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Helvetica Physica Acta, Vol. 53 (1980), Birkhäuser Verlag, Basel

On a systematic search for integrals of the motion*)

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(1. IV. 1980; rev. 23. V. 1980)

Abstract. Noether's theorem stripped of the usual unnecessary complications of explicit transformations of the independent coordinates is proved in a constructive way. This constructive proof makes Noether's theorem more accessible for applications. Integrals of the motion or local conservation equations, also those corresponding to non-manifest symmetries, can be found easily.

As an application, without explicitly using the Galilei symmetry group, the 10 linearly independent Eulerian integrals for a system of N particles with central two-body interaction are derived. Without explicitly using the so-called Jacobi-Schrödinger group, the 12 linearly independent integrals of the motion are found for the special case that the interaction potentials are inversely proportional to the squares of the particle distances.

In an analogous way the n-dimensional isotropic harmonic oscillator is discussed and for the n-dimensional Kepler problem, the conservation of the Runge–Lenz vector is derived.

1. Noether's theorem simplified

In recent years, a lot of publications are devoted to symmetry transformations and integrals of the motion. A great deal of them are based on Noether's theorem (Noether, 1918), which is clearly represented by Hill (1951). Most treatments are unnecessarily complicated. The reason is that mostly infinitesimal transformations (variations) of the independent variables as well as variations of the field variables are considered. It can be shown (Steudel, 1966) that as far as the conservation equations resulting from Noether's theorem are concerned, such infinitesimal transformations are equivalent to variations of the field variables alone.

Let a physical system be fully described by say

$$\varphi_i(x), \qquad i = 1, 2, 3, \ldots m$$

where

$$x = \{x_0, x_1, x_2, \ldots, x_{n-1}\},$$

and

$$x_0 = t$$

represents the time.

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*) Work supported by the Swiss National Science Foundation.

Further, we assume the system to have a Lagrangian density $\mathscr{L}$ which is a function of x, the m-tuple $\varphi=\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$, the mn-tuple

$$\varphi_x=\{\varphi_{i,\alpha} \mid \alpha=0, 1, \ldots, n-1;\ i=1, 2, \ldots, m\},$$

the mn^2-tuple

$$\varphi_{xx}=\{\varphi_{i,\alpha\beta} \mid \alpha, \beta=0, 1, \ldots, n-1;\ i=1, 2, \ldots, m\}, \text{ etc.}$$

Here $\varphi_{i,\alpha}$ stands for $\partial\varphi_i/\partial x_\alpha$, etc. We write

$$\mathscr{L}=\mathscr{L}(x, \varphi, \varphi_x, \varphi_{xx}, \ldots) \tag{1.1}$$

The action or action functional then is

$$A=\int \mathscr{L}(x, \varphi, \varphi_x, \varphi_{xx}, \ldots)\, d^n x, \tag{1.2}$$

where the integration is done over some fixed region G_n in the n-dimensional x-space. The equations of motion now follow from Hamilton's principle or the action principle

$$\delta A \doteq 0, \tag{1.3}$$

and can be written as

$$\frac{\partial\mathscr{L}}{\partial\varphi_i}-d_\alpha\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}+d_\alpha d_\beta\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}\cdots \doteq 0, \tag{1.4}$$

where a summation over repeated Greek indices from 0 through $n-1$ is implied and d_α denotes differentiation with respect to x_α at constant x_β $(\alpha \neq \beta)$ and varying $\varphi(x)$.

Here and in the rest of this paper, the weak identity $\doteq$ is used to indicate identities which only hold for solutions $\varphi_i(x)$ of the equations of motion.

