_1, B_2]$ if there exists $j \in J$ such that $\kappa^{(j)} \supset \{\beta_1, \beta_2\}$'.

Let us then note that any element β in $\Gamma^{(\Delta)}$ can be uniquely decomposed into *-connected component in $\Gamma \cap \mathcal{P}(\mathcal{B}_\Delta)$; indeed if $\beta = \gamma(X) = \beta' \cup \beta''$ with $X \subset \underline{\Lambda}$ and β', β'' *-disconnected then for all j in J

$$\sigma_\beta(\kappa^{(j)}) = 1 = \sigma_{\beta'}(\kappa^{(j)}) = \sigma_{\beta''}(\mathcal{K}^{(j)})$$

and thus β' and β'' are in $\Gamma \cap \mathcal{P}(\mathcal{B}_\Delta)$.

Theorem 4. *Let $\{L, \mathcal{B}, K\}$ be any ferromagnetic system with finite density of sites, finite range interactions such that $\underline{K} = \inf_{B\in\mathcal{B}} K_B > 0$, and such that:*

i) *There exists a sequence of finite domains $\Lambda_i \to \mathcal{L}$ such that $\Gamma \cap \mathcal{P}(\mathcal{B}_{\Lambda_i}) = \Gamma^{(\Lambda_i)}$*
ii) *There exists a family of generators $\{\kappa^{(j)}\}$ for $\mathcal{K}_f$ such that $\sup_{B\in\mathcal{B}} |\{j \in J; \kappa^{(j)} \ni B\}| = a < \infty$.*

Then, for any S in $\mathcal{S}$ there exists a temperature T_0' such that

$$\hat{\tau}^{(+,S)} \geqslant AC^{(+,S)} \qquad \text{for all } T < T_0'$$

where $C^{(+,S)}$ is the geometrical constant introduced in Theorem 1 and A is a positive constant.

Remarks

1) The condition (i) is exactly the condition which is needed for Peierls argument [3, 4]. It is equivalent to $\Gamma^{(f)}=\Gamma_f$ and $\Gamma^{(f)}\cap\mathcal{P}(\mathcal{B}_{\Lambda_i})=\Gamma^{(\Lambda_i)}$. This condition (i) is expected to be always satisfied for $\mathbb{Z}^\nu$-invariant systems with $\Gamma^{(f)}=\Gamma_f$; such a result will be established in Sec. 6 for $\mathcal{L}=\mathbb{Z}^\nu$.
In the following we shall give a generalization of this theorem where condition (i) can be dropped.
2) The condition (ii) is necessary because we have no assumption on $\mathcal{L}$ and $\mathcal{B}$; in particular it is known to be verified for $\mathbb{Z}^\nu$-invariant systems with $\mathcal{L}=\mathbb{Z}^\nu$.
3) Conditions (i) and (ii) are the conditions introduced in (*I*),

Proof of Theorem 4

Let Λ be a parallelipiped with sides $(L_1,\ldots,L_{\nu-1},2M)$, $\beta_\Lambda=\gamma(S_d)\cap\mathcal{B}_\Lambda$, and $\underline{\Lambda}$ be a domain of the sequence introduced in (i) such that $\underline{\Lambda}\supset\Lambda$. If $C^{(+,S)}=0$ it is trivially true (Theorem 2); let us then assume $C^{(+,S)}>0$.

Using the condition (i) we see that any element β in $\Gamma^{(\underline{\Lambda})}$ can be uniquely decomposed into *-connnected components in

$$\Gamma\cap\mathcal{P}(\mathcal{B}_{\underline{\Lambda}})=\Gamma^{(\underline{\Lambda})}$$

Furthermore, any graph $\beta\in\mathcal{P}(\mathcal{B}_{\underline{\Lambda}})$ can be decomposed into *-connected components which we write as:

$$\beta=\bigcup_{i=1}^{q}\bar{\beta}_i\bigcup_{j=1}^{r}\tilde{\beta}_d$$

where

$\bar{\beta}_i$ is *-connected to β_Λ

$\tilde{\beta}_j$ is not *-connected to β_Λ

The LT-expansion of $\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda)$ yields:

$$\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda)=\frac{\displaystyle\sum_{\substack{\beta=(\bar{\beta}_1\cdots\bar{\beta}_q,\tilde{\beta}_1\cdots\tilde{\beta}_r)\\(\pi\bar{\beta}_i)\beta_\Lambda\in\Gamma^{(\underline{\Lambda})};\tilde{\beta}_i\in\Gamma^{(\underline{\Lambda})}}}\prod_{B\in\beta}e^{-2K_B}}{\displaystyle\sum_{\beta\in\Gamma^{(\underline{\Lambda})}}\prod_{B\in\beta}e^{-2K_B}}\leqslant\sum_{\substack{\beta=(\bar{\beta}_1,\ldots,\bar{\beta}_q)\\\beta\beta_\Lambda\in\Gamma^{(\underline{\Lambda})}\\\bar{\beta}_i\ \text{*-conn. to}\ \beta_\Lambda}}e^{-2\underline{K}|\beta|}$$

where $\underline{K}=\inf_{B\in\mathcal{B}}|K_B|>0$ (by assumption on the function K) i.e.

$$\mu_{\underline{\Lambda}}^{(+)}(\beta_{\underline{\Lambda}})\leqslant\sum_{q=C_\Lambda}^{\infty}e^{-2\underline{K}q}N_\Lambda^{(q)}$$

where $C_\Lambda=\min_{\gamma\in\Gamma^{(f)}}|\gamma\cdot\beta_\Lambda|$ is the constant introduced in Theorem 1 and $N_\Lambda^{(q)}$ is the number of graphs with length q such that all its *-connected components are *-connected to β_Λ.

Using Lemma 5 of (*I*) we know that

$$N_\Lambda^{(q)} \leqslant \begin{cases} 4^q \alpha^q & \text{if } q \geqslant |\beta_\Lambda| \\ 4^{|\beta|}\alpha^q & \text{if } q \leqslant |\beta_\Lambda| \end{cases}$$

where $\alpha = a^2$ with a the constant introduced in condition (ii), and thus

$$\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda) \leqslant \sum_{q=C_\Lambda}^{|\beta_\Lambda|-1} 4^{|\beta_\Lambda|}\alpha^q e^{-2q\underline{K}} + \sum_{q=|\beta_\Lambda|}^{\infty} (4\alpha e^{-2\underline{K}})^q$$
$$\leqslant \frac{4^{|\beta_\Lambda|}(\alpha e^{-2\underline{K}})^{C_\Lambda}}{1-4\alpha e^{-2\underline{K}}} \qquad \text{if } 4\alpha e^{-2\underline{K}} < 1$$

By the condition on the function K, $e^{-2\underline{K}} \to 0$ as $T \to 0$ and thus there exists T_0' such that $4\alpha e^{-2\underline{K}} < 1$ if $T < T_0'$.

