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On the unique continuation property for Schrödinger hamiltonians

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Abstract. We prove a unique continuation theorem for differential operators in $\mathbf{R}^n$ of the form $-\Delta+\sum_{j=1}^{n} W_jD_j+V$, where the functions $V, W_1, \ldots, W_n$ are locally unbounded. For example, we can allow $W_j \in L^{2n-1}_{\rm loc}$ and $V\in L^2_{\rm loc}$ if $n=2, 3$, $V\in L^{(2n-1)/3}_{\rm loc}$ in $n\geqslant 4$. We can also treat N-body Schrödinger operators with two particle potentials in $L^p_{\rm loc}(\mathbf{R}^3)$ for $p>2$.

1. Introduction

With the great progress made in the spectral analysis of one-body Schrödinger hamiltonians the last years we are left with, essentially, only one obscure point: the question of (strictly) positive eigenvalues for locally unbounded potentials (see [1], Remark after Theorem 3.1). More precisely, one would like to show that the equation $(-\Delta+V)u=\lambda u$ does not have a nontrivial solution $u\in L^2(\mathbf{R}^n)$ if $\lambda>0$ and V is a real function in the *SR* class (see [1]), or to give a counterexample to this proposition. The result is known to be true if V is locally bounded in the complement of a compact set of measure zero with connected complement and satisfies a condition of decrease at infinity (see [12], Theorem XIII.58 for example). The proof of this assertion involves two steps: first one shows that u must be zero in a neighbourhood of infinity and then one proves that the differential operator $-\Delta+V-\lambda$ has the (weak) unique continuation property (see below), so that u must be identically zero. Both stages use the local boundedness of V. The purpose of this paper is to relax the condition $V\in L^\infty_{\rm loc}$ in the second stage, i.e. to prove the unique continuation property for a class of differential operators with locally unbounded coefficients.

Let $\Omega\subset\mathbf{R}^n$ be open and connected; we denote by $H^s(\Omega)$ the usual Sobolev spaces and by $H^s_{\rm loc}(\Omega)$ their local counterparts (s can be any real number). We adopt Hörmander's [7] notations and conventions; in particular, $D_j=-i\partial_j$, $\partial_j\equiv\partial/\partial x_j$, $j=1,\ldots,n$. Let $P=P(x,D)=\sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha$ be a differential operator on Ω, a_α being complex valued functions on Ω such that $P\Psi$ is well defined for any $\Psi\in H^m_{\rm loc}(\Omega)$ (in the sense of distributions) and $P\Psi\in L^2_{\rm loc}(\Omega)$. We say that P has the unique continuation property in Ω if: $\Psi\in H^m_{\rm loc}(\Omega)$, $P\Psi=0$ and $\Psi \mid U=0$ for some

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open, non-empty $U \subset \Omega$, imply $\Psi = 0$. Let:

$$H_c^s(\Omega) = \{\Psi \in H^s(\mathbf{R}^n) \mid \text{supp } \Psi \text{ is a compact subset of } \Omega\} \tag{1}$$

We say that P has the weak unique continuation property in Ω if: $\Psi \in H_c^m(\Omega)$, $P\Psi = 0 \Rightarrow \Psi = 0$. Remark that 'unique continuation' is a local property (i.e. it is enough to prove it in the neighbourhood of each point) while 'weak unique continuation' is not.

Several authors have proved that various classes of differential operators have the unique continuation property: see the references given in Hörmander [7] Chapter 8. For counterexamples see Plis [10], [11] and Hörmander [8]. The most general results are those obtained by Aronszajn–Krzywicki–Szarski [4] and Hörmander [7] Theorem 8.9.1 (in this theorem, if P_m is elliptic, it is sufficient that the coefficients of P_m be Lipschitz). For second-order elliptic operators in $n \geqslant 3$ dimensions, Plis' counterexample [11] shows that these results are optimal from the point of view of regularity of the higher order coefficients a_α, $|\alpha| = 2$. For $n = 2$, there are much more general results due to Bers and Nirenberg [5]. The lower order coefficients a_α, $|\alpha| = 0$, 1, are required to be bounded but, as we shall show, Hörmander's estimates (see Appendix 1) allow even locally unbounded a_α ($|\alpha| \leqslant 1$). The conditions are not optimal however (for any $n \geqslant 3$), which motivated us to try to improve Hörmander's estimates. This is done in Section 2: we show that Hörmander's inequalities can be improved only 'in certain directions' and that our final estimates are optimal (from the point of view of the method used in this paper). In Section 3 the unique continuation theorems which follow from the estimates of Section 2 are presented. For $n = 3$ they are quite satisfactory, but not for $n \geqslant 4$. We think that they can not be improved by the method of this paper (since the estimates in Section 2 are optimal). We consider only the case

$$P = -\Delta + \sum_{j=1}^{n} W_j D_j + V,$$

which is the operator appearing in non-relativistic quantum mechanics, but exactly the same method works in general.

W. O. Amrein and A. M. Berthier found a general method of proving weak unique continuation property, based on the spectral properties of differential operators with periodic coefficients (see [2] and [12] page 355). However, it seems that the results obtained until now by this method are weaker than ours (in higher dimensions; in fact, it requires $V \in L_{\text{loc}}^{n-2}$ if $n \geqslant 5$, but only $V \in {}_{\text{loc}}^{p}$, $p > n/2$, if $n = 2$, 3, 4; cf. [3]).

Remark. After the completion of this paper (a first version was presented in a seminar at the Central Institute of Physics, Bucharest, in the summer of 1978) we have learned from W. Amrein of a preprint by M. Schechter and B. Simon treating the same problem and leading to unique continuation under local conditions on V similar to those in [3] cited above. Their method is different from ours, being based on inequalities containing L^p-norms of the function. Our inequalities involve only L^2-norms, but of the function and some of its derivatives of fractional order. Our method has the advantage that it works for elliptic operators of any order (even with variable coefficients in the principal part), perturbed by operators of lower order with locally unbounded coefficients.

The proof of the unique continuation property given in this paper is based on

Carleman's remark that the problem can be reduced to that of proving an inequality. We shall present his idea as a lemma, extracted, together with its proof, from the proof of Theorem 8.9.1 in Hörmander [7]:

Lemma 1. *Let $U \subset \mathbf{R}^n$ be open and let $P: H^m_{\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(U)$ be a linear, local operator* (*i.e.* supp $Pu \subset$ supp u *for any* u). *Let $\phi: U \to \mathbf{R}$ be a continuous function such that there exists a function $\varepsilon: (T, \infty) \to (0, \infty)$ with $\varepsilon(\tau) \to 0$ for $\tau \to \infty$ and an integer $j \in [0, m-1]$ with the property: for any $v \in H^m_c(U)$ and any $\tau > T$:*

$$\sum_{|\alpha| \leq j} \|e^{\tau\phi} D^\alpha v\|_{L^2(U)} \leq \varepsilon(\tau) \|e^{\tau\phi} Pv\|_{L^2(U)} \tag{2}$$

Let $u \in H^m_{\mathrm{loc}}(U)$ be such that $|Pu| < k \sum_{|\alpha| \leq j} |D^\alpha u|$ (a.e. *in* U) *for some constant $k < \infty$. If $x_0 \in U$ and u is zero in the intersection of a neighbourhood of x_0 with a neighbourhood of the set $\{x \in U \mid \phi(x) \geq \phi(x_0), x \neq x_0\}$ then u is zero in a neighbourhood of x_0.*

