On the propagator of the relativistic oscillator

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On the propagator of the relativistic oscillator[1])

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Abstract. We propose an expression for the propagator of the scalar relativistic oscillator [1]. It is shown that this expression has several decent properties. On the other hand we did not succeed in obtaining a unique characterization of the propagator in terms of physical requirements.

1. Formulation of the problem

We consider a system of two spinless constituents described by the wave equation of the relativistic FKR oscillator [1, 2] and introduce an external perturbation (e.g. an electromagnetic field). The problem we put ourselves is the following: How do we calculate the effects of such a perturbation on the system?

In order to get a systematic perturbation theory we study the Green's functions of the oscillator and choose the analogue of the Feynman propagator of the free system. We are therefore led to the problem of constructing an appropriate Green's function to be identified with the propagator of the oscillator.

Here we are confronted with serious difficulties arising from the fact that the wave equation of the relativistic oscillator is supplemented by a covariant constraint eliminating relative time excitations [2]. This gives us a restricted class of solutions of the wave equation. We are looking for a propagator which, in addition to the standard requirements (covariance, analyticity, spectral properties, proper free and nonrelativistic limit), also satisfies this constraint.

Unfortunately, we were unable to find an unambiguous definition determining the propagator uniquely. Nevertheless we can propose a well motivated ansatz which fulfils the above requirements. The generalization to spin $\frac{1}{2}$ systems [3] does not introduce special difficulties, up to gymnastics in spin matrices.

In Section 2 we define the FKR oscillator and show that the associated Bethe-Salpeter kernel leads to unphysical modes. In Section 3 we give a characterization of the causal (homogeneous) Green's function and argue that it is not possible to define the inhomogeneous (retarded, advanced, mixed) Green's functions as in the free case just by multiplying these objects with suitable θ-functions. In Section 4 we make an ansatz for the propagator, motivated by solutions of a compatibility condition, and discuss its properties. In section 5 we provide conclusions.

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2. The FKR oscillator and the Bethe–Salpeter kernel

A system of the two spinless constituents may be described by a covariant wave function $\phi(x', x'')$. States with a sharp value p of the four-momentum are of the form:

$$\phi_p(x', x'') = \exp\left[-\frac{i}{2} p(x'+x'')\right]\phi_p(x'-x'').$$

Notation

$$w_x = \tfrac{1}{2}(x'+x''), \qquad z_x = x'-x'', \qquad a_x = \partial_{z_x} - \lambda z_x, \qquad a_x^+ = -\partial_{z_x} - \lambda z_x,$$

$$D_x^{(1)} = -a_x^+ a_x + m_0^2 + \tfrac{1}{4}\Box_{w_x}, \qquad D_x^{(2)} = a_x \partial_{w_x},$$

$$\phi(x', x'') \to \phi(w_x, z_x), \qquad \phi_p(w_x, z_x) = \exp(-ipw_x)\phi_p(z_x).$$

In the following I omit the subscript x whenever this does not lead to confusion. In the above notation the FKR oscillator is defined by:

$$\begin{aligned} &D^{(1)}\phi(w, z) = 0 \quad \text{wave equation,}\\ &D^{(2)}\phi(w, z) = 0 \quad \text{constraint.} \end{aligned} \tag{1}$$

The free case is obtained for $\lambda \to 0$:

$$D^{\binom{1}{2}} \to \tfrac{1}{2}[(\Box_{x'} + m_0^2) \pm (\Box_{x''} + m_0^2)].$$

For a discussion of the properties of the FKR oscillator within the framework of Hamiltonian quantum theory on the null plane I refer to [2].

The timelike part of the mass spectrum is a sequence of linear trajectories:

$$p_n^2 = 4m_0^2 + 8\lambda n, \qquad n = n_1 + n_2 + n_3, \qquad n_i = 0, 1, \ldots \qquad (i = 1, 2, 3). \tag{2}$$

The corresponding eigenfunctions are products of Hermite polynomials:

$$\begin{aligned} \phi_{p,\vec{n}}(z) = \left(\frac{\lambda}{\pi}\right)^{3/4}\left[\frac{2^{-n+1}}{\sqrt{p_n^2}\,n_1!\,n_2!\,n_3!}\right]^{1/2} \exp\left(\frac{\lambda}{2}z^2\right) H_{n_1}\left(\sqrt{\lambda}\left(z^1 + \frac{p_1}{p_+}z^-\right)\right)\\ \times H_{n_2}\left(\sqrt{\lambda}\left(z^2 + \frac{p_2}{p_+}z^-\right)\right) H_{n_3}\left(\sqrt{\frac{\lambda}{p_n^2}}\left(pz - \frac{p_n^2}{p_+}z^-\right)\right) \end{aligned} \tag{3}$$

$$(p_+ = \tfrac{1}{2}(p_0 + p_3),\ z^- = \tfrac{1}{2}(z^0 - z^3)).$$

In the normalization (3) the functions $\phi_{p,\vec{n}}(x', x'') = \exp(-ipw)\phi_{p,\vec{n}}(z)$ are orthonormal in the following metric:

$$\begin{aligned} &-\int_{\sigma'} d\sigma'_\mu \int_{\sigma''} d\sigma''_\nu \tilde{\phi}_{p^1 \to p', \vec{m}}(x', x'')^* [\overleftrightarrow{\partial}'^\mu \overleftrightarrow{\partial}''^\nu - \lambda z^\mu \overleftrightarrow{\partial}''^\nu - \lambda \overleftrightarrow{\partial}'^\mu z^\nu]\phi_{p,\vec{n}}(x', x'')\\ &= (2\pi)^3\,|p'_0 + p_0|\,\delta(\vec{p}' - \vec{p})\delta_{\vec{m},\vec{n}} \end{aligned} \tag{4}$$

for any two spacelike hypersurfaces σ', σ''. Here the adjoint wave function $\tilde{\phi}_{p,\vec{n}}$ is given by

$$\tilde{\phi}_{p,\vec{n}} = \exp\left(-\lambda\frac{(pz)^2}{p_n^2}\right)\phi_{p,\vec{n}}. \tag{5}$$

It satisfies the following equations:

$$\tilde{D}^{(1)}\tilde{\phi}(w, z) = 0, \qquad \tilde{D}^{(2)}\tilde{\phi}(w, z) = 0,$$

$$\tilde{D}^{(1)} = -a^+a + 2\lambda + m_0^2 + \tfrac{1}{4}\Box_w = D^{(1)+} + 2\lambda, \qquad \tilde{D}^{(2)} = -a^+\partial_w = D^{(2)+}. \tag{6}$$

These wave equations, of course, lead to the same mass spectrum as (1).

