Zeitschrift: Helvetica Physica Acta

Band: 53 (1980)

Heft: 1

Artikel: A new proof of the asymptotic nature of perturbation theory in P()_2 models

Autor: Summers, Stephen J.

DOI: https://doi.org/10.5169/seals-115105

Helvetica Physica Acta,
Vol. 53 (1980) Birkhäuser Verlag, Basel

A new proof of the asymptotic nature of perturbation theory in $P(\phi)_2$ models

Stephen J. Summers[1])

Centre de Physique Théorique, CNRS, Marseille

(7. I. 1980)

Abstract. A new method of proving the asymptotic nature of perturbation theory (for Schwinger and generalized Schwinger functions) in $P(\phi)_2$ quantum field models, which does not presume the convergence of a cluster expansion, is presented. The method recaptures all previous such results and, also, results (in application to certain models [Su 1, 2]) not yet attainable by previous methods. An explicit proof is given for $(\phi^4)_2$ in the two phase region to illustrate the essential points. Application to all $P(\phi)_2$ models with mean field limits is discussed.

I. Introduction

In the light of the fundamental importance of perturbation expansions in providing calculable links between quantum field theories and the experimental data they seek to explain, it is necessary to determine the mathematical status of these expansions and their relation to the exact, nonperturbative field theory. In recent years, important progress has been achieved. For weakly coupled $\lambda P(\phi)_2$ models (or for sufficiently large external field), the perturbation expansions for the Schwinger functions [Di] and for the generalized Schwinger functions [EEF, OSe] have been shown to be asymptotic to arbitrary order in the coupling constant λ.[2]) In fact, the perturbation expansions in $(\lambda\phi^4)_2$ for the Schwinger functions [EMS] and the mass and the two-body S-matrix [EE2] are known to be Borel summable, thus enabling the unique reconstruction of the exact quantum theory from the perturbation series. Similar results on the asymptotic nature of perturbation theory [FO, MS 1, EE 1] and its Borel summability [MS 2, EE 2] are known for weakly coupled $(\lambda\phi^4)_3$. Moreover, it has been shown in [GJS 4] that for $(\lambda\phi^4)_2$ deep in the two-phase region (i.e., very large λ), the perturbation series for the generalized Schwinger functions in variables centered at the appropriate classical means are asymptotic to arbitrary order in $\lambda^{-1/2}$.

However, the knowledge of the convergence of some sort of cluster expansion (see, e.g. [GJS 1, Sp, FO, GJS 4]) is a basic assumption in all of these results. And the definition of and the proof of convergence of cluster expansions are extremely complicated enterprises. Thus, we believe it will be of interest to present a new method of proving the asymptotic nature of perturbation theory (for Schwinger and generalized Schwinger functions) in $P(\phi)_2$ models that recaptures the result in all of the models mentioned above and permits the proof in models where the technical problems of proving the convergence of a cluster

[1]) New Address: Fachbereich 5, Universität Osnabrück, Osnabrück BRD.
[2]) It is known [J] that the perturbation series is divergent.

expansion are not yet solved (see [Su 1, 2]). This method has the advantage of being, we believe, substantially simpler than the earlier one, with a minimal amount of technical complications. We mention, however, if it actually be necessary, that cluster expansions are designed to answer many more questions than simply the asymptotic nature of perturbation theory – questions that cannot be answered through the arguments of this paper: what is the particle spectrum, is perturbation theory Borel summable, etc . . .

In order to suggest the range of application of our approach, it is necessary to recall a property emphasized in [GJS 2]. If a semibounded polynomial $P(x)$ has n (n may be infinite, e.g., when $P(x) = \cos x$) local minima ξ_j, and if one rewrites the polynomial in terms of variables centered at these minima,

$$P(x) = \sum_{i=3}^{d} a_i^j(x-\xi_j)^i + (m_c^j)^2(x-\xi_j)^2/2 + E_j,$$

where $d =$ degree of P, then if, as the dominant coupling constant a_d^j becomes small, one has

$$|a_i^j| \ll (m_c^j)^2, \qquad \forall_i \geq 3, \forall_j,$$

$P(x)$ is said to have a mean field limit. Of course, $\lambda P(x)$ (and $P(x) - hx$, when $|h|$ is large enough) trivially has a mean field limit and only one global minimum. Because the interaction parameters are small with respect to the classical mass m_c^j, heuristically one expects that the quantum corrections to the classical picture will be small. Thus, corresponding to the global minima of the polynomial will be (pure) states with means given approximately by the (classical) field values yielding the minima. If there are more than one global minima, the corresponding states will coexist, and the expectations of at least some physical observables in these states will differ. Thus, the quantum field model will manifest phase transitions. Polynomials with mean field limits and (infinitely) many global minima may be manufactured, and their study has already yielded interesting results [GJS 3, GJS 4, Fr. 2, FSS, CR, Ga, Su 1, 2].

The natural context of application of the arguments of this paper is the set of $P(\phi)_2$ models whose interaction polynomials possess a mean field limit and for whose associated (pure) states one can prove that the expectation that the average value of the field lies near any 'wrong' minimum ξ_j of the polynomial, i.e., any minimum other than the one determining the mean of the state, is suitably small.

To briefly (and crudely) anticipate the argument to be presented, let us consider a polynomial with n global minima ξ_j such that

$$a_i^j = O(a_d^1) \equiv O(a), \qquad d \geq i \geq 3. \tag{1.1}$$

Then integration by parts entails that, for example,

$$\langle \phi(x) - \xi_j \rangle^j = -\left\langle \int C(x-y) \sum_{i=3}^{d} i a_i^j : (\phi - \xi_j)^{i-1} : (y)\, dy \right\rangle^j, \tag{1.2}$$

where $\langle \cdot \rangle^j$ is the (pure) state corresponding to the minimum ξ_j, and

$$C(x-y) = (-\Delta + (m_c^j)^2)^{-1}(x, y)$$

is the covariance of the free Euclidean field with mass m_c^j. The space-time integration is controlled by the exponential decay of the free covariance $C(x-y)$.

Thus, we concentrate on

$$\langle :(\phi-\xi_j)^{i-1}:(y)\rangle^j, \qquad i\geq 3. \tag{1.3}$$

If one knows that the probability (in the j-state) that the average value of the field lies near the "correct" minimum ξ_j is suitably large, one expects that (1.3) is small, uniformly in a. Thus, the assumed smallness of the interaction parameters (1.1) entails with (1.2) that

$$\langle \phi(x)\rangle^j = \xi_j + O(a);$$

that is to say, one has an asymptotic expansion to zero'th order in a. Further integration by parts yields an expansion

$$\langle \phi(x)\rangle^j = \xi_j + \sum_{i=1}^{r} \alpha_i^j a^i + \sum_k \langle R_k(\phi-\xi_j)\rangle^j,$$

where the constants $\{\alpha_i^j\}$ are given simply by perturbation theory in the interaction (with bare mass m_c^j)

$$\sum_{i=3}^{d} a_i^j :(\phi-\xi_j)^i:,$$

and the remainders $\langle R_k(\phi-\xi_j)\rangle^j$ contain at least $r+1$ (derivatives of the) interaction polynomials, and, thus, are $O(a^{r+1})$. The main point of the paper is that one can prove the necessary bounds on the remainders, uniform in the coupling constant, without grinding through a cluster expansion.

We shall return to the generality of the approach in Chapter VI, but in order to present the essential elements of the argument in as transparent a manner as possible, we shall carry out the proof in detail for $(\phi^4)_2$ in the two-phase region. This will illustrate how to handle phase transitions and coexisting states in the simplest possible example. The proof in the weak coupling (single-phase) limit is trivial (see [Su 1]). The balance of the paper is organized as follows: Chapter II establishes the technical context and isolates the crucial estimate to be proven, i.e., the uniform estimate on the remainder. Chapter III presents the proof of this estimate, assuming two other bounds. The first bound is proven in Chapter IV, using chessboard estimates and vacuum energy bounds. The second is proven, using a Peierls' argument and convexity properties of the vacuum energy density, in Chapter V.

II. Existence and integration by parts

There is presently a reasonably large number of devices to construct a state corresponding to a semibounded interaction polynomial in two space-time dimensions (see, e.g., [GJ 2] for a brief overview). There are essentially two methods that are valid for arbitrary coupling parameters: a compactness argument (going back to [GJ 1]), applicable to arbitrary semibounded polynomial, and arguments employing correlation inequalities and upper bounds (the correlation inequalities providing monotone convergence), applicable to even $P(\phi)_2$ models with half-Dirichlet boundary conditions [GRS 1] and to arbitrary semibounded polynomials [FS] (the latter provides a construction that coincides with the above compactness construction for 'almost all' values of the external field). The argument that will be presented is applicable to states constructed by any of these methods. We shall explicitly consider states obtained through the most general construction – the compactness argument – and shall at the appropriate places in the proof comment on the applicability to states obtained through the other means of construction. Furthermore, although we shall work only with Euclidean quantum fields, all results will have a natural translation into physical (Minkowski) space through the Osterwalder–Schrader reconstruction theorem [OS].

The existence theorem we shall state and utilize has a long history; see [GJ 2] for references to the literature. We shall state it in its Euclidean form. To do so, we must define the following objects. The finite volume interacting measure for the interaction density P, a semibounded polynomial, is defined by

$$d\phi_\Lambda = e^{-\int_\Lambda :P(\phi):(x)\,dx}\, d\mu(\phi)_C,$$

where $d\mu(\phi)_C$ is the Gaussian (probability) measure with support on $S'(R^2)$ (the space of tempered distributions), with mean zero and covariance $C = (-\Delta + m^2)^{-1}$. m is the bare mass of the model and : : denotes Wick ordering with respect to $d\mu(\phi)$.

If we define the following function spaces

$$L_{1,p} = L_1(R^2) \cap L_p(R^2), \qquad p < \infty$$
$$L_{1,\infty,\epsilon} = L_1(R^2) \cap \{f \mid \|f\|_\infty < \epsilon\},$$

we have from [GJ 2]:

Theorem 2.1. *The infinite volume Schwinger functions*

$$S_n(x_1, \ldots, x_n) = \lim_{\Lambda \uparrow R^2} \frac{\int \prod_{i=1}^n \phi(x_i)\, d\phi_\Lambda}{\int d\phi_\Lambda}$$

exist and are moments of a unique measure $d\phi$ on $S'(R^2)$. Moreover, they satisfy the Osterwalder–Schrader axioms [OS], excluding possibly clustering and Euclidean invariance (however, time-translation invariance holds). Let $1 \le m_i \le d =$ degree P, $1 \le i \le n$, and ϵ be sufficiently small. The generalized Schwinger functions

$$\int \prod_{i=1}^n :\phi^{m_i}: (x_i)\, d\phi$$

are continuous as multilinear forms on $\prod_{i=1}^n L_{1,\, d/d-m_i}$. They are functional derivatives of

$$Z(h_1, \ldots, h_d) = \int \exp\Big(\sum_{j=1}^d :\phi^j: (h_j) \Big)\, d\phi,$$

which is bounded and analytic in $h_j \in L_{1,\, d/d-j}$.

Remarks. 1. The above theorem also obtains with Dirichlet boundary conditions on the state.

2. The restriction on the degree of the Wick monomials in the generalized Schwinger functions (which does not, however, restrict the total degree of the product) will be tacitly assumed in the balance of the paper.

We also have from [GJ 2] the fact that, with the same degree of generality, one can integrate by parts in the infinite volume state:

Theorem 2.2. *The following formula is valid.*

$$\int :\phi^d:(h)R(\phi)\,d\phi = \int d\phi \int dx A(x)\left[\frac{\delta R}{\delta\phi(x)} - R(\phi) :P'(\phi(x)):\right],$$

where

$$A(x) = j\int dy C(x-y)h(y) :\phi^{j-1}:(y),$$

$$R(\phi) = \prod_{i=1}^{n} :\phi^{m_i}: (h_i),$$

$$C = (-\Delta + m^2)^{-1}.$$

The particular model we have chosen in order to illustrate our method is given by the following interaction polynomial:

$$P(x) = \lambda x^4 - \tfrac{1}{4}x^2 - hx - E_c, \tag{2.1}$$

where $0 \le \lambda \ll 1$, $|h| \le \lambda^2$ and $\inf_x P(x) = 0$. (The limitation on h here is *not* essential, but will spare us some calculation.) This polynomial has two local minima

$$\xi_\pm = \pm(8\lambda)^{-1/2} + h + O(h^2), \tag{2.2}$$

and we remark that

$$E_c = -[(64\lambda)^{-1} + h(8\lambda)^{-1/2}] + O(h^2). \tag{2.3}$$

It is important to note that

$$\lambda x^4 - \tfrac{1}{4}x^2 + (64\lambda)^{-1} = \lambda(x-\xi_\pm)^4 \pm (2\lambda)^{1/2}(x-\xi_\pm)^3 + \tfrac{1}{2}(x-\xi_\pm)^2; \tag{2.4}$$

thus, this polynomial possesses a mean field limit.