Therefore

$$-\text{Log}\, \mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda) \geqslant C_\Lambda(2\underline{K} - \text{Log}\, \alpha) - |\beta_\Lambda| \,\text{Log}\, 4$$

and $\hat{\tau}^{(+,S)} \geqslant AC^{(+,S)}$ with $A > 0$ if $T < T_0'$ (note that $\frac{1}{\pi L_i} |\beta_\Lambda|$ is uniformly bounded since $S \in \mathcal{S}$ and the system has finite density of sites and finite range interactions).

Let us note that for $\mathbb{Z}^\nu$-invariant systems conditions (i) implies that there exists a unique symmetric equilibrium state at low temperature [7]. We shall now see that this uniqueness property is in fact sufficient.

To derive this property we introduce the following function on $\mathcal{P}(\mathcal{B}_\Lambda)$

$$\mu_\Lambda'(\beta) = \frac{\sum_{\bar\beta : \beta\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B}}{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B}}$$

$$\phi_\Lambda = \Gamma \cap \mathcal{P}(\mathcal{B}_\Lambda)$$

Lemma. *Let* $\{\mathcal{L}, \mathcal{B}, K\}$ *be a ferromagnetic system with finite range interactions. The limit* $\mu'(\beta) = \lim_{\Lambda \to \mathcal{L}} \mu_\Lambda'(\beta)$ *exists and defines a symmetric equilibrium state.*

Proof

1) Let us first show that

$$\mu'_{\Lambda,K^{(1)}}(\beta) - \mu'_{\Lambda,K^{(2)}}(\beta) \geqslant 0 \quad \text{if} \quad K^{(2)} \geqslant K^{(1)} > 0$$

(Griffith's inequality).

From the definition of μ_Λ' we have:

$$\mu_\Lambda'(\beta) = \prod_{B \in \beta} e^{-2K_B} \frac{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B \sigma_B(\beta)}}{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B}}$$

Using Poisson formulae (e.g. of [3] p. 166) with

$$\phi_\Lambda \subset \mathcal{P}(\mathcal{B}_\Lambda) \qquad \phi_\Lambda^\perp = \{\tilde\beta \in \mathcal{P}(\mathcal{B}_\Lambda); \sigma_\beta(\tilde\beta) = +1\ \forall \beta \in \phi_\Lambda\}$$

i.e.

$$\sum_{\bar\beta \in \phi_\Lambda} f(\bar\beta) = 2^{|\mathcal{B}_\Lambda|} \sum_{\tilde\beta \in \phi_\Lambda^\perp} \tilde f(\tilde\beta)$$

$$\tilde f(\tilde\beta) = 2^{-|\mathcal{B}_\Lambda|} \sum_{\bar\beta \subset \mathcal{B}_\Lambda} \sigma_{\bar\beta}(\tilde\beta) f(\bar\beta)$$

where

$$f(\bar\beta) = \prod_{B \in \bar\beta} e^{-2K'_B} \qquad K'_B = K_B \text{ or } K'_B = K_B \sigma_B(\beta)$$

and

$$\tilde f(\tilde\beta) = 2^{-|\beta_\Lambda|} \prod_{B \in \mathcal{B}_\Lambda} (1 + \sigma_B(\tilde\beta) e^{-2K_B'})$$

$$= 2^{-|\mathcal{B}_\Lambda|} \prod_{B \in \mathcal{B}_\Lambda} (e^{K'_B} + e^{-K'_B}) \prod_{B \in \mathcal{B}_\Lambda} e^{-K_B} \prod_{B \in \tilde\beta} \operatorname{th} K'_B$$

we obtain

$$\mu'_\Lambda(\beta) = \frac{\sum_{\tilde\beta \in \phi_\Lambda^\perp} \sigma_\beta(\tilde\beta) \prod_{B \in \tilde\beta} \operatorname{th} K_B}{\sum_{\tilde\beta \in \phi_\Lambda^\perp} \prod_{B \in \tilde\beta} \operatorname{th} K_B}$$

Therefore:

$$\mu'_{\Lambda,K^{(1)}}(\beta) - \mu'_{\Lambda,K^{(2)}}(\beta)$$

$$= A^2 \sum_{\tilde\beta \in \phi_\Lambda^\perp} \sum_{\beta' \in \phi_\Lambda^\perp} [\sigma_\beta(\tilde\beta) - \sigma_\beta(\beta')] \cdot \prod_{B \in \tilde\beta} \operatorname{th} K_B^{(1)} \prod_{B \in \beta'} \operatorname{th} K_B^{(2)}$$

$$= A^2 \sum_{\beta' \in \phi_\Lambda^\perp} \sum_{\tilde\beta \in \phi_\Lambda^\perp} (1 - \sigma_\beta(\tilde\beta\beta'))\sigma_\beta(\tilde\beta) \prod_{B \in \tilde\beta\beta'} \operatorname{th} K_B^{(2)} \prod_{B \in \tilde\beta} \operatorname{th} K_B^{(1)} [\operatorname{th} K_B^{(2)}]^{\sigma_B(\beta')}$$

$$= A^2 \sum_{\beta'' \in \phi_\Lambda^\perp} (1 - \sigma_\beta(\beta'')) \prod_{B \in \beta''} \operatorname{th} K_B^{(2)} \qquad \mu'_{\Lambda,K^{(3)}}(\beta) \geq 0$$

2) If thus follows from the above inequality that

$$1 \geq \mu'_{\Lambda_2}(\beta) \geq \mu'_{\Lambda_1}(\beta) \quad \text{if} \quad \Lambda_2 \supset \Lambda_1$$

and thus $\mu'(\beta) = \lim_{\Lambda \to \mathcal{L}} \mu'_\Lambda(\beta)$ exists (at least for appropriate chosen sequences $\Lambda \to \mathcal{L}$).

3) To show that $\mu'(\beta)$ defines an equilibrium state we recall*) that it is sufficient to show that:

a) $\mu'(\phi) = 1$

b) $\mu'(\beta) = \mu'(\beta\gamma)$ for all $\beta \in \mathcal{P}_f(\mathcal{L})$, $\gamma \in \Gamma^{(f)}$

*) Proposition 1 of [7].

c) $(D_{K_*}\mu)(\beta) = (D_{K_*}\mu)(\beta\kappa)$ for all $\beta \in \mathcal{P}_f(\mathcal{L})$, $\kappa \in \mathcal{K}_f$
d) μ' defines a positive form.

a) Obvious
b) For all $\gamma = \gamma(X) \in \Gamma^{(f)}$ and $\Lambda \supset X$ we have $\gamma \in \Gamma^{(\Lambda)}$. Furthermore $\beta'' \in \Gamma \cap \mathcal{P}(\mathcal{B})$ and $\gamma \in \Gamma^{(\Lambda)}$ imply $\beta'' \cdot \gamma \in \phi_\Lambda$ which yields:

$$\mu'_\Lambda(\beta) = \mu'_\Lambda(\beta\gamma) \qquad \forall \Lambda \supset X$$

i.e.

