Vacuum state and particle creation in external electromagnetic fields[1)]

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Abstract. It is shown that a recently proposed definition of physical quantum states in external gauge fields (which is also applicable to generic gravitational fields) (i) reproduces some well known results about particle creation in static and time-dependent electromagnetic fields and (ii) provides a formally and physically satisfactory interpretation of a variety of phenomena which so far have been inaccessible to a conventional quantum field theoretical treatment. These phenomena include Klein's paradox, the Schiff-Snyder-Weinberg effect and the occurrence of resonance and Bloch type solutions in potentials which are periodic in time or space.

1. Introduction

During the past decade quantum field theory in external gravitational fields and, to a somewhat lesser degree, quantum field theory in external electromagnetic fields have attracted much interest (for recent reviews on these subjects see e.g. [1] and [2]). Although both enterprises are closely related, the gravitational context is usually considered as posing the more fundamental problem (but also as providing the more exciting perspectives) due to the lack of flat in- and out regions or even of asymptotic flatness in realistic models of space-time. It should be realized, however, that a somewhat milder version of the same problem exists also in the electromagnetic case, if the external field is of infinite duration. In particular strong electromagnetic fields, which are static within a good approximation, are encountered in various circumstances, but to our knowledge no satisfactory quantum field theory in the presence of supercritical field strengths has been formulated so far except for the special case of constant fields.

The embarrassing ambiguity of the notions of "ground state" and of "particle" in curved space-time has provoked several attempts of "vacuum definitions" (e.g. the "adiabatic" [3] and the "quasiclassical" [4] one and the requirement of "Hamiltonian diagonalization" [5]). On the other hand it has been argued [6] that the particle concept should be considered as a secondary one which makes sense only under special conditions (such as high symmetry of the background) and that the physics has to be sought only in the expectation values of observables which mediate the backreaction on the external field. However, even if this is taken for granted, the ambiguity persists in the choice of the state in which the expectation values are to be calculated.

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One obvious requirement that any general philosophy of choice has to fulfil is that in Minkowski space it should yield the unique vacuum state known to exist there. But we feel that in addition a selection rule for quantum states, if it exists at all, should be *universal* in the sense that it be equally applicable to all types of external gauge fields, in particular also to electromagnetic fields. A definition of this type has been proposed in papers [7]–[9], where also some new results on particle creation in external gravitational and electromagnetic fields were reported. To our knowledge this definition is the only one proposed so far that can claim universality in the above sense.

The main purpose of the present paper is to give a detailed derivation of the electromagnetic particle creation effects mentioned in [7] and [8]. Since many of these effects are well known, their verification constitutes a powerful test of any approach to the external field problem. The explicit proof that this test is successfully passed by our approach certainly strengthens the confidence in those of its predictions which are more remote from direct physical insight, in particular those concerning gravitation. Despite its emphasis of methodology, this paper is not confined to the rederivation of old results. The results on particle creation by potentials periodic in time or space appear to be new; moreover our treatments of the Klein paradox and the Schiff-Snyder-Weinberg effect are *quantum-field theoretical* accounts of the creation processes associated with these phenomena in contrast to other approaches which remain on the level of first quantization.

Although the covariant in-out formalism on which this paper is based has also been extended to cover Dirac particles [10], we shall confine ourselves to the linear scalar matter field. This paper is entirely devoted to the identification of physical quantum states and the evaluation of creation rates. The equally important problem of regularization of divergent expectation values of current densities is not treated. Likewise interesting mathematical aspects of the external field problem (e.g. existence of the S-matrix; for a recent review see e.g. [11]) have not been pursued. They would require a level of rigor which is beyond that of rudimentary functional analysis as employed in this paper.

In the following section we repeat some basic facts about quantum field theory in external fields from Ref. [9] and restate the two equivalent versions of the particle definition which is to be used in the subsequent sections of the paper. These deal with the following types of external fields: time-dependent, asymptotically constant, homogeneous electric fields (Sections 3); homogeneous temporally periodic fields (Section 4); static magnetic fields (Section 5); supercritical static electric fields in one space-dimension (Klein's paradox) (Section 6); spherically symmetric potentials including the Coulomb potential (Section 7) and spatially periodic potentials (Section 8). Sections 3, 4, 6 and 7 also contain exactly soluble examples.

2. Scalar particles in a classical electromagnetic field

In this paper we consider a quantized scalar field Φ of charge e and mass m coupled to a background electromagnetic potential described by the 4-vector A^μ. Φ obeys the minimally coupled Klein–Gordon (KG) equation

$$(\Box_e + m^2)\Phi = 0 \tag{2.1}$$

$$\Box_e := \eta^{\mu\nu}(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) \tag{2.2}$$

and the commutation relations

$$[\Phi(x), \Phi^{\dagger}(x')] = iG(x, x') \tag{2.3}$$

where G is the Cauchy two-point function solving the initial-value problem of equation (2.1)

$$\psi(x) = (G * \psi)_{\Sigma}(x) \tag{2.4}$$

$$(G * \psi)_{\Sigma}(x) := -\int_{\Sigma} d\sigma^{\mu}(x') G(x, x')[\overleftrightarrow{\partial}'_{\mu} + 2ieA_{\mu}(x')]\psi(x') \tag{2.5}$$

Equation (2.4) is valid for any solution ψ of equation (2.1) and any spacelike hypersurface Σ.

There are two distinguished solutions $K(x, x')$, $\tilde{K}(x, x')$ of the inhomogeneous equation associated with (2.1),

$$[\Box_e(x) + m^2]\overset{(\sim)}{K}(x, x') = -\delta^{(4)}(x - x') \tag{2.6}$$

which are the integral kernels of the boundary values $(\Box_e + m^2 - (+)i0)^{-1}$ of the resolvent of the minimally coupled KG differential operator. The resolvent is well-defined because this operator is self-adjoint with respect to the scalar product

$$\langle f, g \rangle = \int d^4x f^* g \tag{2.7}$$

(for a more detailed discussion of the self-adjointness condition see Ref. [9]). We call K and $\tilde{K}$ the "propagator" and "antipropagator", respectively. These Green's functions define linear operators $\overset{(\sim)}{P}\uparrow$, $\overset{(\sim)}{P}\downarrow$ acting on the linear space H of classical solutions of equation (2.1):

$$(\overset{(\sim)}{P} \uparrow \psi)(x) := (\overset{(\sim)}{K} * \psi)_{\Sigma}(x) \qquad x > \Sigma \tag{2.8}$$

$$(\overset{(\sim)}{P} \downarrow \psi(x) := -(\overset{(\sim)}{K} * \psi)_{\Sigma}(x) \qquad x < \Sigma \tag{2.9}$$

The subspaces defined by these operators,

$$H^+ := P \uparrow H \tag{2.10}$$

$$H^- := \tilde{P} \uparrow H \tag{2.11}$$

$$H_+ := \tilde{P} \downarrow H \tag{2.12}$$

$$H_- := P \downarrow H \tag{2.13}$$

yield two decompositions of H:

$$H = H^+ \oplus H^- = H_+ \oplus H_- \tag{2.14}$$

These decompositions are orthogonal with respect to the Klein–Gordon charge form

$$(\varphi, \psi) := \int_{\Sigma} d\sigma^{\mu} \varphi^* (i\overleftrightarrow{\partial}_{\mu} - 2eA_{\mu})\psi \tag{2.15}$$

Moreover, $(.,.)$ is positive in $H^+ \cup H_+$ and negative in $H^- \cup H_-$. Thus the formal requirements for the construction of two Fock representations corresponding to the two decompositions (2.14) are fulfilled. We give the 4 subspaces (2.10)–(2.13) the followong physical interpretation:

$$\begin{aligned} H^+ &= \{\text{outgoing particle solutions}\} \\ H_+ &= \{\text{ingoing particle solutions}\} \\ H^- &= \{\text{outgoing antiparticle solutions}\} \\ H_- &= \{\text{ingoing antiparticle solutions}\} \end{aligned} \tag{2.D}$$

As shown in Ref. [9], definition (2.D) can be reformulated in a manner that dispenses with the introduction of the propagators:

> A solution of the KG equation is an $\genfrac{}{}{0pt}{}{\text{outgoing}}{\text{ingoing}}$ particle (antiparticle) mode if it admits an analytical continuation in m^2 such that it remains regular, except in the $\genfrac{}{}{0pt}{}{\text{past}}{\text{future}}$, if m^2 acquires a $\genfrac{}{}{0pt}{}{\text{negative (positive)}}{\text{positive (negative)}}$ imaginary part. (2.D′)

By the term “regular” we mean that the solution under consideration obeys the restrictions characterizing the space of distributions on which $\Box_e$ can be defined and is self-adjoint not only in the formal sense. In all the examples treated in this paper, these restrictions exclude an exponential growth in any direction in terms of the Cartesian coordinates. In the case of singular potentials they also involve certain boundary conditions at the singularity.

Once the subspaces (2.10)–(2.13) have been identified, the calculation of the particle creation rate proceeds in the following well known manner: Choose orthonormal bases $\{{}^+\psi_{\vec\lambda}\}$, $\{{}^-\psi_{\vec\lambda'}\}$, $\{{}_+\psi_{\vec\lambda}\}$, $\{{}_-\psi_{\vec\lambda'}\}$ in H^+, H^-, H_+, H_-, i.e.

$$({}^+\psi_{\vec\lambda}, {}^+\psi_{\vec\lambda'}) = ({}_+\psi_{\vec\lambda}, {}_+\psi_{\vec\lambda'}) = \delta(\vec\lambda, \vec\lambda') \tag{2.16}$$

$$({}^-\psi_{\vec\lambda}, {}^-\psi_{\vec\lambda'}) = ({}_-\psi_{\vec\lambda}, {}_-\psi_{\vec\lambda'}) = -\delta(\vec\lambda, \vec\lambda') \tag{2.17}$$

where $\vec\lambda$ represents a collection of quantum numbers or is a more general type of index. The coefficients ${}^+a_{\vec\lambda}$, ${}^-a^\dagger_{\vec\lambda'}$, ${}_+a_{\vec\lambda}$, ${}_-a^\dagger_{\vec\lambda'}$, defined by the expansion of the field Φ in terms of the in- and outgoing modes modes can be interpreted as particle annihilation or antiparticle creation operators and yield the in- and outgoing Fock representations based on vacua $|0\text{ in}\rangle$ and $|0\text{ out}\rangle$. The expectation value $\langle N_{\vec\lambda}\rangle$ of the number of pairs created in the mode $\vec\lambda$ in the state $|0\text{ in}\rangle$ is determined by

$$\langle N_{\vec\lambda}\rangle\, \delta(\vec\lambda, \vec\lambda') = \langle 0\text{ in}|^+\, a^\dagger_{\vec\lambda} a_{\vec\lambda'}\, |0\text{ in}\rangle \tag{2.18}$$

If the transformation between the in- and outgoing basis is of the special form (as it will happen in all the cases treated in this paper)

$$\begin{aligned} {}^+\psi_{\vec\lambda} &= \alpha_{\vec\lambda +},\, \psi_{\vec\lambda} + \beta_{\vec\lambda -}\psi_{\vec\lambda} \\ {}^-\psi_{\vec\lambda} &= \gamma_{\vec\lambda +},\, \psi_{\vec\lambda} + \varepsilon_{\vec\lambda -}\psi_{\vec\lambda} \end{aligned} \tag{2.19}$$

then

$$\langle N_{\vec{\lambda}} \rangle = |\beta_{\vec{\lambda}}|^2 = |\gamma_{\vec{\lambda}}|^2 = |\alpha_{\vec{\lambda}}|^2 - 1 \tag{2.20}$$

This formula will frequently be used in the subsequent sections.