The interaction in the (bounded) space-time region Λ is defined to be

$$:P_\Lambda: = \int_\Lambda :P(\phi):(x)\,dx,$$

where : : denotes Wick ordering with respect to mass $m^2 = 1$. We comment that by scaling and re-Wick ordering, this interaction is equivalent to [GJS 2]

$$\lambda_0 :\phi^4: + \tfrac{1}{2} :\phi^2:, \qquad \lambda_0 \gg 1.$$

The finite volume interacting measures we shall consider are given by

$$d\phi_\Lambda^\pm = e^{-:P_\Lambda: + \frac{1}{2}:(\phi-\xi_\pm)^2_\Lambda:}\,d\mu(\phi-\xi_\pm), \tag{2.5}$$

where $d\mu(\phi-\xi_\pm)$ is the Gaussian measure with mean $\xi_\pm$ and covariance $C = (-\Delta+1)^{-1}$. We note that the second term in the interaction exponent cancels the mass and the mean of the Gaussian measure in Λ, leaving an external field $\xi_\pm$ exterior to Λ. This term also cancels the quadratic term of the polynomial expressed in the variable $\phi-\xi_\pm\equiv\Psi_\pm$. In particular, at $h=0$,

$$\begin{aligned} :P(\Psi_\pm):&\equiv:P(\phi):-\tfrac{1}{2}:(\phi-\xi_\pm)^2: \\ &=\lambda\,:\Psi_\pm^4:\pm(2\lambda)^{1/2}:\Psi_\pm^3:. \end{aligned} \tag{2.6}$$

The infinite volume states obtained from the measures (2.5), whose existence are assured by Theorem 2.1, are both states associated to the polynomial $P(x)$. Indeed, it is easy to see, using [Fr 1, FS], that both states satisfy the DLR equations for $P(x)$.

We may now state the theorem to be proven. To minimize unnecessary repetition, we will generally state results for the + state in $h\geq 0$ and leave tacit the obvious statement for the − state in $h\leq 0$.

Theorem 2.3. *For all $0\leq h\leq\lambda^2$ and any h, $\{m_i\}$ and r positive integers, there exist coefficients $\{\alpha_i^+(h)\}$, such that for all small enough $\lambda\geq 0$,*

$$\int\prod_{i=1}^{n}:(\phi-\xi_+)^{m_i}:(x_i)\,d\phi^+=\sum_{i=1}^{r}\alpha_i^+(h)\lambda^{i/2}+O(\lambda^{(r+1)/2}).$$

The coefficients $\{\alpha_i^+(h)\}$ are independent of λ and continuous in h. They are, in fact, precisely those given by perturbation theory calculated about the minimum ξ_+. $O(\lambda^{(r+1)/2})$ depends on $N(A)=\sum_{i=1}^n m_i$.

Remarks. 1. Thus, perturbation theory about the appropriate minimum is asymptotic in $\lambda^{1/2}$ to arbitrary order.

2. The $\pm$ state at $h=0$ will be defined as a limit of $\pm$ states as $h\downarrow 0$ ($h\uparrow 0$); see Chapter V. Results from the convergence of the mean field cluster expansion in this model suggest that this additional limit should be unnecessary, i.e., that the $\pm$ boundary conditions placed on the finite volume measures should suffice in picking out the correct state at $h=0$. However, we have not yet been able to eliminate this step.

3. It should be commented, since our states are not necessarily Euclidean invariant, that the result is independent of the choice of the $\{x_i\}$. However, the $\pm$ state at $h=0$ will be shown to satisfy *all* of the Osterwalder–Schrader axioms (including clustering).

Proof. Theorem 2.3 is a statement about distributions, thus to minimize unnecessary technical complications, we shall "smear" the Wick monomials in unit lattice squares, as follows. Place a lattice of unit squares Δ_j, centered at the lattice sites $j\in Z^2$, over R^2. Let

$$:\Psi_+^{m_i}:(\Delta_i)=\int_{\Delta_i}:\Psi_+^{m_i}:(x_i)\,dx_i.$$

Then we shall actually consider

$$\int\prod_{i=1}^{n}:(\phi-\xi_+)^{m_i}:(\Delta_i)\,d\phi^+\equiv\left\langle\prod_{i=1}^{n}:(\phi-\xi_+)^{m_i}:(\Delta_i)\right\rangle^+.$$

The choice of a unit lattice is arbitrary. Any other lattice would suffice as well.

By Theorem 2.2, we may integrate by parts in the + state. Thus, by repeated integration by parts, applied to all the linear factors of the original product of Wick monomials and to the linear factors of the subsequent derivatives of the interaction polynomial $:P(\Psi_+):$ ((2.6)) brought into the integrand and continued until each term in the resulting sum either is a constant on path space or contains at least $r+1$ (derivatives of the) polynomials $:P(\Psi_+):$, one obtains the following expansion:

$$\left\langle \prod_{i=1}^{n} :(\phi-\xi_+)^{m_i}: (\Delta_i) \right\rangle^+ = \sum_{i=1}^{r} \alpha_i^+(h)\lambda^{i/2} + \sum_k \langle R_k(\Psi_+)\rangle^+, \tag{2.7}$$

where a typical term in the finite sum over $\langle R_k(\psi_+)\rangle^+$ is of the form

$$\left\langle \int \left(\prod_{\mu=1}^{M} :P^{(\alpha_\mu)}(\psi_+): (y_\mu)\right) w(y)\, dy \right\rangle^+,$$

and $M \geq r+1$, $P^{(\alpha_\mu)}$ is the α_μth derivative of P, and

$$w(y) = \int v(x, y) \prod_{i=1}^{n} \chi_{\Delta_i}(x_i)\, dx.$$

$\chi_{\Delta_i}(x_i)$ denotes the characteristic function for the unit square Δ_i, and $v(x, y)$ is a product of $N \geq M$ factors $C(x_i - x_j)$, $C(x_i - y_j)$, $C(y_i - y_j)$. It is easy to see that the constants $\alpha_i^+(h)$ are exactly those given by perturbation theory about the minimum ξ_+. We note that because the interaction coefficients of $:P(\psi_+):$ are $O(\lambda^{1/2})$ (see (2.6)) and because we require that each $R_k(\psi_+)$ contains at least $r+1$ (derivatives of the) $:P(\psi_+):$'s, the coefficients of $\langle R_k(\psi_+)\rangle^+$ are $O(\lambda^{(r+1)/2})$.

Theorem 2.3 will be proven if we can prove the following essential bound.

Proposition 2.4. *For all $0 \leq h \leq \lambda^2$, and any positive integers M, $\{\alpha_\mu\}_{\mu=1}^M$, one has for all small enough $\lambda \geq 0$,*

$$\left| \left\langle \int \left(\prod_{\mu=1}^{M} :P^{(\alpha_\mu)}(\psi_+): (y_\mu)\right) w(y)\, dy \right\rangle^+ \right| = O(\lambda^{M/2}).$$

This is the crucial estimate, which shall be proven using the facts that at average values of the field ϕ in a small neighborhood of ξ_+, the interaction is weak, and, for $h \geq 0$, that the probability that the average value of the field lies outside of this neighborhood is small.

III. Estimate of the remainder

In this chapter we will prove Proposition 2.4, assuming two estimates that are proven in following chapters. Before we state these estimates, however, we need another definition. Let

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b] \\ 0, & x \notin [a, b] \end{cases}, \tag{3.1}$$

and let

$$\chi_+(x) = \chi_{[0,\infty)}(x)$$
$$\chi_-(x) = \chi_{(-\infty,0)}(x).$$

We define the average field in a unit lattice square Δ to be

$$\phi(\Delta)=\int_{\Delta}\phi(x)\,dx.$$

Then, we will write

$$\chi_{\pm}(\Delta)=\chi_{\pm}(\phi(\Delta)). \tag{3.2}$$

Note that

$$1=\chi_{+}(\Delta)+\chi_{-}(\Delta).$$

We also wish to define a 'spin configuration' function $\sigma(\cdot)$, which is constant on unit lattice squares and takes only + or − as values. The name arises from the fact that a product such as

$$\prod_{\Delta\subset R^2}\chi_{\sigma(\Delta)}(\Delta)$$

'maps' the Euclidean field $\phi(x)$, whose average in a square Δ, $\phi(\Delta)$, is an unbounded random variable, onto a configuration of spins that are either 'up' or 'down', i.e., discrete random variables. We introduce the 'spin' characteristic functions $\chi_{\pm}(\Delta)$ in order to examine separately those regions of path space $S'(R^2)$ whose elements have average values lying close to the correct (wrong) minimum.

Finally, for a given unit lattice square Δ, we define

$$F^{\sigma(\Delta)}(\Delta)=\prod_{i=1}^{n}c_i:(\phi-\xi_{\sigma(\Delta)})^{m_i}:(\Delta), \tag{3.3}$$

where $\{c_i\}_{i=1}^{n}$ is a set of given coefficients. We denote the total degree of F by

$$N(F(\Delta))=\sum_{i=1}^{n}m_i. \tag{3.4}$$

Then we state the following result.

Proposition 3.1. *Let $\{w_j\}_{j=1}^{\nu}$ be a collection of localized functions such that $w_j\in L_q(\Delta_j)$, for some $q>1$. Then for any collection*

$$\{F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j)\}_{j=1}^{\nu},$$

there exists a constant K such that for all small enough λ and $|h|\le\lambda^2$, if $K(N)=K^NN!$, one has the following estimate:

$$\left|\left\langle\prod_{j=1}^{\nu}F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j)\right\rangle^{\pm}\right|$$

$$\le\prod_{\{j\mid\sigma_{1,j}=\sigma_{2,j}\}}\left[\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\,\|w_j\|_p\right]$$

$$\times\prod_{\{j\mid\sigma_{1,j}\neq\sigma_{2,j}\}}\left[\lambda^{-N(F_j)/2}\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\,\|w_j\|_p\right],$$

for any $p>1$.

To make clear the meaning of this estimate, let us take a simple example. It informs us that while

$$|\langle(\phi-\xi_+)(\Delta)\chi_+(\Delta)\rangle^{\pm}|\leq K, \tag{3.5}$$

we have

$$|\langle(\phi-\xi_+)(\Delta)\chi_-(\Delta)\rangle^{\pm}|\leq O(\lambda^{-1/2}). \tag{3.6}$$

This is reasonable, since (3.5) says if one looks in that part of path space where the average value of the field is near ξ_+, then the expectation of $(\phi-\xi_+)(\Delta)$ is 'small'. But if one looks where the average value of the field is near ξ_-, (3.6) asserts that the expectation of $\phi-\xi_+$ could be large since $|\xi_- - \xi_+| = O(\lambda^{-1/2})$.

The following estimate provides a precise statement of the fact that the probability that the average value of the field actually is close to the wrong minimum is very small.

Proposition 3.2. *There exists a* $\lambda_0>0$ *such that for all* $0\leq\lambda\leq\lambda_0$ *and all* $0\leq h\leq\lambda^2$, *there exists a* $c>0$ *such that*

$$\langle\chi_-(\Delta)\rangle^+\leq e^{-c\lambda^{-1/2}},$$

independently of Δ.

This is proven via a Peierls' argument and some careful analysis in Chapter V.

Proof of Proposition 2.4. Before we launch into the most general case, let us return to the simple example of the introduction. To obtain the uniform bound on the counterpart, in our model, to (1.3), we write first

$$|\langle :(\phi-\xi_+)^{i-1}: (\Delta)\rangle^+| = |\langle :(\phi-\xi_+)^{i-1}: (\Delta)\chi_+(\Delta)\rangle^+ + \langle :(\phi-\xi_+)^{i-1}: (\Delta)\chi_-(\Delta)\rangle^+|. \tag{3.7}$$

Proposition 3.1 entails that the absolute value of the first term on the right hand side is bounded by $K(i-1)$. And application of Hölder's inequality yields for the second term:

$$|\langle:(\phi-\xi_+)^{i-1}: (\Delta)\chi_-(\Delta)\rangle^+| \leq \langle(:(\phi-\xi_+)^{i-1}: (\Delta))^2\chi_-(\Delta)\rangle^{+1/2}\langle\chi_-(\Delta)\rangle^{+1/2}. \tag{3.8}$$

The first factor is estimated through Proposition 3.1 by

$$K(i-1)\lambda^{-(i-1)/2},$$

and the second factor is bounded by $e^{-c\lambda^{-1/2}/2}$, according to Proposition 3.2 $(h\geq 0)$. Thus, (3.7) is bounded by

$$K(i-1)(1+\lambda^{-(i-1)/2}e^{-c\lambda^{-1/2}}),$$

and the necessary uniform bound is proven.