There arise problems in the spacelike part of the mass spectrum. The eigenfunctions and adjoint wave functions are known but for most of them the above metric leads to divergent integrals. A short discussion of the completeness problem of the oscillator on the null plane may be found in [4].

The basic problem investigated here is the construction of the associated inhomogeneous Green's function to be identified with the Feynman propagator which for two free particles is given by

$$G_F^{\text{free}}(x', x'' \mid y', y'') = \frac{1}{i}\,\Delta_F(x' - y'; m_0^2)\,\frac{1}{i}\,\Delta_F(x'' - y''; m_0^2). \tag{7}$$

The free propagator may be characterized by the four inhomogeneous wave equations ($\lambda = 0$)

$$DG_F^{\text{free}} = I^{\text{free}}, \qquad D \in \{D_x^{(1)}, D_x^{(2)}, \tilde{D}_y^{(1)}, \tilde{D}_y^{(2)}\}, \qquad I^{\text{free}} \in \{I^{(1)}, I^{(2)}, \tilde{I}^{(1)}, \tilde{I}^{(2)}\},$$

$$I^{(1)} = \tilde{I}^{(1)} = \frac{1}{2i}[\delta(x' - y') + \delta(x'' - y'')]\,\frac{1}{i}\,\Delta_F(z_x - z_y; m_0^2), \tag{8}$$

$$I^{(2)} = \tilde{I}^{(2)} = \frac{1}{2i}[\delta(x' - y') - \delta(x'' - y'')]\,\frac{1}{i}\,\Delta_F(z_x - z_y; m_0^2).$$

supplemented with suitable boundary conditions. Clearly, the inhomogeneities vanish unless $x' = y'$ or $x'' = y''$. This property suggests that one characterizes the propagator associated with the oscillator by the inhomogeneous wave equations

$$DG_F = I \tag{9}$$

and requires that the inhomogeneities vanish unless $x' = y'$ or $x'' = y''$. This requirement is however inconsistent: in order for (9) to admit a solution the inhomogeneities have to obey an integrability condition. An ansatz of the structure (8) for $I^{(1)}$, $I^{(2)}$, $\tilde{I}^{(1)}$, $\tilde{I}^{(2)}$ fails to satisfy this integrability condition. This implies that the locality requirement on the inhomogeneities has to be weakened ('segment locality' instead of 'point locality', see Section 4).

Alternatively, one might characterize the propagator by the fourth order differential equations

$$B * G_F = G_F * B = 1,$$

$$B(x', x'' \mid y', y'') = -(D_x^{(1)} + 2\lambda - D_x^{(2)})(D_x^{(1)} + D_x^{(2)})\delta(x' - y')\delta(x'' - y''), \tag{10}$$

which are also satisfied by the free propagator ($B^{\text{free}} \sim -(\Box' + m_0^2)(\Box'' + m_0^2)$), and avoid the integrability problem. (The factor 2λ in the differential operator is inserted to guarantee that both ϕ and $\tilde{\phi}$ are eigenstates of B in the sense:

$$B * \phi = \tilde{\phi} * B = 0$$

$$[(D^{(1)} + 2\lambda - D^{(2)})(D^{(1)} + D^{(2)}) = [(\tilde{D}^{(1)} - 2\lambda + \tilde{D}^{(2)})(\tilde{D}^{(1)} - \tilde{D}^{(2)})]^+].$$

We call B the Bethe–Salpeter operator of the oscillator.)

The equations (10) determine G_F up to a solution of the homogeneous equations $B*G=0=G*B$. In order to get a unique solution one has to supplement (10) with suitable boundary conditions such that G_F indeed describes propagation. A convenient method to achieve this consists in solving the corresponding Euclidean problem. In the Euclidean region $B^{-1}(x', x'' \mid y', y'')$ is uniquely determined by the requirement that the Fourier transform with respect to the variables z_x, z_y and w is well defined (at least as a tempered distribution). Explicitly we have found the following expression for this kernel:

$$B^{-1}(x', x'' \mid y', y'') = \frac{1}{(2\pi)^4 i} \int d^4p \exp(-ipw) B^{-1}(z_x \mid p \mid z_y), \tag{11}$$

$$\begin{aligned} B^{-1}(z_x \mid p \mid z_y) = {} & \frac{1}{32\pi^2} \exp\left[\frac{\lambda}{2}(z_x^2 - z_y^2)\right] \int_0^\infty d\tau \sqrt{\tau} \left(\frac{\lambda}{1-e^{-\lambda\tau}}\right)^{3/2} \int_{\xi_1}^{\xi_2} d\xi \\ & \times \exp\left[-\frac{\tau}{4}\left(m_0^2 - \frac{p^2}{4}\right) + \frac{\lambda}{1-e^{-\lambda\tau}}(e^{-\lambda\tau/2} z_x - z_y)^2 - \frac{\xi^2 p^2 \tau}{16} \right. \\ & \left. - \frac{i}{2} \xi \sqrt{\frac{\lambda\tau}{1-e^{-\lambda\tau}}}\, p(e^{-\lambda\tau/2} z_x - z_y)\right], \end{aligned} \tag{12}$$

$$\xi_{1,2} = \pm \frac{2}{\sqrt{\lambda\tau}} \left[\frac{1-e^{-\lambda\tau/2}}{1+e^{-\lambda\tau/2}}\right]^{1/2}, \qquad (z_x - z_y)^2 < 0, \qquad \operatorname{Re} p^2 < 4m_0^2.$$

The inverse B^{-1} describing propagation in the Minkowski region is obtained as the boundary value of the analytic continuation corresponding to the following prescription

$$B^{-1} \to B^{-1}(w^2 - i\varepsilon, wz_x, wz_y, z_x^2, z_x z_y, z_y^2)$$

(motivated by the free case, see Subsection 4.D).