$$\mu'(\beta) = \mu'(\beta\gamma) \qquad \forall \gamma \in \Gamma^{(f)}$$

c)

$$\mu'_\Lambda(\beta) = \frac{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B} \prod_{B \in \beta} e^{-2K_B \sigma_B(\beta)}}{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B}}$$

and

$$e^{-2K_B \sigma_B(\bar\beta)} = ch 2K_B + \sigma_B(\bar\beta) sh 2K_B$$

imply

$$\mu'_\Lambda(\beta) = \prod_{B \in \beta} ch 2K_B \sum_{\tilde\beta \subset \beta} \prod_{B \in \tilde\beta} \text{th}\,(-2K_B)\sigma'_\Lambda(\tilde\beta) = (D_K \sigma'_\Lambda)(\beta)$$

with

$$\sigma'_\Lambda(\beta) = \frac{\sum_{\bar\beta \in \phi_\Lambda} \sigma_\beta(\bar\beta) \prod_{B \in \bar\beta} e^{-2K_B}}{\sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B}}$$

Therefore $(D_{K_*}\mu'_\Lambda)(\beta) = \sigma'_\Lambda(\beta)$

and

$$\sigma'_\Lambda(\beta) = \sigma'_\Lambda(\beta \cdot \kappa) \qquad \text{for all } \kappa \in \mathcal{K}_f$$

(since $\sigma_{\mathcal{H}}(\bar\beta) = 1$ for $\bar\beta \in \phi_\Lambda = \Gamma \cap \mathcal{P}(\mathcal{B}_\Lambda)$ and $\mathcal{H} \in \mathcal{K}_f$)
d) We know [7] that μ' will define a symmetric form iff

$$\sum_{\tilde\beta \subset \beta} \sigma_{\tilde\beta}(\beta_1)\sigma'(\tilde\beta) \geqslant 0 \qquad \forall \beta, \beta_1 \in \mathcal{P}_f(\mathcal{B})$$

which is verified since

$$\sum_{\tilde\beta \subset \beta} \sigma_{\tilde\beta}(\beta_1)\sigma'_\Lambda(\tilde\beta) = \frac{1}{\cdots\cdots} \sum_{\bar\beta \in \phi_\Lambda} \prod_{B \in \bar\beta} e^{-2K_B} \underbrace{\sum_{\tilde\beta \subset \beta} \sigma_{\tilde\beta}(\beta_1 \bar\beta)}_{2^{|\beta|}\delta_{\beta_1\bar\beta \cap \beta, \phi}}$$

Theorem 5. *Let $\{\mathcal{L}, \mathcal{B}, K\}$ be a ferromagnetic system with finite density of* *sites, finite range interaction, such that* $\underline{K}>0$ *and such that the condition* (ii) of Theorem 4 holds.

1) *If there exists a unique symmetric equilibrium state then for any S in* $\mathcal{S}$ *there exists a temperature* T_0' *such that*

$$\hat{\tau}^{(+,S)} \geq A\hat{C}^{(+,S)} \qquad \text{for all } T<T_0'$$

where $\hat{C}^{(+,S)} = \text{Lim sup}_{\Lambda\to\infty} 1/(\Pi L_i)\hat{C}$, $\hat{C}_\Lambda = \min_{\gamma\in\Gamma_f} |\beta_\Lambda \gamma|$, *and A is a positive constant.*

2) *In any case if* $\hat{C}^{(+,S)}>0$ *for some S in* $\mathcal{S}$ *there exists a phase transition associated with a non local order parameter. (This order parameter is* $\hat{\tau}^{(+,S)}$ *if the symmetric state is unique).*

Proof

1) Since μ' defines a symmetric equilibrium state, the uniqueness assumption implies that

$$\mu'(\beta_\Lambda) = \mu^{(+)}(\beta_\Lambda)$$

We can now use the same proof as for Theorem 4 replacing $\mu_{\underline{\Lambda}}^{(+)}(\beta_\Lambda)$ by $\mu_{\underline{\Lambda}}'(\beta_\Lambda)$ and using the fact that any element $\bar{\beta}$ in $\phi_{\underline{\Lambda}}$ can be uniquely decomposed into *-connected component in $\phi_{\underline{\Lambda}}$.

2) If $\hat{C}^{(+,S)}>0$ the proof of part (1) implies

$$\hat{\tau}' = -\lim_{L_i\to\infty} \lim_{M\to\infty} \frac{1}{\Pi L_i} \text{Log}\, \mu'(\beta_\Lambda) \geq A\hat{C}^{(+,S)} > 0 \quad \text{if} \quad T<T_0';$$

on the other hand, we can repeat the proof of Theorem 3 for $\hat{\tau}'$ to conclude that

$$\tau' = 0 \quad \text{for} \quad T>T_0$$

Indeed using the expression of $\mu_{\underline{\Lambda}}'$ in terms of $\phi_{\underline{\Lambda}}^{\perp}$ we can repeat the proof of Theorem 3, replacing K_Λ by $\phi_{\underline{\Lambda}}^{\perp}$; since $\beta \in \phi_{\underline{\Lambda}}^{\perp} \cap \mathcal{P}(\mathcal{B}_\Lambda)$ implies

$$\sigma_\beta(\beta_\Lambda) = \sigma_\beta(\gamma(S_d)) = \sigma_\beta(\gamma(S_d \cap \underline{\Lambda})) = 1$$

we can conclude just as in Theorem 3.

In conclusion $\hat{\tau}'$ is a non local order parameter, which is zero at high temperature and positive at low temperature.

From the definition of μ_Λ' we have

$$\mu_\Lambda'(\beta\gamma) = \mu_\Lambda'(\beta)\, \forall \gamma \in \phi_\Lambda$$

and

$$\mu_\Lambda'(\beta) = \frac{\sum_{\bar{\beta}\in\phi_\Lambda} \prod_{B\in\bar{\beta}} e^{-2K_B} \prod_{B\in\beta} e^{-2K_B\sigma_B(\bar{\beta})}}{\sum_{\bar{\beta}\in\phi_\Lambda} \prod_{B\in\bar{\beta}} e^{-2K_B}}$$

which yields

$$|\text{Log}\, \mu_{\underline{\Lambda}}'(\beta_\Lambda)| \leq 2\bar{K} \min_{\gamma\in\phi_{\underline{\Lambda}}} |\beta_\Lambda \gamma|$$

and

$$|\text{Log}\, \mu'(\beta_\Lambda)| \leq 2\bar{K} \min_{\gamma \in \Gamma_f} |\beta_\Lambda \gamma|$$

therefore

$$|\hat{\tau}'| \leq 2\bar{K}\hat{C}^{(+,S)}$$

(Notice that $C^{(+,S)} \geq \hat{C}^{(+,S)}$ and $\Gamma^{(f)} = \Gamma_f$ implies $\hat{C}^{(+,S)} = C^{(+,S)}$.