In the general case (2.7), each term in the sum

$$\sum_k \langle R_k(\psi_+)\rangle^+$$

can be represented graphically, with the lines of the graph due to the free covariances $C(x-y)$ and the vertices provided by the derivatives of the original Wick monomials and of the interaction polynomial. The basic point is to estimate the vertices uniformly with respect to λ and independent of their position in space-time and to control the integration over vertex positions by the exponential decay of the free covariance, as in standard perturbation theory. It will be helpful to keep the diagrammatic representation in the back of the mind.

In order to avoid some technical problems, we will assume that not only every linear factor of the original product of Wick monomials has been integrated out, but that no two vertices with derivatives of the interaction polynomial are contracted to each other, unless one or both of them have been completely integrated out, i.e., unless one or both are constants on path space. This assumption leads to no loss of generality, since, presented by a remainder term not satisfying this assumption, one simply continues integrating by parts until the recalcitrant vertices are constants on path space. This only increases M and produces more terms.

With this assumption a typical term that must be estimated is of the form

$$O(\lambda^{M_1/2})\left\langle\int\left(\prod_{\mu=1}^{M_2} :P^{(\alpha_\mu)}(\psi_+):(y_\mu)\right)w(y)\,dy\right\rangle^+, \tag{3.9}$$

where $M_1+M_2=M$ and $O(\lambda^{M_1/2})$ contains the interaction coefficients of the M_1 completely integrated out interaction vertices. Furthermore,

$$w(y)=\int v(x,y)\prod_{i=1}^{n}\chi_{\Delta_i}(x_i)\prod_{i=1}^{n}dx_i\prod_{k=1}^{M_1}dx_k,$$

and $v(x, y)$ is a product of $N(\geq M_1+M_2)$ factors $C(x_i-x_j)$ and $C(x_i-y_j)$. As a shorthand we have subsumed by x not only the position variables (x_i) of the original product of Wick monomials but also those (x_k) of the completely integrated interaction vertices. For the x_k variables, there are, of course, no characteristic functions in the expression for $w(y)$. The point of the assumption we have made is that there are *no* factors $C(y_i-y_j)$, joining vertices with uncontracted fields, in the definition of $w(y)$. The usefulness of this fact will be clear soon. At each variable y_μ and x_k we make a localization sum

$$1=\sum_{j\in Z^2}\chi_{\Delta_j}(\cdot)$$

(the x_i variables are already localized), which yields for (3.9):

$$O(\lambda^{M_1/2})\sum_J\left\langle\int\left(\prod_{\mu=1}^{M_2} :P^{(\alpha_\mu)}(\psi_+):(y_\mu)\right)\prod_{\mu=1}^{M_2}\chi_{\Delta_{j_\mu}}(y_\mu)w_J(y)\,dy\right\rangle^+, \tag{3.10}$$

where

$$w_J(y)=\int v(x,y)\prod_{i=1}^{n}\chi_{\Delta_i}(x_i)\prod_{k=1}^{M_1}\chi_{\Delta_{j_k}}(x_k)\,dx.$$

Each $J=\{j_\nu\}_{\nu=1}^N=\{(j_{\nu,1},j_{\nu,2})\}_{\nu=1}^N$ $(j_\nu\in Z^4)$ denotes a choice of unit lattice square localizations for all covariances in $v(x, y)$, and Δ_{j_μ} is the localization of the μth vertex determined by the choice of J. (Note: J is chosen so that covariances contracting to the same vertex must have the same localization in that variable.) Then, in each term of the sum (3.10) and at each unit lattice square Δ_{j_μ}, one performs the spin configuration expansion

$$1=\chi_+(\Delta_{j_\mu})+\chi_-(\Delta_{j_\mu}).$$

Thus, (3.10) becomes

$$O(\lambda^{M_1/2})\sum_J\sum_{\sigma(\cdot)}\left\langle\int\left(\prod_{\mu=1}^{M_2}:P^{(\alpha_\mu)}(\psi_+):(y_\mu)\right)\right.$$
$$\left.\times\prod_{\mu=1}^{M_2}[\chi_{\Delta_{j_\mu}}(y_\mu)\chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu})]w_J(y)\,dy\right\rangle^+,\quad(3.11)$$

where, of course, the sum $\sum_{\sigma(\cdot)}$ is over the possible 'spin configurations' that can be formed by $\sigma(\Delta_{j_\mu})$, $\mu=1,\ldots,M_2$.

In order to eventually control the sum over localizations, we wish to use the fact that the free covariances are exponentially decreasing and locally integrable; indeed, we have for all x and y:

$$\begin{aligned}O\le C(x-y)&=(-\Delta+1)^{-1}(x,y)\\&\le e^{-(1-\delta)|x-y|}(1+c\,|\ln|x-y||),\end{aligned}\quad(3.12)$$

where $\delta>0$ is arbitrary and small. To display the exponential decoupling, we multiply by 1 in such a way that (3.11) becomes

$$O(\lambda^{M_1/2})\sum_J\sum_{\sigma(\cdot)}\prod_{\nu=1}^{N}\exp\{-\text{dist}(\Delta_{j_{\nu,1}},\Delta_{j_{\nu,2}})/2\}$$
$$\times\left\langle\int\left(\prod_{\mu=1}^{M_2}:P^{(\alpha_\mu)}(\psi_+):(y_\mu)\chi_{\Delta_{j_\mu}}(y_\mu)\chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu})\right)w_J'(y)\,dy\right\rangle^+,$$

where

$$w_J'(y)=\prod_{\nu=1}^{N}\exp\{\text{dist}(\Delta_{j_{\nu,1}},\Delta_{j_{\nu,2}})/2\}w_J(y).$$

Using the fact that no two y_μ's are contracted together, we can rewrite the above as

$$O(\lambda^{M_1/2})\sum_J\sum_{\sigma(\cdot)}\prod_{\nu=1}^{N}\exp\{-\text{dist}(\Delta_{j_{\nu,1}},\Delta_{j_{\nu,2}})/2\}$$
$$\times\left\langle\prod_{\mu=1}^{M_2}(:P^{(\alpha_\mu)}(\psi_+):(f_{j_\mu})\chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu}))\right\rangle^+,\quad(3.13)$$

where, if we define

$$\gamma_{j_\mu}=\{j_\nu\mid j_{\nu,i}=\Delta_{j_\mu},\ i=1\text{ or }2\},$$

we have

$$f_{j_\mu}(y_\mu)=\chi_{\Delta_{j_\mu}}(y_\mu)\prod_{j_\nu\in\gamma_{j_\mu}}\exp\{\text{dist}(\Delta_{j_{\nu,1}},\Delta_{j_{\nu,2}})/2\}\int\prod_{(j_\nu)'\in\gamma_{j_\mu}}[C(y_\mu,x_{j_\nu'})\chi_{\Delta_{j_\nu'}})\,dx_{j_\nu'}],$$

and j_ν' denotes that member of the double $(j_{\nu,1},j_{\nu,2})$ that does not index Δ_{j_μ}

(except, of course, in the case that $\Delta_{j_{\nu,1}} = \Delta_{j_{\nu,2}} = \Delta_{j_\mu}$). In other words, the expectation in (3.13) is of smeared fields, the test functions being (essentially) the free covariances that contract to the corresponding vertex, localized in both variables of their argument.

We focus attention now on

$$\left| \left\langle \prod_{\mu=1}^{M_2} [:P^{(\alpha_\mu)}(\psi_+): (f_{j_\mu})\chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu})] \right\rangle^+ \right|. \tag{3.14}$$

If one of the spin characteristic functions is centered at ξ_-, we use Hölder's inequality to bound (3.14) by

$$\left\langle \left(\prod_{\mu=1}^{M_2} :P^{(\alpha_\mu)}(\psi_+): (f_{j_\mu}) \right)^2 \prod_{\mu=1}^{M_2} \chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu}) \right\rangle^{+1/2} \times \langle \chi_-(\Delta) \rangle^{+1/2}. \tag{3.15}$$

If there are only χ_+'s in the expectation, it is left in peace. If $\sigma(\Delta_{j_\mu}) = +$ for all $\mu = 1, \ldots, M_2$, then Proposition 3.1 provides the following bound for (3.14):

$$K_1(5M_2) \prod_{\mu=1}^{M_2} (\lambda + (2\lambda)^{1/2}) K_2^{m(\Delta_{j_\mu})},$$

with constants K_i independent of J ($m(\Delta_{j_\mu})$ is the number of elements in the set γ_{j_μ}), since

$$\|f_{j_\mu}\|_2 = \Bigg\| \chi_{\Delta_{j_\mu}}(y_\mu) \prod_{j_\nu \in \gamma_{j_\mu}} \exp\{\text{dist}(\Delta_{j_{\nu,1}}, \Delta_{j_{\nu,2}})/2\} \times \int \prod_{(j_\nu)' \in \gamma_{j_\mu}} [C(y_\mu, x_{j'_\nu}) \chi_{\Delta_{j'_\nu}}(x_{j'_\nu})\, dx_{j'_\nu}] \Bigg\|_2 \le K_2^{m(\Delta_{j_\mu})},$$

K_2 independent of the choice of J, by (3.12).

The following worst-case bound (i.e., when all $\sigma(\cdot)$'s $\neq +$) for the first factor in (3.15) is also given by Proposition 3.1:

$$K_1(5M_2) \prod_{\mu=1}^{M_2} [\lambda(\lambda^{-(4-\alpha_\mu)/2}) + (2\lambda)^{1/2}(\lambda^{-(3-\alpha_\mu)/2})] K_2^{m(\Delta_{j_\mu})}$$

with constants independent of the choice of J. However, Proposition 3.2 provides that, when $h \ge 0$ and λ is small enough,

$$\langle \chi_-(\Delta) \rangle^{+1/2} \le e^{-c\lambda^{-1/2}/2}, \qquad c > 0,$$

independently of Δ. Observing that

$$\sum_\mu m(\Delta_{j_\mu}) \le N(A) + 3M,$$

for all J, we may estimate the absolute value of (3.9) by

$$O(\lambda^{M_1/2}) K_3^{N(A)} 2^{M_2} K_4(M) [\lambda^{M_2/2} + e^{-c\lambda^{-1/2}} \lambda^{-M_2/2}] \times \sum_J \prod_{\nu=1}^N \exp\{-\text{dist}(\Delta_{j_{\nu,1}}, \Delta_{j_{\nu,2}})/2\}.$$

The factor 2^{M_2} comes from the sum over spin configurations. The sum over localizations is easily bounded by K_5^N, and since $N \le N(A) + 3M$, the total degree of the Wick powers of the field that have been brought into the integrand, we have

$$\left|\left\langle\int\left(\prod_{\mu=1}^{M}:P^{(\alpha_\mu)}(\psi_+):(y_\mu)\right)w(y)\,dy\right\rangle^{+}\right|=O(\lambda^{M/2}).$$

The $O(\lambda^{M/2})$ is, of course, dependent on M and $N(A)$.

This completes the proof of Proposition 2.4, given Propositions 3.1 and 3.2. It is clear that given these two results (or their counterparts) the argument above is quite general.

IV. Vacuum energy estimates

In this chapter, Proposition 3.1 and bounds necessary for the proof of Proposition 3.2 in the next chapter will be demonstrated. The essential bounds are vacuum energy estimates that are uniform in λ, which are attained by examining the subsets of path space determined by the spin characteristic functions $\chi_\pm(\Delta)$. To be more precise, if $\phi_\kappa(x)$ denotes the ultraviolet cutoff field (we use the ultraviolet cutoff of [GJS 4], which has the useful property that $\phi(\Delta)=\phi_\kappa(\Delta)$), we note that the ultraviolet cutoff interaction density

$$:P(\phi_\kappa):(x)-\tfrac{1}{2}:(\phi_\kappa-\xi_+)^2:(x) \tag{4.1}$$

is not uniformly bounded from below as $\lambda\downarrow 0$. When, in fact, the field is close to ξ_-, (4.1) is $-O(\lambda^{-1})$, since

$$P(\xi_-)\approx 0$$

(h is small). However, when $\phi_\kappa\geq 0$, (4.1) *is* uniformly bounded from below as $\lambda\downarrow 0$. Of course, $\chi_+(\Delta)$ restricts only the average field $\phi(\Delta)$, and

$$\phi(x)=\phi(\Delta)+\delta\phi(x),\qquad x\in\Delta,$$

so it will be necessary to control an error term.