Proof. Let $\omega = U \setminus \mathrm{supp}\, u$, it is an open subset of U such that $x_0 \in \bar{\omega} \cap U$ and $u \mid \omega = 0$. Let U_0 be a neighbourhood of x_0 in U such that u is zero in the intersection of U_0 with a neighbourhood of $\{x \in U \mid \phi(x) \geq \phi(x_0), x \neq x_0\}$. It follows that if $x \in U_0$, $\phi(x) \geq \phi(x_0)$ and $x \neq x_0$, then $x \in \omega$. We choose an open neighbourhood V of x_0 such that $\bar{V} \subset U_0$ and $\theta \in C^\infty_0(U_0)$ with $\theta \mid V = 1$; we denote $v = \theta u \in H^m_c(U)$. Then:

$$\begin{aligned} \|e^{\tau\phi} Pv\|_{L^2(U)} &\leq \|e^{\tau\phi} Pv\|_{L^2(V)} + \|e^{\tau\phi} Pv\|_{L^2(U \setminus V)} \\ &\leq k \sum_{|\alpha| \leq j} \|e^{\tau\phi} D^\alpha v\|_{L^2(V)} + \|e^{\tau\phi} Pv\|_{L^2(U \setminus V)} \end{aligned}$$

since $v = u$ on V. Hence:

$$(1 - k\varepsilon(\tau)) \sum_{|\alpha| \leq j} \|e^{\tau\phi} D^\alpha v\|_{L^2(U)} \leq \varepsilon(\tau) \|e^{\tau\phi} Pv\|_{L^2(U \setminus V)}$$

If $x \notin V$ and $x \in \mathrm{supp}\, Pv$, then $x \in U \setminus \omega$ (since P is local) and $x \neq x_0$, so $\phi(x) < \phi(x_0)$. But $(U \setminus V) \cap \mathrm{supp}\, Pv$ is compact and ϕ is continuous, thus there exists $\varepsilon > 0$ with the property $\phi(x) < \phi(x_0) - \varepsilon$ if x $(U \setminus V) \cap \mathrm{supp}\, Pv$. Let $U_\varepsilon = \{x \in U \mid \phi(x) > \phi(x_0) - \varepsilon\}$:

$$\begin{aligned} &(1 - k\varepsilon(\tau)) e^{\tau(\phi(x_0) - \varepsilon)} \sum_{|\alpha| \leq j} \|D^\alpha v\|_{L^2(U_\varepsilon)} \\ &\qquad \leq (1 - k\varepsilon(\tau)) \sum_{|\alpha| \leq j} \|e^{\tau\phi} D^\alpha v\|_{L^2(U_\varepsilon)} \leq \varepsilon(\tau) \|e^{\tau\phi} Pv\|_{L^2(U \setminus V)} \\ &\qquad \leq \varepsilon(\tau) e^{\tau(\phi(x_0) - \varepsilon)} \|Pv\|_{L^2(U)} \end{aligned}$$

But $Pv \in L^2_{\mathrm{loc}}(U)$ and has compact support (P being local), therefore $\|Pv\|_{L^2(U)} < \infty$. We make $\tau \to \infty$ and get $\|v\|_{L^2(U_\varepsilon)} = 0$. Since U_ε is a neighbourhood of x_0, we obtain the result. Q.E.D.

2. The estimates

The purpose of this section is to improve the estimates given in Theorem 2, Appendix 1, in the case $P = \Delta$. The ideal would be to have these estimates with

$m-\frac{1}{2}=\frac{3}{2}$ replaced by 2. But this is impossible, as Theorem 1, Appendix 1, shows. However, the following lemma indicates that one could be able to obtain better estimates 'in one (and only one) direction'. In this section U will be an open, bounded subset of $\mathbf{R}^n$, $n\geq 2$, and $\phi:\bar{U}\to\mathbf{R}$ a function in $C^\infty(\bar{U})$ (see Appendix 1) such that grad $\phi(x)\neq 0$ for any $x\in\bar{U}$.

Lemma 2. *Let Q be a first (resp second) order differential operator in U with continuous, bounded functions as coefficients. If there is $\mu>\frac{1}{2}$ (resp $\mu>-\frac{1}{2}$) such that for some constants c, $\tau_0\in\mathbf{R}$ and any $u\in C_0^\infty(U)$, $\tau\geq\tau_0$:*

$$\tau^\mu\|Q(e^{\tau\phi}u)\|\leq c\,\|e^{\tau\phi}\,\Delta u\| \tag{3}$$

then there is $\lambda:U\to\mathbf{C}$ (resp $\lambda_j:U\to\mathbf{C}$, $j=1,\ldots,n$) such that $Q=\sum_{j=1}^n\lambda(x)\,\partial_j\phi(x)\,D_j$ + zero-order term (resp $Q=\sum_{j,k=1}^n\lambda_j(x)\,\partial_k\phi(x)D_jD_k$ + lower order terms). Moreover, if $\mu>1$ (resp $\mu>0$) then $\lambda=0$ (resp $\lambda_j=0$, $j=1,\ldots,n$).

Proof. We follow Hörmander's proof of the Theorem 8.1.1. from [7], thus we will not give all the details. Let $x_0\in U$; we can suppose $x_0\equiv 0$ and $\phi(0)=0$. Denote $N=\text{grad}\,\phi(0)\neq 0$, take any $\xi\in\mathbf{R}^n$ with $|\xi|=|N|$ and $\xi N=0$ (scalar product in $\mathbf{R}^n$) and choose $\omega\in C^\infty(\mathbf{R}^n)$ such that $\omega(0)=0$ and grad $\omega(0)=\xi+iN$. It follows that $\phi(x)-\text{Im}\,\omega(x)=$ quadratic form in $x+O(|x|^3)$ for $x\to 0$, in particular $\lim_{\tau\to\infty}\tau(\phi(\tau^{-1/2}x)-\text{Im}\,\omega(\tau^{-1/2}x))$ exists, uniformly in x if x runs over a compact set. Suppose $\psi\in C_0^\infty(\mathbf{R}^n)$ and τ is big enough, then take $u(x)=e^{i\tau\omega(x)}\psi(\sqrt{\tau}x)$ in (3). Since $D_je^{i\tau\omega}=e^{i\tau\omega}(D_j+\tau\,\partial_j\omega)$, if $Q=\sum_{|\alpha|\leq 2}a_\alpha(x)D^\alpha$, we obtain:

$$\tau^\mu\left\|e^{\tau(\phi-\text{Im}\,\omega)}\sum_{|\alpha|\leq 2}a_\alpha(D+\tau\,\text{grad}\,(\omega-i\phi))^\alpha\psi(\sqrt{\tau}\,\cdot\,)\right\|$$
$$\leq c\left\|e^{\tau(\phi-\text{Im}\,\omega)}\sum_{j=1}^n(D_j+\tau\,\partial_j\omega)^2\psi(\sqrt{\tau}\,\cdot\,)\right\|$$

Now, make the change of variable $x=\tau^{-1/2}y$ and then let $\tau\to\infty$. Taking into account that $(\text{grad}\,\omega(0))^2=(\xi-iN)^2=0$, it follows that we must have $\sum_{j=1}^n a_j(0)\xi_j=0$ (resp $\sum_{|\alpha|=2}a_\alpha(0)\xi^\alpha=0$), if Q is of the first (resp second) order (see Hörmander, loc. cit., for details). In the first case, clearly we get $a(0)\equiv(a_1(0),\ldots,a_n(0))=\lambda(0)N$. Let us consider the second case, with a slightly changed notation: $Q=\sum_{j,k=1}^n a_{jk}D_jD_k$ + lower order terms. We have obtained $\sum_{j,k=1}^n a_{jk}(0)\xi_j\xi_k=0$ for any $\xi\in\mathbf{R}^n$ orthogonal to N. We can suppose $a_{jk}=a_{kj}$. Clearly $\sum_{j,k=1}^n(\text{Re}\,a_{jk}(0))\xi_j\xi_k=0$, $\sum_{j,k=1}^n(\text{Im}\,a_{jk}(0))\xi_j\xi_k=0$. Let $A=(\text{Re}\,a_{jk}(0))$ be considered as a selfadjoint operator in $\mathbf{R}^n$. Then $\langle\xi,A\eta\rangle=0$ for ξ, $\eta\in\mathbf{R}^n$ such that $\langle\xi,N\rangle=\langle\eta,N\rangle=0$. It follows easily that there is $l\in\mathbf{R}^n$ such that $\text{Re}\,a_{jk}(0)=l_jN_k+l_kN_j$. Similarly for $\text{Im}\,a_{jk}(0)$, and we finally get $a_{jk}(0)=\frac{1}{2}(\lambda_j(0)N_k+\lambda_k(0)N_j)$ for some $(\lambda_1(0),\ldots,\lambda_n(0))\in\mathbf{C}^n$. But then:

$$\sum_{j,k=1}^n a_{jk}(0)D_jD_k=\sum_{j,k=1}^n\lambda_j(0)N_kD_jD_k$$

The fact that we can not have $\mu>1$ (resp. $\mu>0$) unless $a_j(0)=0$ (resp $a_{jk}(0)=0$) is also easily shown. Q.E.D.