Discussion of the properties of B^{-1} (without details):

(i) B^{-1} has the correct free and nonrelativistic limit (Subsection 4.C).

(ii) Though B^{-1} has poles at the correct values $p^2 = 4m_0^2 + 8\lambda n (n = 0, 1, \ldots)$ the residua

$$R_n(z_x \mid p \mid z_y) = \lim_{p^0 \to p_n{}^0} (p_n^2 - p^2) B^{-1}(z_x \mid p \mid z_y)$$

have exotic contributions (relative time excitations). It may be verified that:

$$B * R_n = R_n * B = 0, \quad \text{but} \quad D^{(\frac{1}{2})} R_n \neq 0 \neq \tilde{D}^{(\frac{1}{2})} R_n.$$

As may have been expected the inverse of the Bethe–Salpeter operator propagates physical as well as unphysical modes of the oscillator; B^{-1} therefore does not solve the problem posed here.

3. Causal Green's function

The causal Green's function $G(x', x'' \mid y', y'')$ of the relativistic oscillator may be characterized by the homogeneous wave equations:

$$DG = 0, \qquad D \in \{D_x^{(1)}, D_x^{(2)}, \tilde{D}_y^{(1)}, \tilde{D}_y^{(2)}\} \tag{13a}$$

and the boundary conditions at $x'^0 = x''^0 = y'^0 = y''^0 = 0$:

$$G = \partial^0_{x'} G = \partial^0_{x''} G = \partial^0_{y'} G = \partial^0_{y''} G = 0, \tag{13b}$$
$$-(\partial^0_{x'}\partial^0_{x''} G)(\vec{x}', \vec{x}'' \mid \vec{y}', \vec{y}'') = -(\partial^0_{y'}\partial^0_{y''} G)(\vec{x}', \vec{x}'' \mid \vec{y}', \vec{y}'')$$
$$= \delta(\vec{x}' - \vec{y}')\, \delta(\vec{x}'' - \vec{y}'').$$

The solution of (13a, b) exists and is unique.

In general the causal Green's function should be expressible by a sum over a complete set of eigenfunctions of the four-momentum. It may be decomposed into a timelike and a spacelike part. According to (3)–(5) the timelike part is

$$G^t(x', x'' \mid y', y'') = \sum_{\substack{\vec{n} \\ p_n^0 \gtrless 0}} \int \frac{d^3 p}{(2\pi)^3 2\, |p_n^0|} \phi_{p,\vec{n}}(x', x'') \tilde{\phi}_{p,\vec{n}}(y', y'')^*. \tag{13c}$$

The spacelike part should, at least formally, be of similar structure [4]. G^t may also be written in the form

$$G^t(x', x'' \mid y', y'') = \int \frac{d^4 p}{(2\pi)^3} \exp(-ipw) \sum_{n=0}^{\infty} \delta(p^2 - p_n^2) \phi_n(pz_x, pz_y, z_x^2, z_x z_y, z_y^2),$$
$$\phi_n(pz_x, pz_y, z_x^2, z_x z_y, z_y^2) = \int_{-1}^{1} d\xi \int_{-\infty}^{\infty} d\eta \exp[-i(\xi p z_x - \eta p z_y)] \tag{13d}$$
$$\times \tilde{\phi}_n(\xi, \eta, z_x^2, z_x z_y, z_y^2).$$

In the free case $\tilde{\phi}_n$ vanishes unless $\xi, \eta \in \{\frac{1}{2}, -\frac{1}{2}\}$, but in the oscillator case this is not true anymore. Intuitively the oscillator may be viewed as describing objects extended along the straight line segment connecting the two defining points. The naïve definition of the retarded Green's function suggested by the free case:

$$G_{\text{ret}}(x', x'' \mid y', y'') = \theta(x'^0 - y'^0)\theta(x''^0 - y''^0) G(x', x'' \mid y', y'')$$

is not acceptable, because it violates Lorentz invariance. ($G(x', x'' \mid y', y'')$ does not vanish if say $x' - y'$ is spacelike, but $x'' - y''$ is not.)

4. Our candidate for the propagator

Actually the starting point of our investigation was the observation (see Section 3) that the Green's functions, in spite of the writing, are not really four-point functions but rather two-segment functions and that the inhomogeneities should vanish unless the two lines intersect ('segment locality', which is the appropriate notion for a two-segment function).

We made the following ansatz:

$$I(x', x'' \mid y', y'') = \frac{1}{i} \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \exp[-ip(x(\xi) - y(\eta))]$$
$$\times I(\xi, \eta, z_x^2, z_x z_y, z_y^2),$$
$$x(\xi) = \tfrac{1}{2}(1+\xi)x' + \tfrac{1}{2}(1-\xi)x'', \qquad y(\eta) = \tfrac{1}{2}(1+\eta)y' + \tfrac{1}{2}(1-\eta)y''. \tag{14}$$

(We did not restrict the support of the variables ξ and η, but assumed the most simple segment locality, i.e. independence on p^2.)

Because of the commutation relations

$$[D_x^{(2)}, D_x^{(1)}] = 2\lambda D_x^{(2)}, \qquad [\tilde{D}_y^{(2)}, \tilde{D}_y^{(1)}] = -2\lambda \tilde{D}_y^{(2)}, \tag{15}$$

the following compatibility conditions have to be satisfied:

$$D_x^{(2)} I^{(1)} = (D_x^{(1)} + 2\lambda) I^{(2)}, \qquad \tilde{D}_y^{(2)} \tilde{I}^{(1)} = (\tilde{D}^{(1)} - 2\lambda)\tilde{I}^{(2)}. \tag{16}$$

