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General solution of multichannel partial-wave dispersion relations

II. Noncoincident thresholds, one pole approximation

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Abstract. We find the general solution for the matrix of s-wave amplitudes when there are two channels with different thresholds and the only unphysical singularity is a simple pole. The method of solution uses a representation theorem for matrix R-functions. We distinguish a special solution called the isolated solution which contains only the prescribed pole position and residue matrix as parameters. There are cases where the isolated solution does not exist but other solutions do exist. We also find cases where the diagonal element of the isolated solution which corresponds to elastic scattering in the channel with the lower threshold is not the isolated solution of the equivalent inelastic one-channel problem for that channel.

1. Introduction

In an earlier paper [1] we considered the case of an arbitrary number of coupled two-body channels whose thresholds were coincident. Competing channels were neglected, the unphysical singularities of the S-matrix of s-wave amplitudes were taken to be a finite number of simple poles and the input consisted of the positions of these poles and the residue matrix at each pole. By using a generalization of some parts of Schur-Pick-Nevanlinna interpolation theory to analytic matrices we were able to obtain the general solution for the partial-wave S-matrix, to give necessary and sufficient conditions for a solution to exist and to distinguish different types of soluble problem.

The history of the problem of obtaining solutions of multichannel partial-wave dispersion relations and a discussion of various questions of physical interest which one wishes to answer are given in [1]. The next step beyond [1] is to consider cases where the channel thresholds do not coincide. This generalization alters the nature of the problem and the mathematical methods which seem to be needed for its solution. In this paper we shall solve rigorously the simplest special

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case for which the channel thresholds do not coincide. The simplification consists in taking a single pole as the only unphysical singularity and in considering only two channels and s-waves. The motivation for considering this special case is to obtain an insight into the structure of the probable solution of the general problem. We shall comment in the conclusion on the extension of our results to $l>0$ and to more than two channels.

Comparison of the solution given in this paper with the solution of the one-channel case when the effect of competing channels is specified via the function R (see [2]) shows that there is a similarity between the methods used. The key mathematical result used in this paper is a representation theorem which is proved in the Appendix. It is a generalization of a quite well known result for ordinary complex-valued R-functions [3] to what we call matrix R-functions.

We shall see that, among the solutions of our problem there is a distinguished solution $\mathbf{f}^{(0)}$ which we call the *isolated solution*. Although we do not make the connection with Levinson's theorem explicitly, it is clear that this is the isolated solution in the sense of Hamilton and Tromborg [4]. It contains only the prescribed pole position and residue matrix as parameters. Unlike the one-channel case and the multichannel case with coincident thresholds, it sometimes happens that the isolated solution does not exist, though other solutions do exist.

In [1] we considered the following question which has been extensively discussed in the literature. For the n-channel problem with coincident thresholds we took the isolated solution, obtained from it the inelasticity parameters for each of the n possible elastic scattering processes and considered the n equivalent inelastic one-channel problems (EIOCPs) thus generated. We found that in the one pole case the n diagonal elements of the isolated solution of the n-channel problem were always the isolated solutions of the respective EIOCPs. For the two pole case however we were able to construct an explicit two-channel example for which this was no longer true. For the problem considered in this paper we can take the isolated solution and calculate from it the inelasticity parameter R for elastic scattering in channel 1, the channel with the lower threshold. The function R will be 1 between the thresholds, greater than 1 above the threshold for channel 2. Given R, we have the EIOCP for channel 1. Is the diagonal element $f_{11}^{(0)}$ of the isolated solution of the two-channel problem always the isolated solution of the EIOCP for channel 1? As we shall see, the answer is no, even though there is only one pole. The fact that the channel thresholds are different makes it possible to choose the residue matrix at the pole so that $f_{11}^{(0)}$ has a CDD zero between the two thresholds.

To meet some criticism expressed by Johnson and Warnock [5] we want to point out that in the one-channel case with η- or R-unitarity and in the multichannel case with coincident thresholds our methods can be adapted to cover the case of an extended left-hand cut. Moreover, it is possible to construct in some sense the best approximation to an extended cut by a finite number of poles. Some remarks on the one-channel case are made by Nenciu [6]. Longer and more technical proofs are required, and we do not want to write out the results at the moment. Eventually we hope to be able to solve the multichannel case with noncoincident thresholds and an extended left-hand cut, but the many pole problem has to be solved first.

The plan of the paper is as follows. In Section 2 we solve the s-wave problem with two channels whose thresholds do not coincide and with one pole whose position and residue matrix are prescribed. The reader who is not interested in detailed proofs can go directly to the results at the end of the section. Section 3 gives special examples which illustrate the points mentioned already and discusses the case when the residue matrix at the pole is singular. Section 4 contains concluding comments, while the representation theorem used in Section 2 and other mathematical results are given in an appendix.

2. General solution

We consider a system of two coupled two-body channels with different thresholds. Using the Mandelstam variable s, the thresholds are s_1, s_2, with $s_1 < s_2$. We consider the 2×2 matrix $\mathbf{S}$ whose element S_{ij} is the s-wave S-matrix element for the process $j \to i$. Time-reversal invariance will be assumed to hold, so that $\mathbf{S} = \mathbf{S}^t$.[2]) We define the matrix $\mathbf{f}$ by

$$\mathbf{S} = \mathbb{1}_2 + 2i\mathbf{q}^{1/2}\mathbf{f}\mathbf{q}^{1/2},$$

where $\mathbf{q}$ is the diagonal matrix whose diagonal emements are the channel momenta in the centre-of-momentum frame. From the unitarity of $\mathbf{S}$ we have

$$\mathrm{Im}\,\mathbf{f} = \mathbf{f}^*\mathbf{q}\mathbf{f}, \qquad s > s_2. \tag{1}$$

We now assume that $\mathbf{f}(s)$, for $s > s_2$, is the boundary value from above of a matrix function $\mathbf{f}(z)$ which is analytic in the whole complex plane, except for the cut $[s_1, \infty)$ and a single pole at x_1, where x_1 is real and $x_1 < s_1$. We need to extend the unitarity relation (1) to $\mathbf{f}(s+) = \lim_{\varepsilon\to 0+} \mathbf{f}(s + i\varepsilon)$ for $s_1 \leqslant s \leqslant s_2$. From [7] we see that the required extension is

$$\mathrm{Im}\,\mathbf{f}(s) = \mathbf{f}(s)^*\boldsymbol{\rho}(s)\mathbf{f}(s), \qquad s_1 \leqslant s, \tag{2}$$

where we have shortened $\mathbf{f}(s+)$ to $\mathbf{f}(s)$ and $\boldsymbol{\rho}(s)$ is defined by

$$\boldsymbol{\rho}(s) = \begin{pmatrix} q_1(s) & 0 \\ 0 & q_2(s) \end{pmatrix}, \qquad s_2 \leqslant s, \tag{3a}$$

$$\boldsymbol{\rho}(s) = \begin{pmatrix} q_1(s) & 0 \\ 0 & 0 \end{pmatrix}, \qquad s_1 \leqslant s \leqslant s_2. \tag{3b}$$

To be complete, we list all the properties to be satisfied by the functions $\mathbf{f}(z)$.

(a) $\mathbf{f}(z)$ is analytic on $\mathbb{C} - ([s_1, \infty) \cup \{x_1\})$, where $x_1 < s_1$;
(b) $\mathbf{f}(\bar{z}) = \mathbf{f}(z)^*$;
(c) $\mathbf{f}(z)$ has a simple pole at x_1, with real symmetric residue matrix $\boldsymbol{\Gamma} \neq 0$;
(d) there exists a function $\mathbf{f}(s)$, defined on $[s_1, \infty)$, such that, given $a \geqslant s_1$ and $\varepsilon > 0$, there exists $\delta > 0$ (depending on a and ε) for which

$$\|\mathbf{f}(z) - \mathbf{f}(a)\| < \varepsilon, \qquad z \in \{|z-a| < \delta, \mathrm{Im}\, z > 0\};$$

(e) Equation (2) holds for $\mathbf{f}(s)$;
(f) there exists a non-negative integer k such that $|z|^{-k}\,\|\mathbf{f}(z)\| \to 0$ as $|z| \to \infty$, uniformly in $0 \leqslant \mathrm{Arg}\, z \leqslant \pi$;
(g) $\mathbf{f}(z)^t = \mathbf{f}(z)$.

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[2]) The superscript t on a matrix denotes the transpose, $*$ the adjoint.

We now proceed to obtain such functions $\mathbf{f}(z)$. Define

$$\rho_m(s)=\min\{q_1(s), q_2(s)\}, \qquad s_2<s,$$

and write (2) as

$$\operatorname{Im}\mathbf{f}(s)=\rho_m(s)\mathbf{f}(s)^*\mathbf{f}(s)+\mathbf{f}(s)^*(\boldsymbol{\rho}(s)-\rho_m(s)\mathbb{1}_2)\mathbf{f}(s).$$

Now

$$\boldsymbol{\rho}(s)-\rho_m(s)\mathbb{1}_2\geq 0, \qquad \rho_m(s)\mathbf{f}(s)^*\mathbf{f}(s)\geq 0,$$

and so, by (A.8), (A.7) and (A.1),

$$\|\operatorname{Im}\mathbf{f}(s)\|\geq\rho_m(s)\,\|\mathbf{f}(s)\|^2.$$

However, from (A.4),

$$\|\operatorname{Im}\mathbf{f}(s)\|\leq\|\mathbf{f}(s)\|.$$

The last two inequalities imply that

$$\|\mathbf{f}(s)\|\leq 1/\rho_m(s), \qquad s_2<s.$$

Since

$$q_i(s)\underset{s\to\infty}{\sim}\tfrac{1}{2}s^{1/2}, \qquad i=1,2,$$

it follows that there exists a (finite) positive constant M such that

$$s^{1/2}\,\|f(s)\|\leq M, \qquad s_1\leq s.$$

From (A.2),

$$s^{1/2}\,|f_{ij}(s)|\leq M, \qquad s_1\leq s.$$

We now appeal to the Phragmen-Lindelöf theorem to conclude that

$$|z^{1/2}f_{ij}(z)|\leq M, \qquad |z|\geq r_0, \tag{4}$$

with $r_0>\max\{|x_1|, s_1\}$. This bound is a uniform bound for $0\leq\operatorname{Arg} z\leq 2\pi$. The formulation of the Phragmen-Lindelöf theorem appropriate to our application is given on page 135 of Conway [8], with $a=\frac{1}{2}$ so that the sector has angle 2π. The result given there is clearly unaffected if we exclude from the sector a disc $|z|<r_0$.