In the study of Noether's theorem, one usually considers infinitesimal transformations (variations) of the independent variables x and of the field variables $\varphi_i(x)$:

$$x_\alpha \to x'_\alpha = x_\alpha+\delta^* x_\alpha \tag{1.5}$$

$$\varphi_i(x) \to \varphi'_i(x')=\varphi_i(x)+\delta^*\varphi_i(x) \tag{1.6}$$

and investigates the variation of the action under these infinitesimal transformations. If this variation δA is identical zero or can be written in the form

$$\delta A=\int d^n x\, d_\alpha\psi_\alpha, \tag{1.7}$$

Noether's theorem yields an equation

$$d_\alpha\chi_\alpha \doteq 0. \tag{1.8}$$

which has the form of a *continuity equation.* Here the χ_α's are functions of x, φ, φ_x, φ_{xx}, etc. The quantity χ_0 is called the *density*, the quantities χ_α $(\alpha = 1, 2, \ldots, n-1)$ constitute the components of the *flux*-vector or *current*-density vector.

Integrating (1.8) over some region G_{n-1} in the subspace spanned by the variables $x_1, \ldots, x_{n-1}$, one obtains

$$d/dt \int \chi_0 \, d^{n-1}x \doteq -\int \sum_{\alpha=1}^{n-1} d_\alpha \chi_\alpha \, d^{n-1}x. \tag{1.9}$$

With Green's first theorem one can transform the right-hand side of (1.9) into an integral over the boundary of G_{n-1}. With suitable boundary conditions imposed on φ, the right-hand side of (1.9) will vanish and one obtains

$$d/dt \int \chi_0 \, d^{n-1}x \doteq 0. \tag{1.10}$$

Then $\int \chi_0 \, d^{n-1}x$ is called an *integral of the motion* or *first integral* of the system (1.4). In cases where (1.10) expresses a physical law, such as the energy conservation law, equation (1.10) is termed a *global conservation law*, equation (1.8) being the corresponding *local conservation law*. To find explicit expressions for χ_α, it is necessary to investigate the variations of A precisely. In general, this is not very easy because one has to consider changes of the integrand as well as of the integration bounds. Steudel has proved that if a local conservation equation (1.8) can be derived with the variations (1.5)–(1.6), the same conservation equation can also be derived with an equivalent variation of the field variables alone (Steudel, 1966)

$$\varphi_i(x) \to \varphi_i'(x) = \varphi_i(x) + \delta\varphi_i(x), \tag{1.11}$$

where (see eqs. (1.5), (1.6))

$$\begin{aligned} \delta\varphi_i(x) &= \varphi_i'(x) - \varphi_i(x) = \varphi_i'(x' - \delta^* x) - \varphi_i(x) \\ &= \varphi_i'(x') - \varphi_{i,\alpha}' \delta^* x_\alpha - \varphi_i(x), \end{aligned} \tag{1.12}$$

or

$$\delta\varphi_i(x) = \delta^* \varphi_i(x) - \varphi_{i,\alpha} \delta^* x_\alpha \tag{1.13}$$

This means that in the search for local conservation equations, variations of the field variables only may be considered.

This idea, which is also found in von der Linden 1963, and mentioned in Havas 1978, is rarely used in the literature. Nevertheless, Steudel's result simplifies the construction of local conservation equations considerably. Especially, this is the case if we have to do with so-called hidden symmetries (non-manifest symmetries). The simplification mainly comes from the fact that, as we will see, it is sufficient to discuss variations of the Lagrangian density instead of the action.

Let us consider a variation

$$\varphi_i(x) \to \varphi_i'(x) = \varphi_i(x) + \delta\varphi_i(x). \tag{1.14}$$

Then a first-order Taylor expansion of $\mathcal{L}$ gives

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i} \delta\varphi_i + \frac{\partial\mathcal{L}}{\partial\varphi_{i,\alpha}} \delta\varphi_{i,\alpha} + \frac{\partial\mathcal{L}}{\partial\varphi_{i,\alpha\beta}} \delta\varphi_{i,\alpha\beta} + \cdots. \tag{1.15}$$

(Summation over $i = 1, 2, \ldots, m$ and $\alpha, \beta = 0, 1, 2 \ldots n-1$ is implied.) With

$$\delta\varphi_{i,\alpha} = (\delta\varphi_i)_{,\alpha} = d_\alpha \delta\varphi_i$$

and the chain-rule for differentiation, we also may write

$$\delta\mathscr{L}=\delta\varphi_i\left(\frac{\partial\mathscr{L}}{\partial\varphi_i}-d_\alpha\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}+d_\alpha d_\beta\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}+\cdots\right)$$
$$+d_\alpha\left(\delta\varphi_i\left(\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}-d_\beta\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}\right)+(\delta\varphi_i)_{,\beta}\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}+\cdots\right). \tag{1.16}$$