An analysis of these integrability conditions, carried out in [4], leads to a propagator with a structure closely related to B^{-1}:

$$\begin{aligned} G_F(z_x \mid p \mid z_y) = {} & \frac{1}{32\pi^2} \exp\left[\frac{\lambda}{2}(z_x^2 - z_y^2)\right] \int_0^\infty d\tau \sqrt{\tau} \left(\frac{\lambda}{1 - e^{-\lambda\tau}}\right)^{3/2} \int_{-1}^{1} d\xi \\ & \times \exp\left[-\frac{\tau}{4}\left(m_0^2 - \frac{p^2}{4}\right) + \frac{\lambda}{1 - e^{-\lambda\tau}}(e^{-\lambda\tau/2} z_x - z_y)^2 \right. \\ & \left. - \frac{\xi^2 p^2 \tau}{16} - \frac{i}{2}\xi \sqrt{\frac{\lambda\tau}{1 - e^{-\lambda\tau}}}\, p(e^{-\lambda\tau/2} z_x - z_y)\right], \\ & (z_x - z_y)^2 < 0, \qquad \operatorname{Re} p^2 < 4m_0^2. \end{aligned} \tag{17}$$

The only difference in comparison to B^{-1} (see (12)) is the range of the ξ-integration.

In configuration space this result may be written as

$$G_F(x', x'' \mid y', y'')\left\{\frac{1}{(2\pi)^4} \exp\left[\frac{\lambda}{2}(z_x^2 - z_y^2)\right]\right\}^{-1}$$

$$= \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \eta^3 \exp\left[-\frac{m_0^2}{4}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) + \tau_1\sigma_1^2 + \tau_2\sigma_2^2\right] \tag{18a}$$

$$\begin{aligned} &= 2\int_0^\infty d\tau \tau^{-3} \eta^3 \exp\left[-\frac{m_0^2}{4}\tau + \frac{1}{\tau}(\sigma_1^2 + \sigma_2^2)\right] \\ &\times \left[\frac{-\sigma_1^2 - \sigma_2^2}{\sqrt{-\sigma_1^2}\sqrt{-\sigma_2^2}} K_1\left(\frac{2}{\tau}\sqrt{-\sigma_1^2}\sqrt{-\sigma_2^2}\right) + 2K_0\left(\frac{2}{\tau}\sqrt{-\sigma_1^2}\sqrt{-\sigma_2^2}\right)\right], \end{aligned} \tag{18b}$$

$$\sigma_{1,2} = x(\pm\xi) - y(\pm\eta),$$

$$x(\xi) = \tfrac{1}{2}(1+\xi)x' + \tfrac{1}{2}(1 - \xi' \to \xi)x'', \qquad y(\eta) = \tfrac{1}{2}(1+\eta)y' + \tfrac{1}{2}(1-\eta)y'',$$

$$\eta = \left[\frac{\lambda\tau}{1 - e^{-\lambda\tau}}\right]^{1/2} = \eta_\tau, \qquad \xi = e^{-\lambda\tau/2}\eta_\tau = \xi_\tau \quad \text{in (18b)}$$

$$\eta = \eta_\tau \mid \tau \to \frac{1}{\tau_1} + \frac{1}{\tau_2}, \qquad \xi = \xi_\tau \mid \tau \to \frac{1}{\tau_1} + \frac{1}{\tau_2} \quad \text{in (18a)}$$

K_ν are modified Bessel functions of the second kind:

$$\int_0^\infty dt t^{\nu-1} \exp\left[-\frac{\alpha}{4t} - pt\right] = 2\left(\frac{\alpha}{4p}\right)^{\nu/2} K_\nu(\sqrt{\alpha p}), \qquad \operatorname{Re} p > 0, \qquad \operatorname{Re}\alpha > 0.$$

The representation (18) exists in the region where $\sigma_{1,2}^2 < 0$ for all values of the arguments. By analytical continuation (18) determines G_F in the whole space of complex arguments, except in the singular points. This will be discussed in Subsection 4.E.

In (18a) it is immediately seen that G_F exhibits a sort of minimal coupling; it represents an average of the product of two free particle propagators along the straight lines through corresponding points.

Properties

A. G_F has the correct free limit.
B. G_F has the proper residua at the poles $p^2 = 4m_0^2 + 8\lambda n$ $(n = 0, 1, \ldots)$ in momentum space.
C. G_F has the correct nonrelativistic limit.
D. G_F has good analytical properties.
E. G_F has the proper short distance behaviour.

4.A. *The free limit*

It is immediately seen in (18a) that G_F has the correct free limit.

It is interesting to consider the free case in more details. In Appendix A an investigation of G_F^{free} in terms of the eigenfunctions of the total four-momentum may be found. The most interesting result is the decomposition of G_F^{free} in its timelike and its spacelike part. In particular this decomposition shows that the Fourier transform

$$G_F^{\text{free}}(p \mid z) = i \int d^4 w \exp(ipw) G_F^{\text{free}}(x', x'' \mid y', y'') \qquad (w = w_x - w_y,\ z = z_x - z_y)$$

does not have any cut for $p^2 \leq 0$. In other words, the spacelike part of $G_F^{\text{free}}(p \mid z)$ is an entire function in the variable p_0. This fact should not surprise because for fixed value of z the spacelike part $G_F^{\text{free},s}$ of G_F^{free} is of compact support with respect to the conjugate variable w^0 of p_0:

$$G_F^{\text{free},s}(x', x'' \mid y', y'') = \theta(\tfrac{1}{2}| z^0 | - w^0) G_F^{\text{free}}(x', x'' \mid y', y'').$$

Roughly speaking, the spacelike eigenstates of the four-momentum p propagate with respect to the relative momentum (the conjugate variable of z) rather than with respect to p.

More surprising may be the fact that a similar situation holds in the oscillator case, too. As seen in the representation (17) $G_F(z_x | \, p \, | z_y)$ does not have any cut for $p^2 \leq 0$. We will come back to this in Subsection 4.B and Section 5.

4.B. *The residua of* $G_F(z_x | \, p \, | z_y)$

For $p_n^{02} = 4m_0^2 + \vec{p}^2 + 8\lambda n$, $n = 0, 1, \ldots$:

$$\lim_{p^0 \to p_n{}^0} (p_n^2 - p^2) G_F(z_x | \, p \, | z_y) = \sum_{\sum n_i = n} \phi_{p,\vec{n}}(z_x) \tilde{\phi}_{p,\vec{n}}(z_y)^*, \tag{19}$$

$\phi_{p,\vec{n}}$, $\tilde{\phi}_{p,\vec{n}}$ as in (3), (5).