We now define a new function

$$\mathbf{g}(z)=(z-x_1)\mathbf{f}(z).$$

Instead of a pole at x_1 we now have

$$\mathbf{g}(x_1)=\boldsymbol{\Gamma}.$$

From Eq. (2),

$$\operatorname{Im}\mathbf{g}(s)=(s-x_1)^{-1}\mathbf{g}(s)^*\boldsymbol{\rho}(s)\mathbf{g}(s), \qquad s_1\leq s, \tag{5}$$

while from Eq. (4)

$$|z^{-1/2}g_{ij}(z)|\leq M', \qquad r_0\leq|z|, \qquad 0\leq\operatorname{Arg} z\leq 2\pi.$$

It follows from this bound that each $g_{ij}(z)$ satisfies a once subtracted dispersion relation. Taking the subtraction point at $x_s < s_1$ we have

$$\mathbf{g}(z) = \mathbf{g}(x_s) + \frac{1}{\pi}\int_{s_1}^{\infty} \operatorname{Im}\mathbf{g}(t)\left(\frac{1}{t-z} - \frac{1}{t-x_s}\right)dt. \tag{6}$$

From (6),

$$\operatorname{Im}\mathbf{g}(z) = \frac{\operatorname{Im} z}{\pi}\int_{s_1}^{\infty} \frac{\operatorname{Im}\mathbf{g}(t)}{|t-z|^2}\,dt, \qquad \operatorname{Im} z \neq 0. \tag{7}$$

Since $\boldsymbol{\rho}(s) \geqslant 0$ for $s \geqslant s_1$, it follows from (5) and (A.8) that

$$\operatorname{Im}\mathbf{g}(s) \geqslant 0, \qquad s \geqslant s_1.$$

Eq. (7) then shows that

$$\operatorname{Im}\mathbf{g}(z) > 0, \qquad \operatorname{Im} z > 0.$$

From Lemma A.2, $\det \mathbf{g}(z) \neq 0$ for $\operatorname{Im} z > 0$. Thus we can define

$$\mathbf{h}(z) = -\mathbf{g}(z)^{-1}, \qquad \operatorname{Im} z > 0. \tag{8}$$

Now

$$\operatorname{Im}\mathbf{h}(z) = \frac{1}{2i}(\mathbf{h}(z) - \mathbf{h}(z)^*) = -\frac{1}{2i}(\mathbf{g}(z)^{-1} - \mathbf{g}(z)^{*-1})$$

$$= \mathbf{g}(z)^{*-1}\operatorname{Im}\mathbf{g}(z)\mathbf{g}(z)^{-1},$$

so that, from (A.9),

$$\operatorname{Im}\mathbf{h}(z) > 0, \qquad \operatorname{Im} z > 0.$$

Moreover, for every $s \in \mathbb{R}$ for which $\det \mathbf{g}(s) \neq 0$, $\mathbf{h}(z)$ converges to $\mathbf{h}(s) = -\mathbf{g}(s)^{-1}$ as $z \to s$ from the upper half-plane. This follows from property (d) of $\mathbf{f}$, which also holds for $\mathbf{g}$. Let

$$s_0 = \{s \in \mathbb{R} \mid \det \mathbf{g}(s) = 0\}.$$

Since $g(z)$ is analytic in $(-\infty, s_1)$ it follows that S_0 has no accumulation point in $(-\infty, s_1)$. We now make the stronger assumption that S_0, which we know to be a closed set of Lebesgue measure zero, has no accumulation point in $\mathbb{R}$. It then has a finite number of points in every closed subinterval and so is countable. It follows from (5) that

$$\operatorname{Im}\mathbf{h}(s) = (s - x_1)^{-1}\boldsymbol{\rho}(s), \qquad s \geqslant s_1, \qquad s \notin S_0. \tag{9}$$

A matrix function which is symmetric, whose matrix elements are analytic in the upper half-plane and whose imaginary part is nonnegative (in the matrix sense) there we shall call a matrix R-function. Clearly $\mathbf{g}(z)$ and $\mathbf{h}(z)$ are such functions. In the appendix we prove a representation theorem for matrix R-functions which is a generalization of a result in [3] on complex-valued R-functions. The theorem says that

$$\mathbf{h}(z) = \boldsymbol{\alpha} + \boldsymbol{\beta} z + \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\boldsymbol{\sigma}(t), \tag{10}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\sigma}(t)$ are real symmetric matrices, $\boldsymbol{\beta} \geqslant 0$,

$$\boldsymbol{\sigma}(t_2) - \boldsymbol{\sigma}(t_1) \geqslant 0, \qquad t_2 > t_1, \tag{11}$$

and

$$V(\tilde{\sigma}_{ij}; \mathbb{R}) < \infty,$$

where V denotes the total variation and $\tilde{\sigma}_{ij}$ is defined by

$$\tilde{\sigma}_{ij}(t) = \frac{\sigma_{ij}(t)}{1+t^2}.$$

To get (10) into the form we require, we follow an argument similar to that in Section 3 of [2]. Write the function $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \boldsymbol{\sigma}_d, \tag{12}$$

where $\boldsymbol{\sigma}_c$ is continuous on $\mathbb{R}$ and $\boldsymbol{\sigma}_d$ is a saltus function; it has a jump $\mathbf{R}_i$ at each of the points of an at most countable set $\{\xi_i\}$ and is constant otherwise. It is clear from (11) that each $\mathbf{R}_i \geqslant 0$. The decomposition (12) is given in Section (13.18.6) of [9]. The function $\boldsymbol{\sigma}_c$ may be split into an absolutely continuous part and a singular part. However, our assumption on S_0 eliminates the singular part. For, if $[t_1, t_2]$ is a closed interval contained in $\mathbb{R} - S_0$, then from the Stieltjes inversion formula and the continuity of $\mathbf{h}(z)$ onto $[t_1, t_2]$ from above,

$$\boldsymbol{\sigma}_c(t_2) - \boldsymbol{\sigma}_c(t_1) = \frac{1}{\pi} \int_{t_1}^{t_2} \operatorname{Im} \mathbf{h}(t)\, dt.$$

Now use (9); since $\boldsymbol{\rho}(t)$ is continuous on $[s_1, \infty)$, $\boldsymbol{\sigma}_c(t)$ is differentiable on $\mathbb{R} - S_0$ and

$$\begin{aligned} \boldsymbol{\sigma}_c'(t) &= 0, \qquad t < s_1, \qquad t \notin S_0, \\ \boldsymbol{\sigma}_c'(t) &= \pi^{-1}(t - x_1)^{-1} \boldsymbol{\rho}(t), \qquad t \geqslant s_1, \qquad t \notin S_0. \end{aligned} \tag{13}$$

This fixes the absolutely continuous part of $\boldsymbol{\sigma}_c$. The measure associated with each matrix element of the singular part of $\boldsymbol{\sigma}_c$ is thus concentrated on S_0, which is countable; this is impossible.

Now note that, from the behaviour of $q_i(t)$ as $t \to \infty$,

$$\int_{s_i}^{\infty} \frac{q_i(t)}{1+t^2}\, dt < \infty, \qquad i = 1, 2.$$

Thus, using (12) and (13) and redefining $\boldsymbol{\alpha}$, (10) becomes

$$\begin{aligned} \mathbf{h}(z) = \boldsymbol{\alpha} + \boldsymbol{\beta} z + \sum_{i=1}^{\infty} \mathbf{R}_i \left(\frac{1}{\xi_i - z} - \frac{\xi_i}{1+\xi_i^2} \right) \\ + \frac{1}{\pi} \int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t - x_1)(t - z)}\, dt, \qquad z \in \mathbb{C} - [s_1, \infty). \end{aligned} \tag{14}$$

From the properties of $\boldsymbol{\sigma}$ we have only that

$$\sum_{i=1}^{\infty} \frac{\|\mathbf{R}_i\|}{1+\xi_i^2} < \infty,$$

so that the sum in (14) has to be kept in the form given.

Functions $\mathbf{h}(z)$ of the form (14) will lead to functions $\mathbf{f}(z)$ satisfying the conditions (a) – (g) given earlier provided that certain further conditions are fulfilled. The first of these is that

$$\mathbf{h}(x_1) = -\mathbf{\Gamma}^{-1} \tag{15}$$

whenever $\mathbf{\Gamma}$ is nonsingular. We shall assume that $\det \mathbf{\Gamma} \neq 0$ throughout the rest of this section and consider the case $\det \mathbf{\Gamma} = 0$ in Section 3. The second condition arises from the fact that by its definition (8), $\mathbf{h}(z)$ is either undefined or nonsingular. Now, from (14), $\mathbf{h}(z)$ is undefined only when $z \in \{\xi_i\}$. Taking note of (8), all we need to check is that $\mathbf{h}(s)$ is nonsingular for $s \notin \{\xi_i\}$. Since $\boldsymbol{\rho}(s) > 0$ for $s > s_2$ (equation (3)) we see from (9) and Lemma A.2 that it remains to ensure that $\mathbf{h}(s)$ is nonsingular for $s \leq s_2$. Physically this means "absence of ghosts".

So far, with an obvious generalization of the definition of $\boldsymbol{\rho}$ in (3), the argument applies to any number of coupled channels. From now on we restrict ourselves to two channels only, and prove first a simple lemma.

Lemma 1. *If* $\mathbf{h}(s)$ *is* not *diagonal for* $s_1 < s \leq s_2$, *then* $\mathbf{h}(s)$ *is nonsingular in that interval.*

Proof. From (14) and (3b) we see that

$$\mathbf{h}(s) = \begin{pmatrix} a(s) + i(s-x_1)^{-1} q_1(s) & b(s) \\ b(s) & c(s) \end{pmatrix}, \qquad s_1 < s \leq s_2,$$

with a, b, c real-valued functions. Thus

$$\det \mathbf{h}(s) = a(s)c(s) - b(s)^2 + i(s-x_1)^{-1} q_1(s)c(s).$$

Im det $\mathbf{h}(s)$ can vanish on $(s_1, s_2]$ only if $c(s)$ vanishes. But if $c(\bar{s}) = 0$ and $b(s) \neq 0$, Re det $\mathbf{h}(s) \neq 0$. Thus $b(s) \neq 0$ implies that $\det \mathbf{h}(s) \neq 0$. □

Physically this result means that with coupling present between the channels a stable bound state in channel 2 above the threshold for channel 1 cannot exist.