In Chapter V it will be necessary to consider path space in yet smaller pieces. In fact, we wish to define the 'shrunken' spin characteristic functions that restrain the average value of the field to lie very close to the minima of the polynomial.

$$\chi_{+,s}(\Delta)=\chi_{[\xi_+-\lambda^{1/4}\xi_+,\,\xi_++\lambda^{1/4}\xi_+]}(\phi(\Delta)), \tag{4.2}$$

$$\chi_{-,s}(\Delta)=\chi_{[\xi_--\lambda^{1/4}\xi_+,\xi_-+\lambda^{1/4}\xi_+]}(\phi(\Delta)) \tag{4.3}$$

(see (3.1)). The "peak" characteristic functions are:

$$\chi_{\pm,p}(\Delta)=\chi_\pm(\Delta)-\chi_{\pm,s}(\Delta). \tag{4.4}$$

Lemma 4.1. *Let $0<\eta\leq 10^{-3}$ and $10^3\eta\leq\zeta$. Then there are strictly positive constants $a=a(\zeta)$, $b=b(\zeta)$. such that for all $0\leq\lambda\leq 10^{-2}$, $|h|\leq\lambda^2$, any Δ, $x\in\Delta$, $\sigma(\Delta)$, and any (large) κ,*

$$:P(\phi_\kappa):(x)-\eta/2:(\phi_\kappa-\xi_{\sigma(\Delta)})^2:(x)-\ln\chi_{\sigma(\Delta)}(\Delta)\geq -a\ln^2\kappa-\zeta:\delta\phi_\kappa^2:(x), \tag{i}$$

$$:P(\phi_\kappa):(x)-\eta/2:(\phi_\kappa-\xi_{\sigma(\Delta)})^2:(x)-\ln\chi_{\sigma(\Delta),p}(\Delta)\geq b\lambda^{-1/2}-a\ln^2\kappa-\zeta:\delta\phi_\kappa^2:(x). \tag{ii}$$

Remark. 1. Since $\chi_{\sigma(\Delta),p}(\Delta)$ restrains the field to take values where the polynomial (2.1) is large, (ii) is reasonable.

2. (i) was proven in [GJS 4]. Thus, we will outline only the proof of (ii).

Proof. Define τ by:

$$\tau(x) = :P(\phi_\kappa): (x) - \eta/2 :(\phi_\kappa - \xi_{\sigma(\Delta)})^2: (x) + \zeta :\delta\phi_\kappa^2: (x).$$

Using the fact that the ultraviolet cutoff Wick ordering constants are $O(\ln \kappa)$, one can easily, as in [GJS 4], show that for any $\epsilon > 0$,

$$\tau(x) \geq (1 - \epsilon\lambda)P(\phi_\kappa)(x) - \eta/2(\phi_\kappa - \xi_{\sigma(\Delta)})^2(x) + \zeta\delta\phi_\kappa^2(x) - O(\epsilon^{-1}) \ln^2 \kappa.$$

We wish to show that

$$\tau(x) - \ln \chi_{\sigma(\Delta),p}(\Delta) \geq b\lambda^{-1/2} - O(1) \ln^2 \kappa. \tag{4.5}$$

(4.5) will be demonstrated for $\sigma(\Delta) = -$. The case $\sigma(\Delta) = +$ is similar.

Case 1: $|\phi_\kappa(x) - \xi_-| \leq \lambda^{1/4}\xi_+/2$

When $\phi(\Delta) \in [\xi_- - \lambda^{1/4}\xi_+, \xi_- + \lambda^{1/4}\xi_+]$, $\chi_{-,p}(\Delta) = 0$, so that this range of average field values yields (4.5). Because $\phi_\kappa(x) = \phi(\Delta) + \delta\phi_\kappa(x)$, we thus must have either (a) $\delta\phi_\kappa \geq \lambda^{1/4}\xi_+/2$ or (b) $\delta\phi_\kappa \leq -\lambda^{1/4}\xi_+/2$ (i.e., when $\phi(\Delta) < \xi_- - \lambda^{1/4}\xi_+$ or $\phi(\Delta) > \xi_- + \lambda^{1/4}\xi_+$). In both cases

$$\delta\phi_\kappa^2(x) \geq \lambda^{1/2}\xi_+^2/4.$$

But, for Case 1 field values (we set $\epsilon = 10^{-3}$),

$$(1 - \epsilon\lambda)P(\phi_\kappa)(x) - \eta/2(\phi_\kappa - \xi_-)^2(x) \geq \tfrac{1}{4}(\phi_\kappa - \xi_-)^2(x) - \eta/2(\phi_\kappa - \xi_-)^2(x) \geq 0$$

(follows from (2.4)). Because $-\ln \chi_{\sigma(\Delta),p}(\Delta) \geq 0$ and $\lambda^{1/2}\xi_+^2/4 = O(\lambda^{-1/2})$, (4.5) is confirmed in this case.

Case 2; $|\phi_\kappa(x) - \xi_+| \leq \xi_+/2$

We must have either (a) $\delta\phi_\kappa(x) \geq \xi_+/4$ or (b) $\phi(\Delta) \geq \xi_+/4$. However, when $\phi(\Delta) \geq 0$, $\chi_{-,p}(\Delta) = 0$, so that (4.5) follows trivially in subcase (b). In Case 2,

$$(\phi_\kappa(x) - \xi_-)^2 \leq 9\xi_+^2$$

so that

$$\tau(x) \geq (1 - \epsilon\lambda)P(\phi_\kappa)(x) - \eta 9\xi_+^2/2 + \zeta\delta\phi_\kappa^2(x) - O(\epsilon^{-1}) \ln^2 \kappa.$$

Because $P(\phi_\kappa)(x) \geq 0$, in subcase (a) we have

$$\begin{aligned} \tau(x) &\geq -\eta 9\xi_+^2/2 + \zeta\xi_+^2/16 - O(\epsilon^{-1}) \ln^2 \kappa \\ &\geq b\lambda^{-1} - O(\epsilon^{-1}) \ln^2 \kappa, \end{aligned}$$

for $b > 0$. Thus, (4.5) is proven in Case 2.

Case 3: $\xi_- + \lambda^{1/4}\xi_+ \leq \phi_\kappa(x) \leq \xi_+/2$

A straightforward calculation shows that

$$P(\phi_\kappa) = \lambda(\phi_\kappa - (8\lambda)^{-1/2})^2(\phi_\kappa + (8\lambda)^{-1/2})^2 - h\phi_\kappa - \delta E_c, \tag{4.6}$$

where δE_c, the difference $E_c(h)-E_c(h=0)$, is (for $|h|\leq\lambda^2$) $O(\lambda^{3/2})$. Thus, in Case 3, a glance at (2.2) makes it clear that there exists a $b>0$ such that

$$(1-\epsilon\lambda)P(\phi_\kappa)(x)-\eta/2(\phi_\kappa(x)-\xi_-)^2\geq b\lambda^{-1/2}. \tag{4.7}$$

Therefore, (4.5) obtains in Case 3.

Case 4: $\phi_\kappa\geq 3\xi_+/2$ or $\phi_\kappa\leq\xi_- -\lambda^{1/4}\xi_+$

Referring once again to (4.6), one sees that there exists a $b>0$ such that for all $\phi_\kappa(x)$ in Case 4, (4.7) holds. This completes the proof of Lemma 4.1.

If we define

$$W^{\pm}(x)=:P(\phi):(x)-\eta/2:(\phi-\xi_\pm)^2:(x),$$

$W^{\pm}_\kappa(x)$ by the substitution $\phi\to\phi_\kappa$ and $\delta W^{\pm}_\kappa(x)=W^{\pm}(x)-W^{\pm}_\kappa(x)$, we have

Lemma 4.2. *There are positive constants K and δ such that if $\{m(\Delta)\mid\Delta\subset R^2\}$ is a set of nonnegative integers and $\{\kappa(\Delta)\mid\Delta\subset Y\subset R^2\}$ is a set of positive numbers, then*

$$\left|\int\prod_{\Delta\subset Y}\delta W^{\pm}_{\kappa(\Delta)}(\Delta)^{m(\Delta)}\,d\mu(\psi_\pm)\right|\leq\prod_{\Delta\subset Y}[(4m(\Delta))!\,(K\kappa(\Delta)^{-\delta})^{m(\Delta)}].$$

K and δ are uniform in λ as $\lambda\downarrow 0$.

Proof. As in [DG].

It is now possible to prove the vacuum energy bounds that are necessary.

Proposition 4.3. *For $\eta>0$ sufficiently small, all λ sufficiently small, all $|h|\leq\lambda^2$, there are strictly positive constants $a(\eta)$, $b(\eta)$ such that for $1\leq p\leq 1+\eta/30$,*

$$\int\exp\{-p(:P_\Lambda(\phi):-\tfrac{1}{2}:(\phi-\xi_\pm)^2_\Lambda:)\}\prod_{\Delta\subset\Lambda}\chi_\pm(\Delta)^p\,d\mu(\phi-\xi_\pm)\leq e^{a|\Lambda|}. \tag{i}$$

and

$$\int\exp\{-p(:P_\Lambda(\phi):-\tfrac{1}{2}:(\phi-\xi_\pm)^2_\Lambda:)\}\prod_{\Delta\subset\Lambda}\chi_{\pm,p}(\Delta)^p\,d\mu(\phi-\xi_\pm)\leq e^{-b\lambda^{-1/2}|\Lambda|}. \tag{ii}$$

Remark. In (i) it is actually possible to replace a by $a\lambda^{1/2}$ (see [GJS 4]).

Proof. By Hölder's inequality,

$$\int\exp\{-p(:P_\Lambda(\phi):-\tfrac{1}{2}:(\phi-\xi_\pm)^2_\Lambda:)\}\prod_{\Delta\subset\Lambda}\chi_\pm(\Delta)^p\,d\mu(\phi-\xi_\pm)$$
$$\leq\left(\int\exp\{-pq'(:P_\Lambda(\phi):+\zeta:\delta\phi^2_\Lambda:-\eta/2:(\phi-\xi_\pm)^2_\Lambda:)\}\prod_{\Delta\subset\Lambda}\chi_\pm(\Delta)^{pq'}\,d\mu(\phi-\xi_\pm)\right)^{1/q'}$$
$$\times\left(\int\exp\left\{qp\left(\zeta:\delta\phi^2_\Lambda:+\frac{1-\eta}{2}:(\phi-\xi_\pm)^2_\Lambda:\right)\right\}d\mu(\phi-\xi_\pm)\right)^{1/q}. \tag{4.8}$$

By conditioning with respect to Neumann boundary conditions [GJS 4, GRS 1], the second factor is estimated by

$$\left(\int \exp\left\{pq\left[\zeta :\delta\phi_\Delta^2: + \frac{1-\eta}{2} :(\phi-\xi_\pm)_\Delta^2:\right]\right\} d\mu(\phi-\xi_\pm)_{\partial\Delta}^N\right)^{|\Lambda|/q|\Delta|}$$

(Δ is not necessarily a unit lattice square). A standard calculation yields for this factor (note $\delta\phi = \delta\psi_\pm$):[3])

$$(\det_2[1-pq(2\zeta(1-P_\Delta)+(1-\eta))(-\Delta_\Delta^N+1)^{-1}])^{-|\Lambda|/2q|\Delta|}, \tag{4.9}$$

where P_Δ is the projection in $L_2(\Delta)$ onto χ_Δ, Δ_Δ^N is the Laplacian with Neumann boundary conditions on $\partial\Delta$, the boundary of the lattice square Δ. But because $(1-p_\Delta)\chi_\Delta = 0$, and because $-\Delta_\Delta^N \upharpoonright \{\chi_\Delta\}^\perp \geq \pi^2/|\Delta|$, we have

$$1-pq(2\zeta(1-P_\Delta)+(1-\eta))(-\Delta_\Delta^N+1)^{-1} \geq 1-pq((1-\eta)+2\zeta\,|\Delta|/\pi^2) > 0$$

if we choose $\eta = 10^{-6}$, $q = 1+\eta/30$, $\zeta = 1$, $|\Delta| = 10^{-6}$ (these have been chosen, also, so that the hypothesis of Lemma 4.1 is satisfied). Thus (4.9) is finite and is bounded by

$$e^{K_1|\Lambda|}$$

for some constant K_1.

Lemmas 4.1 and 4.2 and standard arguments [DG] yield the following bound for the first factor of (4.8):

$$e^{K_2|\Lambda|}$$

for a constant uniform in λ. Of course, Lemma 4.1(ii) entails that if $\prod_{\Delta\subset\Lambda}\chi_+(\Delta)$ is replaced by $\prod_{\Delta\subset\Lambda}\chi_{+,p}(\Delta)$, the above bound is replaced by

$$e^{-K_2\lambda^{-1/2}|\Lambda|}.$$

This completes the proof of Proposition 4.3.