Our next purpose is to show that one can effectively obtain the best possible estimates permitted by Lemma 2 and Theorem 1 from Appendix 1. It will be convenient to express these estimates in terms of the operator:

$$p=\frac{1}{2}\left(\frac{\text{grad } \phi}{|\text{grad } \phi|} D+D\frac{\text{grad } \phi}{|\text{grad } \phi|}\right)\equiv\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial_j\phi}{|\text{grad } \phi|} D_j+D_j\frac{\partial_j\phi}{|\text{grad } \phi|}\right) \tag{4}$$

considered as a (symmetric) operator in $L^2(U)$ with $H_c^1(U)$ as domain.

Theorem 1. *Let* $U\subset\mathbf{R}^n$, $n\geq 2$, *be a bounded, open set and* $\phi\in C^\infty(\bar{U})$ *a real function such that* grad $\phi(x)\neq 0$ *for any* $x\in\bar{U}$. *Then the following assertions are equivalent:*

1) *For any* $x\in\bar{U}$, $\xi\in\mathbf{R}^n$ *such that* $\xi\cdot\text{grad}\,(x)=0$ *(scalar product in* $\mathbf{R}^n$*) and* $|\xi|=|\text{grad } \phi(x)|$ *we have:*

$$\sum_{j,k=1}^{n}\partial_j\,\partial_k\phi(x)(\xi_j+i\,\partial_j\phi(x))(\xi_k-i\,\partial_k\phi(x))>0 \tag{5}$$

2) *There are constants* $c<\infty$, $\tau_0\in\mathbf{R}$ *such that for any real* $s\in[0,2]$, $u\in H_c^2(U)$ *and* $\tau\geq\tau_0$:

$$\tau^{3/2-s}\,\|e^{\tau\phi}u\|_{H^s}+\|p^2(e^{\tau\phi}u)\|+\tau\,\|p(e^{\tau\phi}u)\|+\sum_{j=1}^{n}\|pD_j(e^{\tau\phi}u)\|\leq c\,\|e^{\tau\phi}\,\Delta u\| \tag{6}$$

Proof. Each differential operator P with coefficients in $C^\infty(U)$ will be considered as an operator in $L^2(U)$ with $C_0^\infty(U)$ as domain; the restriction of its adjoint (in the Hilbert space sense) to $C_0^\infty(U)$ will be denoted P^* and Re $P\equiv\frac{1}{2}(P+P^*)$. If Q is another such operator, then $P\geq Q$ means $((P-Q)v,v)\geq 0$ if $v\in C_0^\infty(U)$.

Let Φ_j be the operator of multiplication by $\partial_j\phi$ in $L^2(U)$ and $A_j=D_j+i\tau\Phi_j$. Clearly $e^{\tau\phi}D_j=A_je^{\tau\phi}$. If $A^2=\sum_{j=1}^n A_j^2$, then 4) of Theorem 2, Appendix 1, gives:

$$\tau^{3/2-s}\,\|v\|_{H^s}\leq c\,\|A^2v\| \tag{7}$$

for some constant c and any $s\in[0,2]$, $\tau\geq\tau_0$, $v\in C_0^\infty(U)$. From now on we shall denote by the same letter C all the constants. We show now that if $P_0, P_1,\ldots,P_4$ are differential operators of orders $0,1,\ldots,4$ with coefficients in $C^\infty(\bar{U})$, then there is a constant C such that for any $\tau\geq\tau_0$:

$$\sum_{k=0}^{4}\tau^{3-\kappa}\,\text{Re }P_k\leq cA^{2*}A^2 \tag{8}$$

It is enough to consider $P_k=fD^\alpha$ with $|\alpha|=k$. Let $\beta,\gamma\in\mathbf{N}^n$ such that $|\beta|,|\gamma|\leq 2$, $\alpha=\beta+\gamma$. Then if $v\in C_0^\infty(U)$:

$$\begin{aligned}\tau^{3-k}\,|(fD^\alpha v,v)|=\tau^{3-k}\,|(D^\gamma v,D^\beta(fv))|&\leq\tau^{3/2-|\gamma|}\,\|D^\gamma v\|\cdot\tau^{3/2-|\beta|}\,\|D^\beta(fv)\|\\&\leq c\,\|A^2v\|\cdot c\,\|A^2v\|\leq c(v,A^{2*}A^2v)\end{aligned}$$

(where we have used Leibniz formula) which proves (8).

We shall use vectors whose components are operators, with the usual rules for multiplication, etc. For example: $D=(D_1,\ldots,D_n)$, $\Phi=(\Phi_1,\ldots,\Phi_n)$, $\Phi D=\sum_{j=1}^n\Phi_jD_j$, etc. Then $A^2=(D+i\tau\phi)^2=D^2-\tau^2\Phi^2+i\tau(\Phi D+D\Phi)$. Let $a=\Phi D+D\Phi$, it is a symmetric operator in $L^2(U)$ with $C_0^\infty(U)$ as domain. Since

$A^2 = D^2 - \tau^2\Phi^2 + i\tau a$, $A^{2*} = D^2 - \tau^2\Phi^2 - i\tau a$, we get:

$$\begin{aligned} A^{2*}A^2 &= (D^2 - \tau^2\Phi^2)^2 + \tau^2 a^2 + \tau i[D^2, a] - \tau^3 i[\Phi^2, a] \\ &\geqslant (D^2 - \tau^2\Phi^2)^2 + \tau^2 a^2 - cA^{2*}A^2 \end{aligned}$$

where we have used (8) and the fact that $[D^2, a]$ (resp $[\Phi^2, a]$) is a second (resp zero) order operator. Thus:

$$cA^{2*}A^2 \geqslant (D^2 - \tau^2\Phi^2)^2 + \tau^2 a^2 \geqslant \tau^2 a^2 \tag{9}$$

Let $b = \Phi D$ and $L^2 = \sum_{j<k} (D_j\Phi_k - D_k\Phi_j)^2$. One can easily obtain the following relations:

$$D \cdot \Phi^2 \cdot D = b^*b + L^2$$

$$D \cdot \Phi^2 \cdot D = |\Phi| \cdot D^2 \cdot |\Phi| + |\Phi| \operatorname{div} \operatorname{grad} |\Phi|$$

which imply:

$$D^2 = |\Phi|^{-1} b^*b\, |\Phi|^{-1} + |\Phi|^{-1} L^2 |\Phi|^{-1} - |\Phi|^{-1} \Delta |\Phi| \tag{10}$$

The interest of this decomposition is that the first term contains only derivatives in the direction of grad ϕ, while the second contains only derivatives in directions orthogonal to grad ϕ; the third term is unimportant, being of zero order. From (10) we get (with the notation $[A, B]_+ = AB + BA$):

$$\begin{aligned} (D^2 - \tau^2\Phi^2)^2 &= [(|\Phi|^{-1} L^2 |\Phi|^{-1} - \tau^2\Phi^2) + |\Phi|^{-1} b^*b\, |\Phi|^{-1} - |\Phi|^{-1} \Delta |\Phi|]^2 \\ &= (|\Phi|^{-1} L^2 |\Phi|^{-1} - \tau^2\Phi^2)^2 + (|\Phi|^{-1} b^*b\, |\Phi|^{-1})^2 + (|\Phi|^{-1} \Delta |\Phi|)^2 \\ &\quad + [|\Phi|^{-1} L^2 |\Phi|^{-1} - \tau^2\Phi^2, |\Phi|^{-1} b^*b\, |\Phi|^{-1}]_+ \\ &\quad - [|\Phi|^{-1} L^2 |\Phi|^{-1} - \tau^2\Phi^2, |\Phi|^{-1} \Delta |\Phi|]_+ \\ &\quad - [|\Phi|^{-1} b^*b\, |\Phi|^{-1}, |\Phi|^{-1} \Delta |\Phi|]_+ \end{aligned}$$