For the case of two channels we can restrict the representation (14) of $\mathbf{h}(z)$ a little more. We see from (6) that the derivative of $\mathbf{g}(s)$ is

$$\mathbf{g}'(s) = \frac{1}{\pi} \int_{s_1}^{\infty} \frac{\operatorname{Im} \mathbf{g}(t)}{(t-s)^2}\, dt > 0, \qquad s < s_1. \tag{16}$$

Now $\mathbf{g}(s)$ is degenerate if and only if $g_{11}(s) = g_{22}(s)$ and $g_{12}(s) = 0$. Since $\mathbf{g}(z)$ is analytic in $(-\infty, s_1)$, either $\mathbf{g}(s)$ is degenerate on a subset of $(-\infty, s_1)$ which has no accumulation point in $(-\infty, s_1)$ or it is degenerate for all $s < s_1$ (Kato [10] gives this result on page 64 for matrix functions of any order). Now in the first case the eigenvalues $g_1(z)$, $g_2(z)$ of $\mathbf{g}(z)$ are analytic in $(-\infty, s_1)$ (Kato [10], Ch. II, Theorem 6.1). From (16) and Lemma A.1 it follows that $g_1(s)$ and $g_2(s)$ are increasing functions on $(-\infty, s_1]$. In the second case, $\mathbf{g}(s) = g(s)\mathbb{1}_2$ for $s < s_1$ and it is again clear from (16) that $g(s)$ is an increasing function on $(-\infty, s_1]$. It follows that each of the functions $g_1(s)$, $g_2(s)$ can vanish at most once on $(-\infty, s_1]$. If $g_i(s)$ vanishes at ξ_i, we choose the functions g_i so that $\xi_1 \leq \xi_2$ whenever both ξ_1, ξ_2 exist. If now $h_1(s)$, $h_2(s)$ are the eigenvalues of $\mathbf{h}(s)$, then, since $h_i(s) = -1/g_i(s)$, each function $h_i(s)$ is differentiable and increasing on $(-\infty, s_1]$, with the exception of the point ξ_i if it exists. Either $h_i(s)$ does not vanish on $(-\infty, s_1]$ or $h_i(s)$ is positive for $s < \xi_i$, increases to $+\infty$ as $s \uparrow \xi_i$, jumps at ξ_i from $+\infty$ to $-\infty$ and increases from $-\infty$ but remains negative as s increases from ξ_i to s_1.

The points ξ_1, ξ_2, if they exist, are the only points in $(-\infty, s_1]$ at which $\mathbf{h}(s)$ is undefined. By redefining $\boldsymbol{\alpha}$ we can recast the representation (14):

$$\mathbf{h}(z) = \boldsymbol{\alpha} + \frac{\mathbf{R}_1}{\xi_1 - z} + \frac{\mathbf{R}_2}{\xi_2 - z} + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t-x_1)(t-z)}\, dt + \boldsymbol{\Delta}(z), \tag{17}$$

where

$$\boldsymbol{\Delta}(z) = \boldsymbol{\beta} z + \sum_{i=3}^{\infty} \mathbf{R}_i \left(\frac{1}{\xi_i - z} - \frac{\xi_i}{1+\xi_i^2}\right) \tag{18}$$

and

$$\xi_1 \leq \xi_2 \leq s_1, \qquad s_1 < \xi_i\,(i \geq 3), \qquad \mathbf{R}_i \geq 0, \qquad \boldsymbol{\beta} \geq 0. \tag{19}$$

The matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ are of rank 0 or 1. We now prove a substantial theorem.

Theorem 1. *If there exists a function* $\mathbf{h}(z)$ *of the form* (17), *with* $\det \mathbf{h}(s) \neq 0$ *for* $s \in (-\infty, s_1] - \{\xi_1, \xi_2\}$ *and satisfying* (15), *then there exists a function* $\mathbf{h}^{(0)}(z)$ *with* $\boldsymbol{\beta} = 0$ *and* $\mathbf{R}_i = 0$ $(i = 1, 2, \ldots)$, *which also satisfies* (15) *and has* $\det \mathbf{h}^{(0)}(s) \neq 0$ for $s \in (-\infty, s_1]$.

Proof. First we remove the function $\boldsymbol{\Delta}(z)$. Define

$$\mathbf{h}(z; \lambda) = \boldsymbol{\alpha} + \frac{\mathbf{R}_1}{\xi_1 - z} + \frac{\mathbf{R}_2}{\xi_2 - z} + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t-x_1)(t-z)}\, dt + (1-\lambda)\,\boldsymbol{\Delta}(z) + \lambda \boldsymbol{\Delta}(x_1), \qquad 0 \leq \lambda \leq 1.$$

Then

$$\mathbf{h}(z; 0) = \mathbf{h}(z),$$
$$\mathbf{h}(x_1; \lambda) = \mathbf{h}(x_1) = -\boldsymbol{\Gamma}^{-1}, \quad 0 \leq \lambda \leq 1.$$

Looking at $\mathbf{h}(z; 1)$, which we call $\tilde{\mathbf{h}}(z)$, we see that in it the function $\boldsymbol{\Delta}(z)$ in $\mathbf{h}(z)$ has been replaced by the constant $\boldsymbol{\Delta}(x_1)$. Moreover, $\tilde{\mathbf{h}}(z)$ satisfies (15); we now show that $\det \tilde{\mathbf{h}}(s) \neq 0$ for $s \in (-\infty, s_1] - \{\xi_1, \xi_2\}$.

For fixed s, $\mathbf{h}(s; \lambda)$ is continuously differentiable in λ and so are the eigenvalues $h_i(s; \lambda)$, by Theorem 6.8, Ch. II of Ref. [10]. Further,

$$\frac{\partial}{\partial \lambda}\mathbf{h}(s; \lambda) = \boldsymbol{\Delta}(x_1) - \boldsymbol{\Delta}(s) \begin{array}{ll} \geq 0, & s < x_1, \\ \leq 0, & s > x_1, \end{array}$$

since $\boldsymbol{\Delta}'(s) \geq 0$ for $s \leq s_1$, on differentiating (18) and using (19). Thus each eigenvalue $h_i(s; \lambda)$ is a nondecreasing function of λ on $[0, 1]$ when $s < x_1$ and is a nonincreasing function of λ on $[0, 1]$ when $s > x_1$. Further, each $h_i(s; \lambda)$ remains an increasing function of s on $(-\infty, s_1]-\{\xi_i\}$ for fixed $\lambda \in [0, 1]$.

Now if $h_i(s; 0) > 0$ on $(-\infty, s_1]$, take $-M < x_1$ and $\lambda > 0$. From the statements of the previous paragraph,

$$h_i(s; \lambda) > h_i(-M; \lambda) \geq h_i(-M; 0) > 0, -M < s \leq s_1.$$

But M may be chosen arbitrarily large. Similarly, if $h_i(s; 0) :< 0$ on $(-\infty, s_1]$, then for $\lambda < 0$

$$h_i(s; \lambda) < h_i(s_1; \lambda) \leq h_i(s_1; 0) < 0, \qquad s < s_1.$$

Finally, if $h_i(s; 0)$ jumps at ξ_i from $+\infty$ to $-\infty$, we use the argument for positive functions to the left of ξ_i and that for negative functions to the right of ξ_i. We have thus shown that $\det \mathbf{h}(s; \lambda) \neq 0$ for $s \in (-\infty, s_1]-\{\xi_1, \xi_2\}$, $\lambda \in [0, 1]$.

The argument for removing the poles at ξ_1, ξ_2 is similar. Assume that $\xi_1 < x_1$ and define

$$\mathbf{h}[z; \tau] = \mathbf{R}_1 \frac{\tau - x_1}{\xi_1 - x_1} \cdot \frac{1}{\tau - z} + \boldsymbol{\omega}(z), \qquad \tau \leq \xi_1, \tag{20}$$

where

$$\boldsymbol{\omega}(z) = \boldsymbol{\alpha} + \frac{\mathbf{R}_2}{\xi_2 - z} + \frac{1}{\pi} \int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t - x_1)(t - z)} \, dt.$$

Then

$$\mathbf{h}[z; \xi_1] = \tilde{\mathbf{h}}(z),$$

$$\mathbf{h}[x_1; \tau] = \mathbf{h}(x_1) = -\boldsymbol{\Gamma}^{-1}, \qquad \tau \leq \xi_1.$$

We are going to take τ to $-\infty$, thus obtaining a function $\mathbf{h}[z; -\infty]$ in which the term $\mathbf{R}_1/(\xi_1 - z)$ in $\tilde{\mathbf{h}}(z)$ is replaced by the constant $\mathbf{R}_1/(\xi_1 - x_1)$. Moreover, $\mathbf{h}[z; -\infty]$ satisfies (15); we shall show that $\det \mathbf{h}[s; -\infty] \neq 0$ for $s \in (-\infty, s_1]-\{\xi_2\}$.

On differentiating (20),

$$\frac{\partial}{\partial \tau} \mathbf{h}[s; \tau] = \frac{\mathbf{R}_1}{\xi_1 - x_1} \frac{x_1 - s}{(\tau - s)^2} \begin{array}{ll} \leq 0, & s < x_1, \\ \geq 0, & s > x_1. \end{array}$$

The derivative is undefined at $s = \tau$. Thus as τ decreases from ξ_1 to $-\infty$, $h_i[s; \tau]$ increases for $s < x_1$, decreases for $s > x_1$. Further, for each $\tau \leq \xi_1$, $h_i[s; \tau]$ is an increasing function of s on $(-\infty, s_1]$, with of course the jump point excluded.

The eigenvalue $h_1[s; \tau]$, for fixed τ, has a jump from $+\infty$ to $-\infty$ as s passes through τ. The other eigenvalue $h_2[s; \tau]$ has a jump at ξ_2 if $\mathbf{R}_2 \neq 0$; otherwise it is of fixed sign on $(-\infty, s_1]$. The same argument as before shows that $h_2[s; -\infty] \neq 0$ for $s \in (-\infty, s_1]$. The argument for $h_1[s; \tau]$ is almost the same. When $\tau < \xi_1$, $h_1[1; \tau] \leq h_1[1; \xi_1] < 0$ and so $h_1[s; \tau] < 0$ for $\tau < s \leq s_1$. Also, if we choose M so that $-M < \tau$ $(< \xi_1)$, then $h_1[-M; \tau] > h_i[-M; \xi_1] > 0$ and so $h_1[s; \tau] > 0$ for $-M \leq s < \tau$. But M may be chosen arbitrarily large. Thus $\det \mathbf{h}[s; \tau] \neq 0$ for $s \in (-\infty, s_1]-\{\tau, \xi_2\}$. Now take the limit $\tau \to -\infty$.