Before proceeding further, we must pause for further definitions. The vacuum energy density is defined to be

$$\alpha_\infty^\pm = \lim_{\Lambda\uparrow R^2} \alpha_\Lambda^\pm = \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int d\phi_\Lambda^\pm$$

(see (2.5)). This limit is known to exist [Gu 1]. It is, in fact, not difficult to prove directly (see [Su 1, 2] for a somewhat more complicated example) that $\alpha_\infty^+ = \alpha_\infty^-$; thus, we shall drop the superscript. We wish to recall the chessboard estimate [FS]. If F_α is a measurable function of the fields with support in the lattice square Δ_α, then

$$\left|\left\langle \prod_{\alpha\in N} F_\alpha \right\rangle^\pm\right| \leq \exp\left\{\sum_{\alpha\in N} (\alpha_\infty^\pm(F_\alpha)-\alpha_\infty)\,|\Delta|\right\}, \tag{4.10}$$

[3]) $\det_2[1+A] = \exp[\operatorname{tr}\ln(1+A) - \operatorname{tr} A]$.

where N is some index set and

$$\alpha_\infty^\pm(F_\alpha)=\lim_{\Lambda\uparrow R^2}\frac{1}{|\Lambda|}\ln\int\prod_{\Delta_\beta\subset\Lambda}(F_\alpha)_\beta\,d\phi_\Lambda^\pm. \tag{4.11}$$

$(F_\alpha)_\beta$ is the function with support in Δ_β obtained by a series of reflections in lattice lines and translations of the function F_α (see [FS]). We comment that the chessboard estimates are valid for all states constructed by the methods mentioned at the outset of Chapter II.

We shall, thus, be interested in a lower bound on the infinite volume vacuum energy density:

Lemma 4.4. *For arbitrary parameter values,* $\alpha_\infty\geq 0$.

Proof. By Jensen's inequality, if $h\geq 0$,

$$\begin{aligned}\alpha_\Lambda^+&=\frac{1}{|\Lambda|}\ln\int\exp\{-:P_\Lambda(\phi):+\tfrac{1}{2}:(\phi-\xi_+)_\Lambda^2:\}\,d\mu(\phi-\xi_+)\\&=\frac{1}{|\Lambda|}\ln\int\exp\{-:P_\Lambda(\phi+\xi_+):+\tfrac{1}{2}:\phi_\Lambda^2:\}\,d\mu(\phi)\\&\geq\frac{1}{|\Lambda|}\ln\exp\left\{\int-:P_\Lambda(\phi+\xi_+):+\tfrac{1}{2}:\phi_\Lambda^2:\,d\mu(\phi)\right\}\\&=-P(\xi_+)+E_c=0.\end{aligned}$$

Thus, for $h\geq 0$, $\alpha_\infty^+\geq 0$. Similarly, one can show that for $h\leq 0$, $\alpha_\infty^-\geq 0$. But $\alpha_\infty^+=\alpha_\infty^-=\alpha_\infty$, completing the proof.

Proof of Proposition 3.1. By the chessboard inequality,

$$\left|\left\langle\prod_{j=1}^{\nu}F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j)\right\rangle^+\right|\leq\exp\left\{\sum_{j=1}^{\nu}(\alpha_\infty^\pm(F^{\sigma_{1,j}}\chi_{\sigma_{2,j}})-\alpha_\infty)\right\}. \tag{4.12}$$

For arbitrary Λ,

$$\alpha_\Lambda^\pm(F^{\sigma_{1,j}}\chi_{\sigma_{2,j}})=\frac{1}{|\Lambda|}\ln\int\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_\Delta\chi_{\sigma_{2,j}}(\Delta)\,d\phi_\Lambda^\pm, \tag{4.13}$$

and we have a spin configuration throughout Λ with the single spin $\sigma_{2,j}$. If $\sigma_{2,j}=+$, we estimate (4.13) through Hölder's inequality:

$$\begin{aligned}&\frac{1}{|\Lambda|}\frac{1}{p'}\ln\int\left(\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_\Delta\right)^{p'}d\mu(\phi-\xi_+)\\&\quad+\frac{1}{|\Lambda|}\frac{1}{p}\ln\int\exp\{-p[:P_\Lambda(\phi):-\tfrac{1}{2}:(\phi-\xi_+)_\Lambda^2:]\}\prod_{\Delta\subset\Lambda}\chi_{\sigma_{2,j}}(\Delta)\,d\mu(\phi-\xi_+).\end{aligned} \tag{4.14}$$

By Proposition 4.3, the second term is estimated by ln a, if we choose $p\simeq 1+10^{-7}$ (and so that p' is even). The first term can be estimated by the checkerboard estimate [GRS 1, 2]:

$$\frac{1}{|\Lambda|}\frac{1}{p'q}\left(\sum_{\Delta\subset\Lambda}\ln\int(F^{\sigma_{1,j}}(w_j))_\Delta^{p'q}\,d\mu(\phi-\xi_+)\right),$$

where q is independent of $|\Lambda|$ and has been chosen even. If $\sigma_{1,j}=+$, (recall, $\sigma_{2,j}=+$), then this is bounded by

$$\ln\left[\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\,\|w_j\|_p\right], \tag{4.15}$$

$p>1$ (see (3.3) and (3.4)), using a standard argument [DG]. This bound is uniform in λ and Λ. If, however, $\sigma_{1,j}=-$, because $|\xi_- - \xi_+| = O(\lambda^{-1/2})$, the same argument on Gaussian integrals leads to the bound

$$\ln\left[\left(\prod_{i=1}^{n_j}|c_{i,j}|\,K(N(F_j))\right)\lambda^{-N(F_j)/2}\,\|w_j\|_p\right]. \tag{4.16}$$

This bound is uniform in Λ.

If $\sigma_{2,j}=-$, we must shift the mean of the Gaussian measure (as well as that of the second term of the interaction exponent) in order to use the uniform bound of Proposition 4.3. The Gaussian measure is S-quasi-invariant [e.g., Fr 1] and its Radon-Nikodym derivative is given by

$$\frac{d\mu(\phi+f)}{d\mu(\phi)} = e^{-\phi((-\Delta+1)f)-(1/2)\langle f,(-\Delta+1)f\rangle},$$

where $\langle\cdot,\cdot\rangle$ signifies the real L_2 inner product.

In order to define an admissible shift that will also accomplish the desired translation from ξ_+ to ξ_- in Λ, we define:

$$g(x) = \eta_\nu \int (-\Delta+1)^{-1}(x-y)\,\nu\!\left(\frac{x-y}{L}\right)h_1(y)\,dy,$$

where

$$\nu(x)=\begin{cases}0, & \text{if } |x|>\frac{1}{2}\\ 1, & \text{if } |x|\le\frac{1}{4}\end{cases},\qquad 0\le\nu(x)\le 1,\qquad \nu(x)\in C_0^\infty,$$

$$\eta_\nu^{-1} = \int (-\Delta+1)^{-1}(y)\,\nu(y/L)\,dy,$$

$$h_1(y)=\begin{cases}\xi_+, & y\in R^2\backslash(\Lambda\cup N(\partial\Lambda))\\ \xi_-, & y\in\Lambda\cup N(\partial\Lambda)\end{cases},$$

$N(\partial\Lambda)=\{\Delta\subset R^2 \mid \operatorname{dist}(\Delta,\partial\Lambda)\le L\}$, $1\le L<\infty$ and fixed. What is important to notice is that $g\restriction\Lambda=\xi_-$ and $g(x)=\xi_+$ for $x\in R^2\backslash\Lambda$ such that $\operatorname{dist}(x,\partial\Lambda)\ge 2L$.

Then we can write (4.13) as

$$\frac{1}{|\Lambda|}\ln\int\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_\Delta\chi_{\sigma_{2,j}}(\Delta)e^{-:P_\Lambda(\phi):+1/2:(\phi-g)^2_\Lambda:}$$
$$\times\exp\left\{\int_\Lambda(\phi-g)(g-\xi_+)+\tfrac{1}{2}\int_\Lambda(g-\xi_+)^2\right\}d\mu(\phi-\xi_+)$$
$$=\frac{1}{|\Lambda|}\ln\int\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_\Delta\chi_{\sigma_{2,j}}(\Delta)e^{-:P_\Lambda(\phi):+1/2:(\phi-\xi_-)^2_\Lambda:}$$
$$\times\exp\left\{\int_{R^2\backslash\Lambda}(\phi-g)(g-\xi_+)+\tfrac{1}{2}\int_{R^2\backslash\Lambda}(g-\xi_+)^2\right\}d\mu(\phi-g), \tag{4.17}$$

using the above-mentioned properties of g. Now apply Hölder's inequality to estimate (4.17) by

$$\frac{1}{|\Lambda|}\frac{1}{p'}\ln\int\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_{\Delta}^{p'}\exp\left\{p'\int_{R^2\setminus\Lambda}[(\phi-g)(g-\xi_+)+\tfrac{1}{2}(g-\xi_+)^2]\right\}d\mu(\phi-g)$$
$$+\frac{1}{|\Lambda|}\frac{1}{p}\ln\int e^{-p[:P_\Lambda(\phi):-1/2:(\phi-\xi_-)^2_{\Lambda}:]}\prod_{\Delta\subset\Lambda}\chi_{\sigma_{2,j}}(\Delta)^p\,d\mu(\phi-g).$$

The second term is estimated as before, using Proposition 4.3. (It is easy to see that because $g\restriction\Lambda=\xi_-$ and g is continuous, we may indeed directly apply Proposition 4.3.) Hölder's inequality applied again estimates the first term by

$$\frac{1}{|\Lambda|}\frac{1}{p'q}\ln\int\prod_{\Delta\subset\Lambda}(F^{\sigma_{1,j}}(w_j))_{\Delta}^{p'q}\,d\mu(\phi-g)$$
$$+\frac{1}{|\Lambda|}\frac{1}{p'q'}\ln\int\exp\left\{p'q'\int_{R^2\setminus\Lambda}[(\phi-g)(g-\xi_+)+\tfrac{1}{2}(g-\xi_+)^2]\right\}d\mu(\phi-g).$$

Because, in $R^2\setminus\Lambda$, g differs from ξ_+ only in a strip along $\partial\Lambda$, it is easy to see the second term is dominated by

$$\frac{1}{|\Lambda|}LC\,|\partial\Lambda|$$

where C depends on λ. And the first term is estimated, using the arguments utilized previously, by

$$\ln\left[\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\,\|w_j\|_p\right],$$

if $\sigma_{i,j}=-$, or by

$$\ln\left[\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\lambda^{-N(F_j)/2}\,\|w_j\|_p\right],$$

if $\sigma_{1,j}=+$. These estimates, with (4.15), (4.16) and Lemma 4.4, in the limit $\Lambda\uparrow R^2$, yield the proposition.

The arguments in the proof of this proposition are directly applicable to states constructed by the means mentioned previously. In particular, if one considers (half) Dirichlet boundary conditions, one must replace Δ in the definition of g (and the shift formulas) by Δ_Λ^D, the Laplacian operator with (zero) Dirichlet boundary conditions on $\partial\Lambda$. Then, it is possible to push the arguments through (note that because

$$(-\Delta_\Lambda^D+1)^{-1}(x,y)=0$$

when $x\in\partial\Lambda$, $g(x)=0$ if $x\in\partial\Lambda$, so it is clear that $g(x)$ is in the domain of $-\Delta_\Lambda^D+1$ and is an admissible shift).

We will conclude this chapter by stating two results that will be of use in the proof of Proposition 3.2.

Lemma 4.5. *For all sufficiently small λ, all $|h|\le\lambda^2$, and any set $Y\subset R^2$ composed of unit lattice squares Δ, there exists a constant $b>0$ (independent of λ, h,* *Y*) *such that*

$$\left\langle \prod_{\Delta\subset Y} \chi_{\pm,p}(\Delta)\right\rangle \le e^{-b\lambda^{-1/2}|Y|}.$$

Remark. The expectation is in *either* the + or − state.

Proof. Follows trivially from the chessboard estimates, Proposition 4.3 and Lemma 4.4.

Lemma 4.6. *There exist strictly positive constants K, c, such that for all sufficiently small* λ, *all* $|h|\le\lambda^2$ *and any collection*

$$\{F^{\sigma_{1,j}}(\Delta_j)\}_{j=1}^{\nu}$$

of functions defined in (3.3), *one has the following estimate:*

$$\left|\left\langle \prod_{j=1}^{\nu} F^{\sigma_{1,j}}(\Delta_j)\chi_{\sigma_{2,j},p}(\Delta_j)\right\rangle^{\pm}\right| \le \prod_{j=1}^{\nu}\left\{\left(\prod_{i=1}^{n_j}|c_{i,j}|\right)K(N(F_j))\lambda^{-N(F_j)/2}e^{-c\lambda^{-1/2}}\right\}.$$

Remark. We recall that $K(N(F_j))=K^{N(F_j)}N(F_j)!$. See also (3.4). We emphasize that only peak spin characteristic functions are in the expectation.