3. Homogeneous, asymptotically constant electric fields

In this section we deal with electromagnetic potentials of the type $A^\mu = (0, \vec{A}(t))$ where it is assumed that both the direction and the modulus of the electric field become constant for $t \to \pm\infty$. The quantum effects that result from these conditions were first explored by Heisenberg and Euler [12] and later derived by Schwinger ([13], [14]) by very elegant formal techniques. As was shown in a previous paper [9] the formalism that we will employ is in accordance with Schwinger's, but it is more closely adapted to conventional quantum field theory in that two Fock spaces of in- and outgoing particles are constructed explicitly. Thus we are able to show that in the special case considered our particle definition coincides with the "quasiclassical" one which was introduced by Nikishov [15]. A treatment of electron-positron creation by a time-dependent electric field with the use of the quasiclassical particle definition is also due to Marinov and Popov ([16] and references cited therein). Whereas for Dirac particles this definition is equivalent to the requirement of instantaneous diagonalization of the Hamiltonian mentioned in the introduction, no such connection exists in the scalar case.

As can be seen from the example of the constant electric field treated in the end of this section there exist complete systems of solutions of the KG equation minimally coupled to the potential A^μ which show quasiclassical behavior for $t \to \infty$ or $t \to -\infty$. We denote these systems by $\{{}^{\pm}\psi_{m^2\vec{k}}\}$ and $\{{}_{\pm}\psi_{m^2\vec{k}}\}$, respectively:

$${}^{\pm}\psi_{m^2\vec{k}} = (2\pi)^{-3/2} \exp(i\vec{k}\vec{x})\, {}^{\pm}f_{m^2\vec{k}}(t) \tag{3.1}$$

$${}^{\pm}f_{m^2\vec{k}} \xrightarrow{t\to\infty} [2E(t)]^{-1/2} \exp\left[\mp i\int_{t_0}^{t} dt' E(t')\right] \tag{3.2}$$

$$E(t) := [m^2 + (\vec{k} - e\vec{A}(t))^2]^{1/2} \tag{3.3}$$

The modes ${}_{\pm}\psi_{m^2\vec{k}}$, ${}_{\pm}f_{m^2\vec{k}}$ are defined in an analogous manner with "$t \to \infty$" of (3.2) being replaced by "$t \to -\infty$". The normalization has been chosen such that

$$({}^{\pm}\psi_{m^2\vec{k}}, {}^{\pm}\psi_{m^2\vec{k}'}) = ({}_{\pm}\psi_{m^2\vec{k}}, {}_{\pm}\psi_{m^2\vec{k}'}) = \pm\delta(\vec{k} - \vec{k}') \tag{3.4}$$

Eq. (3.4) implies that the "classical S-matrix" is pseudo-unitary:

$${}^{+}\psi_{m^2\vec{k}} = \alpha_{m^2\vec{k}}\, {}_{+}\psi_{m^2\vec{k}} + \beta_{m^2\vec{k}}\, {}_{-}\psi_{m^2\vec{k}} \tag{3.5}$$

$${}^{-}\psi_{m^2\vec{k}} = \beta^*_{m^2\vec{k}}\, {}_{+}\psi_{m^2\vec{k}} + \alpha^*_{m^2\vec{k}}\, {}_{-}\psi_{m^2\vec{k}}$$

$$|\alpha_{m^2\vec{k}}|^2 - |\beta_{m^2\vec{k}}|^2 = 1 \tag{3.6}$$

If ${}^{+}\psi_{m^2\vec{k}}$ is continued analytically in m^2 into the lower half complex plane by substituting $m^2 \to m^2 - i\varepsilon$ in (3.3), (3.2), (3.5), it converges to 0 as $t \to \infty$ whereas it grows exponentially for $t \to -\infty$ (the latter property follows from the asymptotic behavior of the ingoing modes, the branch cut of $E(m^2)$ lying on the negative real m^2-axis). Thus in virtue of definition (2.D′) the $^+\psi$'s are readily identified as outgoing particle modes. A similar inspection shows that $^-\psi \in H^-$, $_+\psi \in H_+$, $_-\psi \in H_-$ (cf. (2.10)–(2.13)). An alternative, but more lengthy proof of this relation consists of using the spectral representation of the propagator K and then checking the propagation properties (2.8), (2.9).

In order to make the discussion of some exactly soluble cases that follows self-contained we list some well known general formulae for creation probabilities. According to (2.20) the number of particles created with canonical momentum $\vec{k}$ is equal to $|\beta_{\vec{k}}|^2$. Because of the homogeneity of the external field the total number of particles created is either zero or infinite. The "Golden Rule" yields the spatial density n of the number of particles created

$$n = (2\pi)^{-3} \int d^3k \, |\beta_{\vec{k}}|^2 \tag{3.7}$$

which is finite if the external field is only of finite duration. We now introduce the absolute probability $c_{\vec{k}}$ of *no* pair with momentum $\vec{k}$ being created and the associated *relative* probability $w_{\vec{k}}$ of pair creation in the same mode. They are related by

$$c_{\vec{k}}(1 + w_{\vec{k}} + w_{\vec{k}}^2 + \cdots) = 1 \Rightarrow c_{\vec{k}} = 1 - w_{\vec{k}} \tag{3.8}$$

The expectation value of the total number of pairs created with momentum $\vec{k}$ is

$$\langle N_{\vec{k}} \rangle = c_{\vec{k}}(0 + w_{\vec{k}} + 2w_{\vec{k}}^2 + \cdots) = \frac{1 - c_{\vec{k}}}{c_{\vec{k}}} \tag{3.9}$$

Hence (cf. (2.20) and (3.6))

$$c_{\vec{k}} = |\alpha_{\vec{k}}|^{-2} \tag{3.10}$$

If the spatial sections of Minkowski space were replaced by 3-dimensional tori the vacuum persistence probability would be

$$|\langle 0 \text{ out} \mid 0 \text{ in} \rangle| = \Pi c_{\vec{k}} \tag{3.11}$$

where the product is over the countable set of allowed momenta. In the standard topology (3.11) generalizes to

$$|\langle 0 \text{ out} \mid 0 \text{ in} \rangle|^2 = \exp\left(-\int d^3x w\right) \tag{3.12}$$

$$w = -(2\pi)^{-3} \int d^3k \ln c_{\vec{k}} \tag{3.13}$$

The left hand side of (3.12) is either unity or zero. Because of the homogeneity of the field one may consider $\exp(-w)$ as the probability that no particles are created in a unit volume. Again, this is finite for a switched external field. The quantity w is related to the effective Lagrangian of the Schwinger formalism [13] by

$$w = 2 \int_{-\infty}^{\infty} dt \, \mathrm{Im}\, \mathscr{L}^{(1)} \tag{3.14}$$

We have calculated the momentum spectrum of the particles created for some 2-dimensional examples. It is straightforward to recover from these the 4-dimensional cases with the field in the x^3-direction by the formal substitution $k \to k^3$, $m^2 \to m^2 + k_1^2 + k_2^2$.

(a) *Time-dependent potential of the Sauter type*

$$A^\mu = (0, V/[1+\exp(-at)])$$

$$\langle N_k \rangle = \frac{\cosh\left(2n\dfrac{E_+ - E_-}{a}\right)\begin{cases} -\cos 2\pi\lambda & (\lambda \text{ real}) \\ +\cosh 2\pi \operatorname{Im}\lambda & (\lambda \text{ complex}) \end{cases}}{\cosh[2\pi(E_+ + E_-)/a] - \cosh[2\pi(E_+ - E_-)/a]}$$

$$E_+ := [m^2 + (k - eV)^2]^{1/2}, \qquad E_- := (m^2 + k^2)^{1/2}, \qquad \lambda(\lambda - 1) = -e^2V^2/a^2$$

As for $k \to \infty$ $N_k \to \exp(-4k/a)$ the number density of particles is finite except in the case $m^2 = 0$, where the integral (3.7) diverges logarithmically at $k = 0$ and $k = eV$. This property is shared by w of (3.14).

(b) *The temporal step potential*

$$A^\mu = (0, V\Theta(t))$$

$$\langle N_k \rangle = \frac{(E_- - E_+)^2}{4E_-E_+} \xrightarrow{k\to\infty} \frac{e^2V^2}{k^2}$$

The convergence properties of the number density are the same as in example a), from which b) emerges in the limit $a \to \infty$.

(c) *The constant electric field* $A^\mu = (0, Ft)$

Since this example is the only nontrivial one as far as the problem of particle definition is concerned, we present it in some more detail.