Conclusion

If there exists a unique symmetric equilibrium state, or if $\Gamma^{(f)} = \Gamma_f$, the condition $\hat{C}^{(+,S)} > 0$ is a necessary and sufficient condition for the existence of a phase transition association with surface tension $\hat{\tau}$; in all cases it is a sufficient condition.

The constant $\hat{C}^{(+,S)}$ is most easily evaluated using a HT–LT duality transformation; indeed any HT–LT duality transformation defined by means of a bijection $d: B \to B^*$ of $\mathcal{B}$ onto $\mathcal{B}^*$ induces a bijection of Γ_f onto $\mathcal{K}_f^*$ and therefore

$$\hat{C}_\Lambda = \min_{\kappa = \mathcal{K}_f^*} |\beta_\Lambda^* \kappa^*|; \qquad \beta_\Lambda^* = \{B^*;\ B \in \beta_\Lambda\}$$

i.e. $\hat{C}_\Lambda$ is the length of the smallest graph on the dual which can be obtained as product of β_Λ^*, with a closed graph κ^* of finite length.

4. Examples

All examples given in (*I*) satisfy the conditions of Theorem 4 and the conclusions remain valid; let us then consider systems which do not satisfy the conditions of Theorem 4, and cannot be discussed using the results of (I).

4.1. 2-*dimensional system with* 4 *body forces*

Let $\mathcal{L} = \mathbb{Z}^2$, $\mathcal{B} = \{B \subset \mathbb{Z}^2,\ B = \text{rectangles with sides of length 1 and 2}\}$ then $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ where

$$\mathcal{S}_0 = \{S \in \mathcal{S};\ |S \cap X| \text{ is even} \quad \forall X \text{ elementary square}\}$$

$$\mathcal{S}_1 = \{S \in \mathcal{S};\ |S \cap X| \text{ is odd} \quad \forall X \text{ elementary square}\}$$

To compute the geometrical constant $\hat{C}^{(+,S)}$, we note that the usual Ising model is a HT–LT dual of this model for the mapping $d: B \mapsto B^*$ represented on Fig. 1 ([3], p. 26).

It is seen easily seen that for any S in $\mathcal{S}_p$, $p = 0, 1$, the graph $\gamma(S_d)^*$ is of the form shown on Fig. 2(a), and thus if $p = 0$ we can take $\kappa^* = \prod_{\substack{i \in [-L/2, L/2] \\ \theta_i = 0}} \kappa_i^*$ to show that $\hat{C}_\Lambda = 2$ and $\hat{C}^{(+,S)} = 0$.

On the other hand if $p = 1$, $\gamma(S_d)^*$ is simply a broken line such as shown on Fig. 3, in which case $\hat{C}_\Lambda = L$ and $\hat{C}^{(+,S)} = 1$.

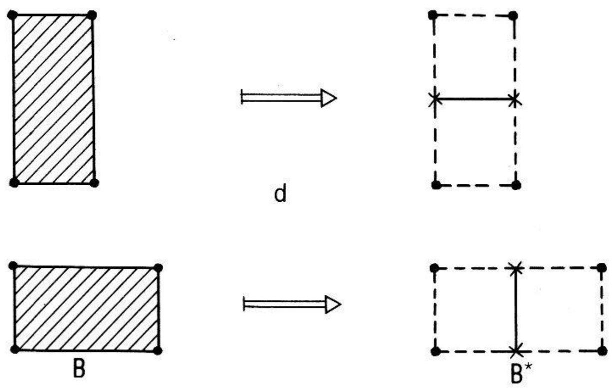

Figure 1
The lattice $\mathscr{L} = \{\bullet\}$, the dual $\mathscr{L}^* = \{X\}$ and the HT–LT duality transformation d.

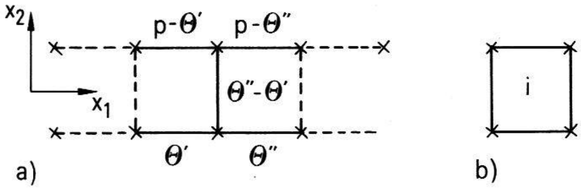

Figure 2
a) $\gamma(S_d)^*$ for $S \in \mathscr{S}_p$; $\theta', \theta'' \in \{0, 1\}$ and the difference are mod 2
b) element $\kappa_i^{(*)}$.

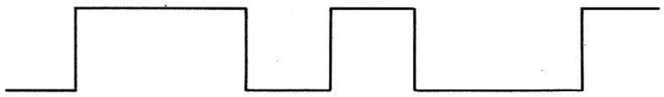

Figure 3
$\gamma(S_d)^*$ for $S \in \mathscr{S}_1$.

In conclusion for any S in $\mathscr{S}_0$, $\hat{\tau}^{(+,S)} = 0$ at all temperatures; for any S in $\mathscr{S}_1$ there exists a temperature T_0 such that $\hat{\tau}^{(+,S)} = 0$ for $T > T_0$ and $\hat{\tau}^{(+,S)} > 0$ for $T < T_0$: *this model has thus a phase transition associated with surface tension* $\hat{\tau}$.

In the same manner, we can also conclude that for any pair of states $p^{(S)}$, $p^{(S')}$ with S, S' in $\mathscr{S}_p$, $\hat{\tau}^{(S,S')} = 0$; on the other hand $\hat{\tau}^{(S,S')} > 0$ if $S \in \mathscr{S}_0$ and $S' \in \mathscr{S}_1$ at low temperature.

Conjecture 1

The surface tension $\hat{\tau}^{(S,S')}$ is different from zero if and only if the states $p^{(S)}$ and $p^{(S')}$ are distinct.

With this conjecture, we arrive at the conclusion that there exists exactly two distinct states at low temperature with symmetry group $\mathscr{S}_0$ and no spontaneous magnetization.

Let us recall that it is in fact known that this model has a first order phase transition with local order parameter $\langle \sigma_X \rangle$, $X =$ elementary square, with a critical temperature given by the critical temperature of the Ising model; moreover it is also known that there exists exactly two states at low temperature with symmetry group $\mathscr{S}_0$. These known results support therefore our conjecture.

4.2. *Triangular system with diluted 3-body forces*

This model was discussed in [9] and is represented on Fig. 4.