Proof. Follows readily from the argument of the proof of Proposition 3.1, Lemma 4.4 and Proposition 4.3(ii).

V. Peierls' estimates

The aim of this chapter is to prove Proposition 3.2. We will first sketch the argument. A Peierls' argument will yield the estimate

$$\langle\chi_+(\Delta_\alpha)\chi_-(\Delta_\beta)\rangle^+ \le e^{-c\lambda^{-1/2}}, \tag{5.1}$$

for some $c>0$, uniformly in Δ_α, Δ_β, and $|h|\le\lambda^2$. The vacuum energy density $\alpha_\infty(\lambda, h)$ is convex in h and, thus, is differentiable in h at all but countably many values of h. Thus, we can pick a sequence (for λ small but fixed) $\{h_n\}$, that converges from above to 0, at every point of which $\partial\alpha_\infty(\lambda, h)/\partial h$ exists. But whenever $\partial\alpha_\infty(\lambda, h)/\partial h$ exists, the state is pure [Gu 2, Si], i.e., for any function $F(\phi)$,

$$\lim_{|x-y|\to\infty}\langle F(\phi(x))F(\phi(y))\rangle^+ - \langle F(\phi(x))\rangle^+\langle F(\phi(y))\rangle^+ = 0.$$

Because the bound in (5.1) is uniform in Δ_α, Δ_β, we have

$$\lim_{\mathrm{dist}(\Delta_\alpha,0)\to\infty}\langle\chi_+(\Delta_\alpha)\rangle^+\langle\chi_-(\Delta_\beta)\rangle^+ \le e^{-c\lambda^{-1/2}} \tag{5.2}$$

at the external field values h_n. Utilization of the convexity in h of the vacuum energy density and further analysis of the first and second Schwinger functions yield the following bound for $h \geq 0$:

$$\langle \chi_+(\Delta) \rangle^+ \geq e^{-K\lambda^{1/4}}, \qquad \forall \Delta, \tag{5.3}$$

for some $K > 0$, and where the + state at $h = 0$ is defined as the limit of the + state at strictly positive external field h, as $h \downarrow 0$. (5.3) and (5.2) clearly yield the desired result at $h = 0$. The proof is completed when we note that the FKG inequalities entail that $\langle \chi_- \rangle^+$ is monotone decreasing in h.

The arguments of this chapter can be readily applied to models with more complicated phase diagrams (see [Su 1, 2]) and to states constructed via the methods previously mentioned. Clearly the most difficult step is the proof of (5.3) or its counterpart. Even in the model considered in detail here, the proof is not trivial. (Because we have destroyed the $\phi \leftrightarrow -\phi$ symmetry of the $h = 0$ model by the introduction of boundary conditions, i.e., the choice of mean of the Gaussian measure, one cannot conclude

$$\langle \chi_+(\Delta) \rangle = \langle \chi_-(\Delta) \rangle = \tfrac{1}{2}.)$$

But we shall return to the generalizibility of the proof later.

We begin with the proof of (5.1).

Proposition 5.1. *There exists a $c > 0$ such that for all $\Delta_\alpha, \Delta_\beta$, all small enough λ and $|h| \leq \lambda^2$,*

$$\langle \chi_+(\Delta_\alpha) \chi_-(\Delta_\beta) \rangle^\pm \leq e^{-c\lambda^{-1/2}}.$$

Proof. Using

$$1 = \chi_+(\Delta) + \chi_-(\Delta)$$

at every $\Delta \subset \Lambda_0$, where Λ_0 is a large square containing Δ_α and Δ_β, we have

$$\langle \chi_+(\Delta_\alpha) \chi_-(\Delta_\beta) \rangle^+ = \sum_{\sigma(\cdot)} \left\langle \prod_\Delta \chi_{\sigma(\Delta)}(\Delta) \right\rangle^+, \tag{5.4}$$

where $\sum_{\sigma(\cdot)}$ is the sum over configurations $\sigma(\cdot)$ such that $\sigma(\Delta_\alpha) = +$, $\sigma(\Delta_\beta) = -$. Following [GJS 3, Fr 2], we estimate (5.4) by

$$\sum_\gamma \left\langle \prod_{(\Delta,\Delta') \in N(\gamma)} \chi_+(\Delta) \chi_-(\Delta') \right\rangle^+,$$

where $N(\gamma)$ is the set of nearest neighbor pairs of unit lattice squares bordering a connected contour γ, consisting of unit lattice lines, separating Δ_α and Δ_β. One has from [GJS 3, Fr 2] that, once one establishes that

$$\left\langle \prod_{(\Delta,\Delta') \in N(\gamma)} \chi_+(\Delta) \chi_-(\Delta') \right\rangle^+ \leq e^{-\delta\lambda^{-1/2}|\gamma|}, \tag{5.5}$$

where $\delta > 0$ and $|\gamma| =$ length of γ ($|\gamma| \geq 4$), the proposition is proven. We will follow [Fr 2] in proving the validity of (5.5). We may assume all pairs in $N(\gamma)$ are mutually disjoint (separating them with Hölder's inequality if they are not).

Writing

$$\chi_+(\Delta) = \chi_{+,s}(\Delta) + \chi_{+,p}(\Delta)$$

(see (4.2)–(4.4)), we have

$$\Big\langle \prod_{(\Delta,\Delta')\in N(\gamma)} \chi_+(\Delta)\chi_-(\Delta')\Big\rangle^+$$

$$\le \sum_{\gamma'} \Big\langle \prod_{(\Delta,\Delta')\in N(\gamma')} \chi_{+,p}(\Delta)\Big\rangle^{+1/2} \Big\langle \prod_{(\Delta,\Delta')\in N(\gamma\setminus\gamma')} \chi_{+,s}(\Delta)\chi_-(\Delta')\Big\rangle^{+1/2}, \quad (5.6)$$

where $\sum_{\gamma'}$ runs over the subsets of γ (regarded as a set of unit lattice lines). However,

$$\chi_{+,s}(\Delta)\chi_-(\Delta') \le e^{(\phi(\Delta)-\phi(\Delta'))-(\xi_+-\lambda^{1/4}\xi_+)}, \quad (5.7)$$

since

$$\chi_{+,s}(\Delta) \le e^{\phi(\Delta)-(\xi_+-\lambda^{1/4}\xi_+)}$$

and

$$\chi_-(\Delta) \le e^{-\phi(\Delta)}.$$

If we choose functions $h^i_{\Delta,\Delta'}$ as in [Fr 2], such that

$$\phi(\Delta)-\phi(\Delta') = \sum_{i=0}^{1} \phi(\partial_i h^i_{\Delta,\Delta'}),$$

we have, using (5.7) and the Gaussian domination bound [FSS, Fr 2]

$$\Big\langle \exp\Big\{\sum_{i=0}^{1} \phi(\partial_i h^i)\Big\}\Big\rangle^+ \le \exp\Big\{\sum_{i=0}^{1} \|h^i\|_2^2\Big\},$$

that

$$\Big\langle \prod_{(\Delta,\Delta')\in N(\gamma\setminus\gamma')} \chi_{+,s}(\Delta)\chi_-(\Delta')\Big\rangle^{+1/2} \le e^{-\delta\lambda^{-1/2}|\gamma\setminus\gamma'|}, \quad (5.8)$$

$\delta > 0$. Thus, with Lemma 4.5, (5.6) and (5.8) imply (5.5) and the proposition.

Remark. Any polynomial with a mean field limit, such that the polynomial forms a relatively large potential barrier between the positions of any two minima (and thus excluding, with high probability, field values between the minima), is amenable to this argument.

We wish to recall a beautiful extension of Guerra's theorem on the consequences of the differentiability of the vacuum energy density. This will be the formulation used.

Theorem 5.2 [FS]. *If $\alpha_\infty(\lambda, h)$ is differentiable in h, then the state at that set of interaction parameter values satisfies the Osterwalder–Schrader axioms, including clustering, and is independent of the classical boundary conditions (free, Dirichlet, periodic, Neumann, half-Dirichlet, etc.).*

Remark. The proof of the independence of boundary conditions in [FS] uses a cluster expansion (whose convergence is independent of the range of coupling constants in the semibounded polynomial $P(x)$ determining the state). However, to conform rigorously to our claim that our proof uses no cluster expansion, it should be remarked that the proof of the property that is of interest to us here, the clustering of the state, depends only on the validity of the chessboard estimates, and utilizes no cluster expansion (see Theorems 4.2 and 4.4 of [FS]).

We begin the proof of (5.3) by showing its validity for $h>0$. The limit $h\downarrow 0$ will be discussed directly thereafter.

Proposition 5.3. *There exists a strictly positive constant K such that for all sufficiently small λ, all $0<h\le\lambda^2$ and any unit lattice square Δ,*

$$\langle\chi_+(\Delta)\rangle^+\ge c^{-K\lambda^{1/4}}.$$

Proof. We note that for any Δ_α, Δ_β,

$$\begin{aligned}\lambda\langle\phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ &= \lambda\langle\phi(\Delta_\alpha)\chi_+(\Delta_\alpha)\phi(\Delta_\beta)\chi_+(\Delta_\beta)\rangle^+\\ &\quad+\lambda\langle\phi(\Delta_\alpha)\chi_-(\Delta_\alpha)\phi(\Delta_\beta)\chi_-(\Delta_\beta)\rangle^+\\ &\quad+\lambda\sum_{\sigma(\Delta_\alpha)\neq\sigma(\Delta_\beta)}\langle\phi(\Delta_\alpha)\chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\phi(\Delta_\beta)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta)\rangle^+.\end{aligned}\tag{5.9}$$

We may estimate the last terms in (5.9) by

$$\begin{aligned}&\left|\lambda\sum_{\sigma(\Delta_\alpha)\neq\sigma(\Delta_\beta)}\langle\phi(\Delta_\alpha)\chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\phi(\Delta_\beta)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta)\rangle^+\right|\\ &\qquad\le\sum_{\sigma(\Delta_\alpha)\neq\sigma(\Delta_\beta)}(\lambda^2\langle\phi(\Delta)^4\rangle^+)^{1/2}\langle\chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta)\rangle^{+1/2}.\end{aligned}\tag{5.10}$$

But, by Proposition 3.1, since

$$\lambda^2\langle\phi(\Delta)^4\rangle^+=\lambda^2\langle\phi(\Delta)^4\chi_+(\Delta)\rangle^+ +\lambda^2\langle\phi(\Delta)^4\chi_-(\Delta)\rangle^+,$$

we have $(\lambda^2\langle\phi(\Delta)^4\rangle^+)^{1/2}\le K_1$, for K_1 a constant uniform in small λ. Thus, by Proposition 5.1, (5.10) is estimated by

$$K_2e^{-c\lambda^{-1/2}}.$$

Therefore,

$$\begin{aligned}\lambda\langle\phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ &\ge\lambda\langle\phi(\Delta_\alpha)\chi_+(\Delta_\alpha)\phi(\Delta_\beta)\chi_+(\Delta_\beta)\rangle^+\\ &\quad+\lambda\langle\phi(\Delta_\alpha)\chi_-(\Delta_\alpha)\phi(\Delta_\beta)\chi_-(\Delta_\beta)\rangle^+-K_2e^{-c\lambda^{-1/2}}.\end{aligned}$$

Recalling

$$\chi_\pm(\Delta)=\chi_{\pm,s}(\Delta)+\chi_{\pm,p}(\Delta),$$

and using Lemma 4.6, one has

$$\begin{aligned}\lambda\langle\phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ &\ge\lambda\langle\phi(\Delta_\alpha)\chi_{+,s}(\Delta_\alpha)\phi(\Delta_\beta)\chi_{+,s}(\Delta_\beta)\rangle^+\\ &\quad+\lambda\langle\phi(\Delta_\alpha)\chi_{-,s}(\Delta_\alpha)\phi(\Delta_\beta)\chi_{-,s}(\Delta_\beta)\rangle^+-K_3e^{-c\lambda^{-1/2}}\\ &\ge\lambda(1-\lambda^{1/4})^2\xi_+^2\langle\chi_{+,s}(\Delta_\alpha)\chi_{+,s}(\Delta_\beta)\rangle^+\\ &\quad+\lambda(\xi_-+\lambda^{1/4}\xi_+)^2\langle\chi_{-,s}(\Delta_\alpha)\chi_{-,s}(\Delta_\beta)\rangle^+-K_3e^{-c\lambda^{-1/2}}\end{aligned}\tag{5.11}$$