In the first term, as a factor we recognize the equation of motion (1.4). Therefore we have the weak identity

$$\delta L\doteq d_\alpha\left(\delta\varphi_i\left(\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}-d_\beta\frac{\partial\mathscr{L}}{\varphi_{i,\alpha\beta}}\right)+(\delta\varphi_i)_{,\beta}\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}+\cdots\right) \tag{1.17}$$

If now by guessing or by systematic investigation, one finds that for a special choice of the variation $\delta\varphi_i$ as function of φ and its derivatives, one has the strong identity

$$\delta L=d_\alpha\psi_\alpha, \tag{1.18}$$

from (1.17) and (1.18) follows

$$d_\alpha\left(\psi_\alpha-\delta\varphi_i\left(\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}-d_\beta\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}\right)-(\delta\varphi_i)_{,\beta}\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}+\cdots\right)\doteq 0, \tag{1.19}$$

which has the structure of a local conservation law.

Infinitesimal transformations (1.14) which yield an expression of the form (1.18) are termed *Noetherian variations.* A simplified version of Noether's theorem now is:

Every Noetherian variation yields a local conservation equation.

In the applications of Noether's theorem it is essential to make a good choice for the Noetherian variations. In most literature, those applications are restricted to manifest symmetries. For instance if the Lagrangian density $\mathscr{L}$ does not depend on the field φ_i explicitly, i.e. if

$$\frac{\partial\mathscr{L}}{\partial\varphi_i}=0, \tag{1.20}$$

the infinitesimal shift of the field φ_i

$$\delta\varphi_i=\varepsilon_i\rightarrow\delta\mathscr{L}=0, \tag{1.21}$$

is a Noetherian variation. With (1.17) it yields

$$d_\alpha\left(\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha}}-d_\beta\frac{\partial\mathscr{L}}{\partial\varphi_{i,\alpha\beta}}+\cdots\right)\doteq 0, \tag{1.22}$$

which is the equation of motion written in the form of a local conservation equation.

Another manifest symmetry is met in those cases where $\mathscr{L}$ does not contain explicitly the independent variable x_α, i.e. when

$$\frac{\partial\mathscr{L}}{\partial x_\alpha}=0. \tag{1.23}$$

Then the infinitesimal shift $\delta x_\alpha = \varepsilon_\alpha$ leaves the action invariant. With (1.11) we find the corresponding Noetherian variation

$$\delta\varphi_i = \varepsilon_\alpha d_\alpha \varphi_i = \varepsilon_\alpha \varphi_{i,\alpha} \tag{1.24}$$

that yields

$$\delta\mathcal{L} = \varepsilon_\alpha d_\alpha \mathcal{L}. \tag{1.25}$$

Then (1.24), (1.25) together with (1.17) give

$$d_\gamma s_{\gamma\alpha} \doteq 0, \tag{1.26}$$

where

$$s_{\gamma\alpha} = \varphi_{i,\alpha}\left(\frac{\partial\mathcal{L}}{\partial\varphi_{i,\gamma}} - d_\beta \frac{\partial\mathcal{L}}{\partial\varphi_{i,\gamma\beta}}\right) + \varphi_{i,\alpha\beta}\frac{\partial\mathcal{L}}{\partial\varphi_{i,\gamma\beta}} + \cdots - \delta_{\alpha\gamma}\mathcal{L}, \tag{1.27}$$

and $\delta_{\alpha\gamma}$ is the usual Kronecker symbol

$$\delta_{\alpha\gamma} = 0\ (\alpha \neq \gamma); \qquad \delta_{\alpha\gamma} = 1\ (\alpha = \gamma). \tag{1.28}$$

If $\mathcal{L}$ only depends on derivatives up to the first order, (1.27) reads

$$s_{\gamma\alpha} = \varphi_{i,\alpha}\frac{\partial\mathcal{L}}{\partial\varphi_{i,\gamma}} - \delta_{\alpha\gamma}\mathcal{L},$$

which in many physical situations is the usual expression for the energy-momentum tensor. In those cases equation (1.26) expresses the conservation of energy and of momentum.