Sketch of the proof: Substitution in (17):

$$r = \exp\left(-\frac{\lambda\tau}{2}\right).$$

Consider a new function: for $z^2<0$ $(z=z_x-z_y)$, Re $p^2>0$

$$G'_F(z_x|\,p\,|z_y)=\frac{1}{8\pi^{3/2}}\frac{1}{\sqrt{p_n^2}}\exp\left[\frac{\lambda}{2}(z_x^2-z_y^2)\right]\frac{\sqrt{\lambda}i}{\sin\left(\frac{\pi}{2\lambda}\left(m_0^2-\frac{p^2}{4}\right)\right)}\oint^{\gamma}dr(1-r^2)^{-3/2}$$

$$\times(-r)^{-1+(1/2\lambda)(m_0{}^2-p^2/4)}$$

$$\times\exp\left\{\frac{\lambda}{1-r^2}\left[r\left(z_x-\frac{pz_x}{p^2}p\right)-\left(z_y-\frac{pz_y}{p^2}p\right)\right]^2\right\}.$$

The integration path runs as shown in Fig. 1.

$G'_F(p^2, pz_x, pz_y, z_x^2, z_xz_y, z_y^2)$ is meromorphic in the Re $p^2>0$ half plane and $G_F-G'_F$ is holomorphic there. q.e.d.

(19) are the expected residua, because the timelike part of the causal Green's function may be written as (see (13c, d)):

$$G^t(z_x|\,p\,|z_y)=\sum_{\vec{n}}2\pi i\,\delta(p^2-p_n^2)\phi_{p,\vec{n}}(z_x)\tilde{\phi}_{p,\vec{n}}(z_y)^*.$$

It is interesting to note that, as $G_F(z_x|\,p\,|z_y)$ is a meromorphic function in p_0, by the theorem of Mittag–Leffler G_F is determined by its residua up to an entire function:

$$G_F(z_x|\,p\,|z_y)=\sum_{\vec{n}}\frac{\phi_{p,\vec{n}}(z_x)\tilde{\phi}_{p,\vec{n}}(z_y)^*}{p_n^2-p^2}+E(z_x|\,p\,|z_y),$$

where $E(z_x|\,p\,|z_y)$ is entire in p_0.

Note however that in $E(z_x|\,p\,|z_y)$ not only the spacelike contributions are contained but also those which arise from the ambiguity in the off-shell extrapolation (compare with the free case in Appendix A).

In fact we found an appropriate off-shell extrapolation of the residua making

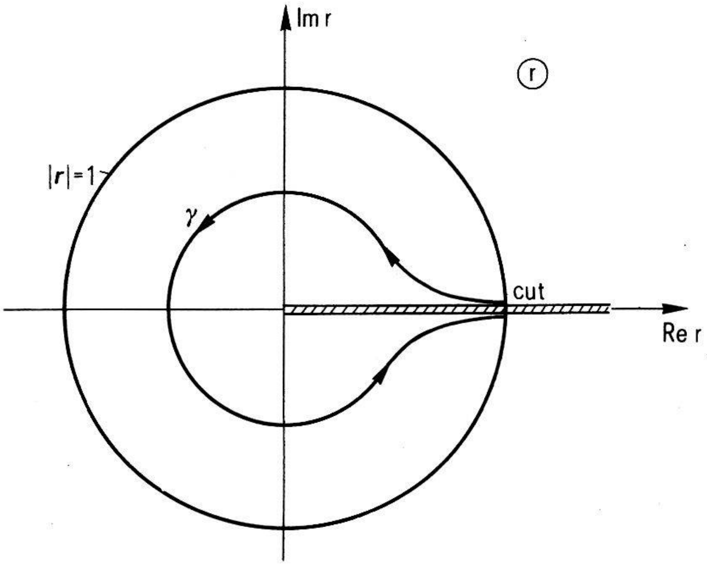

Figure 1
The integration curve γ in the r-plane which is cut along the positive real axis.

$E(z_x|\,p\,|z_y)$ zero. For $(z_x-z_y)^2<0$, Re $p^2<4m_0^2$:

$$G_F(z_x|\,p\,|z_y)=\int_{-1}^{1}d\xi\int_{-\infty}^{\infty}d\eta\,\exp\left[-\frac{i}{2}p(\xi z_x-\eta z_y)\right]\sum_{n=0}^{\infty}\frac{\mathcal{S}_n(\xi,\eta,z_x^2,z_xz_y,z_y^2)}{p_n^2-p},\tag{20}$$

$$\int_{-1}^{1}d\xi\int_{-\infty}^{\infty}d\eta\,\exp\left[-\frac{i}{2}p(\xi z_x-\eta z_y)\right]\mathcal{S}_n(\xi,\eta,z_x^2,z_xz_y,z_y^2)\,|_{p^2=p_n^2}$$

$$=\sum_{\sum n_i=n}\phi_{p,\vec{n}}(z_x)\tilde{\phi}_{p,\vec{n}}(z_y)^*.$$

This may be proven by using the generating function for products of Hermite polynomials.

Actually the result in the beginning of this subsection is a consequence of (20) or, properly speaking, it may be taken as a proof that the interchangings of summation and integrations in the proof of (20) are allowed.

Thus, in addition to the characterization of the propagator by its locality properties we have found an alternative characterization by means of the above off-shell extrapolation which is minimal in its p^2-dependence and, in this sense, canonical.

From (20) further information may be taken without any calculations: For $(z_x-z_y)^2<0$ the inhomogeneities

$$I=DG_F$$
$$(D\in\{D_x^{(1)},D_x^{(2)},\tilde{D}_y^{(1)},\tilde{D}_y^{(2)}\},\qquad I\in\{I^{(1)},I^{(2)},\tilde{I}^{(1)},\tilde{I}^{(2)}\})$$

have to vanish unless the two lines $\{x(\xi)|\,|\xi|\leq 1\}$ and $\{y(\eta)|\,|\eta|\geq 1\}$ do intersect (segment-locality).