Using (8), we obtain that the last two terms are $\geqslant -cA^{2*}A^2$. On the other hand:

$$\begin{aligned} \tau^2[\Phi^2, |\Phi|^{-1} b^*b\, |\Phi|^{-1}]_+ &= \tau^2(|\Phi|\, b^*b\, |\Phi|^{-1} + |\Phi|^{-1} b^*b\, |\Phi|) \\ &= \tau^2(2b^*b + (\text{terms of order} \leqslant 1)) \\ &\leqslant 2\tau^2 b^*b + cA^{2*}A^2 \\ &= \tfrac{1}{2}\tau^2 a^2 + \tau^2 \text{ (terms of order } \leqslant 1) \\ &\quad + cA^{2*}A^2 \leqslant cA^{2*}A^2 \end{aligned}$$

where we have used (8) again and the second inequality from (9). Accordingly:

$$\begin{aligned} (D^2 - \tau^2\Phi^2)^2 &\geqslant (|\Phi|^{-1} L^2 |\Phi|^{-1} - \tau^2\Phi^2)^2 + (|\Phi|^{-1} b^*b\, |\Phi|^{-1})^2 \\ &\quad + (|\Phi|^{-1} \Delta |\Phi|)^2 + [|\Phi|^{-1} L^2 |\Phi|^{-1}, |\Phi|^{-1} b^*b\, |\Phi|^{-1}]_+ - cA^{2*}A^2 \end{aligned}$$

We use this inequality and (9) in order to get:

$$cA^{2*}A^2 \geqslant (|\Phi|^{-1} b^*b\, |\Phi|^{-1})^2 + [|\Phi|^{-1} L^2 |\Phi|^{-1}, |\Phi|^{-1} b^*b\, |\Phi|^{-1}]_+ \tag{11}$$

But:

$$(|\Phi|^{-1} b^*b\, |\Phi|^{-1})^2 = b^*b\, |\Phi|^{-4} b^*b + (\text{terms of order} \leqslant 3) \geqslant \lambda (b^*b)^2 - cA^{2*}A^2$$

where $\lambda = \inf |\Phi|^{-4} > 0$ is a strictly positive constant. And:

$$[|\Phi|^{-1} L^2 |\Phi|^{-1}, |\Phi|^{-1} b^*b |\Phi|^{-1}]_+ = 2 |\Phi|^{-1} b^* |\Phi|^{-1} L^2 |\Phi|^{-1} b |\Phi|^{-1} + (\text{terms of order} \leqslant 3) \geqslant 2 |\Phi|^{-1} b^* |\Phi|^{-1} L^2 |\Phi|^{-1} b |\Phi|^{-1} - cA^{2*}A^2$$

Then, from (11):

$$cA^{2*}A^2 \geqslant (b^*b)^2 + |\Phi|^{-1} b^* |\Phi|^{-1} L^2 |\Phi|^{-1} b |\Phi|^{-1} \geqslant (b^*b)^2 \tag{12}$$

since $L^2 \geqslant 0$. Applying (10) again we obtain

$$\begin{aligned}|\Phi|^{-1} b^* |\Phi|^{-1} L^2 |\Phi|^{-1} b |\Phi|^{-1} &= |\Phi|^{-1} b^*D^2b |\Phi|^{-1} - |\Phi|^{-1} b^* |\Phi|^{-1} b^*b \\ &\quad \cdot |\Phi|^{-1} b |\Phi|^{-1} + |\Phi|^{-1} b^* |\Phi|^{-1} \Delta |\Phi| b |\Phi|^{-1} \\ &= b^* |\Phi|^{-1} D^2 |\Phi|^{-1} b \\ &\quad - b^*b |\Phi|^{-4} b^*b + (\text{terms of order} \leqslant 3) \\ &= b^*D \cdot |\Phi|^{-2} \cdot Db - b^*b |\Phi|^{-4} b^*b \\ &\quad + (\text{terms of order} \leqslant 3) \\ &\geqslant \sqrt{\lambda} b^*D^2b - \mu(b^*b)^2 - cA^{2*}A^2\end{aligned}$$

where $\mu = \sup |\Phi|^{-4} > 0$. Using (12) (two times) we finally get:

$$cA^{2*}A^2 \geqslant (b^*b)^2 + b^*D^2b \tag{13}$$

Notice that $a = 2 |\Phi| p + (\text{terms of zero order})$, thus:

$$\tau^2a^2 = 4\tau^2p |\Phi|^2 p + \tau^2(\text{terms of order} \leqslant 1) \geqslant 4\mu^{-1/2}\tau^2p^2 - cA^{2*}A^2 \tag{14}$$

because of (8). Also, since $b = |\Phi| p + (\text{terms of zero order})$:

$$\begin{aligned}(b^*b)^2 + b^*D^2b &= p^2 |\Phi|^4 p^2 + D \cdot p^2 \cdot D + (\text{terms of order} \leqslant 3) \\ &\geqslant \mu^{-1}p^4 + D \cdot p^2 \cdot D - cA^{2*}A^2\end{aligned} \tag{15}$$

From (9), (14), (13) and (15) we get:

$$cA^{2*}A^2 \geqslant p^4 + D \cdot p^2 \cdot D + \tau^2p^2 \tag{16}$$

This is equivalent to: for any $v \in C_0^\infty(U)$ and any $\tau \geqslant \tau_0$:

$$\|p^2v\|^2 + \sum_{j=1}^n \|pD_jv\|^2 + \tau^2 \|pv\|^2 \leqslant c \|A^2v\|$$

Taking $v = e^{\tau\phi}u$, $u \in C_0^\infty(U)$, we obtain the part of the inequality (6) which is not given by Hörmander inequalities (Theorem 2, Appendix 1). Q.E.D.

Remark. By Lemma 2 and Theorem 1 from Appendix 1 show that the inequality (6) cannot be improved from the point of view of the powers of τ and of the differential operators which appear in the left-hand side.

We shall use this theorem only in the case of radial functions $\phi = \phi(|x|)$. It is clear that in this case p becomes the radial momentum ($x = (x_1, \ldots, x_n)$, x_j = operator of multiplication by x_j):

$$p = \frac{1}{2}\left(\frac{x}{|x|} D + D\frac{x}{|x|}\right) = -i\left(\frac{\partial}{\partial r} + \frac{n-1}{2r}\right) \tag{17}$$

Corollary. *Let $U \subset \mathbf{R}^n$, $n \geq 2$, be a bounded, open set, such that $0 \notin \bar{U}$. Let $a = \inf\{|x| \mid x \in U\}$, $b = \sup\{|x| \mid x \in U\}$ and let $\phi : [a, b] \to \mathbf{R}$ be a function of class C^∞ such that $\phi'(r) \neq 0$, $\forall r \in [a, b]$. Then the following statements are equivalent:*

1)

$$\phi''(r) + \frac{\phi'(r)}{r} > 0 \quad \text{if} \quad r \in [a, b] \tag{18}$$

2) *There are constants $c < \infty$, $\tau_0 \in \mathbf{R}$ such that for any $s \in [0, 2]$, $u \in H_c^2(U)$ and $\tau \geq \tau_0$:*

$$\tau^{3/2-s} \|e^{\tau\phi}u\|_{H^s} + \|p^2(e^{\tau\phi}u)\| + \sum_{j=1}^{n} \|pD_j(e^{\tau\phi}u)\| + \tau \|p(e^{\tau\phi}u)\| \leq c \|e^{\tau\phi}\Delta u\| \tag{19}$$

where p is the radial momentum (17) and we have denoted by ϕ the function $U \ni x \mapsto \phi(|x|)$ also.