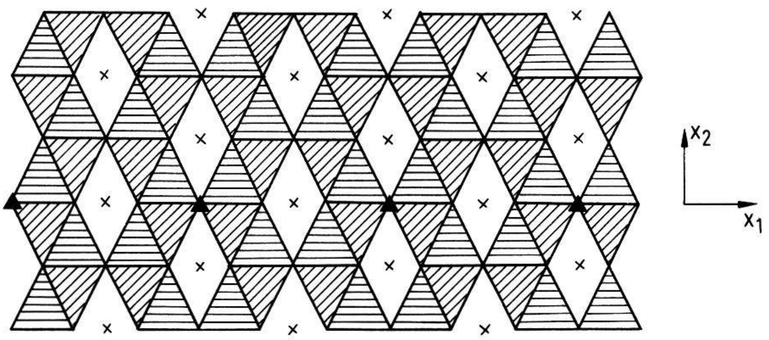

Figure 4
◬ 3-body forces,
$\{\bullet\} = \mathscr{L} \supset \mathscr{L}_1 = \{\blacktriangle\}$; $\{X\}$ = dual lattice = $\mathscr{L}^*$.

Again in this example $\mathscr{S}$ is of infinite order and can be decomposed as $\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1$

$$\mathscr{S}_0 = \{S \in \mathscr{S};\ \mathscr{L}^1 \cap S = \phi\}$$

$$\mathscr{S}_1 = \{S \in \mathscr{S};\ \mathscr{L}^1 \subset S\}$$

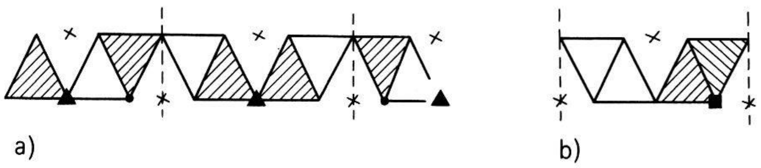

Figure 5
a) $S \in \mathscr{S}_1$ b) $S \in \mathscr{S}_0$
$\{\bullet, \blacktriangle, \blacksquare\} = S \cap \{x_2 = 0\}$
{◬} $\subset \gamma(S_d)$.

To compute $\hat{C}^{(+,S)}$ we note that the usual Ising model is again a HT–LT dual in the general sense of Ref. [3; Section 2.6] for the mapping $b \mapsto b^*$ shown in Fig. 6.

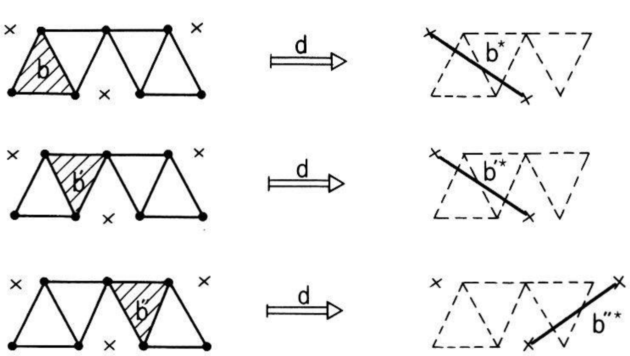

Figure 6
HT-LT duality defined by a bijection, i.e. $b^* \neq b'^*$ but $\sigma_{b^*} = \sigma_{b'^*}$.

For any S in $\mathscr{S}_0$ the graph $\gamma(S_d)^*$ is a closed graph consisting of 'double lines;' for any S in $\mathscr{S}_1$ it is a closed graph with 'simple lines'

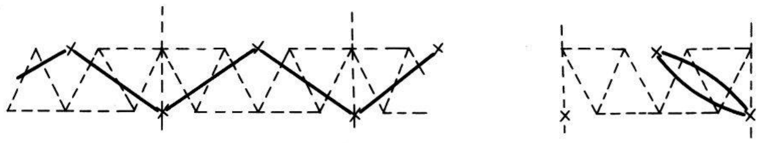

Figure 7
$\gamma(S_d)^*$ for the example Fig. 5.

It thus follows that $\hat{C}_\Lambda = 0$ if $S \in \mathcal{S}_0$ and $\hat{C}_\Lambda = L$ if $S \in \mathcal{S}_1$. As consequence for any S in $\mathcal{S}_0$, $\hat{\tau}^{(+,S)} = 0$ at all temperatures; for any S in $\mathcal{S}_1$ $\hat{\tau}^{(+,S)} > 0$ at low temperatures; *this model has thus a phase transition associated with surface tension* $\hat{\tau}$.

Let us recall that it is known [9] that this model has a first order phase transition with spontaneous magnetization on the sublattice $\mathcal{L}^1$ given by the spontaneous magnetization of the Ising model (with th $K^{\text{Ising}} = (\text{th } K)^2$).

Repeating the same argument, we can again conclude that $\hat{\tau}^{(S,S')} = 0$ if $S, S' \in \mathcal{S}_p$, $p = 0, 1$, and $\hat{\tau}^{(S,S')} > 0$ at low temperature if $S \in \mathcal{S}_0$, $S' \in \mathcal{S}_1$. With the same conjecture as before we would thus be led to conclude that there exists only two distinct states at low temperature with symmetry group $\mathcal{S}_0$.

5. HT–HT duality transformation

In this section, we shall discuss HT–HT duality transformation; this will enable us to see that the proof Theorem 5 reduces to proving Theorem 4 on the dual system. This will imply that the non local order parameter $\hat{\tau}'$ is the surface tension on the dual model.

We recall [3] that the system $\{\mathcal{L}^*, \mathcal{B}^*, K^*\}$ is a HT–HT dual system for $\{\mathcal{L}, \mathcal{B}, K\}$ if there exists a surjective mapping $d: B \mapsto B^*$ of $\mathcal{B}$ onto $\mathcal{B}^*$, such that the induced mapping $D: \beta \mapsto \beta^*$ of $\mathcal{P}(\beta)$ onto $\mathcal{P}(\mathcal{B}^*)$, yields a bijection of $\mathcal{K}_f$ into $\mathcal{K}_f^*$ and such that $e^{K^*_{B^*}} = \prod_{B \in d^{-1}B^*} e^{K_B}$. We then know that D is an isomorphism between the groups $\mathcal{K}_f$ and $\mathcal{K}_f^*$.

5.1. *Construction of a* HT–HT *dual system*

Let $\{\gamma^{(j)}\}_{j \in J} \subset \Gamma_f$ be a family of generators for Γ_f, i.e. for all $\gamma \in \Gamma_f$ there exists $I \subset J$ finite such that $\gamma = \prod_{i \in I} \gamma^{(i)}$; in the following we shall assume that the following condition is satisfied

$$\sup_{B \in \mathcal{B}} |\{j \in J;\ \gamma^{(j)} \ni B\}| \leq b < \infty$$

(which is analogous to the condition (ii) of Theorem 4).

We then define $\{\mathcal{L}^*, \mathcal{B}^*, K^*\}$ by:

a) $\mathcal{L}^* = J$, i.e. $x_j^* = j$
b) $d: B \mapsto B^* = \{j \in J;\ \gamma^{(j)} \ni B\}$
c) $K^*_{B^*} = K_B$

Property 1

The system $\{\mathcal{L}^*, \mathcal{B}^*, K^*\}$ is a HT–HT dual for $\{\mathcal{L}, \mathcal{B}, K\}$ such that $\Gamma^{*(f)} = \Gamma_f^*$; the duality transformation d is a bijection.