The classical action for a scalar particle (solution of the Hamilton-Jacobi equation) which separates in Cartesian coordinates is $\pm kx \pm S_k(t)$, where S_k is the phase of ${}^+f_k$ appearing in (3.2):

$$S_k = \int^t dt'[m^2 + (k - eFt')^2]^{1/2} \xrightarrow{t\to\pm\infty} \pm(z^2/4 + a \ln|z|)$$

$$z := (2eF)^{1/2}(t - k/eF)$$

$$a := m^2/2eF$$

The following exact solutions of the KG equation are asymptotically quasiclassical and orthonormal:

$${}^+f_{m^2k} = 2^{-3/4}(eF)^{-1/4}e^{i\delta}E^*(-a, -z)$$

$$\xrightarrow{t\to\infty} z^{-1/2}\exp[-i(z^2/4 + a(\ln z)]$$

$$\delta = \frac{\pi}{4} + \frac{1}{2}\arg\Gamma(\tfrac{1}{2} - ia)$$

E is the parabolic cylinder function defined in [17], p. 693. The transformation coefficients of (3.5) are

$$\alpha = (2\pi)^{1/2} \exp(-\pi a/2)/\Gamma(1/2 + ia)$$
$$\beta = i \exp(-\pi a)$$

Hence the number of particles created, $|\beta_k|^2 = \exp(-2\pi a)$, is independent of k and the number density diverges. However an argument due to Nikishov [15] renders a more detailed description of the creation process possible: If the canonical momentum of a particle is equal to k, then its kinetical momentum $k - eFt$ vanishes at $\tau = k/eF$. Hence ${}^{+}\psi_k$ describes a particle which is created at $\tau(k) = k/eF$ and $|\beta_k|^2$ is the total number of particles created at $\tau(k)$. Thus in our case (3.13) can be rewritten as

$$w = (2\pi)^{-1} \int dk \ln c_k = (2\pi)^{-1} eF \int d\tau \ln c$$

and we obtain

$$2 \operatorname{Im} \mathscr{L}^{(1)}(x) = -(2\pi)^{-1} eF \ln [1 + \exp(-2\pi a)]$$
$$= (2\pi)^{-1} eF \sum_{1}^{\infty} (-)^{n-1} n^{-1} \exp(-n\pi m^2/eF)$$

With the substitutions mentioned it is easy to rederive the 4-dimensional result

$$2 \operatorname{Im} \mathscr{L}^{(1)}(x) = e^2 F^2 (2\pi)^{-3} \sum_{1}^{\infty} (-1)^{n-1} n^{-2} \exp(-n\pi m^2/eF)$$

([13], [18]).

4. Homogeneous electric fields periodic in time

Periodic external fields are of special interest as they do not admit a quasiclassical behavior of the wavefunctions even asymptotically. Thus the need for a more general particle concept than the quasiclassical one is evident in this case. We show in the following that the "in-out" formalism is perfectly applicable also in this situation. We consider the potential $A^\mu = (0, \vec{A}(t))$ with $\vec{A}(t)$ periodic in t with period T. Our ansatz for the physical modes of the scalar field is analogous to that of (3.1):

$$\psi_{m^2\vec{k}} = (2\pi)^{-3/2} \exp(i\vec{k}\vec{x}) f_{m^2\vec{k}}(t) \tag{4.1}$$

As the KG equation is of second order, there are 2 linearly independent solutions for fixed values of m^2 and $\vec{k}$. Due to the invariance of the potential under time translations by the interval T, there exist solutions $\psi^{(B)}_{m^2\vec{k}}$ with the property

$$f^{(B)}_{m^2\vec{k}}(t+T) = \exp[iA(m^2, \vec{k})] f^{B}_{m^2\vec{k}}(t) \tag{4.2}$$

This is a consequence of the well known Bloch theorem (see e.g. [19]) familiar from solid state physics, and we will refer to the modes (4.2) as "Bloch waves". Usually, i.e. in the spatially variable static analog of the potential treated here, only the case of real A is considered. We first impose this reality condition on the solutions (6.2), too, but will eventually relax it. For fixed m^2, the condition implies a band structure of the $\vec{k}$ spectrum in complete analogy to the energy spectrum of electrons in a crystal. A/T may be regarded as the "quasi-energy" of the Bloch waves. Since f obeys a *real* ordinary differential equation, both $f^{(B)}$ and $f^{(B)*}$ represent Bloch waves with characteristic phase A and $-A$, respectively. For fixed $\vec{k}$, the reality condition for A implies a band structure of the m^2 spectrum. Within a band, A is a real analytic function of m^2. We can continue this function analytically into the complex m^2-plane and derive the property

$$A(m^2-i\varepsilon, \vec{k}) = A^*(m^2+i\varepsilon, \vec{k}) \quad (m^2 \text{ real}) \tag{4.3}$$

The relations (4.2) and (4.3) are all we need to apply definition (2.D′) to the Bloch waves with real quasi-energy. For (4.2) and (4.3) imply that either $f^{(B)}$ of $f^{(B)*}$ is exponentially damped in t after an analytic continuation into the lower half of the complex m^2-plane. We denote this particular Bloch wave by ${}^+f$, since it is an outgoing particle mode according to (2.D′). Denoting the other modes defined by (2.D′) in an analogous manner, we conclude from (4.3)

$${}^+f = {}_+f = {}^-f^* = {}_-f \quad \text{if } A \text{ real} \tag{4.4}$$

Thus no particles are created in the modes (4.2) with real A. This is no surprise, since due to the strict periodicity of the external field the motion of a particle is "quasi-free" and it is impossible to distinguish "ingoing" and "outgoing" modes by a different asymptotic behavior for $t \to -\infty$ and $t \to +\infty$.

The situation is different, however, if A is complex. There is no physical reason to exclude the corresponding modes from our considerations, as long as m^2 and $\vec{k}$ are real. Although the exponential time-dependence implied by (4.2) excludes these modes from the generalized domain of self-adjointness of $\Box_e$, the charge form still exists and can easily be shown to fulfill

$$(\psi^{(B)}, \psi^{(B)}) = 0 \quad \text{if } A \text{ complex} \tag{4.5}$$

As A is locally an analytic function of m^2 also for these modes, we conclude

$$\psi^{(B)} \in H^+ \cap H^- \quad \text{if } \operatorname{Im} A > 0 \tag{4.6}$$

$$\psi^{(B)} \in H_+ \cap H_- \quad \text{if } \operatorname{Im} A < 0 \tag{4.7}$$

Thus not only the difference between particles and antiparticles gets lost in these modes, but also the overlap between the spaces of ingoing and outgoing solutions, i.e. the classical S-matrix (2.19) and the Bogolyubov transformation implied by it do not exist any more. The same phenomenon also occurs in a deep square well potential, which will be discussed in Section 7. The only consistent interpretation of (4.6), (4.7) appears to be that a finite number of "pseudo-particles" with zero charge is created in every "resonant" mode (A complex) in a finite time. Because of the homogeneity of the external field, and since the resonant modes form a continuum, the expectation value of the number density of particles created per unit time is finite as in the case of the constant electric field treated in 3.c).

5. Static magnetic fields

It is well known that magnetic fields, although they induce various vacuum polarization phenomena [20], do not give rise to spontaneous pair creation.

Intuitively speaking the reason is that a magnetic field cannot separate the virtual pairs that constitute the physical vacuum. This fact is reflected by the in-out formalism in the following way.

Consider the potential $A^{\mu}=(0, \vec{A}(\vec{x}))$. If $\vec{A}(\vec{x})$ is a sufficiently "well-behaved" function, the negative of the minimally coupled Laplacian

$$-\Delta_e=(i\partial_\alpha-eA_\alpha)(i\partial_\alpha-eA_\alpha) \tag{5.1}$$

defines a self-adjoint operator on a certain domain $D\subset L^2(R^3)$, where $L^2(R^3)$ is the Hilbert space of square integrable functions on the physical space. (More specifically, from theorems of Ref. [21] one can infer that if A_α is continuously differentiable and $\partial_\alpha A_\alpha=0$ then $-\Delta_e$ is essentially self-adjoint on the space of smooth functions with compact support $C_0^\infty(R^3)$, i.e. $D\supset C_0^\infty(R^3)$ is unique and the completion of $C_0^\infty(R^3)$ with respect to the norm

$$\|\psi\|^2_{-\Delta_e}=\|\psi\|^2-\int d^3x\psi^*\Delta_e\psi.)$$

If $-\Delta_e$ is self-adjoint it is also positive, and we may choose a complete system $\{\varphi_{\vec{\lambda}}\}$ of eigendistributions of $-\Delta_e$ obeying

$$-\Delta_e\varphi_{\vec{\lambda}}=\lambda^2\varphi_{\vec{\lambda}} \tag{5.2}$$

$$\int d^3x\varphi^*_{\vec{\lambda}}\varphi_{\vec{\lambda}'}=\delta(\vec{\lambda},\vec{\lambda}') \tag{5.3}$$

As the self-adjointness of $-\Delta_e$ implies that of the minimally coupled d'Alembertian

$$\Box_e=\frac{\partial^2}{\partial t^2}-\Delta_e \tag{5.4}$$

a complete generalized orthonormal basis of eigendistributions for $\Box_e$ is given by

$$^{\pm}\psi_{m^2\vec{\lambda}}=(4\pi E)^{-1/2}\exp(\mp iEt)\varphi_{\vec{\lambda}} \tag{5.5}$$

$$E:=(m^2+\lambda^2)^{1/2}, \qquad \lambda^2>-m^2 \tag{5.6}$$

The orthonormality relations are

$$\langle{}^{+}\psi_{m^2\vec{\lambda}}, {}^{-}\psi_{m'^2\vec{\lambda}'}\rangle=0 \tag{5.7}$$

$$\langle{}^{\pm}\psi_{m^2\vec{\lambda}}, {}^{\pm}\psi_{m'^2\vec{\lambda}'}\rangle=\delta(m^2-m'^2)\,\delta(\vec{\lambda},\vec{\lambda}') \tag{5.8}$$

For $m^2>0$, to which we restrict ourselves, all physically meaningful solutions of the minimally coupled KG equations can be obtained by superposition of the modes $\psi_{m^2\vec{\lambda}}$. (This is not true in the case $m^2<0$, since then even in the absence of any external field exponentially damped or anti-damped solutions exist which signalize the well known tachyon instability.) With the observation that the modes (5.5) diverge exponentially either as $t\to\infty$ or $t\to-\infty$ as soon as m^2 becomes complex, it is now easy to verify that ${}^{+}\psi({}^{-}\psi)$ are particle (antiparticle) solutions as defined by (2.D′) and that there is no distinction between the ingoing and outgoing modes. The latter fact implies the stability of the vacuum as anticipated.