Proof. We have only to see what condition (5) looks like in this case. Since $\operatorname{grad} \phi(|x|) = \phi'(|x|)(x/|x|)$ the conditions on ξ are $\xi \cdot x = 0$ and $|\xi| = |\phi'(|x|)|$. Then:

$$\partial_j \, \partial_k \phi(|x|) = \phi''(|x|) \frac{x_j x_k}{|x|^2} + \phi'(|x|) \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right)$$

and the left member of (5) is:

$$\begin{aligned} &\phi'' \frac{(x \operatorname{grad} \phi)^2}{|x|^2} + \frac{\phi'}{|x|} |\xi + i \operatorname{grad} \phi|^2 - \phi' \frac{(x \operatorname{grad} \phi)^2}{|x|^3} \\ &= \phi''(\phi')^2 + \frac{\phi'}{|x|} 2(\phi')^2 - \frac{\phi'}{|x|} (\phi')^2 \\ &= \left(\phi'' + \frac{\phi'}{|x|} \right) (\phi')^2 \quad \text{Q.E.D.} \end{aligned}$$

3. The main results

This section contains two unique continuation theorems: the first is based only on Hörmander's inequalities and gives an 'almost good' result for N-body Schrödinger operators (cf. Corollary 2 to Theorem 2). The second theorem, which is based on inequality (6), improves this result in the case of one body hamiltonians.

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be open and connected and let $V, W_1, \ldots, W_n$ be some complex, measurable functions on Ω such that:*

For each $U \Subset \Omega$ (i.e. U open, with compact closure $\bar{U}$, and $\bar{U} \subset \Omega$) multiplication by V (resp W_j, $j = 1, \ldots, n$) is a bounded operator from $H_c^{3/2}(U)$ (resp. $H_c^{1/2}(U)$) into $L^2(U)$ with norm convergent to zero when the diameter of U goes to zero. (20)

Then, if $\psi \in H_{\text{loc}}^2(\Omega)$, $(-\Delta + \sum_{j=1}^n W_j D_j + V)\psi = 0$ on Ω and $\psi(x) = 0$ on an open, non-empty subset of Ω, it follows that $\psi \equiv 0$.

Proof. Let $P=-\Delta+\sum_{j=1}^{n} W_j D_j + V$, such that $P: H^2_{\text{loc}}(\Omega)\to L^2_{\text{loc}}(\Omega)$ is linear and local. Let ϕ be any function satisfying all the conditions of Theorem 1. We denote by $N(U)$ (resp $N_j(U)$, $j=1,\ldots,n$) the norm of the operator of multiplication by V (resp W_j) defined on $H^{3/2}_c(U)$ (resp $H^{1/2}_c(U)$) and with values in $L^2(U)$. Then, if $v\in H^2_c(U)$:

$$\begin{aligned}\|e^{\tau\phi}Pv\| &\geq \|e^{\tau\phi}\,\Delta v\|-\sum_{j=1}^{n}\|e^{\tau\phi}W_jD_jv\|-\|e^{\tau\phi}Vv\|\\ &\geq \|e^{\tau\phi}\,\Delta v\|-\sum_{j=1}^{n}N_j(U)\|e^{\tau\phi}D_jv\|_{H^{1/2}}-N(U)\|e^{\tau\phi}v\|_{H^{3/2}}\\ &\geq \|e^{\tau\phi}\,\Delta v\|-\sum_{j=1}^{n}N_j(U)(\|e^{\tau\phi}v\|_{H^{3/2}}+c\tau\,\|e^{\tau\phi}v\|_{H^{1/2}})\\ &\quad -N(U)\,\|e^{\tau\phi}v\|_{H^{3/2}}\geq\left[1-c\left(\sum_{j=1}^{n}N_j(U)+N(U)\right)\right]\cdot\|e^{\tau\phi}v\|_{H^{3/2}}\end{aligned}$$

where we have used (6) for $s=\frac{1}{2},\frac{3}{2}$. Accordingly, if U is small enough we obtain:

$$\|e^{\tau\phi}v\|+\|e^{\tau\phi}\text{ grad }v\|\leq c\tau^{-1/2}\,\|e^{\tau\phi}Pv\| \tag{21}$$

for any $\tau\geq\tau_0$, $v\in H^2_c(U)$ (see Theorem 2, Appendix 1).

From now on the proof of the unique continuation property is standard; we give the details for completeness. It is enough to prove that if ψ is zero in some open ball $B\Subset\Omega$, then ψ is zero in a neighbourhood of $\bar{B}$ (Ω being connected). Let x_0 be a point on the boundary of B. Take another ball B_0 such that $\bar{B}_0\subset\bar{B}$, $x_0\in\bar{B}_0$ and radius $(B_0)<$radius (B). Take the center of B_0 as origin of the coordinates in $\mathbf{R}^n$. Let $\phi:(0,\infty)\to\mathbf{R}$ be a C^∞, decreasing function such that $\phi'(r)\neq 0$ and $\phi''(r)+[\phi'(r)/r]>0$, $\forall r>0$ (for example $\phi(r)=r^{-\beta}$, $\beta>0$). Let U be a small, open neighbourhood of x_0, such that (21) is valid (we have the corollary of Theorem 1). Since ψ is zero in B, which is a neighbourhood of the set $\bar{B}_0\backslash\{x_0\}\equiv\{x\mid\phi(x)\geq\phi(x_0),\ x\neq x_0\}$, Lemma 1 implies that ψ is zero in a neighbourhood of x_0. Q.E.D.

Conditions which assure the validity of (20) are given by Schechter [13]. For example:

Corollary 1. *Let $V, W_1,\ldots,W_n\in L^2_{\text{loc}}$ be such that for any (small) compact $K\subset\Omega$ the expressions ($j=1,\ldots,n$):*

$$\begin{aligned}&\sup_{y\in K}\int_K |V(x)|^2\,\omega_3^{(n)}(x-y)\,dx;\\ &\sup_{y\in K}\int_K |W_j(x)|^2\,\omega_1^{(n)}(x-y)\,dx\end{aligned} \tag{22}$$

are finite and go to zero when the diameter of K goes to zero (where $\omega_1^{(n)}(x)=|x|^{1-n}$ for any $n\geq 2$, and $\omega_3^{(n)}(x)=1$ if $n=2$; $=|\ln|x||$ if $n=3$; $=|x|^{3-n}$ if $n\geq 4$). Then all the conditions of Theorem 2 are verified.

Proof. We use Theorem 7.3 of Schechter [13], more precisely the inequality (7.14). If $U\Subset\Omega$ and $\bar{U}=K$, then the norm of the operator $V\mid U: H^{3/2}_c(U)\to$

$L^2(U)$ is smaller then the norm of the operator $V\chi_K: H^{3/2}(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ (χ_K is the characteristic function of K). The norm of this last operator is dominated by $CN_3(V\chi_K)$ (Schechter's notations) where C is a constant depending on n only. But $(B(y;1)=\{x\in\mathbf{R}^n \mid |x-y|<1\})$:

$$N_3(V\chi_K)=\sup_{y\in\mathbf{R}^n}\int_{K\cap B(y;1)}|V(x)|^2\,\omega_3^{(n)}(x-y)\,dx$$

Thus, if $K_\varepsilon=\{x\mid$ there is $z\in K$ such that $|x-z|<\varepsilon\}$, we get

$$H_3(V\chi_K)\leqslant\sup_{y\in K_\varepsilon}\int_{K_\varepsilon}|V(x)|^2\,\omega_3^{(n)}(x-y)\,dx$$
$$+\sup_{y\notin K_\varepsilon}\int_{K\cap B(y;1)}|V(x)|^2\,\omega_3^{(n)}(x-y)\,dx$$

In the integrand of the second term $\omega_3^{(n)}(x-y)\leqslant C(\varepsilon)$, where $\varepsilon\mapsto C(\varepsilon)$ is a decreasing function near zero; therefore the second term is bounded by $C(\varepsilon)\,\|V\|_{L^2(K)}$. The first term is smaller then $\frac{1}{2}\eta$ (choose some $\eta>0$) if diameter $(K_\varepsilon)\leqslant 2\varepsilon+$diameter $(K)<\delta(\eta)$. Fix $\varepsilon>0$ such that $\varepsilon<\frac{1}{2}\delta(\eta)$ and then take diameter $(K)<\delta(\eta)-2\varepsilon$ and such that $C(\varepsilon)\,\|V\|_{L^2(K)}<\frac{1}{2}\eta$. It then follows that $N(V\chi_K)\leqslant\eta$. The case of W_j is treated similarly. Q.E.D.