We stress here that it is not necessary to have an *a priori* knowledge of the Noetherian variations or symmetries of the system. With a suitable ansatz for the variations $\delta\varphi_i$ and using equation (1.15) one can systematically figure out conditions for $\delta\varphi_i$ being a Noetherian variation. As an example assume that the Lagrangian density (1.1) only depends on x, φ and φ_x, and that it is a homogeneous polynomial in φ and φ_x of degree 2. Then the equation of motion (1.4) is a linear second (or first) order partial differential equation and all coefficients are linear polynomials in φ and φ_x. Taking now as an ansatz

$$\delta\varphi_i = \alpha_{ij}(x)\varphi_j + \beta_{ij\alpha}(x)\varphi_{j,\alpha},$$

and substituting it in (1.15) one can find out conditions on $\alpha_{ij}(x)$ and $\beta_{ij\alpha}(x)$ for $\delta\varphi_i$ to be Noetherian. The density and flux density of the corresponding local conservation equation then are polynomial expressions in φ and φ_x of second degree.

In the following sections, we will give some illustrations for the use of the method described above. These illustrations are chosen out of the field of classical mechanics. Examples for continuous systems can be found elsewhere (Kobussen, 1973 and 1976).

2. Integrals of the motion of the N-body problem

Consider N particles in an Euclidean space $\mathbf{R}_3$. The particles will be numbered from $K=1$ through $K=N$. The coordinates of the Kth particle are denoted by $\mathbf{r}_K=(x_K, y_K, z_K)$, its mass being m_K. The total kinetic energy of the system is

$$T=\frac{1}{2}\sum_{K=1}^{N} m_K\dot{\mathbf{r}}_K\cdot\dot{\mathbf{r}}_K, \tag{2.1}$$

where the dot $\cdot$ denotes the usual inner product in $\mathbf{R}_3$. We assume the potential energy U only depends on the mutual distances of the particles:

$$U=U(r_{KL}), \tag{2.2}$$

where

$$r_{KL}=|\mathbf{r}_K-\mathbf{r}_L|=((\mathbf{r}_K-\mathbf{r}_L)\cdot(\mathbf{r}_K-\mathbf{r}_L))^{1/2} \tag{2.3}$$

is the mutual distance of the particles K and L.

The dynamics of the system now follows from the Lagrangian

$$\mathscr{L}=T-U=\frac{1}{2}\sum_{K=1}^{N} m_K\dot{\mathbf{r}}_K\cdot\dot{\mathbf{r}}_K-U. \tag{2.4}$$

A first-order Taylor expansion of (2.4) yields

$$\delta\mathscr{L}=d/dt\sum_{K=1}^{N} m_K\dot{\mathbf{r}}_K\cdot\delta\mathbf{r}_K-\sum_{K=1}^{N} m_K\ddot{\mathbf{r}}_K\cdot\delta\mathbf{r}_K-\sum_{\substack{K,L=1\\K\neq L}}^{N}\sum 2\frac{\partial U}{\partial r_{KL}}\,r_{KL}^{-1}(\mathbf{r}_K-\mathbf{r}_L)\cdot\delta\mathbf{r}_K. \tag{2.5}$$

Then, the equation of motion is

$$m_K\ddot{\mathbf{r}}_K+\sum_{\substack{L=1\\K\neq L}}^{N} 2\frac{\partial U}{\partial r_{KL}}\,r_{KL}^{-1}(\mathbf{r}_K-\mathbf{r}_L)\doteq 0, \tag{2.6}$$

and from (2.5) remains

$$\delta\mathscr{L}\doteq d/dt\sum_{K=1}^{N} m_K\dot{\mathbf{r}}_K\cdot\delta\mathbf{r}_K. \tag{2.7}$$