(see (4.2)–(4.4)). Because $\chi_{\pm,s}(\Delta_\beta)=1-\chi_\mp(\Delta_\beta)-\chi_{\pm,p}(\Delta_\beta)$, one concludes from

(5.11) that

$$\begin{aligned}\lambda\langle\phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ \geq{} & \lambda(1-\lambda^{1/4})^2\xi_+^2\langle\chi_{+,s}(\Delta_\alpha)\rangle^+ \\ & -\lambda(1-\lambda^{1/4})\xi_+^2\langle\chi_{+,s}(\Delta_\alpha)\chi_-(\Delta_\beta)\rangle^+ \\ & -\lambda(1-\lambda^{1/4})^2\xi_+^2\langle\chi_{+,s}(\Delta_\alpha)\chi_{+,p}(\Delta_\beta)\rangle^+ \\ & +\lambda(\xi_-+\lambda^{1/4}\xi_+)^2\langle\chi_{-,s}(\Delta_\alpha)\rangle^+ \\ & -\lambda(\xi_-+\lambda^{1/4}\xi_+)^2\langle\chi_{-,s}(\Delta_\alpha)\chi_+(\Delta_\beta)\rangle^+ \\ & -\lambda(\xi_-+\lambda^{1/4}\xi_+)^2\langle\chi_{-,s}(\Delta_\alpha)\chi_{-,p}(\Delta_\beta)\rangle^+ \\ & -K_3e^{-c\lambda^{-1/2}} \\ \geq{} & \lambda(1-\lambda^{1/4})^2\xi_+^2\langle\chi_{+,s}(\Delta_\alpha)\rangle^+ \\ & +\lambda(\xi_-+\lambda^{1/4}\xi_+)^2\langle\chi_{-,s}(\Delta_\alpha)\rangle^+ - K_4e^{-c\lambda^{-1/2}},\end{aligned} \tag{5.12}$$

where we have used Lemma 4.6 and Proposition 5.1 in the last inequality. By (2.2), for $|h|\leq\lambda^2$,

$$\lambda(\xi_-+\lambda^{1/4}\xi_+)^2=\lambda(\xi_+-\lambda^{1/4}\xi_+)^2+O(\lambda^{5/2}),$$

and by Lemma 4.5,

$$\begin{aligned}\langle\chi_{+,s}(\Delta)\rangle^+ + \langle\chi_{-,s}(\Delta)\rangle^+ &= 1-\langle\chi_{+,p}(\Delta)\rangle^+ - \langle\chi_{-,p}(\Delta)\rangle^+ \\ &\geq 1-2e^{-c\lambda^{-1/2}};\end{aligned}$$

therefore, (5.12) implies

$$\begin{aligned}\lambda\langle\phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ &\geq \lambda(1-\lambda^{1/4})^2\xi_+^2 - O(\lambda^{5/2}) \\ &\geq \omega_+^2 - K_5\lambda^{1/4}\end{aligned} \tag{5.13}$$

$(\omega_+\equiv\lambda^{1/2}\xi_+)$.

As previously mentioned, $\alpha_\infty(\lambda, h)$ is differentiable in h almost everywhere, so we can choose a sequence $\{h_n\}$ of external field values converging to 0 from above such that

$$\left.\frac{\partial\alpha_\infty(\lambda, h)}{\partial h}\right|_{h=h_n}$$

exists for each n. Because (5.13) is independent of Δ_α and Δ_β, we have from Theorem 5.2 that

$$\lambda^{1/2}\langle\phi(\Delta)\rangle_n^+ \geq \omega_+ - K_6\lambda^{1/4}, \qquad \forall\Delta, \tag{5.14}$$

where $\langle\cdot\rangle_n$ denotes the state evaluated at $h=h_n$. It is easy to see that the FKG inequalities (see, e.g. [GRS 1]) entail that $\langle\phi(\Delta)\rangle^+$ is monotone increasing in h. Thus, (5.14) implies

$$\lambda^{1/2}\langle\phi(\Delta)\rangle^+ \geq \omega_+ - K_6\lambda^{1/4} \tag{5.15}$$

for all $h>0$. (We note that because the state for $(\phi^4)_2$ is known to be pure for $h\neq 0$ [Si], we could conclude (5.15) directly from (5.13). But we wish to maintain the arguments as general as possible, and the FKG inequalities as used are applicable to arbitrary semibounded polynomials.)

Further, one has that

$$\lambda^{1/2}\langle\phi(\Delta)\rangle^+ \leq \lambda^{1/2}\langle\phi(\Delta)\chi_{+,s}(\Delta)\rangle^+ + \lambda^{1/2}\langle\phi(\Delta)\chi_{-,s}(\Delta)\rangle^+ + K_7 e^{-c\lambda^{-1/2}}$$
$$\leq (1+\lambda^{1/4})\omega_+\langle\chi_{+,s}(\Delta)\rangle^+ - (1-\lambda^{1/4})\omega_+\langle\chi_{-,s}(\Delta)\rangle^+ + K_8\lambda^{5/2}.$$

Noting that

$$\langle\chi_{+,s}(\Delta)\rangle^+ \leq 1 - \langle\chi_{-,s}(\Delta)\rangle,$$

this yields

$$\lambda^{1/2}\langle\phi(\Delta)\rangle^+ \leq (1+\lambda^{1/4})\omega_+ - (1+\lambda^{1/4})\omega_+\langle\chi_{-,s}(\Delta)\rangle^+$$
$$-(1-\lambda^{1/4})\omega_+\langle\chi_{-,s}(\Delta)\rangle^+ + K_8\lambda^{5/2}.$$

Therefore, by using (5.15), we conclude that for all $h>0$,

$$\langle\chi_{-,s}(\Delta)\rangle^+ \leq \frac{(1+\lambda^{1/4})\omega_+ - \omega_+ + K_9\lambda^{1/4}}{(1+\lambda^{1/4})\omega_+ + (1-\lambda^{1/4})\omega_+}$$
$$\leq K_{10}\lambda^{1/4}.$$

This entails that

$$\langle\chi_+(\Delta)\rangle^+ = 1 - \langle\chi_-(\Delta)\rangle^+ \geq 1 - \langle\chi_{-,s}(\Delta)\rangle^+ - e^{-c\lambda^{-1/2}}$$
$$\geq 1 - K_{11}\lambda^{1/4},$$

whenever $h>0$, which yields the desired conclusion.

Propositions 5.1 and 5.3 yield Proposition 3.2 for $h>0$, as already described. To extend the result to $h=0$, we must define the + state at $h=0$ (resp., − state) through a limit of + (−) states as $h\downarrow 0$ ($h\uparrow 0$). In particular, we define the + state at $h=0$ by

$$\left\langle\prod_{i=1}^{\nu}\phi(f_i)\right\rangle^+ = \lim_{n\to\infty}\left\langle\prod_{i=1}^{\nu}\phi(f_i)\right\rangle_n^+, \qquad f_i \geq 0.$$

By the monotonicity of the Schwinger functions in h (second Griffiths' inequality [GRS 1]), this limit exists and is independent of the particular choice of sequence $\{h_n\}\downarrow 0$. We show in the appendix, furthermore, that this limit defines a unique probability measure for which the functional $Z(f_1)$ (see Theorem 2.1) is bounded and analytic in $f_1 \in L_{1,4/3}$, and for which the generalized Schwinger functions exist and are continuous on $\Pi\, \mathscr{L}_{2,\Sigma} \supset \Pi\, S(R^2)$ (see Appendix). The Schwinger functions of this measure satisfy all the Osterwalder–Schrader axioms (including clustering) and are independent of the classical boundary conditions. (The remark following Theorem 5.2 is applicable here.) We comment that although the second Griffiths' inequality is known only for even polynomial interaction, the independence of the choice of sequence $\{h_n\}$ can be, to a large extent, recovered for arbitrary semibounded polynomials (see Appendix).

Therefore, Propositions 5.1 and 5.3 entail Proposition 3.2 at $h=0$. The proof of Theorem 2.3 is, thus, complete.

VI. Discussion

We have seen, in the simplest possible case evincing a phase transition, a new proof of the asymptotic nature of the perturbation expansions for the generalized

Schwinger functions. We would now like to discuss in a bit more detail its application to all $P(\phi)_2$ models with mean field limits.

In weakly coupled $\lambda P(\phi)_2$, since the Wick ordering lower bounds are uniform as $\lambda \downarrow 0$, one has, as analog to Proposition 3.1, the estimate

$$\left|\left\langle \prod_{j=1}^{\nu} F^j(\Delta_j)\right\rangle\right| \leq \prod_{j=1}^{\nu} \left[\left(\prod_{i=1}^{n_j} |c_{i,j}|\right) K(N(F_j))\right],$$

where K is uniformly bounded as $\lambda \downarrow 0$ and

$$F(\Delta) = \prod_{i=1}^{n} c_i :\phi^{m_i}: (\Delta).$$

Thus, the proof of the asymptotic nature of perturbation theory in such models is straightforward (see [Su 1]). But in models manifesting phase transitions, the corresponding Wick lower bounds will not be uniform as the appropriate coupling constants approach zero, and a counterpart to Proposition 3.2 is necessary.

We shall summarize the argument in the context of a semibounded polynomial with a mean field limit and n local minima ξ_j. A glance at the proof of Proposition 3.1 suggests that one can prove the following estimate

$$\left|\left\langle \prod_{j=1}^{\nu} F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j)\right\rangle\right| \leq \prod_{\{j \mid \sigma_{1,j}=\sigma_{2,j}\}} \left[\left(\prod_{i=1}^{n_j} |c_{i,j}|\right) K(N(F_j))\, \|w_j\|_p\right]$$
$$\times \prod_{\{j \mid \sigma_{1,j}\neq\sigma_{2,j}\}} \left[|\xi_{\sigma_{1,j}} - \xi_{\sigma_{2,j}}|^{N(F_j)} \left(\prod_{i=1}^{n_j} |c_{i,j}|\right) K(N(F_j))\, \|w_j\|_p\right], \qquad (6.1)$$

where $\sigma_{1,j}, \sigma_{2,j}$ take values in $\{1, \ldots, n\}$ (see (3.3)), and where the constant K is uniformly bounded as the dominant coupling constant a_d goes to zero (see discussion in Chapter I). The Peierls' argument in Proposition 5.1 would yield

$$\langle \chi_{\sigma_1}(\Delta_\alpha)\chi_{\sigma_2}(\Delta_\beta)\rangle \leq \exp\{-c\, |\xi_{\sigma_1} - \xi_{\sigma_2}|\} \qquad (6.2)$$

uniformly in Δ_α and Δ_β (see remark following the proof of Proposition 5.1). Thus, if one can prove the estimate

$$\langle \chi_{\sigma_1}(\Delta)\rangle^{\sigma_1} \geq K, \qquad (6.3)$$

for K^{-1} a constant uniformly bounded as $a_d \downarrow 0$ when the interaction parameters $\{a_i\}_{i=3}^{d}$ are restrained in some region of parameter space, then by choosing appropriate sequences in parameter space (chosen such that the vacuum energy density is differentiable in the external field at every point in the sequence) one can show, using (6.2),

$$\langle \chi_{\sigma_2}(\Delta)\rangle^{\sigma_1} \leq K^{-1} \exp\{-c\, |\xi_{\sigma_1} - \xi_{\sigma_2}|\}, \qquad (6.4)$$

with parameters restrained in the (closure of the) aforesaid region of parameter space (of course, at the boundary of this region the σ_1 state is understood to be the suitable limit state). With (6.1) and (6.4) the argument of Chapter III can be set in motion to produce the proof of the 'asymptocity' of perturbation theory about the σ_1 th minimum. (Strict asymptoticity in the coupling constant a_α requires that the subdominant couplings $\{a_i^{\sigma_1}\}_{i=3}^{d-1}$ are suitable functions of a_d.)

The strongly model-dependent bound (6.3) must be verified in individual models. The argument presented in Chapter V can be immediately applied to models such as

$$:P(\phi):=\lambda\ :R(\phi):-\tfrac{1}{4}:\phi^2:,$$

where $R(\phi)$ is an even polynomial, or

$$:P(\phi):=\lambda\ :R(\phi):+\lambda\varepsilon\ :Q(\phi):-\tfrac{1}{4}:\phi^2:;$$

where ϵ is a sufficiently small parameter and $Q(\phi)$ is an odd polynomial (see [Fr 2]). The analog to Proposition 3.2 for a model with a more complicated phase diagram (with phase transition lines for $h\neq 0$ and a triple point) has been proven in [Su 1, 2], and the proof of the asymptotic nature of perturbation theory has been carried out in detail there.