6. Static electric fields I: Klein's paradox

For our purposes it is no essential restriction to consider a static potential $A^\mu = (A^0(x), 0)$ in 2 space-time dimensions instead of a 4-dimensional one that depends only on one spatial coordinate. We assume

$$A^0(x) \xrightarrow{x\to\pm\infty} A^0_\pm + F_\pm x \tag{6.1}$$

where $A^0_\pm, F_\pm$ (the electric fields at spatial infinity) are constants. In order to exclude bound states, which will be treated in Section 7, from our discussion, we assume furthermore

$$\lim_{x\to\infty} eA^0(x) - \lim_{x\to-\infty} eA^0(x) > 2m \tag{6.2}$$

A typical form of the potential A^0 (with $F_+ = F_- = 0$) is shown in Fig. 1. The hatched region shows the energy interval which is forbidden for a classical particle, in the dependence of x. Quantum mechanically, a particle with energy $eA^0_- + m < E < eA^0_+ - m$ (we will refer to this as the "critical" interval) may tunnel through the forbidden zone, however. In doing this, its current changes its direction so that the reflection coefficient is greater than 1. This is the famous "paradox" discovered by Klein [22]. Although it was recognized quite early as an indication of particle creation ([12, [23]) (the simplest explanation being Einstein's famous argument interrelating induced and spontaneous emission of a quantum-mechanical system), no satisfactory quantum field-theoretical description of this process has been given in the literature. A thorough investigation including various numerical examples is due to Fulling [24], but remains inconclusive with respect to the question of particle interpretation. Bongaarts and Ruijsenaars [25] have even concluded that the Klein paradox "corresponds to a situation which cannot be described properly within the framework of a field theory with an external potential", because a unitary S-matrix no longer exists under condition (6.2). We feel that this latter phenomenon, which is common to most "creative" external fields with non-compact support, should not be overdramatized, and that the Fock space description of the creation process, which we

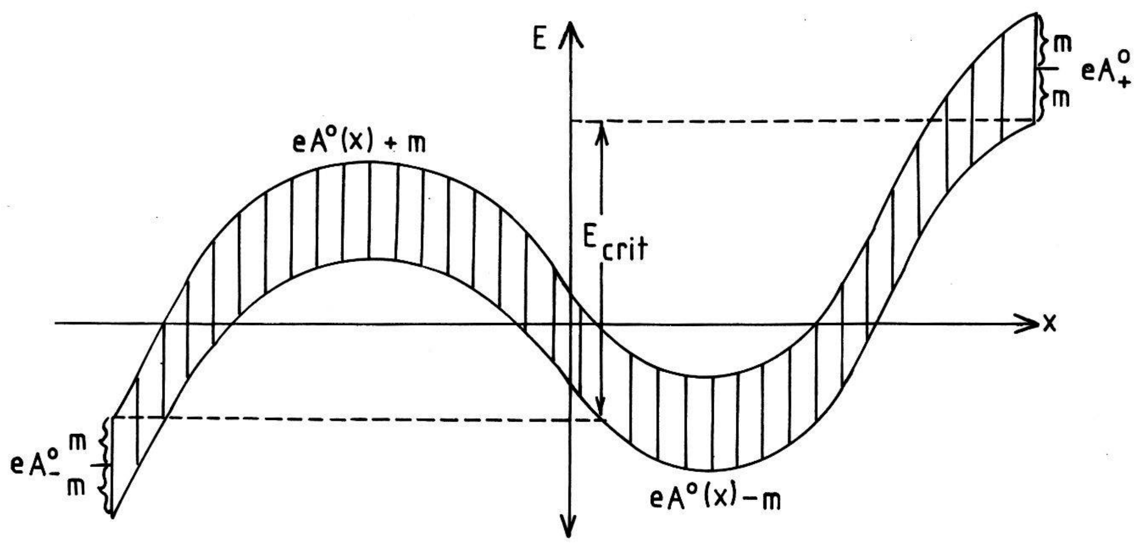

Figure 1
Forbidden energies (hatched) of a classical particle in a potential $A^0(x)$ exhibiting Klein's paradox.

obtained, is completely satisfactory from the physical point of view. Mathematically it should be taken as an indication that a more general formulation of quantum mechanics namely the C^*-algebra approach is needed [26].

As the KG equation of our problem is mathematically equivalent to the one of Section 3 (with the sign of m^2 reversed), the following systems of solutions $\{\psi\uparrow_{m^2E}, \psi\downarrow_{m^2E}\}$, $\{\uparrow\psi_{m^2E}, \downarrow\psi_{m^2E}\}$ are obtained from (3.1) by obvious changes of notation:

$$\psi\uparrow(\downarrow) = (2\pi)^{-1/2} \exp(-iEt) f\uparrow(\downarrow)(x) \tag{6.3}$$

$$f\uparrow(\downarrow)(x) \xrightarrow{x\to\infty} [2k(x)]^{-1/2} \exp\left[+(-)i\int^x dx' k(x)\right] \tag{6.4}$$

$$\text{if } k(x) := [(E - eA^0(x))^{1/2} - m^2]^{1/2} \quad \text{is (positive) real}$$

$$\xrightarrow{x\to\infty} [2k(x)]^{-1/2} \exp\left[+(-)\int^x dx' \kappa(x')\right]$$

$$\text{if } \kappa(x) := [m^2 - (E - eA^0(x))^2]^{1/2} \quad \text{is (positive) real}$$

Let $\uparrow(\downarrow)\psi_{m^2E}$ and $\uparrow(\downarrow)f_{m^2E}$ be defined in an analogous manner with "$x \to \infty$" being replaced by "$x \to -\infty$" in (6.4). If $\lim_{x\to\pm\infty} k(x) =: k_\pm$ is (positive) real – the limit may also be infinite –, then current conservation implies

$$f\downarrow_{m^2E} = \alpha_{m^2E} \downarrow f_{m^2E} + \beta_{m^2E} \uparrow f_{m^2E} \tag{6.5}$$

$$f\uparrow_{m^2E} = \beta^*_{m^2E} \downarrow f_{m^2E} + \alpha^*_{m^2E} \uparrow f_{m^2E}$$

$$|\alpha|^2 - |\beta|^2 = 1 \tag{6.6}$$

If k_+ and/or k_- is imaginary, the α and β coefficients still exist and can be obtained from those of (6.5) by analytic continuation in E (however they do no longer obey equation (6.6)).

We consider first the case $F_+ = F_- = 0$. Here definition (2.D′) can be used directly to identify the in- and outgoing physical states in the following way: Write

$$E(k_-, m^2) = \pm(k_-^2 + m^2)^{1/2} + eA_-^0 \tag{6.7}$$

$$k_+(k_-, m^2) = [k_-^2 + 2e(\pm(k_-^2 + m^2)^{1/2} + eA_-^0)(A_-^0 - A_+^0) + e^2(A_+^0 - A_-^0)^2]^{1/2} \tag{6.8}$$

and consider the matrix coefficients defined in (6.5) as functions of k_- and m^2. Continue the solution $\psi\uparrow_{m^2E}(k_+ \text{ real})$ analytically to complex values of m^2 while keeping k_- fixed at a real value, i.e. consider that solution ψ of the KG equation with complex m^2, which is defined by the asymptotic behavior

$$\psi \xrightarrow{x\to\infty} [4\pi k_+(k_-, m^2)]^{-1/2} \exp[-iE(k_-, m^2)t + ik_+(k_-, m^2)x] \tag{6.9}$$

This solution converges to 0 as $x \to \infty$, if $\mathrm{Re}(E - m) > eA_-^0$ and $\mathrm{Im}(m^2) < 0$ or if $\mathrm{Re}(E + m) < eA_-^0$ and $\mathrm{Im}(m^2) > 0$ (because $A_-^0 - A_+^0 < 0$). In both cases it vanishes also for $t \to \infty$. Therefore $\psi\uparrow$ is an outgoing particle mode in the first case and an outgoing antiparticle mode in the second case. A similar discussion applies to k_+ negative imaginary. In the same manner, possibly by interchanging the roles of k_+ and k_-, the physical character of all the solutions (6.3) and their "reflected" versions can be determined. The complete result is listed in Table 1.

E	$\psi\uparrow$	$\psi\downarrow$	$\uparrow\psi$	$\downarrow\psi$
$> eA^0_+ + m$	$\longleftarrow H^+ = H_+ \longrightarrow$			
$(eA^0_+ - m, eA^0_+ + m)$	-	$H^+ = H_+$	-	-
$(eA^0_- + m, eA^0_+ - m)$	H^+	H_+	H^-	H_-
$(eA^0_- - m, eA^0_- + m)$	-	-	$H^- = H_-$	-
$< eA^0_- - m$	$\longleftarrow H^- = H_- \longrightarrow$			

Table 1
Classification of the solutions in the static potential (2.1), (2.2) according to their energies.

As can be inferred from Table 1, our definition predicts pair creation in the "critical" energy region and is thus in accordance with what one expects physically. Moreover, an application of the principle of stationary phase, i.e. differentiation of the phase of the wave-functions with respect to the parameter E, shows that wave-packets built from the $\genfrac{}{}{0pt}{}{\text{ingoing}}{\text{outgoing}}$ particle (antiparticle) modes are concentrated at large negative (positive values of x as $\genfrac{}{}{0pt}{}{t\to\infty}{t\to-\infty}$. Hence the terms "ingoing" and "outgoing" attributed to the modes of Table 1 apply also in their intuitive meaning. The charges of these solutions will be evaluated below.

We now turn to the case where not both F_+ and F_- are zero. Since here k_+ and/or k_- is infinite, the construction of suitable analytic continuations is not obvious and we have not succeeded in applying definition (2.D′) directly. Therefore we have to calculate the propagator K first and then to use definition (2.D). There are at least two methods to calculate K. The most straightforward one is to make use of the special relationship already mentioned of our problem to the time-dependent case of Section 3. From this we conclude that K has *spatial* propagation properties (if a "charge form" is defined on timelike hypersurfaces) analogous to the chronological ones on which our particle definition is based. These propagation properties can readily be inferred from (2.D′) after interchanging space and time (care has to be taken however of the fact that the propagator of this section corresponds to the antipropagator of Section 3, since the mathematical equivalence is only established by the substitution $m^2\to -m^2$). More specifically, "future" has to be replaced by "the direction of increasing coordinate x". The propagator is completely determined by the functions it propagates and can be evaluated using the "temporal charge" of these functions:

$$K(x, y) = -i\int_{-\infty}^{\infty} dE'\Big[\Theta(x^1-y^1)\frac{1}{\alpha^*_{E'}}\psi\uparrow_{E'}(x)\uparrow\psi^*_{E'}(y) + \Theta(y^1-x^1)\frac{1}{\alpha^*_{E'}}\downarrow\psi_{E'}(x)\psi\downarrow^*_{E'}(y)\Big] \quad (6.10)$$

Alternatively we can start from the spectral representation of the resolvent $-(\square_e + m^2 - i0)^{-1}$, as already mentioned in Section 3, and calculate K according to the formula

$$K(x, y) = \int_{-\infty}^{\infty} d(m'^2) \int dE \sum_{\alpha} \frac{\psi_{m^2E\alpha}(x)\psi^*_{m^2E\alpha}(y)}{m'^2 - m^2 + i0} \tag{6.11}$$

by integration in the complex m^2-plane. $\{\psi_{m^2E\alpha}\}$ denotes a complete system of eigendistributions of $\Box_e$ orthonormal with respect to the scalar product (2.7) (the label α assumes two values at most). Since the calculation is rather involved, we quote only the result

$$\begin{aligned} K(x, y) \xrightarrow{x^1, y^1 \to \infty} & -i\int_{-\infty}^{\infty} dE \frac{\exp[-iE(x^0 - y^0)]}{4\pi[k(x^1)k(y^1)]^{1/2}} \left[\Theta(x^1 - y^1) \exp\left(i\int_{y^1}^{x^1} k(x')\,dx'\right)\right. \\ & \left. + \Theta(y^1 - x^1) \exp\left(-i\int_{y^1}^{x^1} k\right) - \frac{\beta}{\alpha^*} \exp\left(i\int^{x^1} k + i\int^{y^1} k\right)\right] \\ \xrightarrow{x^1(y^1) \to +(-)\infty} & -i\int_{-\infty}^{\infty} dE \frac{\exp[-iE(x^0 - y^0)]}{4\pi[k(x^1)k(y^1)]^{1/2}} \frac{1}{\alpha^*} \exp\left[i\int^{x^1} k - i\int^{y^1} k\right] \end{aligned} \tag{6.12}$$

by which (6.10) is implied.