Now we will look for integrals of the motion of which the leading term, i.e. the term that comes from the right-hand side of (2.7), is linear in the coordinates $\mathbf{r}_K$ and velocities $\dot{\mathbf{r}}_K$. Therefore, we introduce the ansatz

$$\delta\mathbf{r}_K=\mathbf{a}_K, \tag{2.8}$$

$\mathbf{a}_K$ being an infinitesimal vector, possibly explicitly depending on t, that does not depend on $\mathbf{r}_L$ nor on $\dot{\mathbf{r}}_L$ $(L=1, 2, \ldots N)$. With (2.4) we find

$$\delta\mathscr{L}=\sum_{K=1}^{N} m_K\dot{\mathbf{r}}_K\cdot\dot{\mathbf{a}}_K-\sum_{\substack{K,L=1\\K\neq L}}^{N}\sum 2\frac{\partial U}{\partial r_{KL}}\,r_{KL}^{-1}(\mathbf{r}_K-\mathbf{r}_L)\cdot\mathbf{a}_K.$$

The second term of this expression vanishes if one takes

$$\mathbf{a}_K = \mathbf{a} \tag{2.10}$$

for all $K = 1, \ldots, N$ identical. Then we have

$$\begin{aligned}\delta\mathcal{L} &= \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{a}} \\ &= d/dt \sum_{K=1}^{N} m_K \mathbf{r}_K \cdot \dot{\mathbf{a}} - \sum_{K=1}^{N} m_K \mathbf{r}_K \cdot \ddot{\mathbf{a}}.\end{aligned} \tag{2.11}$$

Thus, $\delta\mathcal{L}$ is a total time derivative

$$\delta\mathcal{L} = d/dt \sum_{K=1}^{N} m_K \mathbf{r}_K \cdot \dot{\mathbf{a}}, \tag{2.12}$$

if a satisfies

$$\ddot{\mathbf{a}} = 0 \tag{2.13}$$

or

$$\mathbf{a} = \boldsymbol{\alpha} + t\boldsymbol{\varepsilon} \tag{2.14}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ being arbitrary constant infinitesimal vectors. With (2.8), (2.10) from (2.7) follows

$$\delta\mathcal{L} \doteq d/dt \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot \mathbf{a} \tag{2.15}$$

The equations (2.12) and (2.15) together yield

$$d/dt \sum (\mathbf{a} \cdot m_K \dot{\mathbf{r}}_K - \dot{\mathbf{a}} \cdot m_K \mathbf{r}_K) \doteq 0. \tag{2.16}$$

Taking either $\boldsymbol{\alpha}$ or $\boldsymbol{\varepsilon}$ zero, instead of (2.16), with (2.14) we have the two vectorial conservation equations

$$d/dt\, \mathbf{P} \doteq 0; \qquad \mathbf{P} = \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K, \tag{2.17}$$

$$d/dt\, \mathbf{G} \doteq 0; \qquad \mathbf{G} = \sum_{K=1}^{N} m_K \mathbf{r}_K - t\mathbf{P}. \tag{2.18}$$

Next we will look for integrals of the motion of which the leading term is quadratic in the coordinates and the velocities. In this search, we also restrict ourselves to those integrals of which the leading term can be split up in the sum of one-particle terms. This leads to the ansatz

$$\delta\mathbf{r}_K = A_K \mathbf{r}_K + B_K \dot{\mathbf{r}}_K, \tag{2.19}$$

where A_K and B_K are possibly time dependent infinitesimal 3×3 matrices. For this ansatz, we investigate the variation δT of the kinetic energy T first. From