If $\xi_1 > x_1$, we start with τ at ξ_1 and move it to $+\infty$. Exactly the same procedure will remove the pole at ξ_2 if it is present. □

The theorem we have just proved shows that in considering whether there exist functions $\mathbf{h}(z)$ which lead to functions $\mathbf{f}(z)$ satisfying all the specified conditions, we can confine ourselves to functions $\mathbf{h}^{(0)}(z)$ of the form

$$\mathbf{h}^{(0)}(z)=\boldsymbol{\alpha}+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)(t-z)}\,dt. \tag{21}$$

For if such a function does not exist, nor do others of the more general form (17). We shall call $\mathbf{h}^{(0)}(z)$ a restricted function. We must still require that $\mathbf{h}^{(0)}(z)$ satisfies (15) and that $\det \mathbf{h}^{(0)}(s)\neq 0$ for $s\leqslant s_2$. When these conditions are satisfied we shall call $\mathbf{h}^{(0)}(z)$ the *isolated solution* of our problem (though strictly speaking it is $\mathbf{f}^{(0)}(z)$ which should be so called). Clearly, $\boldsymbol{\alpha}$ is fixed by (15) and the isolated solution is a distinguished solution which is fixed by the input data $(x_1, \boldsymbol{\Gamma})$ and contains no disposable parameters.

We now define

$$\mathbf{A}=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)(t-s_1)}\,dt>0,\qquad B=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)^2}\,dt>0. \tag{22}$$

Since $\mathbf{A}>0$, $\mathbf{A}^{-1/2}$ exists and we can define

$$\mathbf{M}=-\mathbf{A}^{-1/2}(\boldsymbol{\Gamma}+\mathbf{B})\mathbf{A}^{-1/2}. \tag{23}$$

The next theorem gives the key result of the paper.

Theorem 2. *A necessary and sufficient condition that there exists a function* $\mathbf{h}^{(0)}(z)$ *which satisfies* (15) *and has*

$$\det \mathbf{h}^{(0)}(s)\neq 0,\qquad s\leqslant s_1, \tag{24}$$

is that each eigenvalue m_i of the matrix $\mathbf{M}$ *defined in* (23) *satisfies either*

$$m_i\geqslant 0\quad\text{or}\quad m_i<-1. \tag{25}$$

Proof. Define

$$\mathbf{I}(z)=\mathbf{A}^{-1/2}\mathbf{h}^{(0)}(z)\mathbf{A}^{-1/2}. \tag{26}$$

The condition (24) becomes

$$\det \mathbf{I}(s)\neq 0,\qquad s\leqslant s_1. \tag{27}$$

From (21) and (26),

$$\mathbf{I}(-\infty)=\lim_{s\to-\infty}\mathbf{I}(s)=\mathbf{A}^{-1/2}\boldsymbol{\alpha}\mathbf{A}^{-1/2}. \tag{28}$$

Further,

$$\mathbf{I}(s_1)=\mathbf{A}^{-1/2}(\boldsymbol{\alpha}+\mathbf{A})\mathbf{A}^{-1/2}=\mathbf{I}(-\infty)+\mathbb{1}_2. \tag{29}$$

Let $I_1(s)$, $I_2(s)$ be the eigenvalues of $\mathbf{I}(s)$. They are analytic in $(-\infty, s_1)$ and since from (21)

$$\mathbf{h}^{(0)\prime}(s)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)(t-s)^2}\,dt>0,\qquad s<s_1,$$

they are increasing functions of s on $(-\infty, s_1]$. From (29),

$$I_i(s_1) = I_i(-\infty) + 1.$$

Condition (27) is equivalent to $I_i(s) \neq 0$, $s \leqslant s_1$. For this to hold it is necessary and sufficient that

$$I_i(-\infty) \geqslant 0 \quad \text{or} \quad I_i(-\infty) < -1. \tag{30}$$

But $\mathbf{h}^{(0)}(z)$ has to satisfy (15) and so, by (21) and (22),

$$\boldsymbol{\alpha} = -\boldsymbol{\Gamma}^{-1} - \mathbf{B}. \tag{31}$$

From (28) the matrix $\mathbf{M}$ defined in (23) is indeed $\mathbf{I}(-\infty)$ and (25) is just the necessary and sufficient condition (30). □

It is easy to see that, if $-\boldsymbol{\Gamma}^{-1} > 0$, then (25) becomes

$$-\boldsymbol{\Gamma}^{-1} \geqslant \mathbf{B} = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t-x_1)^2}\, dt, \tag{32a}$$

while if $-\boldsymbol{\Gamma}^{-1} < 0$, (25) becomes

$$-\boldsymbol{\Gamma}^{-1} < \mathbf{B} - \mathbf{A} = -\frac{(s_1 - x_1)}{\pi}\int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t-x_1)^2(t-s_1)}\, dt. \tag{32b}$$

The conditions (32a, b) may be compared directly with the conditions for the one-channel case which are given in Section 4.1 of [2] (where the variable s was scaled so that $s_1 = 1$).

We have still to make sure that $\det \mathbf{h}^{(0)}(s) \neq 0$ for $s_1 < s \leqslant s_2$. From Lemma 1, since

$$h_{12}^{(0)}(z) = \alpha_{12} = (-\boldsymbol{\Gamma}^{-1})_{12},$$

it is sufficient that $\Gamma_{12} \neq 0$. However, if $\Gamma_{12} = 0$ we must be more careful. The restricted function $\mathbf{h}^{(0)}(z)$ satisfying (15) is then diagonal for all z; from (21) and (31) it is

$$\mathbf{h}^{(0)}(z) = -\boldsymbol{\Gamma}^{-1} + \frac{(z-x_1)}{\pi}\int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t-x_1)^2(t-z)}\, dt. \tag{33}$$

Now $h_{11}^{(0)}(s)$ cannot vanish for $s > s_1$ since $q_1(s) > 0$ there. To have $h_{11}^{(0)}(s) \neq 0$ for $s \leqslant s_1$ it is necessary and sufficient that

$$-\Gamma_{11}^{-1} \geqslant \frac{1}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2}\, dt \quad \text{or} \quad -\Gamma_{11}^{-1} < -\frac{(s_1-x_1)}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2(t-s_1)}\, dt. \tag{34}$$

At the same time we must ensure that $h_{22}^{(0)}(s) \neq 0$, not just for $s \leqslant s_1$ but for $s \leqslant s_2$. For this it is necessary and sufficient that

$$-\Gamma_{22}^{-1} \geqslant \frac{1}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2}\, dt \quad \text{or} \quad -\Gamma_{22}^{-1} < -\frac{(s_2-x_1)}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2(t-s_2)}\, dt. \tag{35}$$

It is worth looking more closely at what happens when (34) holds for Γ_{11} and

$$-\frac{(s_2-x_1)}{\pi}\int_{s_2}^{\infty}\frac{q_2(t)}{(t-x_1)^2(t-s_2)}\,dt\leqslant-\Gamma_{22}^{-1}<-\frac{(s_1-x_1)}{\pi}\int_{s_2}^{\infty}\frac{q_2(t)}{(t-x_1)^2(t-s_1)}\,dt. \quad (36)$$

Then $h_{22}^{(0)}(s)$ has a zero in $(s_1, s_2]$ and it is no longer true that $\det \mathbf{h}^{(0)}(s)\neq 0$ for $s\leqslant s_2$. Thus there is no isolated solution of our problem. However, there are other solutions! For take $\xi>s_2$, $\varepsilon>0$,

$$\mathbf{R}_\varepsilon=\begin{pmatrix}\varepsilon & \frac{1}{2}\varepsilon\\ \frac{1}{2}\varepsilon & \varepsilon\end{pmatrix}>0 \quad (37)$$

and define

$$\mathbf{h}^{(\xi,\varepsilon)}(z)=-\mathbf{\Gamma}^{-1}+\frac{\mathbf{R}_\varepsilon(z-x_1)}{(\xi-x_1)(\xi-z)}+\frac{(z-x_1)}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)^2(t-z)}\,dt. \quad (38)$$

Then $\mathbf{h}^{(\xi,\varepsilon)}(x_1)=-\mathbf{\Gamma}^{-1}$ as required and, from considerations of continuity, if we fix ξ and keep ε sufficiently small, then $\det \mathbf{h}^{(\xi,\varepsilon)}(s)\neq 0$ for $s\leqslant s_1$. Moreover $\mathbf{h}^{(\xi,\varepsilon)}(z)$ is nondiagonal and so $\det \mathbf{h}^{(\xi,\varepsilon)}(s)\neq 0$ for $s_1<s\leqslant s_2$, by Lemma 1. Thus $\mathbf{h}^{(\xi,\varepsilon)}(z)$ leads to a solution of our problem. We put these results in the form of a lemma.

Lemma 2. *If $\mathbf{\Gamma}$ is diagonal then* (34) *and* (35) *give a necessary and sufficient condition that the restricted function $\mathbf{h}^{(0)}(z)$ given in* (33) *leads to a solution of our problem* (*the isolated solution*). *If however* (34) *continues to hold for Γ_{11} but Γ_{22} satisfies* (36) *instead of* (35) *then the restricted function* (33) *does not lead to a solution, but other solutions do exist.* (*For example, for fixed $\xi>s_2$ and $\mathbf{R}_\varepsilon$ defined by* (34), (38) *defines a function $\mathbf{h}^{(\xi,\varepsilon)}(z)$ which leads to a solution for all sufficiently small ε.*)

We promised in Section 1 to collect the results of this section together at the end. Recall that we set out to find functions satisfying conditions (a)–(g) given at the beginning of the section and that we have assumed that $\det \mathbf{\Gamma}\neq 0$.

(I) (See Theorem 1) Suppose that a function $\mathbf{h}(z)$ exists of the form (17), which satisfies $\mathbf{h}(x_1)=-\mathbf{\Gamma}^{-1}$ (equation (15)) and has $\det \mathbf{h}(s)\neq 0$ for $s\in(-\infty, s_1]-\{\xi_1, \xi_2\}$. Then there exists a restricted function $\mathbf{h}^{(0)}(z)$ of the form (21) which also satisfies (15) and has $\det \mathbf{h}^{(0)}(s)\neq 0$ for $s\in(-\infty, s_1]$.

(II) (See Theorem 2) A necessary and sufficient condition that there exists a restricted function $\mathbf{h}^{(0)}(z)$ satisfying the conditions in (I) is that each eigenvalue m_i of the matrix $\mathbf{M}$ defined by (22) and (23) satisfies either

$$m_i\geqslant 0 \quad \text{or} \quad m_i<-1.$$

(III) If $\Gamma_{12}\neq 0$ and the condition just given in (II) is satisfied, then $\mathbf{h}^{(0)}(z)$ is uniquely determined by x_1, $\mathbf{\Gamma}$ as

$$\mathbf{h}^{(0)}(z)=-\mathbf{\Gamma}^{-1}+\frac{(z-x_1)}{\pi}\int_{s_1}^{\infty}\frac{\boldsymbol{\rho}(t)}{(t-x_1)^2(t-z)}\,dt$$

and gives the isolated solution $\mathbf{f}^{(0)}(z)$ of the problem via

$$\mathbf{f}^{(0)}(z) = -\mathbf{h}^{(0)}(z)^{-1}/(z-x_1).$$

If however $\Gamma_{12} = 0$ then the condition given in (II) is sufficient for solutions of our problem to exist, but a more restrictive condition on Γ_{22} is required for the isolated solution to exist. Details are given in Lemma 2.