This may be verified explicitly in our case. For $(z_x-z_y)^2<0$ the inhomogeneities are of the form

$$I(x',x''\,|\,y',y'')=\sum_{\pm}\int_0^{\infty}d\tau\delta(x(\pm\xi_\tau)-y(\pm\eta_\tau))I_\pm(\tau,z_x^2,z_xz_y,z_y^2)\tag{21}$$

with

$$\eta_\tau=\left[\frac{\lambda\tau}{1-e^{-\lambda\tau}}\right]^{1/2}\in[1,\infty),\qquad \xi_\tau=e^{-\lambda\tau/2}\eta_\tau\in[0,1].$$

They clearly satisfy segment locality, as suggested from the very beginning. (The inhomogeneities of B^{-1} do not satisfy segment locality!)

It is interesting to consider the support properties of the variables ξ_τ and η_τ in (21) in more details. That ξ_τ is concentrated on the interval [0, 1] was to be expected from the qualitative picture of the oscillator as describing an object extended along the straight line between x' and x''. More surprising is the support property of η_τ: it suggests that the adjoint oscillator describes an object concentrated on the straight line through the points y' and y'' outside the segment between them.

4.C. *The nonrelativistic limit*

The nonrelativistic Hamiltonian H for the oscillator is

$$i\partial_{w^0} = H(\vec{p}) = \frac{1}{m_0}\left\{\frac{\vec{p}^2}{4} - \Delta_z + \lambda^2\vec{z}^2 - 3\lambda\right\}.$$

$H(\vec{p})$ is selfadjoint in the usual nonrelativistic metric. A complete set of orthonormal eigenfunctions is given by: ($n = n_1 + n_2 + n_3$, $n_i = 0, 1, \ldots$ ($i = 1, 2, 3$))

$$\mathcal{S}_{\vec{n}}(\vec{z}) = \left(\frac{\lambda}{\pi}\right)^{3/4} \frac{2^{-n/2}\exp\left(-\frac{\lambda}{2}\vec{z}^2\right)}{[n_1!\, n_2!\, n_3!]^{1/2}} H_{n_1}(\sqrt{\lambda}z^1)H_{n_2}(\sqrt{\lambda}z^2)H_{n_3}(\sqrt{\lambda}z^3)$$

$$= \lim_{c\to\infty} \sqrt{m_0 c}\,\phi_{p,\vec{n}}(\vec{z})$$

(for $\phi_{p,\vec{n}}$ see (3)). They satisfy

$$H(\vec{p})\mathcal{S}_{\vec{n}}(\vec{z}) = \varepsilon_n \mathcal{S}_{\vec{n}}(\vec{z}), \qquad \varepsilon_n = \frac{\vec{p}^2}{4m_0} + \frac{2\lambda n}{m_0} = \lim_{c\to\infty}[cp_n^0 - 2m_0c^2].$$

The resolvent is given by:

$$R(\vec{z}_x|\,\varepsilon, \vec{p}\,|\vec{z}_y) = \sum_{\vec{n}} \frac{\mathcal{S}_{\vec{n}}(\vec{z}_x)\mathcal{S}_{\vec{n}}(\vec{z}_y)^*}{\varepsilon_n - \varepsilon} = \left(\frac{\lambda}{\pi}\right)^{3/2} \frac{m_0}{4}\int_0^\infty d\tau$$

$$\times (1 - e^{-\lambda\tau})^{-3/2} \exp\left\{\frac{m_0\tau}{4}\left(\varepsilon - \frac{\vec{p}^2}{4m_0}\right)\right.$$

$$\left. - \frac{\lambda}{2(1 - e^{-\lambda\tau})}[(e^{-\lambda\tau/2}\vec{z}_x - \vec{z}_y)^2 + (\vec{z}_x - e^{-\lambda\tau/2}\vec{z}_y)^2]\right\}. \tag{22}$$

The representation (22) is valid in the region $\mathrm{Re}\,\varepsilon < p^2/4m_0$. The free limit is the correct one:

$$R^{\text{free}}(t, \vec{w}, \vec{z}) = \frac{1}{(2\pi)^4 i}\int d\varepsilon \int d^3p \exp(-i\varepsilon t - ip_j w^j) \lim_{\lambda\to 0} R(\vec{z}_x|\,\varepsilon, \vec{p}\,|\vec{z}_y)$$

$$= \theta(t)\left(\frac{m_0}{2\pi i t}\right)^3 \exp\left[\frac{i m_o}{2t}(\vec{x}_1^2 + \vec{x}_2^2)\right],$$

$$\vec{x}_{1,2} = \vec{w} \pm \tfrac{1}{2}\vec{z}, \qquad \vec{z} = \vec{z}_x - \vec{z}_y.$$

It is immediately shown that our candidate for the propagator of the oscillator has the correct nonrelativistic limit:

$$R(\vec{z}_x|\,\varepsilon, \vec{p}\,|\vec{z}_y) = \lim_{c\to\infty} 4m_0^2 c G_F(\vec{z}_x|\,p\,|\vec{z}_y) \tag{23}$$

if in the expression (17) for G_F p^2 is replaced by $p^2 = 4m_0^2c^2 + 4m_0\varepsilon - \vec{p}^2$.

4.D *Analytical properties*

The representation (18b) shows that the analytical continuation of $G_F(x', x'' | y', y'')$ is well defined for any complex values of the arguments except in points, for which there exists a $\tau (0 \le \tau < \infty)$ with $\sigma_1^2 \ge 0$ and/or $\sigma_2^2 \ge 0$.

The free case tells us how to take boundary values, namely by giving $-\sigma_1^2$ and $-\sigma_2^2$ a positive imaginary part: $-\sigma_{1,2}^2 \to -\sigma_{1,2}^2 + i\varepsilon$. This may equivalently be described by giving the variable w^2 in the function G_F of the invariant arguments a negative imaginary part:

$$G_F \to G_F(w^2 - i\varepsilon, wz_x, wz_y, z_x^2, z_x z_y, z_y^2). \tag{24}$$

Of course, the prescription (24) is only motivated, not proven by considering the free case.