In order to derive the propagation properties of K in the ordinary sense we evaluate $K * \psi\uparrow_E$ and suppose at first k_+ and k_- to be real. As the integrand in (6.10) is not continuously differentiable at $x^1 = y^1$, we separate the integral

$$\begin{aligned} (K * \psi\uparrow_E)(x) &= -\int_{-\infty}^{\infty} K(x, y)\left[\frac{\overleftrightarrow{\partial}}{\partial y^0} + 2ieA^0(y^1)\right]\psi\uparrow_E(y^1)\,dy^1 \\ &= -\int_{-\infty}^{x^1} - \int_{x^1}^{\infty} \end{aligned} \tag{6.13}$$

into two parts, which we can calculate by partial integration. Let x^1 be large enough so that the asymptotic behavior (6.4) of the wave functions may be employed. Then (we postpone the E'-integration of (6.10))

$$\begin{aligned} \int_{-\infty}^{x^1} dy \uparrow f^*_{E'}(y)[E + E' - 2eA^0(y)]f\uparrow_E(y) &= \frac{1}{E' - E} \uparrow f^*_{E'} \frac{\overleftrightarrow{d}}{dy} f\uparrow_E \bigg|_{-\infty}^{x^1} \\ = \frac{i}{E' - E}\Bigg\{ & \frac{1}{2(kk')^{1/2}}\left[-\beta(k - k')\exp\left(i\int^{x^1}(k + k')\right)\right. \\ & \left. + \alpha^*(k + k')\exp\left(i\int^{x^1}(k - k')\right)\right] - \frac{1}{2(kk')^{1/2}} \\ \times & \left[\beta^*(k' - k)\exp\left(-i\int^{y}(k + k')\right) + \alpha^*(k + k')\exp\left(-i\int^{y}(k - k')\right)\right]\Bigg|^{-\infty}\Bigg\} \end{aligned} \tag{6.14}$$

Since

$$\lim_{x \to \pm\infty} (k')^{-1/2} \frac{\overleftrightarrow{d}}{dx} k^{-1/2} = 0$$

we could omit some terms. If k' is real, we may infer from

$$k \pm k' = (E - E')\frac{E + E' - 2eA^0}{k \mp k'}, \qquad \int^{-\infty} \frac{dy}{k'(y)} = \infty, \qquad \left|\frac{\partial k}{\partial E}\right| < \infty$$

and the local boundedness of α, β regarded as functions of E that the contribution at $y \to -\infty$ is equal to $s_- a^*(\pi/2)\,\delta(E-E')$ with

$$s_\pm := \operatorname{sgn}\left[E - \lim_{x\to\pm\infty} eA^0(x)\right] \tag{6.15}$$

If k' is imaginary, the contribution from $y \to -\infty$ vanishes leaving the rest unchanged. Analogously, we get for the second part of (6.13)

$$\int_{x^1}^{\infty} f{\downarrow}^*_{E'}(y)(E+E'-2eA^0)f{\uparrow}_E(y)\,dy$$
$$= i\frac{k-k'}{2(E-E')}[k'(x^1)k(x^1)]^{-1/2}\exp\left[i\int^{x^1}(k+k')\right] \tag{6.16}$$

Adding both parts and performing the E'-integration, we obtain

$$(K*\psi{\uparrow}_E)_{y^0}(x) = i\int_{-\infty}^{\infty} dE'\,\frac{\exp[-iE'(x^0-y^0)-iEy^0]}{(2\pi)^{3/2}[2k(x^1)]^{1/2}}\,\frac{\exp\left(i\int^{x^1} k\right)}{E'-E+is_-\cdot 0}$$
$$= \begin{cases} \Theta(x^0-y^0)\psi\uparrow_E(x) & \text{if } s_->0 \\ -\Theta(y^0-x^0)\psi\uparrow_E(x) & \text{if } s_-<0 \end{cases} \tag{6.17}$$

It is now evident that our calculation is correct even for imaginary $k = i\kappa$. Since an exponential decay of the wavefunction can occur only in one spatial direction, the term proportional to $\delta(E-E')$ persists if k' is real. The "action" of the propagator on the remaining solutions and also that of the antipropagator can be derived in an analogous manner as the one just employed. The results coincide with those derived under the more special condition $F_+ = F_- = 0$ and given in Table 1. For sake of completeness we mention that under this last condition there is yet another method of deriving Table 1: By a suitable choice of integration contours in the complex m^2-plane one can arrive at the *chronologically* ordered representation of K directly without having to make recourse to the spatially ordered representation (6.10).

The charge forms of the solutions (6.3) and of ${\uparrow}\psi$, ${\downarrow}\psi$ can be calculated by the method used in equation (6.14). They are

$$(\psi{\uparrow}_E, \psi{\uparrow}_{E'}) = (\psi{\downarrow}_E, \psi{\downarrow}_{E'}) = 2n\,\delta(E-E')s_-\begin{cases}|\alpha|^2\\|\beta|^2\end{cases} \tag{6.18a}$$

$$({\uparrow}\psi_E, {\uparrow}\psi_{E'}) = ({\downarrow}\psi_E, {\downarrow}\psi_{E'}) = 2\pi\,\delta(E-E')s_+\begin{cases}|\alpha|^2\\|\beta|^2\end{cases} \tag{6.18b}$$

$$({\uparrow}\psi_E, \psi{\uparrow}_{E'}) = ({\downarrow}\psi_E, \psi{\downarrow}_{E'})^* = 2\pi\,\delta(E-E')s_+\begin{cases}\alpha^*\\0\end{cases} \tag{6.18c}$$

$$({\uparrow}\psi_E, \psi{\downarrow}_{E'}) = ({\downarrow}\psi_E, \chi{\uparrow}_{E'})^* = 2\pi\,\delta(E-E')s_-\begin{cases}0\\\beta\end{cases} \tag{6.18d}$$

$$(\psi{\uparrow}_E, \psi{\downarrow}_{E'}) = 2\pi\,\delta(E-E')s_-\alpha\beta \tag{6.18e}$$

$$({\uparrow}\psi_E, {\downarrow}\psi_{E'}) = -2\pi\,\delta(E-E')s_+\alpha^*\beta \tag{6.18f}$$

In (6.18a–d) the upper value is valid if $s_+ = s_-$, the lower one if $s_+ = -s_-$. Equations (6.18a) and (6.18b) are valid also for the solutions that describe total reflection to the left or to the right (k_+ or k_- imaginary), if one substitutes for α, β the corresponding analytical continuations of the coefficients of (6.5) (in this case $|\alpha| = |\beta|$).

The formulae (6.18) enable us to construct the in- and outgoing Fock representation according to Table 1. In the following we shall only be interested in the critical energy region because it is only there that the two representations differ. Denoting the physical modes normalized with respect to the charge form by ${}_\pm\psi$, ${}^\pm\psi$, we have

$$\begin{aligned} {}^+\psi &= (\alpha^*/\beta)_+\psi - (1/\beta)_-\psi \\ {}^-\psi &= (1/\beta)_+\psi - (\alpha/\beta)_-\psi \end{aligned} \tag{6.19}$$

Therefore an expected number of

$$\langle N_E \rangle = |\beta_E|^{-2} \tag{6.20}$$

particles is created with energy E. This number equals exactly the "transmission coefficient" (negative ratio of the transmitted to the ingoing current) of the wavefunction $\psi\downarrow$. This is the expected relationship between induced and spontaneous emission mentioned at the beginning of this section. The expected number of particles created per unit time is

$$\frac{d\langle N \rangle}{dt} = (2\pi)^{-1} \int dE \langle N_E \rangle \tag{6.21}$$

where the integral extends over the critical energy region. The probability of no particles being created in the unit time is $\exp(-w)$ with

$$w = (2\pi)^{-1} \int dE \ln(1 + \langle N_E \rangle) = 2 \int dx^1 \operatorname{Im} \mathscr{L}^{(1)}(x) \tag{6.22}$$

(cf. (3.12)–(3.14)).

The canonical Hamiltonian of the scalar field,

$$\begin{aligned} H &= \tfrac{1}{2} \int d^3x [(\dot\phi + ieA^0)(\dot\phi^\dagger - ieA^0) + \vec\nabla\phi\vec\nabla\phi^\dagger + m^2\phi^\dagger\phi + j^0 A^0] \\ j^0 &= e\phi^\dagger(i\overleftrightarrow{\partial}_0 - 2eA_0)\phi \end{aligned} \tag{6.23}$$

has, after symmetrization, the following representation in terms of the ingoing expansion coefficients of ϕ:

$$\begin{aligned} 2H = &\int_{eA^0_+ + m}^{\infty} dE\,E \sum_{i=1}^{2} \{{}_+a^\dagger_{Ei}, {}_+a_{Ei}\} + \int_{eA^0_+ - m}^{eA^0_+ + m} dE\,E \{{}_+a^\dagger_E, {}_+a_E\} \\ &+ \int_{eA^0_- + m}^{eA^0_+ - m} dE\,E (\{{}_+a^\dagger_E, {}_+a_E\} - \{{}_-a^\dagger_E, {}_-a_E\}) \\ &+ \int_{eA^0_- - m}^{eA^0_- + m} dE\,|E| \{{}_-a^\dagger_E, {}_-a_E\} + \int_{-\infty}^{eA^0_- - m} dE\,|E| \sum_{i=1}^{2} \{{}_-a^\dagger_{Ei}, a_{Ei}\} \end{aligned} \tag{6.24}$$

The representation in terms of the ${}^+a_E$, ${}^-a_E$ is similar. The index i in the sums appearing in the first and the last integral labels two orthogonal modes with energy E, e.g. $\psi_{E1} = \uparrow\psi_E$, $\psi_{E2} = \psi\downarrow_E$. The contribution of the critical energy interval $eA^0_- + m < E < eA^0_+ - m$ to H is such that H cannot be made positive by a gauge transformation as it could if $eA^0_+ - eA^0_- < 2m$. Thus a lower bound for H no longer exists as soon as the Klein paradox occurs. Therefore neither $|0\text{ in}\rangle$ nor $|0\text{ out}\rangle$ (which can be made zero energy eigenstates by normal ordering in the in- and out-representation, respectively) are states of lowest energy. Lower energy states can always be obtained by filling these states with particles or antiparticles in the critical energy region.