3. Special cases

We begin by discussing a case mentioned in the introduction where the element $f_{11}^{(0)}$ of the isolated solution of the two-channel problem is *not* the isolated solution of the equivalent inelastic one-channel problem for channel 1. Write $\mathbf{h}(x_1)$ instead of $-\mathbf{\Gamma}^{-1}$ and consider a two-channel problem for which

$$h_{11}(x_1) > \frac{1}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2}\,dt \quad \text{or} \quad h_{11}(x_1) < -\frac{(s_1-x_1)}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2(t-s_1)}\,dt, \tag{39}$$

$$-\frac{(s_2-x_1)}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2(t-s_2)}\,dt$$

$$\leqslant h_{22}(x_1) < -\frac{(s_1-x_1)}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2(t-s_1)}\,dt, \tag{40}$$

$$h_{12}(x_1) \neq 0.$$

We need to verify that (25) holds. Note that if we start with $\mathbf{M}$ diagonal and then take $M_{12} \neq 0$ the smaller eigenvalue is decreased and the larger eigenvalue increased. From (39) and (40), $M_{11} > 0$ or $M_{11} < -1$, while $M_{22} < -1$. In the case $M_{11} > 0$, (25) is satisfied for any $h_{12}(x_1)$; when $M_{11} < -1$, (25) will hold provided $|h_{12}(x_1)|$ is sufficiently small. Then the isolated solution exists and is given by (33):

$$h_{11}^{(0)}(z) = h_{11}(x_1) + \frac{(z-x_1)}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2(t-z)}\,dt, \tag{41}$$

$$h_{22}^{(0)}(z) = h_{22}(x_1) + \frac{(z-x_1)}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2(t-z)}\,dt, \tag{42}$$

$$h_{12}^{(0)}(z) = h_{12}(x_1).$$

Moreover, from (40) and (42) we see that $h_{22}^{(0)}(s) = 0$ at a point $s_0 \in (s_1, s_2]$.

Now

$$h_{22}^{(0)}(z) \to h_{22}(x_1) - \frac{1}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t-x_1)^2}\,dt,$$

a constant, as $|z| \to \infty$, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant 2\pi$, and

$$\operatorname{Im}(-h_{22}^{(0)}(s)^{-1}) \underset{s\to\infty}{\sim} (\text{constant})s^{-1/2}.$$

Thus we may write a dispersion relation for $-h_{22}^{(0)}(z)^{-1}$:

$$-h_{22}^{(0)}(z)^{-1} = -h_{22}(x_1)^{-1} + \frac{\gamma(z-x_1)}{z-s_0} + \frac{(z-x_1)}{\pi}\int_{s_2}^{\infty} \frac{\operatorname{Im}(-h_{22}^{(0)}(t)^{-1})}{(t-x_1)(t-z)}\,dt, \tag{43}$$

with $\gamma<0$. We now compute the inelasticity parameter $R_1^{(0)}(s)$ for elastic scattering in channel 1 corresponding to the isolated solution:

$$\begin{aligned} R_1^{(0)}(s) &= q_1(s)^{-1}\,\mathrm{Im}\,(-f_{11}^{(0)}(s)^{-1}) \\ &= (s-x_1)q_1(s)^{-1}\,\mathrm{Im}\,[h_{11}^{(0)}(s)-h_{12}^{(0)}(s)^2/h_{22}^{(0)}(s)] \\ &= 1+(s-x_1)q_1(s)^{-1}h_{12}(x_1)^2\,\mathrm{Im}\,(-h_{22}^{(0)}(s)^{-1}), \end{aligned} \tag{44}$$

the second term being >0 for $s>s_2$.

Consider now the EIOCP for channel 1. The residue of the pole at x_1 is Γ_{11}; thus the isolated solution, if it exists, is given by

$$h_1^{(0)}(z)=h_1(x_1)+\frac{(z-x_1)}{\pi}\int_{s_1}^{\infty}\frac{q_1(t)R_1^{(0)}(t)}{(t-x_1)^2(t-z)}\,dt, \tag{45}$$

where

$$h_1(x_1)=-\Gamma_{11}^{-1}=h_{11}(x_1)-h_{12}(x_1)^2/h_{22}(x_1). \tag{46}$$

For the isolated solution to exist, we must have

$$h_1^{(0)}(-\infty)\geqslant 0 \quad \text{or} \quad h_1^{(0)}(s_1)<0.$$

From (45) this condition is

$$h_1(x_1)\geqslant\frac{1}{\pi}\int_{s_1}^{\infty}\frac{q_1(t)R_1^{(0)}(t)}{(t-x_1)^2}\,dt \quad \text{or} \quad h_1(x_1)<-\frac{(s_1-x_1)}{\pi}\int_{s_1}^{\infty}\frac{q_1(t)R_1^{(0)}(t)}{(t-x_1)^2(t-s_1)}\,dt. \tag{47}$$

Now using (46), (44) and (39) we see that (47) is satisfied provided $|h_{12}(x_1)|$ is sufficiently small.

We wish to compare

$$f_{11}^{(0)}(s)=-\frac{h_{22}^{(0)}(s)}{(s-x_1)\det\mathbf{h}^{(0)}(s)}$$

with

$$f_1^{(0)}(s)=-\frac{1}{(s-x_1)h_1^{(0)}(s)}$$

for $s\geqslant s_1$. Since the former has a zero at $s=s_0$ which the latter does not have, it is clear that the functions cannot be the same. In detail, from (41), (45), (46), (44) and (43),

$$\begin{aligned} &\frac{\det\mathbf{h}^{(0)}(z)}{h_{22}^{(0)}(z)}-h_1^{(0)}(z) \\ &\quad= h_{11}^{(0)}(z)-h_{12}^{(0)}(z)^2/h_{22}^{(0)}(z)-h_1^{(0)}(z) \\ &\quad= h_{11}(x_1)-h_1(x_1)-\frac{(z-x_1)}{\pi}\int_{s_2}^{\infty}\frac{q_1(t)(R_1^{(0)}(t)-1)}{(t-x_1)^2(t-z)}\,dt-\frac{h_{12}(x_1)^2}{h_{22}^{(0)}(z)} \\ &\quad= h_{12}(x_1)^2\left[-h_{22}^{(0)}(z)^{-1}+h_{22}(x_1)^{-1}-\frac{(z-x_1)}{\pi}\int_{s_2}^{\infty}\frac{\mathrm{Im}\,(-h_{22}^{(0)}(t)^{-1})}{(t-x_1)(t-z)}\,dt\right] \\ &\quad= \frac{h_{12}(x_1)^2\gamma(z-x_1)}{z-s_0}\neq 0. \end{aligned}$$

Thus the diagonal component $f_{11}^{(0)}$ of the isolated solution of the two-channel problem does not coincide with the isolated solution of the EIOCP for channel 1.

Note that the case where $h_{12}(x_1)=0$ is in some sense singular. If we take the limit of $\mathbf{f}^{(0)}(z)$ as $h_{12}(x_1)\to 0$, we find that $\mathbf{f}^{(0)}(z)$ has a pointwise limit for $z\neq s_0$ which is

$$\tilde{f}_{ii}(z)=-\frac{1}{(z-x_1)h_{ii}^{(0)}(z)},\qquad \tilde{f}_{12}(z)=0.$$

The pointwise limit for $z\neq s_0$ is the 'formal' isolated solution when $h_{12}(x_1)=0$, which has a pole in $\tilde{f}_{22}(z)$ at $z=s_0$. But at $z=s_0$ only $f_{11}^{(0)}$ has a limit, namely zero, and this is obviously not $\tilde{f}_{11}(s_0)$. It has been argued that the 'formal' isolated solution is to be considered as the *physical* solution of the problem, even when it is not strictly a solution. Thus, in the case considered above, it would be claimed that when $h_{12}(x_1)=0$ but (39) and (40) hold, the pole at x_1 (which is an approximation to the forces acting) produces a bound state in channel 2 at $s=s_0$. But this viewpoint has to take account of the singular nature of the case with $h_{12}(x_1)=0$. There is a *qualitative* difference between the case with $h_{12}(x_1)=0$ and cases with $h_{12}(x_1)\neq 0$, however small, when $h_{11}(x_1)$ and $h_{22}(x_1)$ satisfy (39) and (40). When $h_{12}(x_1)=0$ (strictly no coupling between the channels via the driving forces), the 'formal' isolated solution is uncoupled at all energies. The component $\tilde{f}_{22}$ has the pole (bound state) at $z=s_0$ which we have discussed; however $\tilde{f}_{11}$ is a genuine solution of the one-channel problem and shows no sign of the pole in $\tilde{f}_{22}$ at $z=s_0$. But when $h_{12}(x_1)\neq 0$, however small, this is not true. The isolated solution of the two-channel problem is a genuine solution for which $f_{11}^{(0)}$ has a zero at $z=s_0$. The pole is no longer present in $f_{22}^{(0)}$, though $f_{22}^{(0)}$ grows large at $z=s_0$ if $|h_{12}(x_1)|$ is small.