It would be interesting to investigate the topology of the singularities of G_F in more details; we do not intend to do this here.

In any case, it seems to be very convenient in applications to make calculations always in the Euclidean world (the representations we have given are well defined there) and to come back to the Minkowski world only at the very end.

4.E. *Short distance behaviour*

The representations we have given do not simplify if only two of the points x', x'', y', y'' coincide.

A case where the leading contribution may be extracted in a simple way is the following: for $x' - y' \to 0$, $x'' - y'' \to 0$

$$G_F(x', x'' y', y'') = \frac{1}{(2\pi)^4} \frac{1}{-(x'-y')^2 + i\varepsilon} \frac{1}{-(x''-y'')^2 + i\varepsilon} + \text{l.s.t.} \tag{25}$$

The less singular terms (l.s.t.) do not factorize. (We do not know them explicitly.) Therefore only the most singular contribution of the 'both-sided' short distance behaviour of G_F is the same as for free particles.

Of some interest is the high-momentum behaviour of

$$G_F(z_x | p | z_y) = G_F(p^2, pz_x, pz_y, z_x^2, z_x z_y, z_y^2).$$

It may be shown that for $z^2 < 0$, $|\arg(-p^2)| < \pi$, $|p^2| \to \infty$:

$$\begin{aligned} G_F(p^2, pz_x, pz_y, z_x^2, z_x z_y, z_y^2) &\sim \frac{1}{4\pi^2(-p^2)} \exp\left[\frac{\lambda}{2}(z_x^2 - z_y^2)\right] \int_0^\infty d\tau \\ &\times \frac{1}{\sqrt{\tau}} \left(\frac{\lambda}{1-e^{-\lambda\tau}}\right)^{3/2} \exp\left[-\frac{\tau}{4} m_0^2 + \frac{\lambda}{1-e^{-\lambda\tau}} (e^{-\lambda\tau/2} z_x - z_y)^2\right] \\ &\times \left\{ \exp\left[-\frac{i}{2} \sqrt{\frac{\lambda\tau}{1-e^{-\lambda\tau}}}\, p(e^{-\lambda\tau/2} z_x - z_y)\right] \right. \\ &\left. + \exp\left[\frac{i}{2} \sqrt{\frac{\lambda\tau}{1-e^{-\lambda\tau}}}\, p(e^{-\lambda\tau/2} z_x - z_y)\right] \right\} \end{aligned} \tag{26}$$

4.F. *Problems in the search of* G_F^{-1}

We did not succeed in obtaining an explicit inverse G_F^{-1} of G_F. At first glance one might think that G_F^{-1} should be equal to the Bethe-Salpeter operator B (see

(10)). However:

$$B * G_F = 1 - R, \qquad G_F * B = 1 - \tilde{R}. \tag{27}$$

The operators R and $\tilde{R}$ may be calculated explicitly (their integral kernels satisfy the same line-segment locality as the inhomogeneities). In a formal sense there naturally arises a perturbation series for G_F^{-1}:

$$\begin{aligned} G_F^{-1} &= (1-R)^{-1} * B = B + R * B + R * R * B + \cdots \\ G_F^{-1} &= B * (1-\tilde{R})^{-1} = B + B * \tilde{R} + B * \tilde{R} * \tilde{R} + \cdots \end{aligned} \tag{28}$$

Of course, in order to give (28) a meaning, the convergence of the series should be proven.

5. Conclusion

There does not seem to exist a canonical definition for a two-segment function. We therefore have not succeeded in characterizing the propagator of the relativistic FKR oscillator in a unique way. In fact we have found two functions describing propagation which are connected with the oscillator: G_F (see Section 4) and B^{-1} (see Section 2).

G_F is the better candidate because of the following points:

(1) Its locality properties are maximal (see beginning of Section 4).
(2) It is the only one which is closely related to the causal Green's function (see Subsection 4.B and Appendix B).
(3) It is characterizable by a simple off-shell extrapolation of the residua (see Subsection 4.B) which is minimal in its p^2-dependence and which leads to a unique expression in the Euclidean region.

On the contrary B^{-1} has ghosts in its residua and inhomogeneities not satisfying segment locality. The advantage of B^{-1} is that the inverse is known explicity (by definition), which is not the case for G_F.

As an application we give in Appendix B a characterization of the causal Green's function in terms of the analytic continuation of the function G_F. Moreover we are able to separate the timelike and spacelike contributions explicity.

The generalization of these results to spin 1/2 systems is not difficult.

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Appendix A: The free two-particle propagator

$$G_F^{\text{free}}(x', x'' \mid y', y'') = G_F^{\text{free}}(x'-y', x''-y''),$$

$$G_F^{\text{free}}(x_1, x_2) = \frac{1}{i}\Delta_F(x_1; m_0^2)\frac{1}{i}\Delta_F(x_2; m_0^2), \qquad \Delta_F(x; m_0^2) = \int \frac{d^4p}{(2\pi)^4}\frac{\exp(-ipx)}{m_0^2 - p^2 - i\varepsilon}.$$

We intend to investigate G_F^{free} with regard to the individual contributions of the timelike and spacelike eigenfunctions of the four-momentum. These functions are given by (in terms of null plane variables [2]):

$$\phi_{p,\boldsymbol{\kappa}}(x', x'') = \exp(-ipw)\phi_{\boldsymbol{\kappa}}(z), \qquad \phi_{\boldsymbol{\kappa}}(z) = \exp(-i\kappa z),$$

$$w = \tfrac{1}{2}(x', x''), \qquad z = x' - x'', \qquad \kappa z = \kappa_- z^- + \kappa_L z_L + \kappa_\tau 2^\tau, \qquad z_L = \mathbf{pz} = p_+ z^+ + p_\tau z^\tau,$$

$$p_+\kappa_- = p_\tau \kappa^\tau + \tfrac{1}{2}p_T^2\kappa_L - \tfrac{1}{2}m^2(\boldsymbol{\kappa})\kappa_L, \qquad m^2(\boldsymbol{\kappa}) = (\tfrac{1}{4} - \kappa_L^2)^{-1}(m_0^2 + \kappa_T^2).$$