Proof. Let us first note that the condition on the family of generators implies that for all B in $\mathcal{B}$, $|B^*|\leq b$.

To show that d is a bijection let $X^{(j)}\subset\mathcal{L}$ be such that $\gamma^{(j)}=\gamma(X^{(j)})$, $j\in J$; the relation $B_1^*=B_2^*$ reads:

$$\{j\in J;\ \gamma^{(j)}\ni B_1\}=\{j\in J;\ \gamma^{(j)}\ni B_2\}$$

i.e.

$$\{j\in J;\ \sigma_{B_1}(X^{(j)})=-1\}=\{j\in J;\ \sigma_{B_2}(X^{(j)})=-1\}$$

Suppose now that $B_1\neq B_2$; then there exists $x\in B_1$ such that $x\notin B_2$, i.e. $B_1\in\gamma(x)$, $B_2\notin\gamma(x)$. Since $\gamma(x)\in\Gamma_f$ there exists $J_x\subset J$ finite such that

$$\gamma(x)=\prod_{j\in J_x}\gamma^{(j)}$$

$$\sigma_{B_1}(x)=-1=\prod_{j\in J_x}\sigma_{B_1}(X^{(j)})$$

$$\sigma_{B_2}(x)=+1=\prod_{j\in J_x}\sigma_{B_2}(X^{(j)})$$

Therefore there would exist $j\in J_x$ such that $\sigma_{B_1}(X^{(j)})\neq\sigma_{B_2}(X^{(j)})$ which is in contradiction with the condition $B_1^*=B_2^*$. In conclusion d is a bijection.

To show that d defines a HT–HT duality transformation, we first note that for all $j\in J$, $B\in\gamma^{(j)}$ iff $B^*\ni x_j^*$, i.e. $B\in\gamma^{(j)}$ iff $B^*\in\gamma^*(x_j^*)$; therefore the image of $\gamma^{(j)}$ under D is $\gamma^*(x_j^*)$ and the image of the product $\gamma^{(j)}\gamma^{(k)}$ is $\gamma^*(x_j^*)\gamma^*(x_k^*)$: i.e. the image of the generators of Γ_f are the generators of $\Gamma^{*(f)}$. Since d is a bijection we conclude that D induces a bijection of Γ_f into $\Gamma^{*(f)}$ and thus a bijection of $\Gamma_f^\perp\cap\mathcal{P}_f(\mathcal{B})$ into $\Gamma^{*(f)\perp}\cap\mathcal{P}_f(\mathcal{B}^*)$. But

$$\Gamma^{*(f)\perp}\cap\mathcal{P}_f(\mathcal{B}^*)=\mathcal{K}^*\cap\mathcal{P}_f(\mathcal{B}^*)=\mathcal{K}_f^*$$

while

$$\begin{aligned}\Gamma_f^\perp\cap\mathcal{P}_f(\mathcal{B})&=\{\beta\in\mathcal{P}_f(\mathcal{B});\ \sigma_\beta(\gamma)=+1\ \forall\gamma\in\Gamma_f\}\\&=\{\beta\in\mathcal{P}_f(\mathcal{B});\ \prod_{B\in\beta}\sigma_B(X)=+1\ \forall X\in\mathcal{P}_f(\mathcal{L})\}\\&=\mathcal{K}_f\end{aligned}$$

thus D induces a bijection of $\mathcal{K}_f$ into $\mathcal{K}_f^*$ and d is a HT–HT duality transformation.

Finally since $\Gamma_f\cong\Gamma^{*(f)}$ and d is a bijection then $\Gamma\cong\Gamma^*$ and thus $\Gamma_f\cong\Gamma_f^*=\Gamma^{*(f)}$.

5.2. *Examples*

1) Consider the system with diluted 3-body forces on the triangular lattice discussed in Section 4.2 and let us take as family of generators for Γ_f the elements $\{\gamma^{(j)}\}$ shown on Fig. 8.

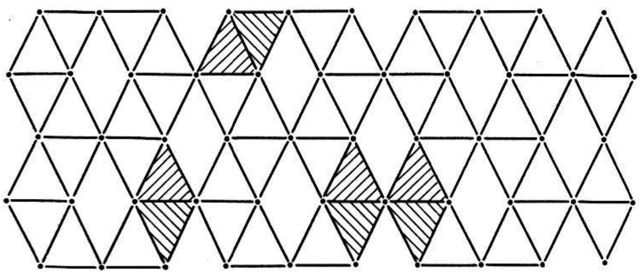

Figure 8
Family of generators for Γ_f.

Following then the general construction we obtain the HT–HT dual shown on Fig. 9. We should notice that $\Gamma^{(f)} \neq \Gamma_f$ but $\Gamma^{*(f)} = \Gamma_f^*$.

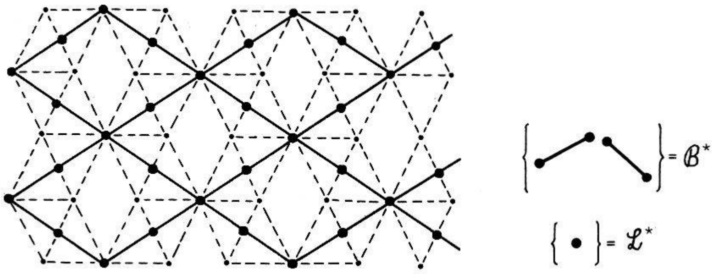

Figure 9
A dual model for the model of Fig. 4.

2) Let us consider the example of Section 4.1 and take as family of generators for Γ_f the elements $\{\gamma^{(i)}\}$ shown on Fig. 10. The HT–HT dual obtained by our construction is the usual Ising model.

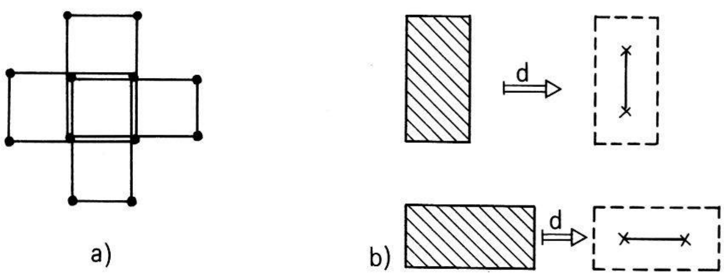

Figure 10
a) generators for Γ_f
b) HT–HT duality transformation

3) As last example we shall consider the *2-dimensional gauge model* shown on Fig. 11 (the 3-dimensional case was discussed in (*I*)).

This system has a group $\mathscr{S}$ of infinite order as well as a non trivial gauge group, i.e. $\mathscr{S}_f \neq \{\phi\}$. It is easy to see that for all S in $\mathscr{S}$, $\hat{\tau}^{(+,S)} = 0$: *there exists no phase transition associated with surface tension;* with our conjecture of Section 4.1 we are led to conclude that there exist a unique state at all temperatures.