We now consider an unusual limit of the case we have been considering. Keep s_1 fixed and take s_2 out to $+\infty$, but adjust $h_{12}(x_1)$ and $h_{22}(x_1)$ as functions of s_2 so that the zero s_0 in $h_{22}^{(0)}(s)$ remains fixed (and $s_0>s_1$, of course). For this purpose we take the masses in channel 2 to be equal, so that

$$q_2(s)=\tfrac{1}{2}(s-s_2)^{1/2}.$$

Everything can then be calculated analytically; we sketch the results. From (42),

$$h_{22}^{(0)}(s)=h_{22}(x_1)+\frac{1}{2[(s_2-x_1)^{1/2}+(s_2-s)^{1/2}]}-\frac{1}{4(s_2-x_1)^{1/2}},\ s\leqslant s_2, \tag{48}$$

$$h_{22}^{(0)}(s)=h_{22}(x_1)+\frac{(s_2-x_1)^{1/2}+i(s-s_2)^{1/2}}{2(s-x_1)}-\frac{1}{4(s_2-x_1)^{1/2}},\ s\geqslant s_2. \tag{49}$$

The condition (40) on $h_{22}(x_1)$ becomes

$$\frac{1-\phi(s_2)}{1+\phi(s_2)}<\alpha\leqslant 1,$$

where

$$\alpha=-4(s_2-x_1)^{1/2}h_{22}(x_1),\qquad \phi(s_2)=\left(\frac{s_2-s_1}{s_2-x_1}\right)^{1/2}.$$

The zero s_0 of $h_{22}^{(0)}(s)$ is given by

$$s_0=\frac{4\alpha}{(1+\alpha)^2}s_2+\left(\frac{1-\alpha}{1+\alpha}\right)^2 x_1.$$

From (44) and (49),

$$q_1(s)(R_1^{(0)}(s)-1)=(s-x_1)h_{12}(x_1)^2\,\mathrm{Im}\,h_{22}^{(0)}(s)/|h_{22}^{(0)}(s)|^2$$
$$=\frac{8(s_2-x_1)(s-s_2)^{1/2}h_{12}(x_1)^2}{(1+\alpha)^2-4\alpha(s_2-x_1)/(s-x_1)}.$$

To keep s_0 fixed as $s_2\to\infty$ we choose α so that

$$\alpha=\frac{[(s_2-x_1)^{1/2}-(s_2-s_0)^{1/2}]^2}{s_0-x_1}\underset{s_2\to\infty}{\sim}\frac{(s_0-x_1)}{4s_2},$$
$$h_{22}(x_1)=-\frac{[(s_2-x_1)^{1/2}-(s_2-s_0)^{1/2}]^2}{4(s_2-x_1)^{1/2}(s_0-x_1)}\underset{s_2\to\infty}{\sim}-\frac{(s_0-x_1)}{16s_2^{3/2}}. \tag{50}$$

At the same time, for reasons which will soon become clear, we choose $h_{12}(x_1)$ so that

$$16s_2^{3/2}h_{12}(x_1)^2\xrightarrow[s_2\to\infty]{}k>0.$$

We need to check that (25) is satisfied as $s_2\to\infty$. Since (39) holds we have a fixed M_{11} (equation (23)) satisfying

$$M_{11}>0\quad\text{or}\quad M_{11}<-1. \tag{51}$$

Again using (22) and (23)

$$|M_{12}|=\frac{|h_{12}(x_1)|}{A_{11}^{1/2}A_{22}^{1/2}}\underset{s_2\to\infty}{\sim}\frac{k^{1/2}}{2A_{11}^{1/2}s_2^{1/2}},$$
$$M_{22}=-\frac{(1+\alpha)[(s_2-x_1)^{1/2}+(s_2-s_1)^{1/2}]}{2(s_2-x_1)^{1/2}}\underset{s_2\to\infty}{\sim}-1-\frac{(s_0-s_1)}{4s_2},$$

since

$$A_{22}=\frac{1}{2[(s_2-x_1)^{1/2}+(s_2-s_1)^{1/2}]},\qquad B_{22}=\frac{1}{4(s_2-x_1)^{1/2}}.$$

The eigenvalues of $\mathbf{M}$, as $s_2\to\infty$, have the behaviour

$$m_1\sim M_{11}+\frac{k}{4A_{11}(1+M_{11})s_2},$$
$$m_2\sim-1-\left(\frac{s_0-s_1}{4}+\frac{k}{4A_{11}(1+M_{11})}\right)\frac{1}{s_2}.$$

Since (51) holds, (25) is satisfied for all finite s_2 provided that, when $M_{11}<-1$, k is restricted to

$$0<k<-(1+M_{11})(s_0-s_1)A_{11}. \tag{52}$$

It may also be arranged that the isolated solution of the EIOCP for channel 1 always exists as $s_2 \to \infty$. A rather tedious calculation shows that

$$h_1(x_1) - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)R_1(t)}{(t-x_1)^2}\,dt$$
$$= h_{11}(x_1) - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2}\,dt + \frac{4(s_2-x_1)^{1/2}h_{12}(x_1)^2}{\alpha(1+\alpha)}, \tag{53}$$

while

$$h_1(x_1) + \frac{(s_1-x_1)}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)R_1(t)}{(t-x_1)^2(t-s_1)}\,dt$$
$$= h_{11}(x_1) + \frac{(s_1-x_1)}{\pi}\int_{s_1}^{\infty} \frac{q_1(t)}{(t-x_1)^2(t-s_1)}\,dt$$
$$+ \frac{4(s_2-x_1)^{1/2}[(s_2-s_1)^{1/2}+(s_2-x_1)^{1/2}]h_{12}(x_1)^2}{\alpha[(1+\alpha)(s_2-s_1)^{1/2}+(1-\alpha)(s_2-x_1)^{1/2}]}. \tag{54}$$

The final term on the right side of (53) and of (54) approaches the same constant $k/(s_0-x_1)$ as $s_2 \to \infty$. Thus if the first condition in (39) holds, k is unrestricted; if the second holds, we need to restrict k to

$$0 < k < -(1+M_{11})(s_0-x_1)A_{11},$$

which is a weaker condition than (52).

We now look at what happens to $f_{11}^{(0)}(s)$ for $s \geq s_1$, as $s_2 \to \infty$. With $h_{22}(x_1)$ chosen as in (50), the expression (48) for $h_{22}^{(0)}(s)$ becomes

$$h_{22}^{(0)}(s) = -\frac{[(s_2-x_1)^{1/2}-(s_2-s_0)^{1/2}]^2}{4(s_2-x_1)^{1/2}(s_0-x_1)}$$
$$+ \frac{[(s_2-x_1)^{1/2}-(s_2-s)^{1/2}]^2}{4(s_2-x_1)^{1/2}(s-x_1)}, \qquad s \leq s_2.$$

For fixed s,

$$h_{22}^{(0)}(s) \underset{s_2\to\infty}{\sim} \frac{s-s_0}{16 s_2^{3/2}}.$$

Thus

$$f_{11}^{(0)}(s) = -\frac{h_{22}^{(0)}(s)}{(s-x_1)[h_{11}^{(0)}(s)h_{22}^{(0)}(s)-h_{12}(x_1)^2]}$$
$$\xrightarrow[s_2\to\infty]{} \frac{s-s_0}{(s-x_1)[k-(s-s_0)h_{11}^{(0)}(s)]}$$
$$= \frac{-1}{(s-x_1)[h_{11}^{(0)}(s)+k/(s_0-s)]}.$$

In the limit we have taken, $f_{11}^{(0)}(s)$ does not approach the isolated solution of the one-channel elastic scattering problem with a simple pole at x_1, the prescribed residue being $-[h_{11}(x_1)+k/(s_0-x_1)]^{-1}$. It approaches instead a solution with a CDD zero at $s=s_0$.

The example we have given is pathological in the sense that the elements Γ_{12}, Γ_{22} of the residue matrix and the elements $f_{12}^{(0)}(s)$ and $f_{22}^{(0)}(s)$ of the isolated solution all $\to\infty$ (in absolute value) as $s_2\to\infty$. In a sense this does not matter. It could be argued that we have given a simple model of a situation in which there is elastic scattering in a single channel, but the physical amplitude is not the isolated solution of the appropriate dispersion relation. Instead it has a CDD zero in the physical region which is the manifestation of what could perhaps be called a 'confined bound state'. It should be noted that a similar situation has been treated in potential scattering in Refs. [11, 12]. The aim of our example has been to show that if 'confined' channels exist then the 'physical' solution of a partial-wave dispersion relation for the unconfined channels may not be the isolated one.

We consider next what happens when the prescribed residue matrix is singular. Then det $\mathbf{\Gamma}=0$, but $\mathbf{\Gamma}$ is of rank 1 ($\mathbf{\Gamma}\neq\mathbf{0}$). Write $\mathbf{\Gamma}$ in the form

$$\mathbf{\Gamma}=\Gamma\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix},\qquad 0\leq\theta\leq\pi,\qquad \Gamma\neq 0.$$

With $\mathbf{0}$ defined by

$$\mathbf{0}=\begin{pmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{pmatrix},\tag{55}$$

define a function $\tilde{\mathbf{g}}$ in terms of the function $\mathbf{g}$ considered in Section 2 by

$$\tilde{\mathbf{g}}(z)=\mathbf{0}^t\mathbf{g}(z)\mathbf{0}.$$

Then we look for functions $\tilde{\mathbf{g}}(z)$ satisfying

$$\tilde{\mathbf{g}}(x_1)=\mathbf{0}^t\mathbf{\Gamma}\mathbf{0}=\Gamma\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

$$\operatorname{Im}\tilde{\mathbf{g}}(s)=(s-x_1)^{-1}\tilde{\mathbf{g}}(s)^*\tilde{\boldsymbol{\rho}}(s)\tilde{\mathbf{g}}(s),\qquad s\geq s_2,$$

with

$$\tilde{\boldsymbol{\rho}}(s)=\mathbf{0}^t\boldsymbol{\rho}(s)\mathbf{0}\geq 0,\tag{56}$$

by (A.8).

To get such functions we write as before

$$\tilde{\mathbf{g}}(z)=-\tilde{\mathbf{h}}(z)^{-1},$$

but now $\tilde{\mathbf{h}}(z)$, which is a matrix R-function, has the form

$$\tilde{\mathbf{h}}(z)=\kappa(x_1-z)^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\mathbf{h}(z),\tag{57}$$

where $\kappa>0$ and $h(z)$ has the form (17), but with just one pole, ξ_1 say, with $\xi_1\leq s_1$. It is easy to verify that

$$\tilde{\mathbf{g}}(z)\xrightarrow[z\to x_1]{}\begin{pmatrix}0 & 0\\ 0 & -h_{22}(x_1)^{-1}\end{pmatrix},$$

so that $h_{22}(x_1)$ must be fixed as

$$h_{22}(x_1) = -\Gamma^{-1}. \tag{58}$$

However, $h_{11}(x_1)$ and $h_{12}(x_1)$ are not fixed. The arguments of Theorem 1 show that if there is a function $\tilde{\mathbf{h}}(z)$ of the form (57) with given $\mathbf{h}(x_1)$ and with $\det \tilde{\mathbf{h}}(s) \neq 0$ for $s \in (-\infty, s_1] - \{x_1, \xi_1\}$, then there is a restricted function $\tilde{\mathbf{h}}^{(0)}(z)$ of the form

$$\tilde{\mathbf{h}}^{(0)}(z) = \kappa(x_1 - z)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{h}^{(0)}(z),$$

with $\mathbf{h}^{(0)}(z)$ of the form (21), $\mathbf{h}^{(0)}(x_1) = \mathbf{h}(x_1)$ and $\det \tilde{\mathbf{h}}^{(0)}(s) \neq 0$ for $s \in (-\infty, s_1] - \{x_1\}$. One eigenvalue of $\tilde{\mathbf{h}}^{(0)}(s)$ behaves like $\kappa/(x_1 - s)$ for s near x_1 and so jumps from $+\infty$ to $-\infty$ there. The other eigenvalue goes through $h_{22}(x_1) = -\Gamma^{-1}$. We have not tried to work out the conditions on $\mathbf{h}(x_1)$ which ensure that neither of these eigenvalues vanishes for $s \in (-\infty, s_1]$. The conditions will depend on κ and are rather complicated.