We close this section with some exactly soluble examples. Mathematically they are trivially related to those of Section 3. The physics is completely different, however.

(a) *Sauter's potential* $A^0 = V(1+\exp(-ax))^{-1}$

The distribution of particles created is

$$\langle N_E\rangle = \frac{\cosh[2\pi(k_- + k_+)/a] - \cosh[2\pi(k_- - k_+)/a]}{\cosh\left(2\pi\dfrac{k_- - k_+}{a}\right)\begin{cases} -\cos 2\pi\lambda & (\lambda \text{ real}) \\ +\cosh \operatorname{Im} 2\pi\lambda & (\lambda \text{ complex})\end{cases}}$$

$$\lambda = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{e^2V^2}{a^2}\right)^{1/2}$$

It is symmetric around its finite maximum at $E = eV/2$ with

$$\langle N_E\rangle \xrightarrow{E\to m} \sim (E-m)^{1/2}$$

There is no divergence as $m \to 0$.

(b) *The step potential* $A^0 = V\Theta(x)$

$$\langle N_E\rangle = 4k_-k_+/(k_+ - k_-)^2 \xrightarrow{E\to eV/2} (eV/2 - 2m^2/eV)^2/(E - eV/2)^2$$

The divergence at $E = eV/2$ reflects the occurrence of spontaneous emission already at the *classical* level: At this energy value $\psi\downarrow$ has no ingoing component, and the current density is everywhere pointing away from $x = 0$. w is finite. The convergence properties survive the transition to 4 dimensions ($k \to k^3$, $m^2 \to m^2 + k_1^2 + k_2^2$).

(c) *The constant electric field in the static gauge* $A^0 = Fx$

$f\downarrow_E$ is obtained from ${}^+f_k$ of ex. c) of Section 3 by the substitutions $m^2 \to -m^2$, $k \to E$, $t \to x$. Hence

$$\langle N_E\rangle = \exp\left(-2\pi\frac{m^2}{eF}\right)$$

Introducing the "classical location of creation" $\xi(E)=E/eF$ we can obtain $2\,\mathrm{Im}\,\mathscr{L}^{(1)}$ by a similar argument as in Section 3.

7. Static electric fields II: spherically symmetric potentials, singularities, and the Schiff-Snyder-Weinberg effect

7.1. *General Discussion*

We open this section with a general treatment of the potential $A^\mu=(A^0(r),0)$, $\lim_{r\to\infty}A^0(r)=0$. Since we will be interested in quasiclassical modes, we need the classical action S for the motion of a charged scalar particle in the potential A^μ. Separation of the Hamilton-Jacobi equation in polar coordinates yields

$$S(t,r,\vartheta,\varphi)=Et\pm\int^r dr'[(E-eA^0(r'))^2-m^2-\vec{L}^2/r'^2]^{1/2}$$
$$\pm\int^\vartheta d\vartheta'(\vec{L}^2-L_3^2\sin^2\vartheta')^{1/2}-L_3\varphi \tag{7.1}$$

Obviously the integration constants E, $\vec{L}^2$, L_3 are the energy, the square and the third component of the angular momentum of the classical particle, and S becomes complex for the classically forbidden values of r and ϑ.

The solutions of interest of the KG equation are of the form

$$\psi_{m^2El\mu}=\exp(-iEt)\,Y_{l\mu}(\vartheta,\varphi)f_{m^2El}(r) \tag{7.2}$$

($Y_{l\mu}$ are the spherical harmonics). They are trivially quasiclassical in t and φ, and also in ϑ for $\vartheta\to0$ and $\vartheta\to\pi$. However even if $A^0=0$ there do not exist solutions (7.2) which are quasiclassical for all values of r. Quasiclassical behavior is possible only for $r\to0$. As the radial part of S is purely imaginary for $r\to0$, f_{m^2El} is real. (In the free-field case f_{m^2El} is a spherical Bessel function. The spherical Hankel functions are quasiclassical for $r\to\infty$, but correspond to a point source of the KG field at the origin.)

Quite generally there exists only one solution f_{m^2El} of the source-free KG equation for fixed values of m^2, E, l, $|E|>m$. Its asymptotic behavior is

$$f_{m^2El}\xrightarrow{r\to\infty}(2kr)^{-1}\left[\alpha_{m^2El}\exp\left(i\int^r k(r')\,dr'\right)+\text{c.c.}\right]$$
$$k:=(E^2-m^2)^{1/2},\qquad k(r)=[(E-eA^0(r))^2-m^2]^{1/2} \tag{7.3}$$

We can continue f_{m^2El} analytically into the region $|E|<m$ with the phase convention

$$k(r)=i[m^2-(E-eA^0(r))^2]^{1/2}=:i\kappa(r) \tag{7.4}$$

Then, of course, the "c.c." term of (7.3) is no longer the complex conjugate of the first term in the bracket, nor is the analytical continuation of $\alpha^*_{m^2E}$ the complex conjugate of that of α_{m^2E} (we will go on using the same symbols, however). For physical reasons we will consider only those solutions in the region $|E|<m$ which describe bound states, i.e. for which $\alpha^*_{m^2E}=0$.

As in the previous section, version (2.D′) of our particle definition cannot be applied in a straightforward manner to the wavefunctions introduced so far. We therefore stick to (2.D) and the evaluation of the propagator via the representation (6.11). Again we will not present the details of the latter calculation, but only note for later reference the following intermediate results: If $|E|>m$,

$$\int_0^\infty drr^2 f^*_{m^2} f_{m'^2} = \lim_{r\to\infty} \frac{1}{m'^2-m^2}\, r^2\left(f^*_{m^2} \frac{\overleftrightarrow{d}}{dr} f_{m'^2}\right)$$
$$= \frac{\pi}{k} |\alpha_{m^2}|\, \delta(m^2-m'^2)$$

If ψ_{m^2} is a bound state,

$$\int_0^\infty drr^2 |f_{m^2}|^2 = \frac{1}{2k}\, \alpha_{m^2} \frac{\partial \alpha^*_{m'^2}}{\partial(m'^2)}\bigg|_{m'^2=m^2} \tag{7.6}$$

(Note that neither in (7.5) nor in (7.6) there is a contribution of the lower boundary of integration, as we are dealing with solutions of the *homogeneous* KG equation.)

The result obtained for the propagator by integration of (6.11) in the complex m^2-plane is

$$K(x,x') \xrightarrow{r,r'\to\infty} -i\int_{-\infty}^{\infty} dE \frac{\exp[-iE(x^0-x'^0)]}{2\pi} \sum_{l,\mu} Y_{l\mu}(\vartheta,\varphi) Y^*_{l\mu}(\vartheta',\varphi')$$
$$\times \frac{1}{2krr'}\left[\Theta(r-r')\exp\left(i\int_{r'}^{r} k\right) + \Theta(r'-r)\exp\left(-i\int_{r'}^{r} k\right) + \frac{\alpha}{\alpha^*}\exp\left(i\int^{r} k + i\int^{r'} k\right)\right] \tag{7.7}$$

Similarly as in (6.12) we recognize here a "radially ordered" representation of K, as opposed to the usual "chronologically ordered" one. For $|E|<m$ the integrand of (7.7) is defined by

$$\alpha^* \to \alpha^* - i\cdot \operatorname{sgn}\left(\frac{\partial\alpha^*}{\partial(m^2)}\right)\cdot 0 \tag{7.8}$$

Proceeding analogously as in (6.13), (6.14) we obtain the "action" of K on ψ,

$$(K*\psi_{El\mu})_{x'^0}(x) \xrightarrow{r\to\infty} Y_{l\mu}(\vartheta,\varphi)\int_{-\infty}^{\infty} dE' \frac{\exp[-iE'(x^0-x'^0)-iEx'^0]}{4\pi kr}$$
$$\times\left\{\frac{i}{E'-E}\left[\alpha_E \exp\left(i\int^{r} k\right) + \alpha^*_E \exp\left(-i\int^{r} k\right)\right] + \left[\exp\left(-i\int^{r} k'\right) + \frac{\alpha_{E'}}{\alpha^*_{E'}}\exp\left(i\int^{r} k'\right)\right].\right.$$
$$\left.\times \lim_{r'\to\infty} \frac{1}{E'-E} \frac{\exp\left(i\int^{r'} k'\right)}{2k'} \frac{\overleftrightarrow{d}}{dr'}\left[\alpha_E \exp\left(i\int^{r'} k\right) + \alpha^*_E \exp\left(-i\int^{r'} k\right)\right]\right\} \tag{7.9}$$

If $|E|>m$, $\lim_{r'\to\infty} = \pi\alpha^*_E \operatorname{sgn} E\, \delta(E-E')$, and the second term in the integral in (7.9) can be absorbed into the first one by setting the denominator equal to $E'-E+\operatorname{sgn} E\cdot 0$, If $|E|<m$, $K*\psi_E$ exists only if the bound state condition $\alpha^*_E=0$

is fulfilled. Then

$$\lim_{r'\to\infty} = i\,\delta_{EE'}\frac{\partial\alpha^*_{E'}}{\partial E'}$$

Therefore the second term in the integral in (7.9) is equal to 0 if $E'\neq 0$, and it does not exist if $E'=E$ (since then $\alpha^*_{E'}=0$). However in the latter case it has a unique interpretation as a distribution: Equation (7.8) implies that

$$\alpha^*_{E'}\xrightarrow{E'\to E}\left.\frac{\partial\alpha^*_{E'}}{\partial E'}\right|_{E'=E}\cdot(E'-E-is_{m^2}\cdot s_E\cdot 0) \tag{7.10}$$

with

$$s_{m^2}:=\operatorname{sgn}\left.\frac{\partial\alpha^*}{\partial(m'^2)}\right|_{m'^2=m^2,E} \tag{7.11}$$

$$s_E:=\operatorname{sgn}\left.\frac{\partial\alpha^*}{\partial E'}\right|_{E'=E,m^2} \tag{7.12}$$

Hence in the case that E is an energy eigenvalue the principal part of the pole at $E'=E$ due to $(\alpha^*_{E'})^{-1}$ occurring in the second term in the integral in (7.9) has to be dropped, and the whole term is equal to

$$-s_{m^2}s_E\alpha_E\exp\left(i\int^r k\right)\pi\,\delta(E-E')$$

This can again be absorbed into the first term by setting the denominator there equal to $E'-E-is_{m^2}\cdot s_E\cdot 0$.