It is straightforward to verify that the mapping d of Fig. 11(b) defines a HT–HT duality transformation, the model obtained by this transformation is a

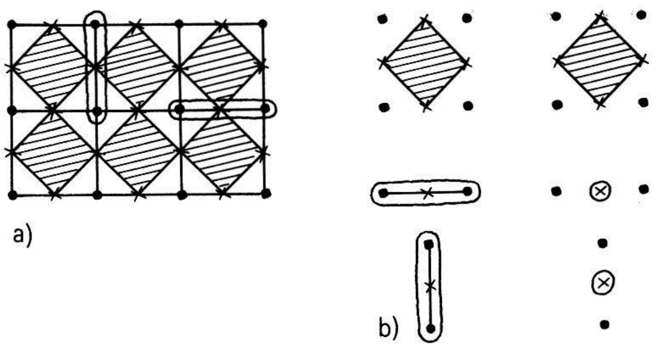

Figure 11
a) 2-dim. gauge model $\mathscr{L} = \{\bullet, X\}$
b) The HT–HT duality transformation $\mathscr{L}^* = \{X\}$ '$\mathscr{B}^* = \{\Leftrightarrow, \otimes\}$.

model on $\mathbb{Z}^2$ with alternating 4-body forces and external field. (It is the HT–LT dual of the usual Ising model in field).

Since the transformation d defines also a LT–LT transformation [3], we conclude [3, p. 214] that there exists a bijection between the equilibrium states of the gauge model and the equilibrium states of this dual model. Using then Theorem 2 of [7] we conclude that the gauge model has a unique $\mathbb{Z}^2$-invariant equilibrium state. (Since it is the HT–LT dual of the usual Ising model in Field). This result supports therefore our conjecture 1.

Let us finally remark that this HT–HT duality transformation corresponds to '*fixing the gauge*' for the original model.

5.3. *HT–HT duality and surface tension*

We shall now show that the function μ'_Λ introduced in Theorem 5 corresponds to $\mu^{*(+)}_{\Lambda^*}$ for the HT–HT dual model.

Let $\{\mathscr{L}, \mathscr{B}, K\}$ be a system and $\{\gamma^{(j)}\}$ be a family of generators for Γ_f satisfying the condition of Section 5.1 (i.e. the number of generators containing a given B is always smaller or equal to a finite constant); furthermore we shall assume that there exists a sequence of finite domains $\Lambda \to \mathscr{L}$ such that the condition $\gamma \in \Gamma \cap \mathscr{P}(\mathscr{B}_\Lambda)$ implies that $\gamma = \prod_{i \in I} \gamma^{(i)}$ with $\gamma^{(i)} \subset \mathscr{B}_\Lambda$ for all $i \in I$.

Let Λ be a domain in the above sequence and

$$\Lambda^* = \{x_i^* \in \mathscr{L}^*;\ \gamma^{(i)} \subset \mathscr{B}_\Lambda\}$$

(Λ^* is then finite)

$$\begin{aligned}\mathscr{B}^*_{\Lambda^*} &= \{B^* \in \mathscr{B}^*;\ \exists x_j^* \in \Lambda^* \text{ s.t. } B^* \in \gamma^*(x_j^*)\}\\ &= \{B \in \mathscr{B};\ \exists j \in J \text{ s.t. } \gamma^{(j)} \subset \mathscr{B}_\Lambda,\ B \in \gamma^{(j)}\}\\ &= \{\mathscr{B}_\Lambda\}^*\end{aligned}$$

$$\begin{aligned}\phi^*_{\Lambda^*} &= \Gamma^* \cap \mathscr{P}(\mathscr{B}_{\Lambda^*}) = \Gamma_f^* \cap \mathscr{P}(\mathscr{B}_{\Lambda^*})\\ &\cong (\Gamma_f \cap \mathscr{P}(\mathscr{B}_\Lambda))^* = (\phi_\Lambda)^*\end{aligned}$$

But

$$\begin{aligned}\phi^*_{\Lambda^*} &= \{\gamma^*(X^*); |X^*|<\infty \quad \gamma^*(X^*) \subset \mathscr{B}^*_{\Lambda^*}\}\\ &= \left\{\gamma^* = \prod_{i\in I} \gamma^*(x_i^*); I<\infty \quad \gamma^* \subset \mathscr{B}^*_{\Lambda^*}\right\}\\ &= \left\{\gamma = \prod_{i\in I} \gamma^{(i)}; I<\infty \quad \gamma \subset \mathscr{B}_{\Lambda}\right\}^*\\ &= \left\{\gamma = \prod_{i\in I} \gamma^{(i)}; I<\infty \quad \gamma^{(i)} \subset \mathscr{B}_{\Lambda}\right\}^*\end{aligned}$$

i.e.

$$\phi^*_{\Lambda^*} = \left\{\gamma^* = \prod_{i\in I} \gamma^*(x_i^*); \; x_i^* \in \Lambda^*\right\}$$

$$(\phi_\Lambda)^* \cong \phi^*_{\Lambda^*} = \Gamma^{*(\Lambda^*)}$$

We have thus

$$\mu'_\Lambda(\beta) = \frac{\sum\limits_{\bar\beta:\beta\bar\beta\in\phi_\Lambda} \prod\limits_{B\in\bar\beta} e^{-2K_B}}{\sum\limits_{\bar\beta\in\phi_\Lambda} \prod\limits_{B\in\bar\beta} e^{-2K_B}} = \frac{\sum\limits_{\bar\beta^*:\beta^*\bar\beta^*\in\Gamma^{*(\Lambda^*)}} \prod\limits_{B^*\in\bar\beta^*} e^{-2K_{B^*}}}{\sum\limits_{\bar\beta^*\in\Gamma^{*(\Lambda^*)}} \prod\limits_{B\in\bar\beta^*} e^{-2K_{B^*}}}$$

i.e.

$$\mu'_\Lambda(\beta) = \mu^{*(+)}_{\Lambda^*}(\beta^*)$$

and

$$\mu'(\beta) = \mu^{*(+)}(\beta^*)$$

To conclude this discussion, we should remark that it follows from a general theorem on HT–HT duality transformation ([3], p. 214) that $\mu'(\beta)$ is a symmetric equilibrium state for $\{\mathscr{L}, \mathscr{B}, K\}$.

6. $\mathbb{Z}^\nu$-Invariant crystal lattice

A $\mathbb{Z}^\nu$-invariant crystal lattice is by definition a system with lattice $\mathscr{L} = \mathbb{Z}^\nu$ and $\mathbb{Z}^\nu$-invariant interactions.*) We shall furthermore consider only *finite range interactions.*

These particular systems were studied in a very original manner in [4]; in particular for ferromagnetic interactions it was shown that they exhibit a phase transition associated with local order parameter if and only if the reduced system has a non trivial symmetry group.