Another class of solutions for the case when $\mathbf{\Gamma}$ is of rank 1 is obtained from solutions of the type (57) by letting $\kappa \to \infty$. From (57),

$$\begin{aligned}\tilde{\mathbf{g}}(z) &= -\tilde{\mathbf{h}}(z)^{-1} \\ &= \left[\det \mathbf{h}(z) + \frac{\kappa h_{22}(z)}{x_1 - z}\right]^{-1}\begin{pmatrix} -h_{22}(z) & h_{12}(z) \\ h_{12}(z) & -h_{11}(z) - \kappa/(x_1 - z) \end{pmatrix}.\end{aligned}$$

Now multiply by $(x_1 - z)/\kappa$ and let $\kappa \to \infty$; then

$$\tilde{\mathbf{g}}(z) \to \begin{pmatrix} 0 & 0 \\ 0 & -h_{22}(z)^{-1} \end{pmatrix}. \tag{59}$$

Condition (58) must of course hold; if this class of solutions is not empty, there is an isolated solution

$$\tilde{\mathbf{g}}^{(0)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & -h_{22}^{(0)}(z)^{-1} \end{pmatrix}.$$

In this case it is easy to give the conditions on Γ for the isolated solution to exist, namely

$$-\Gamma^{-1} \geqslant \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\tilde{\rho}_{22}(t)}{(t - x_1)^2}\, dt$$

or

$$-\Gamma^{-1} < -\frac{(s_1 - x_1)}{\pi}\int_{s_1}^{\infty} \frac{\tilde{\rho}_{22}(t)}{(t - x_1)^2(t - s_1)}\, dt, \qquad \theta \neq \frac{\pi}{2},$$

$$-\Gamma^{-1} < -\frac{(s_2 - x_1)}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{(t - x_1)^2(t - s_2)}\, dt, \qquad \theta = \frac{\pi}{2}.$$

From (55) and (56),

$$\tilde{\rho}_{22}(s) = \rho_{11}(s)\cos^2\theta + \rho_{22}(s)\sin^2\theta.$$

We have the exceptional case $\theta = \pi/2$ for which a stronger condition is required on Γ in the case when $\Gamma > 0$. It is similar to the exceptional case with $\mathbf{\Gamma}$ diagonal which we met in Section 2 when $\det \mathbf{\Gamma} \neq 0$.

It is not difficult to verify that if one starts from general solutions of the form (17) for $\det \mathbf{\Gamma} \neq 0$, with $\boldsymbol{\alpha}$ fixed by $-\mathbf{\Gamma}^{-1}$ and the other quantities ($\boldsymbol{\beta}$, ξ_i, $\mathbf{R}_i$) fixed, and then takes the limit in which one eigenvalue goes to zero, only solutions of the form (59) result. The more general solutions of the form (57) can be obtained by a special limiting procedure in which one of the poles (ξ_1 say) in $\mathbf{h}(z)$ is taken to be close to x_1 and $\xi_1 \to x_1$ as the eigenvalue of $\mathbf{\Gamma}$ goes to zero. Thus reducing the rank of the residue matrix involves tricky limiting procedures.

4. Conclusion

We have now obtained the results given in the abstract. One can convince oneself that our methods can also be applied to cases where there are more than two channels, though there will be more and more subcases to consider as the number of channels increases.

When $l > 0$ we need to go to a modified amplitude matrix. For $l = 1$, the matrix $\mathbf{f}$ introduced at the beginning of Section 2 can be put in the form

$$\mathbf{f}(s) = \mathbf{Q}(s)\mathbf{F}(s)\mathbf{Q}(s),$$

where

$$\mathbf{Q}(s) = \begin{pmatrix} q_1(s) & 0 \\ 0 & q_2(s) \end{pmatrix}$$

and $\lim_{s \to s_i} \mathbf{F}(s)$ exists for $i = 1, 2$, uniformly in angle as usual. The function $q_i(s)$, when continued to complex values of the argument, has $[s_i, \infty)$ as a cut, so that

$$q_2(s) = i\kappa_2(s), \qquad s_1 \leq s \leq s_2,$$

with $\kappa_2(s) > 0$. We need to assume that the other singularities of $q_2(s)$ lie to the left of s_1. Now

$$f_{12}(s) = q_1(s)q_2(s)F_{12}(s),$$

and this behaviour of f_{12} near the thresholds s_1, s_2 cannot be reproduced by means of the solutions we have found. Thus a partial-wave dispersion relation for $l = 1$ (and of course for higher l) has to be formulated for the matrix of reduced amplitudes $\mathbf{F}_l$ which is defined for general l in [7] for example.

Proceeding further with the $l = 1$ case and using the matrix $\mathbf{F}$ we have the unitarity relation

$$\operatorname{Im} \mathbf{F}(s) = \mathbf{F}(s)^* \boldsymbol{\rho}(s)\mathbf{F}(s), \qquad s \geq s_1,$$

where

$$\boldsymbol{\rho}(s) = \begin{pmatrix} q_1(s)^3 & 0 \\ 0 & q_2(s)^3 \end{pmatrix}, \qquad s \geq s_2, \tag{60a}$$

$$\boldsymbol{\rho}(s) = \begin{pmatrix} q_1(s)^3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad s_1 \leq s < s_2. \tag{60b}$$

Defining $\mathbf{G}(z)$ by

$$\mathbf{G}(z) = (z - x_1)\mathbf{F}(z),$$

the same argument as for s-waves shows that $\mathbf{G}$ satisfies an unsubtracted dispersion relation and that $\mathbf{G}$ is a matrix R-function, as is

$$\mathbf{H}(z) = -\mathbf{G}(z)^{-1}.$$

The representation theorem then gives

$$\mathbf{H}(z) = \boldsymbol{\alpha} + \boldsymbol{\beta} z + \sum_{i=1}^{\infty} \mathbf{R}_i \left(\frac{1}{\xi_i - z} - \frac{\xi_i}{1 + \xi_i^2} \right) + \frac{1}{\pi} \int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{t - x_1} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right),$$

with $\boldsymbol{\beta} \geqslant 0$, $\mathbf{R}_i \geqslant 0$. In contrast to the s-wave case, because of the behaviour of $\boldsymbol{\rho}(t)$ for large t, the integral must be kept in the form shown. Incidentally we note that cases with $l > 1$ and just one pole have no solutions, since the integral no longer converges. In general, a necessary condition for solutions to exist is that the number of poles $N \geqslant l$ (compare the result for the one-channel case in [2]).

Returning to the one pole $l = 1$ case, we have $\mathbf{G}(s) > 0$ and $\mathbf{G}'(s) > 0$ for $s < s_1$, from the unsubtracted dispersion relation. Further $g_{ij}(s) \to 0$ as $s \to -\infty$. Thus each eigenvalue $H_i(s)$ increases from $-\infty$ to some negative value as s increases from $-\infty$ to s_1. It is therefore necessary that $\mathbf{H}(x_1) < 0$ and so the prescribed $\boldsymbol{\Gamma} > 0$. The terms with $\xi_1, \xi_2 (\leqslant s_1)$ do not appear. The same argument as before shows that if there is a solution, then there is an isolated solution. A very short calculation shows that, provided $\boldsymbol{\Gamma}$ is nondiagonal, a necessary and sufficient condition for the isolated solution to exist is that

$$-\boldsymbol{\Gamma}^{-1} < -\frac{(s_1 - x_1)}{\pi} \int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t - x_1)^2 (t - s_1)} \, dt.$$

The methods of Section 2 can in fact go part of the way to solving the case with $l = 1$ and two poles (or more generally, cases where the number of poles is $(l + 1)$). For the $l = 1$ case with poles at x_1, x_2 we define

$$\mathbf{G}(z) = (z - x_1)(z - x_2)\mathbf{F}(z).$$

Then $\mathbf{G}(z)$ satisfies a once subtracted dispersion relation and so both $\mathbf{G}(z)$ and $\mathbf{H}(z) = -\mathbf{G}(z)^{-1}$ are matrix R-functions as before. Then $\mathbf{H}(z)$ has the representation (17) but with an integral of the form

$$\frac{1}{\pi} \int_{s_1}^{\infty} \frac{\boldsymbol{\rho}(t)}{(t - x_1)(t - x_2)(t - z)} \, dt,$$

$\boldsymbol{\rho}(t)$ being given by (60a, b). However the argument that, if there is a solution then there is an isolated solution, breaks down. It is specific to the case of one pole, and the two pole case requires much more study (even for the one-channel case the analysis is very elaborate; see Section 4.2 of Ref. [2]).

The main difficulty is to extend the methods of the present paper to cases where the number of poles is greater than $(l + 1)$; encouraging work is in progress.

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Appendix

We begin with some elementary results on matrices. The *norm* of $\mathbf{M}$ is defined by

$$\|\mathbf{M}\| = \max_{\|u\|=1} \|\mathbf{M}u\| = \max_{\substack{\|u\|=1 \\ \|v\|=1}} |(v, \mathbf{M}u)|.$$

When $\mathbf{M}$ is normal ($\mathbf{M}^*\mathbf{M} = \mathbf{M}\mathbf{M}^*$).

$$\|\mathbf{M}\| = \max_i \{|\lambda_i|\},$$

where λ_i are the eigenvalues of $\mathbf{M}$ (see Kato [10], Eq. I-(6.67)). For any matrix $\mathbf{M}$, $\mathbf{M}^*\mathbf{M}$ is hermitian and

$$\|\mathbf{M}^*\| = \|\mathbf{M}\|, \qquad \|\mathbf{M}^*\mathbf{M}\| = \|\mathbf{M}\|^2. \tag{A.1}$$

Also, if e_i is the vector in $\mathbb{C}^n$ whose only non-zero component is 1 in the ith place, then

$$|M_{ij}| = |(e_i, \mathbf{M}e_j)| \leq \|e_i\| \, \|\mathbf{M}e_j\| \leq \|\mathbf{M}\|. \tag{A.2}$$

We define the real and imaginary parts of a matrix $\mathbf{M}$ by

$$\operatorname{Re}\mathbf{M} = \tfrac{1}{2}(\mathbf{M} + \mathbf{M}^*),$$

$$\operatorname{Im}\mathbf{M} = \frac{1}{2i}(\mathbf{M} - \mathbf{M}^*).$$

Then Re $\mathbf{M}$ and Im $\mathbf{M}$ are hermitian matrices and

$$\mathbf{M} = \mathrm{Re}\,\mathbf{M} + i\,\mathrm{Im}\,\mathbf{M}.$$

If $\mathbf{M}^t = \mathbf{M}$, then

$$(\mathrm{Re}\,\mathbf{M})_{ij} = \tfrac{1}{2}(M_{ij} + \bar{M}_{ij}) = \mathrm{Re}\,M_{ij} \tag{A.3}$$

and similarly

$$(\mathrm{Im}\,\mathbf{M})_{ij} = \mathrm{Im}\,M_{ij}.$$

Further

$$\|\mathrm{Im}\,\mathbf{M}\| \leqslant \tfrac{1}{2}(\|\mathbf{M}\| + \|\mathbf{M}^*\|) = \|\mathbf{M}\|, \tag{A.4}$$

by (A.1). The same inequality holds for $\|\mathrm{Re}\,\mathbf{M}\|$.