To conclude the discussion, we wish to remark that although the $(\phi^4)_3$ model is much more singular than $P(\phi)_2$ models, one should be able, using arguments of [FR] and estimates in [FO, MS 1], to verify the validity of Theorems 2.1 and 2.2 without the use of a cluster expansion (a phase-space cell expansion will be necessary, nevertheless). Then, with arguments of [SS], chessboard estimates can be proven, and the proof of the asymptotic nature of perturbation theory presented above could be used.

Acknowledgements

The author is pleased to acknowledge useful criticism by Prof. Konrad Osterwalder.

Appendix

In this appendix we shall outline the proof of the existence and properties, which were claimed at the end of Chapter V, of the limit states at $h=0$. Theorem 2.1 entails that

$$Z(f_1)=\int e^{(\phi-\xi_+)(f_1)}\,d\phi^+$$

is bounded and analytic in $f_1\in L_{1,4/3}$. In particular, this is true at each point h_n of the sequence $\{h_n\}\downarrow 0$ chosen to define the + state at $h=0$. It is easy to see that the FKG inequalities entail that $Z(f_1)(f_1\geq 0)$ is monotone increasing in h. Therefore, $\{Z(f_1)_{h_n}\}$ is a uniformly bounded family of analytic functions, which, by Vitali's theorem, converges uniformly on $L_{1,4/3}$ (possibly through a subsequence of $\{h_n\}$) to an analytic limit $Z^+(f_1)$. And, due to the monotonicity in h, $Z^+(f_1)$ is independent of the choice of sequence $\{h_n\}$ that satisfies

(i) $\{h_n\}\downarrow 0$
(ii) $\partial\alpha_\infty(\lambda, h)/\partial h|_{h=h_n}$ exists, $\forall_n$.

Moreover, one sees that the limit determines a unique measure on $S'(R^2)$, which is independent of the choice of sequence $\{h_n\}\downarrow 0$ and the classical boundary conditions, and whose Schwinger functions satisfy all of the Osterwalder–Schrader axioms, including clustering. The measure is obtained from Minlos' theorem [Mi], once it is remarked that the uniform convergence $Z(f_1)_{h_n} \to Z^+(f_1)$ entails that $J^+(f_1) \equiv Z(if_1)$ satisfies

(i) $J^+(0) = 1$,
(ii) J^+ is continuous on $L_{1,4/3} \supset S(R^2)$,
(iii) J^+ is of positive type.

(i)–(iii) follow from the corresponding properties of the $Z(if_1)_{h_n}$. The measure $d\phi^+$ then generated is the unique measure for which

$$J^+(f_1) = e^{-i\langle \xi_+, f_1\rangle} \int e^{i\phi(f_1)}\, d\phi^+.$$

The remaining properties are immediate consequences of the following slight generalization of Theorem 5.2.

Corollary A.1. *If the Schwinger functions of a state are continuous from the right (or the left) in the external field, then the state satisfies the Osterwalder–Schrader axioms, including clustering, and is independent of the classical boundary conditions.*

Proof. Implicit in the proof of Theorems 4.1, 4.2 and 4.4 of [FS]. See also [Su 1].

Remark. Again, the proof of the independence of boundary conditions depends on the convergence of a cluster expansion, but the remainder of the theorem does not. We further comment that only the one-sided continuity of $\langle \phi(x) \rangle$ is required.

Since the Schwinger functions of the + state (the functional derivatives of $Z^+(f_1)$) at $h = 0$ are, by definition, continuous from the right in h, the desired result follows at once.

To establish the existence of the generalized Schwinger functions of the limit state, we note that since Propositions 3.1 and 3.2 have been shown to be valid for $h > 0$, the appropriately simplified argument of Chapter III (no integration by parts is necessary) yields the following bound, for fixed $1 \le j \le 4$ and $N \in \mathbb{Z}^+$, the positive integers:

$$\begin{aligned} |\mathcal{S}_j^{(N)}(f_j)| &\equiv \left| \int \prod_{\nu=1}^{N} :(\phi - \xi_+)^j:(f_{j_\nu})\, d\phi^+ \right| \\ &\le (jN)!\, K^N\, |f_j|_p^N, \end{aligned} \tag{A.1}$$

where K is a constant uniform in $\{h_n\}$ and $|\cdot|_p$ is given by

$$|f_j|_p = \sum_{\Delta \subset R^2} \|f_j \chi_\Delta\|_p, \qquad p > 1.$$

Denoting the Banach space defined with this norm by $\mathcal{L}_{p,\Sigma}$, one remarks that $L_{1,4/4-j} \supset \mathcal{L}_{j,\Sigma} \supset S(R^2)$. (A.1) entails that the family $\{\mathcal{S}_j^{(N)}(f_j)_{h_n}\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $\mathcal{L}_{j,\Sigma}$. Thus, it converges (possibly through a subsequence) to a limit $\mathcal{S}_j^{(N)}(f_j)^+$ continuous on $\mathcal{L}_{j,\Sigma}$. As there are only countably many $\mathcal{S}_j^{(N)}$, $1 \le j \le 4$, $N \in \mathbb{Z}^+$, one can find a subsequence so that all $\mathcal{S}_j^{(N)}(f_j)^+$ exist.

We comment further that, once the existence of the generalized Schwinger functions has been established, as above, one can copy the argument of [GJ 2] to prove that one can integrate by parts in the limit states, i.e., Theorem 2.2 is valid for the limit states. Here the argument simply goes through a sequence of states at $\{h_n\}$, for which Theorem 2.2 holds, instead of through a sequence of finite volume states as the volume grows to infinity.

We remark that none of the arguments above have utilized any property special to ϕ^4.

REFERENCES

[CR] F. CONSTANTINESCU and H. M. RUCK, *Phase transitions in a continuous three states model with discrete gauge symmetry*, Ann. Phys., *115*, 474–495 (1978).

[Di] J. DIMOCK, *Perturbation series asymptotic to Schwinger functions in $P(\phi)_2$*, Commun. Math. Phys., *35*, 347–356 (1974).

[DG] J. DIMOCK and J. GLIMM, *Measures on Schwartz distribution space and applications to $P(\phi)_2$ field theories*, Adv. Math., *12*, 58–83 (1974).

[EE 1] J.-P. ECKMANN and H. EPSTEIN, *Time-Ordered products and Schwinger functions*, Commun. Math. Phys., *64*, 95–130 (1979).

[EE 2] J.-P. ECKMANN and H. EPSTEIN, *Borel summability of the mass and the S-matrix in ϕ^4 models*, Commun. Math. Phys., *68*, 245–258 (1979).

[EEF] J.-P. ECKMANN, H. EPSTEIN and J. FRÖHLICH, *Asymptotic perturbation expansion for the S-matrix and the definition of time-ordered functions in relativistic quantum field models*, Ann. Inst. H. Poincaré, *25*, 1–34 (1976).

[EMS] J.-P. ECKMANN, J. MAGNEN and R. SENEOR, *Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories*, Commun. Math. Phys., *39*, 251–271 (1975).

[FR] J. FELDMAN and R. RACZKA, *The relativistic field equations of the $(\lambda\phi^4)_3$ quantum field theory*, Ann. Phys., *108*, 212–229 (1977).

[FO] J. FELDMAN and K. OSTERWALDER, *The Wightman axioms and the mass gap for weakly coupled $(\phi^4)_3$ quantum field theories*, Ann. Phys., *97*, 80–135 (1976).

[Fr 1] J. FRÖHLICH, *Schwinger functions and their generating functionals, I and II*, Helv. Phys. Acta, *47*, 265–306 (1974) and Adv. Math., *23*, 119–181 (1977).

[Fr 2] J. FRÖHLICH, *Phase transitions, Goldstone bosons, and topological super-selection rules*, Acta Phys. Austr., Suppl. 15, 133–269 (1976).

[FS] J. FRÖHLICH and B. SIMON, *Pure states for general $P(\phi)_2$ theories: construction, regularity and variational equality*, Ann. Math., *105*, 493–526 (1977).

[FSS] J. FRÖHLICH, B. SIMON and T. SPENCER, *Infrared bounds, phase transitions and continuous symmetry breaking*, Commun. Math. Phys., *50*, 79–95 (1976).

[Ga] K. GAWEDZKI, *Existence of three phases for a $P(\phi)_2$ model quantum field*, Commun. Math. Phys., *59*, 117–142 (1978).

[GJ 1] J. GLIMM and A. JAFFE, *The $\lambda(\phi^4)_2$ quantum field theory without cutoffs, III, The physical vacuum*, Acta Math., *125*, 203–261 (1970).

[GJ 2] J. GLIMM and A. JAFFE, *Two and three body equations in quantum field models*, Commun. Math. Phys., *44*, 293–320 (1975).

[GJS 1] J. GLIMM, A. JAFFE and T. SPENCER, The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions, in *Constructive Quantum Field Theory*, edited by G. Velo and A. Wightman, Lecture Notes in Physics, Vol. 25, Springer-Verlag, Berlin, 1973.

[GJS 2] J. GLIMM, A. JAFFE and T. SPENCER, Existence of phase transitions for quantum fields, in *Les Méthodes Mathématiques de la Théorie Quantique des Champs*, Proceedings of the 1975 Marseille Conference.

[GJS 3] J. GLIMM, A. JAFFE and T. SPENCER, *Phase transitions for ϕ_2^4 quantum fields*, Commun. Math. Phys., *45*, 203–216 (1975).

[GJS 4] J. GLIMM, A. JAFFE and T. SPENCER, *Convergent expansion about mean field theory, I and II*, Ann. Phys., *101*, 610–630 and 631–669 (1976).

[Gu 1] F. GUERRA, *Uniqueness of the vacuum energy density and Van Hove phenomenon in the infinite volume limit for two-dimensional self-coupled Bose fields*, Phys. Rev. Lett., *28*, 1213 (1972).

[Gu 2] F. GUERRA, External field dependence of magnetization and long range order in quantum field theory, in *Quantum Dynamics: Models and Mathematics*, edited by L. Streit, *Acta Phys. Austr., Suppl. 16*, (1976).

[GRS 1] F. GUERRA, L. ROSEN and B. SIMON, *The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics, I and II*, Ann. Math., *101*, 111–189 and 191–259 (1975).

[GRS 2] F. GUERRA, L. ROSEN and B. SIMON, *Boundary conditions for the $P(\phi)_2$ Euclidean field theory*, Ann. Inst. H. Poincaré, *25*, 231–334 (1976).

[J] A. JAFFE, *Divergence of perturbation theory for bosons*, Commun. Math. Phys., *1*, 127–149 (1965).

[MS 1] J. MAGNEN and R. SENEOR, *The infinite volume limit of the $(\phi^4)_3$ model*, Ann. Inst. H. Poincaré, *24*, 95–159 (1976).

[MS 2] J. MAGNEN and R. SENEOR, *Phase space cell expansion and Borel summability for the Euclidean ϕ_3^4 theory*, Commun. Math. Phys., *56*, 237–276 (1977).

[Mi] R. A. MINLOS, *Generalized random processes and their extension in measure*, Trudy Moskov. Mat. Obsc., *8*, 497–518 (1959), *see also:* I. M. GEL'FAND and N. Ya. VILENKIN, *Generalized Functions*, Vol. 4, Academic Press, New York, 1964.

[OS] K. OSTERWALDER and R. SCHRADER, *Axioms for Euclidean Green's Functions, I and II*, Commun. Math. Phys., *31*, 83–112 (1973), and *42*, 281–305 (1975).

[OSe] K. OSTERWALDER and R. SENEOR, *The scattering matrix is non-trivial for weakly coupled $P(\phi)_2$ models*, Helv. Phys. Acta, *49*, 525–535 (1976).

[SS] E. SEILER and B. SIMON, *Nelson's symmetry and all that in the Yukawa$_2$ and $(\phi^4)_3$ field theories*, Ann. Phys., *97*, 470–518 (1976).

[Si] B. SIMON, *Correlation inequalities and the mass gap in $P(\phi)_2$, I. Domination by the two point function*, Commun. Math. Phys., *31*, 127–136 (1973), *II. Uniqueness of the vacuum for a class of strongly coupled theories*, Ann. Math., *101*, 260–267 (1975).

[Sp] T. SPENCER, *The mass gap for the $P(\phi)_2$ quantum field model with a strong external field*, Commun. Math. Phys., *39*, 63–76 (1974).

[Su 1] S. SUMMERS, Harvard University Thesis, 1979.

[Su 2] S. SUMMERS, *On the phase diagram of a two-dimensional Bose quantum field model*, Marseille preprint, CNRS, CPT/79/1138 (1979).