Using Sobolev inequalities we see that (22) are verified if $W_j\in L^{2n}_{\text{loc}}(\Omega)$ (any $n\geqslant 2$) and: $V\in L^{2n/3}_{\text{loc}}(\Omega)$ for $n\geqslant 4$. In the case $n=2$, 3, one must use Hölder inequalities, which give $V\in L^2_{\text{loc}}(\Omega)$ for $n=2$ and $V\in L^q_{\text{loc}}(\Omega)$, for some $q>2$, if $n=3$. We see that Hörmander's inequalities do not give a good result in any dimension, the result for $n=3$ being however, 'almost good' (the potentials $V\in L^2_{\text{loc}}(\mathbf{R}^3)$ constitute a standard class in the quantum mechanical scattering theory). In the following we shall improve the preceding results. Notice, however, the following consequence of Corollary 1 (use Lemma 7.7 of [13] in the same way as in the proof of Lemma 7.4, [13]):

Corollary 2. *Let $V_{ij}\in L^q_{\text{loc}}(\mathbf{R}^3)$ for some $q>2$, $i, j=1,\ldots,N$, and let:*

$$H=-\sum_{i=1}^{n}\Delta_i+\sum_{1\leqslant i<j\leqslant N}V_{ij}(\vec{x}_i-\vec{x}_j)$$

where Δ_i is the laplacian with respect to the ith variable $\vec{x}_i\in\mathbf{R}^3$ in the cartesian product $\mathbf{R}^{3N}=\mathbf{R}^3\times\cdots\times\mathbf{R}^3$. Then: $\psi\in H^2_{\text{loc}}(\mathbf{R}^{3N})$, $H\psi=0$ and $\psi(x)=0$ on an open, non-empty subset of $\mathbf{R}^{3N}$, imply $\psi\equiv 0$.

The following theorem improves, in a certain direction, the result of Theorem 2:

Theorem 3. *Let $\Omega\subset\mathbf{R}^n$ $(n\geqslant 2)$ be open and connected, let $V\in L^q_{\text{loc}}(\Omega)$ for some $q\geqslant 2$ with $q\geqslant(2n-1)/3$ and $W_1,\ldots,W_n\in L^{2n-1}_{\text{loc}}(\Omega)$. If $\psi\in H^1_{\text{loc}}(\Omega)$, $(-\Delta+\sum_{j=1}^n W_jD_j+V)\psi=0$ (as a distribution on Ω) and $\psi(x)=0$ on an open, non-empty subset of Ω, then $\psi=0$ on Ω.*

Proof. From the regularity theorem for elliptic equations it follows that $\psi\in H^2_{\text{loc}}(\Omega)$. Then, as in the proof of Theorem 2, we see that it is enough to prove the inequality (21) for U of the form $U=U_{a,b}=\{x\in\mathbf{R}^n \mid a<|x|<b\}$, where $0<$

$a<b<\infty$ and $b-a$ is as small as we want, and for ϕ having the properties stated in the corollary of Theorem 1. Let p be defined by (17), fix some $0<a_0<b_0<\infty$ and denote U_{a_0,b_0} by U_0.

It follows from the lemma proved in Appendix 2 that if

$$\frac{1}{r}+\frac{1}{q}=\frac{1}{2}, \qquad q\geq \max\left(2, \frac{2n-1}{3}\right), \qquad n=2,3,4,\ldots$$

then there is a constant $c<\infty$ such that for any $v\in C_0^\infty(U_0)$:

$$\|v\|_{L^r(U_0)}\leq c\left(\|v\|_{H^{3/2}}+\sum_{j=1}^{n}\|pD_jv\|\right) \tag{23}$$

Taking $U=U_{a,b}$, $a_0<a<b<b_0$, we get for any $u\in C_0^\infty(U)$:

$$\|e^{\tau\phi}Vu\|\leq \|V\|_{L^q(U)}\|e^{\tau\phi}u\|_{L^r(U_0)}\leq c\,\|V\|_{L^q(U)}\Big(\|e^{\tau\phi}u\|_{H^{3/2}} + \sum_{j=1}^{n}\|pD_j(e^{\tau\phi}u)\|\Big)\leq c\,\|V\|_{L^q(U)}\|e^{\tau\phi}\Delta u\| \tag{24}$$

where we have used (19) with $s=\frac{3}{2}$, the constant c being independent of U $(\subset U_0)$, $\tau(\geq\tau_0)$ and $u(\in C_0^\infty(U))$.

Applying again the Lemma from Appendix 2 we see that, if

$$\frac{1}{r}+\frac{1}{2n-1}=\frac{1}{2},$$

then there is a constant $c<\infty$ such that for any $v\in C_0^\infty(U_0)$:

$$\|v\|_{L^r(U_0)}\leq c(\|v\|_{H^{1/2}}+\|pv\|) \tag{25}$$

Thus, if U is as before and $u\in C_0^\infty(U)$:

$$\begin{aligned}\|e^{\tau\phi}W_jD_ju\| &\leq \|W_jD_j(e^{\tau\phi}u)\|+c\tau\,\|W_je^{\tau\phi}u\|\\ &\leq \|W_j\|_{L^{2n-1}(U)}(\|D_j(e^{\tau\phi}u)\|_{L^r(U_0)}+c\tau\,\|e^{\tau\phi}u\|_{L^r(U_0)})\\ &\leq c\|W_j\|_{L^{2n-1}(U)}(\|D_j(e^{\tau\phi}u)\|_{H^{1/2}}+\|pD_j(e^{\tau\phi}u)\|\\ &\quad+\tau\,\|e^{\tau\phi}u\|_{H^{1/2}}+\tau\,\|p(e^{\tau\phi}u)\|)\\ &\leq c\,\|W_j\|_{L^{2n-1}(U)}(\tau\,\|e^{\tau\phi}u\|_{H^{1/2}}+\|e^{\tau\phi}u\|_{H^{3/2}}\\ &\quad+\tau\,\|p(e^{\tau\phi}u)\|+\|pD_j(e^{\tau\phi}u)\|)\leq c\,\|W_j\|_{L^{2n-1}(U)}\|e^{\tau\phi}\Delta u\|\end{aligned} \tag{26}$$

where (19) was used again and the constant C is independent of U, τ, u as before.

Using (24) and (26) we obtain (P is the same as in the proof of Theorem 2):

$$\|e^{\tau\phi}Pu\|\geq \|e^{\tau\phi}\Delta u\|-\sum_{j=1}^{n}\|e^{\tau\phi}W_jD_ju\|-\|e^{\tau\phi}Vu\|$$

$$\geq\left(1-c\sum_{j=1}^{n}\|W_j\|_{L^{2n-1}(U)}-c\,\|V\|_{L^q(U)}\right)\|e^{\tau\phi}\Delta u\|$$

for any $u\in C_0^\infty(U)$ and $\tau\geq\tau_0$. Accordingly, if $b-a$ is small enough, we get (21), which finishes the proof. Q.E.D.

Remark. It is easily seen that in the above proof there is no need to put a restriction on the size of U if $q>(2n-1)/3$ in the case of V and

$$\frac{1}{r}+\frac{1}{2n-1}>\frac{1}{2}$$

in the case of W_j. In particular, for $n=3$ one obtains the following nice inequality:

Corollary. *Assume that all the assertions of the corollary to Theorem* 1 *are verified and* $n=3$. *Let* $V\in L^2(U)$ *and* W_1, W_2, $W_3\in L^r(U)$ *for some* $r>5$. *Then there are constants* c, $\tau_0\in\mathbf{R}$ *such that for any* $s\in[0,2]$, $\tau\geq\tau_0$, $u\in H^2_c(U)$:

$$\tau^{3/2-s}\,\|e^{\tau\Phi}u\|_{H^s}+\|p^2(e^{\tau\Phi}u)\|+\|p\cdot\operatorname{grad}(e^{\tau\Phi}u)\| \\ +\tau\,\|p(e^{\tau\Phi}u)\|\leq c\,\|e^{\tau\Phi}(-\Delta+\vec{W}\operatorname{grad}+V)u\|.$$