A *hermitian* matrix $\mathbf{H}$ is called *positive* (written $\mathbf{H} > 0$) if

$$(u, \mathbf{H}u) > 0, \qquad \|u\| > 0. \tag{A.5}$$

If $>$ is replaced by $\geqslant$ in (A.5), then $\mathbf{H}$ is called *nonnegative* ($\mathbf{H} \geqslant 0$). Clearly $\mathbf{H} > 0$ if and only if all its eigenvalues are positive. If $\mathbf{H} > 0$ it follows that

$$H_{ii} > 0, \tag{A.6a}$$

$$\|\mathbf{H}\| = \text{largest eigenvalue of } \mathbf{H}, \tag{A.6b}$$

$$\mathrm{tr}\,\mathbf{H} > \|\mathbf{H}\| \quad \text{if } \dim \mathbf{H} > 1, \tag{A.6c}$$

$$|H_{ij}| < \mathrm{tr}\,\mathbf{H}. \tag{A.6d}$$

Further, if $\mathbf{B} \geqslant 0$, $\mathbf{C} \geqslant 0$ and $\mathbf{A} = \mathbf{B} + \mathbf{C}$, then

$$(u, \mathbf{A}u) = (u, \mathbf{B}u) + (u, \mathbf{C}u),$$

so that

$$\mathbf{A} \geqslant 0, \qquad \|\mathbf{A}\| \geqslant \max\{\|\mathbf{B}\|, \|\mathbf{C}\|\}. \tag{A.7}$$

Next, if $\mathbf{H} \geqslant 0$, then

$$\mathbf{M}^*\mathbf{H}\mathbf{M} \geqslant 0 \quad \text{for all } \mathbf{M}, \tag{A.8}$$

and, if $\mathbf{H} > 0$ and $\mathbf{M}$ is nonsingular, then

$$\mathbf{M}^*\mathbf{H}\mathbf{M} > 0. \tag{A.9}$$

Before going to the main representation theorem we prove two lemmas.

Lemma A.1. *Let $\mathbf{H}(x)$ be a 2×2 hermitian matrix defined on an interval centred on x_0 and differentiable at x_0. If $\mathbf{H}(x_0)$ is nondegenerate and $\mathbf{H}'(x_0) > 0$, then the eigenvalues $h_i(x)$ of $\mathbf{H}(x)$, which are differentiable at x_0, satisfy $h_i'(x_0) > 0$, $i = 1, 2$.*

Proof. That the $h_i(x)$ are differentiable at x_0 may be seen from the last part of Theorem 5.4, Ch. II of Ref. [10]. If u_i are the eigenvectors (normalized to 1) of $\mathbf{H}(x_0)$ corresponding to the eigenvalues $h_i(x_0)$, then a straightforward calculation (well known from perturbation theory for nondegenerate eigenvalues) shows that

$$h_i'(x_0) = (u_i, \mathbf{H}'(x_0)u_i) > 0. \quad \square$$

Lemma A.2. *Let* $\mathbf{M}$ *be a matrix for which* $\operatorname{Im}\mathbf{M}>0$. *Then* $\det\mathbf{M}\neq 0$.

Proof. Suppose that $\det\mathbf{M}=0$. Then there is a vector $u_0\neq 0$ with $\mathbf{M}u_0=0$. Then

$$0=(u_0,\mathbf{M}u_0)=(u_0,\operatorname{Re}\mathbf{M}u_0)+i(u_0,\operatorname{Im}\mathbf{M}u_0).$$

Since $\operatorname{Re}\mathbf{M}$ and $\operatorname{Im}\mathbf{M}$ are hermitian matrices, it follows that $(u_0,\operatorname{Im}\mathbf{M}u_0)=0$, which contradicts the assumption that $\operatorname{Im}\mathbf{M}>0$. □

We come now to the representation theorem, which to our knowledge is stated and proved here for the first time. Recall from Section 2 that a *matrix R-function* is a symmetric matrix function whose matrix elements are analytic in the upper half-plane and whose imaginary part is nonnegative (in the matrix sense) there.

Representation Theorem for Matrix R-Functions. The matrix function $\mathbf{h}(z)$ is an R-function if and only if it is representable in the form

$$\mathbf{h}(z)=\boldsymbol{\alpha}+\boldsymbol{\beta}z+\int_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\boldsymbol{\sigma}(t),\ \operatorname{Im}z>0, \tag{A.10}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}(t)$ are real symmetric matrices, $\boldsymbol{\beta}\geqslant 0$ and $\boldsymbol{\sigma}(t)$ is a nondecreasing function in the matrix sense:

$$\boldsymbol{\sigma}(t_2)-\boldsymbol{\sigma}(t_1)\geqslant 0,\qquad t_2>t_1, \tag{A.11}$$

whose matrix elements satisfy

$$V(\tilde{\sigma}_{ij};\mathbb{R})<\infty, \tag{A.12}$$

where

$$\tilde{\sigma}_{ij}(t)=\frac{\sigma_{ij}(t)}{1+t^2}$$

and V denotes the total variation.

Proof. We begin with a theorem of F. Riesz and Herglotz (see [13], page 389). If $H(\zeta)$ is a complex-valued function analytic on $|\zeta|<1$ and having $\operatorname{Re}H(\zeta)\geqslant 0$ there, then $H(\zeta)$ admits a representation

$$H(\zeta)=i\operatorname{Im}H(0)+\int_0^{2\pi}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\,d\tau(\theta), \tag{A.13}$$

where $\tau(\theta)$ is a nondecreasing function. Moreover $\operatorname{Re}H(\zeta)$ satisfies the growth condition

$$\sup_{r<1}\int_0^{2\pi}\operatorname{Re}H(re^{i\theta})\,d\theta<\infty. \tag{A.14}$$

Equations (A.13) and (A.14) also come from Theorem 11.19 of Rudin [14].

Now consider a symmetric matrix function $\mathbf{H}(\zeta)$, analytic on $|\zeta|<1$ and having $\operatorname{Re}\mathbf{H}(\zeta)\geqslant 0$ there. From (A.3), $\operatorname{Re}\mathbf{H}(\zeta)$ is a matrix function each of whose elements is a real harmonic function. Thus $\operatorname{tr}\operatorname{Re}\mathbf{H}(\zeta)$ is a real-valued harmonic function and $\operatorname{tr}\operatorname{Re}\mathbf{H}(\zeta)\geqslant 0$ on $|\zeta|<1$, by (A.6a). It follows that $H(\zeta)=\operatorname{tr}\mathbf{H}(\zeta)$ has the representation (A.13) and satisfies the growth condition (A.14). Using (A.3) and (A.6d) (slightly modified for nonnegative matrices) we have

$$|\mathrm{Re}\, H_{ij}(\zeta)| = |(\mathrm{Re}\,\mathbf{H}(\zeta))_{ij}| \leq \mathrm{tr}\,\mathrm{Re}\,\mathbf{H}(\zeta) = \mathrm{Re}\, H(\zeta),$$

$$\sup_{r<1} \int_0^{2\pi} |\mathrm{Re}\, H_{ij}(re^{i\theta})|\, d\theta \leq \sup_{r<1} \int_0^{2\pi} \mathrm{Re}\, H(re^{i\theta})\, d\theta < \infty.$$

Thus the real part of each matrix element of $\mathbf{H}(\zeta)$ satisfies the growth condition of the first part of Theorem 11.19 of [14] and we can write

$$(\mathrm{Re}\,\mathbf{H}(\zeta))_{ij} = \int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\right) d\tau_{ij}(\theta), \tag{A.15}$$

where τ_{ij} is now a function of bounded variation, so that

$$V(\tau_{ij};[0, 2\pi]) < \infty,$$

V being the total variation. The connection between complex Borel measures on $\mathbb{R}$ or on a closed interval in $\mathbb{R}$ and functions of bounded variation is given in Theorem 8.14 of [14]. It follows from (A.15) that

$$\mathbf{H}(\zeta) = i\,\mathrm{Im}\,\mathbf{H}(0) + \int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\, d\boldsymbol{\tau}(\theta). \tag{A.16}$$

We now follow the argument on page 390 of [9]. The function $\mathbf{h}(z)$ is a matrix R-function if and only if $\mathbf{H}(\zeta)$ is symmetric, is analytic on $|\zeta|<1$ and has $\mathrm{Re}\,\mathbf{H}(\zeta) \geq 0$ there, where

$$\zeta = (z-i)/(z+i) \quad \text{and} \quad \mathbf{H}(\zeta) = -i\mathbf{h}(z).$$

The representation (A.10) is derived from (A.16) via the relation between ζ and z just given and the relation

$$t = -\cot(\theta/2),$$

using the identity

$$\frac{1+tz}{t-z} = \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)(1+t^2).$$

Note that $\boldsymbol{\beta}$ in (A.10) is the sum of the jumps of the function $\boldsymbol{\tau}$ at the points $\theta = 0$ and $\theta = 2\pi$. The function $\boldsymbol{\sigma}(t)$ in (A.10) is related to $\boldsymbol{\tau}(\theta)$ in (A.16) by

$$(1+t^2)^{-1}\, d\boldsymbol{\sigma}(t) = d\boldsymbol{\tau}(\theta),$$

and so (A.12) is satisfied by $\boldsymbol{\sigma}(t)$.

Finally, the representation (A.10) is unique if the function $\boldsymbol{\sigma}(t)$ is normalized in some way, say,

$$\boldsymbol{\sigma}(0) = 0, \qquad \boldsymbol{\sigma}(t) = \tfrac{1}{2}[\boldsymbol{\sigma}(t+0) + \boldsymbol{\sigma}(t-0)].$$

Equation (A.11) follows from the Stieltjes inversion formula

$$\boldsymbol{\sigma}(t_2) - \boldsymbol{\sigma}(t_1) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{t_1}^{t_2} \mathrm{Im}\,\mathbf{h}(t+i\varepsilon)\, dt \geq 0.$$

It is easily checked that

$$\boldsymbol{\beta} = \lim_{y\to\infty} \mathrm{Im}\,\mathbf{h}(iy)/y \geq 0. \quad \square$$