Now it is easy to see that

$$G_F^{\text{free}}(x_1, x_2) = \frac{1}{(2\pi)^4 i}\int d^4p \exp(-ipw) G_F^{\text{free}}(p \mid z), \qquad w = w_x - w_y, \quad z = z_x - z_y,$$

$$G_F^{\text{free}}(p \mid z) = \frac{1}{(2\pi)^3}\int \frac{d^2\kappa_T d\kappa_L}{2(\frac{1}{4} - \kappa_L^2)}\frac{J(p, z, \boldsymbol{\kappa})}{m^2(\boldsymbol{\kappa}) - p^2 - i\varepsilon(m^2)}, \qquad \varepsilon(m^2) = \begin{cases} +1, & m^2(\boldsymbol{\kappa}) > 0 \\ -1, & m^2(\boldsymbol{\kappa}) \le 0 \end{cases}$$

where J may be decomposed into the separate contributions of the timelike and of the spacelike spectrum: Let be

$$\phi_{\boldsymbol{\kappa}}^t(z) = \theta(\tfrac{1}{4} - \kappa_L^2)\phi_{\boldsymbol{\kappa}}(z), \qquad \phi_{\boldsymbol{\kappa}}^{s,\pm}(z) = \theta(\pm\kappa_L - \tfrac{1}{2})\phi_{\boldsymbol{\kappa}}(z).$$

Then: $J = J^t + J^{s,+} + J^{s,-}$,

$$J^t(p, z, \boldsymbol{\kappa}) = \phi_{\boldsymbol{\kappa}}^t(z_x)\phi_{\boldsymbol{\kappa}}^t(z_y)^*\left\{\theta(z^-)\exp\left[\frac{i}{4|p_+|}(p^2 - m^2)z^-\right] + \theta(-z^-)\exp\left[-\frac{i}{4|p_+|}(p^2 - m^2)z^-\right]\right\},$$

$$J^{s,\pm}(p, z, \boldsymbol{\kappa}) = \pm\phi_{\boldsymbol{\kappa}}^{s,+}(z_k)\phi_{\boldsymbol{\kappa}}^{s,\pm}(z_y)^*\left\{\exp\left[\frac{i}{4p_+}(p^2 - m^2)z^-\right] - \exp\left[-\frac{i}{4p_+}(p^2 - m^2)z^-\right]\right\}$$
$$\times\{\theta(z^-)\theta(\pm p_+) + \theta(-z^-)\theta(\mp p_+)\}.$$

Therefore we may look at $J^t(p, z, \boldsymbol{\kappa})$ as an extension of $\phi^t(z_x)\phi^t(z_y)^*$ away from the mass shell $p^2 = m^2(\boldsymbol{\kappa})$. Further:

$$\lim_{p^2 \to m^2} J^{s,\pm} = 0$$

such that $G_F^{\text{free}}(p \mid z)$ does not have any cut for $p^2 \leq 0$. Explicitly:

$$G_F^{\text{free}}(p \mid z) = \frac{1}{8\pi^2} \int_{4m_0^2}^{\infty} dt \frac{1}{t - p^2 - i\varepsilon}$$

$$\times \frac{1}{\sqrt{(pz)^2 - tz^2}} \sin\left[\frac{1}{2}\sqrt{\frac{t - 4m_0^2}{t}}\sqrt{(pz)^2 - tz^2}\right], \qquad z^2 < 0.$$

Appendix B: The causal Green's function

For a characterization of the causal Green's function G we refer to (13) in Section 3. It is not difficult to get a characterization of the timelike part G^t of G in terms of the analytic continuation of G_F. The observation is based on the fact that (13d) may be rewritten as (because of

$$(p_n^2 - p^2 - i\varepsilon)^{-1} - (p_n^2 - p^2 + i\varepsilon)^{-1} = 2\pi i \delta(p^2 - p_n^2))$$

$$G^t = G_F^+ - G_F^-$$

$$G_F^{\pm}(x', x'' \mid y', y'') = \frac{1}{(2\pi)^4 i} \int d^4p \exp(-ipw) G_F(z_x \mid p \mid z_y) \mid \operatorname{Im} p^2 = \pm\varepsilon$$

$G_F(z_x) \mid p \mid z_y)$ according to (17).

Of course:

$$D^{(1)}_{(2)}[G_F^+ - G_F^-] = 0 = \tilde{D}_y^{(1)}{}_{(2)}[G_F^+ - G_F^-].$$

Note that the analytical continuation of G_F^+ is not to be identified with G_F: the two functions take different boundary values. This has to do with the fact that in fixing the boundary values of G_F the spacelike eigenfunctions play a role. In contrast to this the analytical continuation of G_F^+ is everywhere fixed by the substitution $p_0^2 \to p_0^2 + i\varepsilon$ in $G_F(z_x \mid p \mid z_y)$.

We now introduce four functions

Definition. $G_{(a)(b)}(x', x'' \mid y', y'')(a, b \in \{+, -\})$ is the analytical continuation of the representation (18b) with the following convention for taking boundary values: give the null components $-\sigma_1^0$ and $-\sigma_2^0$ the imaginary part $ai\varepsilon$ and $bi\varepsilon$ respectively. Equivalent prescription: replace in $G_F(w^2, wz_x, wz_y, z_x^2, z_xz_y, z_y^2)$ w^0 by $w^0 \mp i\varepsilon$ for $(a, b) = \genfrac{}{}{0pt}{}{(+,+)}{(-,-)}$ and z_x^0 by $z_x^0 \mp i\varepsilon$ (or z_y^0 by $z_y^0 \pm i\varepsilon$) for $(a, b) = \genfrac{}{}{0pt}{}{(+,-)}{(-,+)}$.

Then it may be easily verified that

$$G^t = G_{(+)(+)} + G_{(-)(-)}.$$

Our main result here is motivated by the free case: The full causal Green's function is

$$G = G_{(+)(+)} - G_{(+)(-)} - G_{(-)(+)} + G_{(-)(-)}.$$

It follows that the spacelike part G^s is:

$$G^s = -G_{(+)(-)} - G_{(-)(+)}.$$

Proof. [4].