4. Appendix 1: Hörmander's inequalities

In this appendix we shall state some of Hörmander's results. The first theorem is a slightly strengthened form of Theorem 8.1.1 from [7] and, in fact, follows from the same proof. The second is essentially Theorem 8.3.1 from [7]. Let $\Omega\subset\mathbf{R}^n$ $(n\geq 1)$ be an open, bounded set and $P=\sum_{|\alpha|\leq m}a_\alpha(x)D^\alpha$ a differential operator of order m such that $a_\alpha\in L^\infty(\Omega)$ for any $|\alpha|\leq m$ and $a_\alpha\in C^1(\bar{\Omega})$ if $|\alpha|=m$ ($C^k(\bar{\Omega})$, $k\in\mathbf{N}$ or $k=\infty$, is the set of functions on $\bar{\Omega}$ which can be extended to functions in $C^k(\mathbf{R}^n)$). Let $P_m(x,\zeta)=\sum_{|\alpha|=m}a_\alpha(x)\zeta^\alpha$, $x\in\Omega$, $\zeta\in\mathbf{C}^n$; we use the notation

$$P_m^{(j)}(x,\xi)=\frac{\partial}{\partial\xi_j}P_m(x,\xi);\qquad P_{m,j}(x,\xi)=\frac{\partial}{\partial x_j}P_m(x,\xi)$$

Theorem 1. *Suppose that for some function* $\phi:\Omega\to\mathbf{R}$ *of class* C^2, *some integer* $j\in[0,m]$ *and some real number* μ *there are constants* c, $\tau_0\in\mathbf{R}$ *such that for any* $u\in C_0^\infty(\Omega)$ *and any* $\tau\geq\tau_0$:

$$\tau^{2\mu}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\|e^{\tau\phi}D^\alpha u\|^2\leq c\,\|e^{\tau\phi}Pu\|^2$$

Then $\mu\leq m-j$. *If, moreover, there are* $x\in\Omega$ *and* $\xi\in\mathbf{R}^n$ *such that* $\operatorname{grad}\phi(x)\neq 0$ *and* $P_m(x,\xi+i\operatorname{grad}\phi(x))=0$, *then* $\mu\leq m-j-\frac{1}{2}$. *Suppose now that the inequality is true for* $\mu=m-j-\frac{1}{2}$. *Then, if* $x\in\Omega$, $\xi\in\mathbf{R}^n$, $\sigma\in\mathbf{R}\backslash\{0\}$ *are such that* $P_m(x,\zeta)=0$, *where* $\zeta=\xi+i\sigma\operatorname{grad}\phi(x)$, *we have:*

$$|\zeta|^{2j}\,|\sigma|^{2(m-1-j)}\leq 2c\Bigg[\sum_{j,k=1}^{n}\frac{\partial^2\varphi(x)}{\partial x_j\,\partial x_k}P_m^{(j)}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} \\ +\frac{1}{\sigma}\sum_{k=1}^{n}\operatorname{Im}(P_{m,k}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)})\Bigg]$$

Theorem 2. *Assume* $P_m(x_0,D)$ *elliptic, for any* $x_0\in\bar{\Omega}$, *i.e.* $x_0\in\bar{\Omega}$, $\xi\in\mathbf{R}^n$, $P_m(x_0,\xi)=0\Rightarrow\xi=0$. *Let* $\phi:\bar{\Omega}\to\mathbf{R}$, $\phi\in C^\infty(\bar{\Omega})$, *such that* $\operatorname{grad}\phi(x)\neq 0$ *for* $x\in\bar{\Omega}$. *Then the following statements are equivalent:*

1) *For any* $x\in\bar{\Omega}$, $\xi\in\mathbf{R}^n$, $\sigma\in\mathbf{R}\backslash\{0\}$ *such that* $P_m(x,\zeta)=0$, $\zeta=\xi+i\sigma$

grad $\phi(x)$, *we have:*

$$\sum_{j,k=1}^{n} \frac{\partial^2\phi(x)}{\partial x_j \partial x_k} P_m^{(j)}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} + \frac{1}{\sigma}\sum_{k=1}^{n} \operatorname{Im} P_{m,k}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} > 0$$

2) *There is an integer* $j\in[0,m]$ *such that for some constants* c, $\tau_0\in\mathbf{R}$ *and any* $u\in C_0^\infty(\Omega)$, $\tau\geq\tau_0$:

$$\tau^{m-j-1/2}\sum_{|\alpha|=j}\|e^{\tau\phi}D^\alpha u\|\leq c\,\|e^{\tau\phi}Pu\|$$

3) *There are constants* c, $\tau_0\in\mathbf{R}$ *such that for any* $\tau\geq\tau_0$ *and* $u\in C_0^\infty(\Omega)$:

$$\sum_{|\alpha|\leq m}\tau^{m-|\alpha|-1/2}\|e^{\tau\phi}D^\alpha u\|\leq c\,\|e^{\tau\phi}Pu\|$$

4) *There are constants* $c<\infty$, $\tau_0\in\mathbf{R}$ *such that for any real* $s\in[0,m]$, *any* $u\in H_c^m(\Omega)$ *and* $\tau\geq\tau_0$:

$$\|e^{\tau\phi}u\|_{H^s}\leq c\tau^{s+(1/2)-m}\,\|e^{\tau\phi}Pu\|$$

We must prove only 3⇒4. Recall that for $u\in L^2(\mathbf{R}^n)\equiv H^0(\mathbf{R}^n)$ and $s\in\mathbf{R}$:

$$\|u\|_{H^s}^2\equiv\int_{\mathbf{R}^n}(1+|\xi^2|)^s\,|\hat{u}(\xi)|^2\,d\xi$$

where $\hat{u}$ is the Fourier transform of u; $\|u\|\equiv\|u\|_{H^0}$. The space $H_c^s(\Omega)$ is defined in the introduction. Clearly:

$$\|u\|_{H^s}\leq\|u\|_{H^t}^{s/t}\,\|u\|_{H^0}^{1-s/t}$$

for any $u\in H_c^s(\Omega)$ and $0\leq s\leq t$, which shows that it is enough to prove the inequality from 4 for $s=0$ and $s=m$. In order to do this, we use the inequality from 3 and Leibniz formula:

$$\tau^{-|\alpha|}D^\alpha(e^{\tau\phi}u)=\sum_{\beta\leq\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\tau^{-|\alpha|}D^{\alpha-\beta}e^{\tau\phi}\cdot D^\beta u$$
$$=\tau^{-|\alpha|}e^{\tau\phi}D^\alpha u+\sum_{\beta<\alpha}O(\tau^{-|\beta|})e^{\tau\phi}D^\beta u$$

Remark. It would be interesting to find a characterization of type 1) for functions ϕ which give rise to inequalities of type 4) but with $\tau^{s-(m-1/2)}$ replaced by τ^{s-a}, $a<m-\frac{1}{2}$. An example of such a function for $P=-\Delta$, is $\phi(x)=\pm\ln|\vec{x}|$ ($e^{\tau\phi(x)}=|x|^{\pm\tau}$). It does not verify 1) but it verifies 4) with $\tau^{s-3/2}$ replaced by τ^{s-1}.

5. Appendix 2: an inequality of Sobolev type

The inequality which we shall prove here is similar to the inequalities which appear in the theory of anisotropic Sobolev spaces [6]. Let $0<a<b<\infty$ and $U=\{x\in\mathbf{R}^n \mid a<|x|<b\}$. We denote by S the surface of the unit sphere in $\mathbf{R}^n$ and $d\omega$ its usual volume element. We introduce polar coordinates $\rho=|x|$ and $\omega=x/|x|$, therefore $x\mapsto(\rho,\omega)$ is a diffeomorphism of U onto $I\times S$, $I=(a,b)\subset\mathbf{R}$. $L^2(I\times S)$

will be the space constructed with the product measure $d\rho\otimes d\omega$ of the usual Lebesgue measure $d\rho$ on I and the above measure $d\omega$ on S. Then the operator $T: L^2(U)\to L^2(I\times S)$, $(Tf)(\rho,\omega)=\rho^{(n-1)/2}f(\rho\omega)$ is unitary and $TpT^{-1}=-i(\partial/\partial\rho)$ (p is defined by (17)). The operator TL^2T^{-1}, $L^2=\sum_{j<k}(D_jX_k-D_kX_j)^2$, acts only on the variable ω, being the usual spherical laplacian. We shall identify $p\equiv TpT^{-1}$ and $L^2\equiv TL^2T^{-1}$. Moreover, we denote by the same letter p the self-adjoint extension of p (which has been considered until now as defined only on C_0^∞) defined by periodic boundary conditions (one can also work with $\sqrt{p^*p}$ instead of p, or with the square root of the Friedrichs extension of p^2). More generally, if K is any Hilbert space, then p is identified with the operator $p\otimes 1$ in $L^2(I;K)\equiv L^2(I)\otimes K$. Similarly for L^2: it will be considered as a self-adjoint operator in $L^2(S)$ (i.e. we identify TL^2T^{-1} with $1\otimes L^2$ in $L^2(I\times S)\equiv L^2(I)\otimes L^2(S)$) and we shall use its square root $\Delta=\sqrt{L^2}$.