Now the charge forms of the solutions (7.2) are given by

$$(\psi_{El\mu},\psi_{E'l'\mu'})=\delta_{ll'}\,\delta_{\mu\mu'}\operatorname{sgn}E\frac{\pi}{k}|\alpha|^2\,\delta(E-E')\quad\text{if }|E|>m \tag{7.13a}$$

$$(\psi_{El\mu},\psi_{E'l'\mu'})=-\delta_{ll'}\,\delta_{\mu\mu'}\,\delta_{EE'}\frac{1}{2k}\,\alpha\frac{\partial\alpha^*}{\partial E}\quad\text{for bound states} \tag{7.13b}$$

Equations (7.13) and the positivity of the right hand side of (7.6) imply the sign $q(E)$ of the charge of the wavefunction ψ_E:

$$\begin{aligned} q(E)&=\operatorname{sgn}E &&\text{if }|E|>m\\ q(E)&=-s_{m^2}s_E &&\text{if } E \text{ is on eigenvalue}\end{aligned} \tag{7.14}$$

Therefore (7.9) altogether is reduced to

$$(K*\psi_{El\mu})_{x'^0}(x)=iY_{l\mu}(\vartheta,\varphi)\int_{-\infty}^{\infty}\frac{dE'}{2\pi}\frac{\exp\left[-iE'(x^0-x'^0)-iEx'^0\right]}{E'-E+iq(E)\cdot 0}f_{El\mu}(r) \tag{7.15}$$

Finally the evaluation of (7.15) and of the analog integral for $\tilde{K}*\psi$ reveals:

$$\begin{aligned} \psi_E&\in H^+\cap H_+ &&\text{if } q(E)=+1\\ \psi_E&\in H^-\cap H_- &&\text{if } q(E)=-1\end{aligned} \tag{7.16}$$

Thus, according to our definition, no particles are created corresponding to the modes introduced so far, although in the case $\sup_r|eA^0(r)|>2m$ there exist solutions of the Klein paradox type. They correspond, however, to a meson source and will be discussed in the example below.

If the potential is "deep" enough, there will in general also exist bound state modes with *complex* E, $-m<\operatorname{Re} E<m$. This is the "resonance" phenomenon discovered by Schiff, Snyder and Weinberg [27].

The modes occur in pairs $\{\psi_E, \psi_{E^*}\}$ and their charge forms are

$$(\psi_E, \psi_E)=\frac{1}{E^*-E}\lim_{r\to\infty}\psi_E^*\frac{\overleftrightarrow{d}}{dr}\psi_E=0 \tag{7.17a}$$

$$(\psi_E, \psi_{E^*})\neq 0 \tag{7.17b}$$

It is interesting to classify also these modes according to the particle definition, although it is clear from (7.17) that they do no fit into the construction of a Fock space. Although the resonances do not belong to the generalized domain of self-adjointness of $\Box_e$ and hence do not contribute to the construction of the resolvent as indicated in (6.11), they are "acted upon" by the propagator in a well-defined manner conveyed by equation (7.9). The only difference to our previous calculation is that now the second term of the integrand appearing there vanishes and that the pole of the first term is situated off the real axis. As a consequence

$$\psi_E\in H^+\cap H^- \quad \text{if } \operatorname{Im} E<0$$

$$\psi_E\in H_+\cap H_- \quad \text{if } \operatorname{Im} E>0 \tag{7.18}$$

Thus we encounter a similar situation as that discussed at the end of Section 4: The spaces spanned by the in- and outgoing resonance modes have only a trivial intersection. The only reasonable physical interpretation of this mathematical pecularity seems to be that here much "more" particles are created than in those situations which fit into the usual scheme. This is also suggested by the fact that the "resonances" constitute *classical* examples of *spontaneous* emission by the external field (another example are the modes with $E=eV/2$ in example b) of Section 6). It is plausible to interpret $\exp(-2\operatorname{Im} E)$ as the probability that no pairs of quantum number E are created out of the vacuum during one unit of time, as was proposed by Popov [28]. Because of the Bose statistics we expect the number of pairs created to increase exponentially in time subject to the oversimplifying assumption, of course, that the back reaction and the mutual interaction of the particles are neglected. Calculations of Migdal [29] indicate, however, that the vacuum remains stable if the Coulomb self-interaction of the scalar field is taken into account.

In order to supplement the general framework given above by concrete examples and to provide a more detailed physical description of the creation process we conclude this section with a discussion of the spherical square well and Coulomb potentials.

7.2. *The spherical square well* $A^0=V\Theta(a-r)$, $V<0$

We consider only s-wave ($l=0$) solutions (7.2) with

$$f_{E0}(r)=(k_1r)^{-1}\sin k_1r \quad \text{for } r<a \tag{7.19}$$

$$k_1=[(E-eV)^2-m^2]^{1/2} \tag{7.20}$$

$$\alpha=(k_1^{-1}k\sin k_1a-i\cos k_1a)\exp(-ika) \tag{7.21}$$

(α was defined in (7.3).) The bound state condition is

$$k_1 \operatorname{ctg} k_1 a = ik \tag{7.22}$$

If the depth of the potential is increased "adiabatically" starting from $V=0$, energy eigenvalues $E_n^{\pm}$ separate from the positive and negative energy continuum, respectively, at the values $V_n^{\pm}$ defined by

$$eV_n^{\pm} = \pm m - [m^2 + (n+(1/2)^2)\pi^2 a^{-2}]^{1/2} \tag{7.23}$$

As V decreases, the eigenvalues $E_n^{\pm}$ dive into the energy gap $(-m, m)$ until they coalesce at a critical value V_n^{crit}, where $E_n^+ = E_n^-$. For $V < V_n^{\text{crit}}$ both eigenvalues become complex with $E_n^+ = (E_n^-)^*$. These complex "energy" eigenvalues correspond to the "resonance" modes introduced in the general part of this section. Numerical treatments of the gedanken experiment just described can be found in [27] and [24]. According to [28] the threshold behavior of the imaginary part of E_n is proportional to $(eV_n^{\text{crit}} - eV)^{1/2}$, if $a \ll m^{-1}$. If $a \gg m^{-1}$ then $|eV_1^+| \ll m$, $|2m + eV_1^-| \ll m$ and $|2m + eV_1^{\text{crit}}| \ll m$. Therefore if the radius of the "well" is large in comparison with the Compton wavelength, particle creation occurs almost immediately after the critical depth of the potential $eV = -2m$ is reached. This result is plausible because of the energy balance "potential energy = negative rest energy of a pair". For $a \lesssim m^{-1}$ this reasoning no longer applies because particles are not strictly localizable.

Classically, the resonance modes can be interpreted as follows. Equal amounts of positive and negative charge are continuously created at the potential "wall" $r = a$. The negative charge flows outwards, its current decreasing exponentially for $r \to \infty$. Similarly the positive charge flows inwards with its current vanishing at $r = 0$. Thus the charges remain concentrated in the vicinity of $r = a$ and grow exponentially in time. Quantum mechanically, the resonance modes represent "pairs" rather than "particles" and "antiparticles": The particle component of the pair is bound inside the "wall" with potential energy $-eV$. The antiparticle component is situated just outside the "wall". It is only in this way that the energy balance mentioned above can be maintained.

As already mentioned in the beginning of this section there exist wavefunctions with asymptotic behavior $\sim r^{-1} \exp[-i\int^r k(r')\,dr']$ and showing induced emission in the critical energy region $eV + m < E < -m$. They are solutions of the inhomogeneous KG equation with a point source

$$\rho(x) = \exp(-iEt)\,\delta^{(3)}(\vec{x}) \tag{7.24}$$

This could make one expect that the particle creation rate of this classical source is enhanced by the external field because of the Klein paradox. In the following we show that the interplay between source and external field is different.

According to the general formalism for c-number sources (see e.g. [30]), the presence of the source necessitates a distinction between the algebras of the in- and outgoing fields Φ^{in}, Φ^{out} (and not only of their representations):

$$\Phi^{\text{out}} = \Phi^{\text{in}} - \int d^4x'\, G(x, x')\rho(x') \tag{7.25}$$

where G is the Cauchy propagator introduced in (2.4). Equation (7.25) implies that the in- and outgoing annihilation operators will differ also by c-numbers, apart from the Bogoliubov transformation imposed by the external field. E.g. in the absence of an external field the source (7.24) implies the following transformation of the annihilation operators defined by plane-wave modes:

$$a_{\text{out},\vec{k}} = a_{\text{in},\vec{k}} + (4\pi E)^{-1/2}\,\delta(E-k^0) \quad \text{if } E>m \tag{7.26}$$

Therefore the creation rate is

$$\frac{d\langle N\rangle}{dt} = \frac{k}{2\pi} \tag{7.27}$$

and coincides with the total radial current of the classical solution

$$\int d^4x' K^{\text{ret}}(x,x')\rho(x') = -(4\pi r)^{-1}\exp(-iEt \pm ikr) \quad \text{if } E\begin{cases} >m \\ <-m \end{cases} \tag{7.28}$$

K^{ret} is the retarded Green's function of the scalar field.