*) For all $a \in \mathbb{Z}^\nu$, $B \in \mathscr{R}$ then $\tau_a B \in \mathscr{B}$ and $K_{\tau_a B} = K_B$ where τ_a denotes translation by the vector a.

In this section, we would like to investigate whether our definition generalizes the definition of a phase transition with local order parameter; therefore we would like to see whether systems which satisfy the conditions of [4] will also satisfy our general criterion for the existence of a phase transition.

Let us first give some properties of $\mathbb{Z}^\nu$-invariant crystal lattices.

Property 2

Let $\{\mathbb{Z}^\nu, \mathcal{B}, K\}$ be a $\mathbb{Z}^\nu$-invariant crystal lattice; then

a) $\mathcal{S}_f=\{\phi\}$ i.e. for all S in $\mathcal{S}$, $|S|=0$ or ∞.
b) With Λ a parallelipiped with sides $(L_1, \ldots, L_\nu)$

$$\Gamma^{(f)} \cap \mathcal{P}(\mathcal{B}_\Lambda)=\Gamma^{(\Lambda)}$$

(i.e. $\forall X \in \mathcal{P}_f(\mathcal{L})$ the condition $\sigma_B(X)=+1$ for all $B \notin \mathcal{B}_\Lambda$ implies $X \subset \Lambda$).

Proof
a) Was announced in [10]; for a proof see [4].
b) Follows from the invariance under translation.

Remarks

1) Property 2b) was introduced as a conjecture in [7]; it is expected to remain true if $\mathcal{L}$ is not $\mathbb{Z}^\nu$ but only $\mathbb{Z}^\nu$-invariant.
2) It should be recalled that for general lattice systems the relation

$$\Gamma^{(f)} \cap \mathcal{P}(\mathcal{B}_\Lambda)=\Gamma^{(\Lambda)}$$

implies $\mathcal{S}_f=\{\phi\}$ ([3], p. 71).

We thus see that any $\mathbb{Z}^\nu$-invariant crystal lattice such that $\Gamma^{(f)}=\Gamma_f$ will satisfy the condition to apply Peierls argument ([3], theorem p. 75).

3) Any $\mathbb{Z}^\nu$-invariant crystal lattice with $\Gamma^{(f)}=\Gamma_f$ will thus satisfy the hypothesis of Theorem 4 and also of Theorem 5 of (*I*).

Theorem 6. *Let $\{\mathbb{Z}^\nu, \mathcal{B}, K\}$ be a $\mathbb{Z}^\nu$-invariant ferromagnetic system with $\Gamma^{(f)}=\Gamma_f$. If for all B in $\mathcal{B}$, $|B|=2$, then there exists a phase transition associated with surface tension.*

Proof. Let

$$S=\mathbb{Z}^\nu, \underline{\Lambda} \in \mathcal{P}_f(\mathcal{L}) \qquad X \subset \ddot{\Lambda} \qquad \Lambda \subset \underline{\Lambda};$$

then

$$2\,|\gamma(X) \cap \beta_\Lambda| \leq \gamma(X)$$

Indeed if $|\gamma(X) \cap \beta_\Lambda|=0$ then it is true; if $B \in \gamma(X) \cap \beta_\Lambda$ let $x_1=B \cap X$ and $x_0=B \setminus x$, then $B_1=\tau_{(x_1-x_0)}B \notin \beta_\Lambda$. If $B_1 \notin \gamma(X)$ then $x_2=B_1 \setminus x \in X$ and $B_2=\tau_{(x_2-x_0)}B \in \beta_\Lambda$ and so on.

Since X is finite there exists $n<\infty$ such that $B_n=\tau_{(x_n-x_0)}B \in \gamma(X)$ and

$B_n \notin \beta_\Lambda$. It is easy to verify that $B, B' \in \gamma(X) \cap \mathcal{B}_\Lambda$, $B \neq B'$ implies $B_n \neq B'_n$, and thus $2|\gamma(X) \cap \beta_\Lambda| \leqslant \gamma(X)$.

Consequently $|\gamma(X) \cdot \beta_\Lambda| \geqslant |\beta_\Lambda|$ for all $X \subset \underline{\Lambda}$.

Using the condition $\Gamma^{(f)} = \Gamma_f$ we have:

$$C^{(+,S)} = \lim_{\Lambda \to \mathbb{Z}^\omega} \frac{1}{\pi L_i} C_\Lambda$$

$$C_\Lambda = \min_{x \in \mathcal{P}_f(\mathcal{L})} |\gamma(X)\beta_\Lambda| = \beta_\Lambda$$

Since the interactions are $\mathbb{Z}^\nu$-invariant we conclude that $C^{(+,S)} > 0$ and $\hat{\tau}^{(+,S)} > 0$ at low temperature.

Conclusion

For *pair interaction the criterion* $\Gamma^{(f)} = \Gamma_f$ *for the existence of a spontaneous magnetization at low temperature implies also our criterion for a phase transition.*

Conjecture 2

Theorem 6 remains valid without the restriction to pair interactions.

If this conjecture is satisfied the criterion $\Gamma^{(f)} = \Gamma_f$ for the existence of a spontaneous magnetization is equivalent to our criterion for a phase transition.

Remark. It is important to notice that the criterion $\Gamma^{(f)} = \Gamma_f$ is valid only for $\mathbb{Z}^\nu$-invariant crystal lattices while the criterion $\hat{C}^{(+,S)} > 0$ is always valid.

Let us then consider the case $\Gamma^{(f)} \neq \Gamma_f$. The following property is an easy consequence of the results in [4].

Property 3

The reduction procedure of Ref. [4] is a HT–HT duality transformation.

In other words, the HT–HT duality transformation appears as an extension of the reduction procedure which is applicable to arbitrary systems.

It was shown in [4] that if the reduced system has $\mathcal{S}^* \neq \{\phi\}$ then the original system has a first order phase transition with local order parameter $\rho^{(+)}(D^{-1}x^*)$. (We denote by D^{-1} the mapping which is written as D in [4] to be consistent within our notations).

Let us note that for any $X^* \subset \mathcal{L}^*$ which is quasi-periodic, we can construct $X \subset \mathcal{L}$ such that

$$\sigma_{x^*}(X^*) = \sigma_{D^{-1}x^*}(X)$$

In particular for all $S^* \in \mathcal{S}^*$ we can construct $S \in \mathcal{S}$ such that

$$\gamma(S_d)^* = \gamma^*(S_d^*)$$

and therefore:

$$\hat{\tau}^{*(+,S^*)} = \hat{\tau}^{(+,S)} \qquad \Gamma^{*(f)} = \Gamma_f^*$$

Conclusion

If the Conjecture 2 is satisfied, the criterion of [4] to prove the existence of a first order phase transition implies that the surface tension is non zero at low temperature.

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