If K is any (separable) Hilbert space, the spaces $L^r(I;K)$ are well defined, $1\leqslant r\leqslant\infty$. Let $L^2_s(I;K)$ be the domain of $|p|^s$ ($s\geqslant 0$) provided with the graph norm. Then Sobolev inequalities tell us that $L^2_s(I;K)\subset L^r(I;K)$ continuously if $1/r\geqslant\frac{1}{2}-s$, with strict inequality if $s=\frac{1}{2}$ ($r\in[1,\infty]$ always).

Let $K=L^2(S)$ and K_t be the domain of Δ^t ($t\geqslant 0$) provided with the graph norm. Applying again Sobolev inequalities we get $K_t\subset L^r(s)$ continuously if $1/r\geqslant\frac{1}{2}-t'$, with strict inequality if $t'=\frac{1}{2}$ ($t'=t/(n-1)$; $r\in[1,\infty]$).

Let $0\leqslant s_0<s_1<\infty$, $o\leqslant t_0<t_1<\infty$, we want to say something about the L^r-properties of functions in $L^2_{s_0}(I;K_{t_1})\cap L^2_{s_1}(I;K_{t_0})$. We shall use freely the results of interpolation theory as given, for example, in Lions–Magenes [9]. If X, Y are Hilbert spaces with $X\subset Y$ continuously and densely, then:

$$L^2_{s_0}(I;X)\cap L^2_{s_1}(I;Y)\subset L^2_\mu(I;[X,Y]_{(s_1-\mu)/(s_1-s_0)})$$

continuously, for any $\mu\in[s_0,s_1]$. In particular:

$$L^2_{s_0}(I;K_{t_1})\cap L^2_{s_1}(I;K_{t_0})\subset L^2_\mu(I;K_{[(s_1-\mu)/(s_1-s_0)]t_1+[(\mu-s_0)/(s_1-s_0)]t_0})\tag{27}$$

continuously if $\mu\in[s_0,s_1]$.

On the other hand, if $1/q+1/r=\frac{1}{2}$ and $q\geqslant\max(1/s,1/t')$, with strict inequality when the right hand side is 2, then the Sobolev inequalities stated above give:

$$\|v\|_{L^r(I\times S)}=\|\,\|v\|_{L^r(S)}\,\|_{L^r(I)}\leqslant c\,\|\,\|v\|_{K_t}\,\|_{L^r(I)}=c\,\|v\|_{L^r(I;K_t)}\leqslant c\,\|v\|_{L_s^2(I;K_t)}$$

such that $L^2_s(I;K_t)\subset L^r(I\times S)$ continuously. Using (27) we obtain for any $\mu\in[s_0,s_1]$:

$$L^2_{s_0}(I;K_{t_1})\cap L^2_{s_1}(I;K_{t_0})\subset L^r(I\times S)\tag{28}$$

if $1/q+1/r=\frac{1}{2}$ and

$$q\geqslant\max\left(\frac{1}{\mu},\left(\frac{s_1-\mu}{s_1-s_0}t_1'+\frac{\mu-s_0}{s_1-s_0}t_0'\right)^{-1}\right)$$

with strict inequality when the right member is 2. Now we shall vary μ such as to obtain a minimum value in the right member. If:

$$g(\mu)=\min\left(\mu,\frac{s_1-\mu}{s_1-s_0}t_1'+\frac{\mu-s_0}{s_1-s_0}t_0'\right)$$

this is equivalent with finding $\max_{s_0 \leq \mu \leq s_1} g(\mu)$. Let:

$$\alpha = \frac{t_1' - t_0'}{s_1 - s_0} > 0, \qquad \beta = \frac{s_1 t_1' - s_0 t_0'}{s_1 - s_0}$$

then $g(\mu) = \min(\mu, -\alpha\mu + \beta)$. Since $-\alpha s_0 + \beta = t_1'$, if $t_1' \leq s_0$ then $\max_{s_0 \leq \mu \leq s_1} g(\mu) = g(s_0) = t_1'$. Similarly: $-\alpha s_1 + \beta = t_0'$, therefore if $s_1 \leq t_0'$ then $\max g = g(s_1) = s_1$. If $t_0' < s_1$ and $t_1' < s_0$, max g is obtained at the point of intersection of the two lines $\mu \to \mu$ and $\mu \to -\alpha\mu + \beta$. This point has coordinate:

$$\mu_{\text{int}} = \frac{s_1 t_1' - s_0 t_0'}{t_1' - t_0' + s_1 - s_0} \equiv g(\mu_{\text{int}}) = \max_{s_0 \leq \mu \leq s_1} g(\mu)$$

In conclusion, the conditions on q under which (28) is true, are:

$$\left.\begin{array}{l}
1)\ t_1 \leq (n-1)s_0 \Rightarrow q \geq \dfrac{n-1}{t_1} \left(\text{strict inequality if } \dfrac{n-1}{t_1} = 2\right) \\[2ex]
2)\ (n-1)s_1 \leq t_0 \Rightarrow q \geq \dfrac{1}{s_1} \left(\text{strict inequality if } \dfrac{1}{s_1} = 2\right) \\[2ex]
3)\ (n-1)s_0 < t_1 \quad \text{and} \quad t_0 < (n-1)s_1 \\[1ex]
\qquad \Rightarrow q \geq \dfrac{1}{\left(1 - \dfrac{t_1}{t_1 - t_0}\right) s_0 + \dfrac{t_1}{t_1 - t_0} s_1} + \dfrac{n-1}{\left(1 - \dfrac{s_1}{s_1 - s_0}\right) t_0 + \dfrac{s_1}{s_1 - s_0} t_1}
\end{array}\right\} \qquad (29)$$

(strict inequality if the right hand side is 2)

Now, we go back to $\mathbf{R}^n$, using the remarks made at the beginning of this section. We state only a weaker result than that which follows from (28), (29). Clearly $\|Tv\|_{L^2(I;K_t)} \leq c\,\|v\|_{H^t(U)}$ for any $v \in C_0^\infty(U)$. And, for any Hilbert space K and any $s \geq 0$, there exists $\lambda > 0$ such that $\|\,|p|^s v\|_{L^2(I;K)} \geq \lambda\,\|v\|_{L^2(I;K)}$ for any $v \in \mathring{L}_s^2(I; K) =$ closure of $C_0^\infty(I; K)$ in $L_s^2(I; K)$. These two remarks give the following:

Lemma. *If U and p are as in the beginning of this section and $0 \leq s_0 < s_1 < \infty$, $0 \leq t_0 < t_1 < \infty$, and if q is chosen such that (29) is verified, then there exists a constant $c < \infty$ such that for any $v \in C_0^\infty(U)$, with $1/r + 1/q = \frac{1}{2}$:*

$$\|v\|_{L^r(U)} \leq c\,\|\,|p|^{s_0} v\|_{H^{t_1}(U)} + c\,\|\,|p|^{s_1} v\|_{H^{t_0}(U)}$$

We obtain (23) if $s_0 = 0$, $s_1 = 1$, $t_0 = 1$, $t_1 = \frac{3}{2}$ and (25) if $s_0 = 0$, $s_1 = 1$, $t_0 = 0$, $t_1 = \frac{1}{2}$.

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