We now simply compare the creation rate (7.27) with that for the same source in the presence of the external potential $A^0(r)$. By integration in the complex E-plane one obtains

$$\int d^4x' K^{\text{ret}}(x,x')\rho(x') \xrightarrow{r\to\infty} -(4\pi r\alpha^*)^{-1}\exp\left[-iEx^0 \pm i\int^r k(r')\,dr'\right]$$
$$\text{if } E\begin{cases} >m \\ <-m \end{cases} \tag{7.29}$$

This result is valid under the general assumptions made at the beginning of this section. It is even valid for $|E|<m$ with the analytical continuation (7.4). However it diverges for energy eigenvalues E, thus indicating resonance. Since the connection between the current of (7.29) and the quantum theoretical emission rate is the same as in the case $A^0=0$, the emission rate is different from (7.27) by the factor $|\alpha|^{-2}$. Therefore in our example it is enhanced for $E>eV/2$ and diminished for $E<eV/2$ (as a consequence a real source will create different amounts of particles and antiparticles). So there is no connection with the Klein paradox. The reason is that the source (7.24) not only creates outgoing spherical waves, but also absorbs the waves reflected from the potential barrier (in the critical energy region the latter is the main effect). One can imagine, however, a source which is switched on for a period small enough so that the waves reflected from the barrier are not absorbed but totally reflected at $r=0$. Since the reflection coefficient of the potential is greater than 1 in the critical energy region, the amplitude and charge of the reflected waves will increase exponentially in time. It is in this way that a "resonance" comes about.

7.3 *The Coulomb Potential* $A^0=-Q/r$, $Q>0$

It has been known for a long time [31] that relativistic quantum mechanics in the Coulomb potential develops an ambiguity beyond a critical value of the central charge Q. Although eventually we will not escape the conclusion that definite predictions about particle creation in the most common potential of physics cannot be made unless the Coulomb singularity is replaced by a more realistic potential, a comprehensive, though not too rigorous, treatment of the

Coulomb potential seems appropriate to elaborate the exact reason for the partial breakdown of our formalism.

The radial part $S_1(r)$ of the classical action (7.1) has the two asymptotic limits

$$S_1(r) \to (e^2Q^2 - \vec{L}^2)^{1/2} \ln r \quad \text{for } r \to 0 \tag{7.30}$$

$$S_1(r) \to kr + EeQk^{-1} \ln r \quad \text{for } r \to \infty \tag{7.31}$$

As in the free case the KG equation has solutions which become quasiclassical for $r \to 0$. However not all these solutions do belong to a generalized domain of self-adjointness of $\Box_e$. In the following we give a list of the admissible eigendistributions of $\Box_e$. Eventually it will turn out that all modes exhibiting particle creation are excluded by the self-adjointness condition and thus *no* particle creation is predicted by our definition for the *unregularized* Coulomb potential.

A crucial parameter for self-adjointness is the coefficient λ occurring in the radial part of the KG equation for the solutions (7.2):

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2EeQ}{r} - \frac{\lambda(\lambda+1)}{r^2} + E^2 - m^2\right] f_{m^2El}(r) = 0 \tag{7.32}$$

$$\lambda(\lambda+1) = l(l+1) - e^2Q^2 \tag{7.33}$$

The following cases have to be distinguished:

(a) $\lambda(\lambda+1) \geq 0$. Here the space of admissible solutions is unique. It is spanned by

$$f_{m^2El+} = \exp(-ikr)(2kr)^{\lambda_+} M\left(i\frac{EeQ}{k} + \lambda_+ + 1, 2\lambda_+ + 2, 2ikr\right) \tag{7.34}$$

$$\lambda_+ = -\tfrac{1}{2} + [(l+\tfrac{1}{2})^2 - e^2Q^2]^{1/2} \geq 0 \tag{7.35}$$

M denotes the confluent hypergeometric function. The coefficient α of (7.3) is

$$\alpha = -i\frac{\Gamma(2\lambda_+ + 2)}{\Gamma\left(i\frac{EeQ}{k} + \lambda_+ + 1\right)} \exp\left[-\frac{\pi}{2}\frac{EeQ}{k} + i\left(\frac{EeQ}{k}\ln 2k - \lambda_+ \frac{\pi}{2}\right)\right] \tag{7.36}$$

The function (7.34) is real and defined only for $|E| > m$ as it stands, but it can be continued analytically to values $|E| < m$ using (7.4). The eigenvalues are

$$E^+_{m^2l,N} = \frac{m(N + \lambda_+ + 1)}{[e^2Q^2 - (N + \lambda_+ + 1)^2]^{1/2}} \tag{7.37}$$

(b) $-1/4 \leq \lambda(\lambda+1) \leq 0$. Classically, a particle with this value of the parameter λ falls into the singularity. Quantum mechanically the collapse is still prohibited (there are no oscillations of the wavefunction for $r \to 0$). But now the functions (7.34) no longer define the only possible domains of self-adjointness. Instead they could be replaced by f_{m^2El-} defined by

$$\lambda_+ \to \lambda_- = -\lambda_+ - 1 \tag{7.38}$$

or any linear combination $f_{m^2El+} + \alpha f_{m^2El-}$ with *real* parameter α. Thus there is a one-parameter sequence of self-adjoint extensions of the differential operator $\Box_e$ initially defined on $C_0^\infty(R^4)$. However, among these the one corresponding to the choice of the f_{m^2El+} is distinguished by the regularity property

$$f(r) \xrightarrow{r\to 0} r^{\lambda_+} \quad (\lambda_+ \geq -1/2)$$

Moreover it is this extension which yields the observed spectrum of the hydrogen atom (the Balmer series). In this "physical" extension we have $q(E)=1$ for all bound states (cf. (7.14)). The lowest energy eigenvalue is reached for $Q=1/2$ with $E^+_{m^200}=m2^{-3/2}$. (If no boundary conditions at $r=0$ are introduced, (7.32) possesses solutions lying in $L^2[(0,\infty), r^2\,dr]$ even for arbitrary *complex* values of $E^2-m^2 \in C-R^+$.)

(c) $\lambda(\lambda+1)<-1/4$: We define $\lambda_\pm$ by

$$\lambda_\pm := -1/2 \pm i[e^2Q^2-(l+1/2)^2]^{1/2} \tag{7.39}$$

and keep the notation $f_{m^2El\pm}$ for the analytical continuations of the functions defined originally by (7.34), (7.38).

The Klein paradox is now manifest in the "superradiant" mode f_{m^2El-} with

$$f_{m^2El-} \xrightarrow{r\to 0} (2kr)^{-1/2} \exp(-iv \ln 2kr)$$

$$v = \operatorname{Im} \lambda_+$$

The transmission coefficient (defined in analogy to that of Section 6) of this function is

$$\exp[\pi(EeQ/k+v)] \sinh(2\pi v)/\cosh\left[\pi\left(\frac{EeQ}{k}-v\right)\right]$$

But since this solution corresponds to a source at the origin it is not admissible for a self-adjoint extension. Admissible eigendistributions must behave like

$$r^{-1/2} \sin(v \ln r + \delta) \quad \text{for } r \to 0$$

with a universal constant δ. Thus the self-adjoint extensions can be parametrized by the phase shift δ of the total reflection at the origin. In contrast to case b) none of these extensions is preferred physically. The boundary condition in $r=0$ can be fulfilled by one bound state at most (whose energy must be $E=-m$). In contrast to the square well potential there are no "resonance" solutions obeying the boundary condition. All the existing resonance solutions describe an "eruption" of particles from the origin or their collapse into the origin and have complex "eigenvalues" lying in the series (7.37).

Since the singularity in the origin certainly is an inadmissible idealisation of the actual field of a charged particle, the negative prediction about particle creation in the Coulomb potential should not come as a surprise. If the potential is modified so as to remain finite in $r=0$, "admissible" resonance modes of the same type as in the square well potential will occur which do correspond to particle creation. For an extensive discussion of this process in realistic nuclear potentials as well as of the exciting perspectives of its experimental verification see [32], [33].

8. Static electric fields III: periodic potentials

The subject of this last section is the potential $A^\mu = (A^0(\vec{x}), \vec{0})$ with A^0 periodic in x^1, x^2, x^3. In virtue of the Bloch theorem (cf. 4.6) there exist solutions

$$\psi^{(B)}_{m^2}m^2\vec{\alpha} = (2\pi)^{-3/2}\exp[-iE(m^2, \vec{\alpha})t]e^{i\vec{\alpha}\vec{x}}u_{\vec{\alpha}}(\vec{x}) \tag{8.1}$$

where u has the same periodicity properties as A^0. The quasimomenta α^i are the analogs of the expression A/T of Section 4.

The self-adjointness of $\Box_e$ is respected only by those solutions (8.1) for which E, α_1, α_2, α_3 are real. Eventually we shall allow E to become complex however. Restricting E to real values yields the "energy bands" known from solid state physics. Within an energy band, E is a real analytic function of m^2 if the α_i are kept fixed. Therefore, upon an analytical continuation into the complex m^2-plane,

$$E(m^2 + i\varepsilon, \vec{\alpha}) = E^*(m^2 - i\varepsilon, \vec{\alpha}) \quad (m^2 \text{ real}) \tag{8.2}$$

Consequently, according to definition (2.D′), $\psi^{(B)}_{m^2\vec{\alpha}} \in H^+(H^-)$, if $E(m^2, \alpha)$ is real and if Im $E(m^2 - i\varepsilon, \alpha) < (>) 0$. Moreover, if we denote by $H^+_{\text{s.a.}}$ the linear space of the outgoing particle solutions with real values of E and use an analogous notation for the corresponding restrictions of the other subspaces of physical solutions, we have

$$H^+_{\text{s.a.}} = H_{+\text{s.a.}}, \; H^-_{\text{s.a.}} = H_{-\text{s.a.}} \tag{8.3}$$

Thus no particles are created in the modes with real E.

It is possible that there exist "resonance" type solutions (8.1) with m^2, $\vec{\alpha}$ real, but E complex. They are expected to occur if the variation of A^0 exceeds $2m/e$. The formal properties of these resonance modes are the same as those of the corresponding modes discussed in Sections 4 and 7: The total charge is zero, and

$$\psi^{(B)} \in H^+ \cap H^- \quad \text{if Im } E < 0 \tag{8.4}$$

$$\psi^{(B)} \in H_+ \cap H_- \quad \text{if Im } E > 0 \tag{8.5}$$

Again, this modes correspond to the creation of "pseudoparticles" at a finite rate in every mode.

9. Conclusion

The problem of defining "natural" quantum states in "generic" external gauge fields is too intriguing as that formal consistency alone could be accepted as a sufficient criterion of validity of any solution proposed. Only critical examination of the formalism in as many concrete examples as possible and, eventually, comparison with experiment will show what the right answer is (if there exists any at all). Therefore this article should be regarded mainly as a "test report". We feel that the covariant in-out formalism proposed not only has proved to be a serious candidate for the solution of the problem of vacuum definition but has also produced some new results of direct physical interest. The limitations encountered in Section 7 are inherent in the external field approach itself, which, after all, is only a semiclassical approximation.

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