(2.1) and (2.19) one deduces

$$\delta T = \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot (\dot{A}_K \mathbf{r}_K + (A_K + \dot{B}_K)\dot{\mathbf{r}}_K + B_K \ddot{\mathbf{r}}_K)$$

$$= d/dt \sum_{K=1}^{N} m_K (\tfrac{1}{2}\mathbf{r}_K \cdot \dot{A}_K \mathbf{r}_K + \tfrac{1}{2}\dot{\mathbf{r}}_K \cdot B_K \dot{\mathbf{r}}_K) \tag{2.20}$$

$$+ \sum_{K=1}^{N} m_K (\tfrac{1}{2}\dot{\mathbf{r}}_K \cdot (\dot{A}_K - \dot{A}_K^T)\mathbf{r}_K + \tfrac{1}{2}\dot{\mathbf{r}}_K \cdot (B_K - B_K^T)\ddot{\mathbf{r}}_K$$
$$- \tfrac{1}{2}\mathbf{r}_K \cdot \ddot{A}_K \mathbf{r}_K + \dot{\mathbf{r}}_K \cdot (A_K + \tfrac{1}{2}\dot{B}_K)\dot{\mathbf{r}}_K). \tag{2.21}$$

Thus, δT is the total derivative

$$\delta T = d/dt \sum_{K=1}^{N} m_K (\tfrac{1}{2}\mathbf{r}_K \cdot \dot{A}_K \mathbf{r}_K + \tfrac{1}{2}\dot{\mathbf{r}}_K \cdot B_K \dot{\mathbf{r}}_K) \tag{2.22}$$

if

$$\dot{A}_K = \dot{A}_K^T, \qquad \ddot{A}_K = 0, \tag{2.23}$$

$$B_K = B_K^T, \tag{2.24}$$

and

$$A_K + \tfrac{1}{2}\dot{B}_K = -(A_K + \tfrac{1}{2}\dot{B}_K)^T. \tag{2.25}$$

The conditions (2.23)–(2.25) are satisfied if one takes

$$A_K = S_K + P_K + Q_K t, \tag{2.26}$$

$$B_K = R_K - 2P_K t - Q_K t^2, \tag{2.27}$$

where P_K, Q_K, R_K, S_K are constant infinitesimal matrices satisfying the symmetry conditions

$$S_K = -S_K^T \tag{2.28}$$

$$P_K = P_K^T; \qquad Q_K = Q_K^T; \qquad R_K = R_K^T \tag{2.29}$$

With (2.26)–(2.29) we then have

$$\delta T = d/dt \sum_{K=1}^{N} (\tfrac{1}{2}\mathbf{r}_K \cdot Q_K \mathbf{r}_K + \tfrac{1}{2}\dot{\mathbf{r}}_K \cdot (R_K - 2P_K t - Q_K t^2)\dot{\mathbf{r}}_K), \tag{2.30}$$

the variation (2.19) being

$$\delta \mathbf{r}_K = A_K \mathbf{r}_K + B_K \dot{\mathbf{r}}_K = (S_K + P_K + Q_K t)\mathbf{r}_K + (R_K - 2P_K t - Q_K t^2)\dot{\mathbf{r}}_K. \tag{2.31}$$

Our next step is the investigation of the variation δU of the potential energy (2.2). One easily verifies

$$\delta U = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum \frac{\partial U}{\partial r_{KL}} r_{KL}^{-1}(\mathbf{r}_K - \mathbf{r}_L) \cdot (\delta \mathbf{r}_K - \delta \mathbf{r}_L). \tag{2.32}$$

In general, δU will neither vanish nor be a total time derivative. Therefore, we will simplify the expression (2.32) by assuming that the matrices P_K, Q_K, R_K and

S_K are identical for all K. In that case, we can drop the subscript K in A_K, B_K, P_K, Q_K, R_K and S_K. Equation (2.32) then reads

$$\delta U = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum \frac{\partial U}{\partial r_{KL}} r_{KL}^{-1}((\mathbf{r}_K-\mathbf{r}_L)\cdot A(\mathbf{r}_K-\mathbf{r}_L)+(\mathbf{r}_K-\mathbf{r}_L)\cdot B(\dot{\mathbf{r}}_K-\dot{\mathbf{r}}_L)) \tag{2.33}$$

If one chooses $A=-A^T$, $B=0$, i.e.

$$S_K = S = -S^T \tag{2.34}$$

$$P_K = Q_K = R_K = 0, \tag{2.35}$$

one easily sees that

$$\delta U = 0. \tag{2.36}$$

Thus, we have found the Noetherian variation

$$\delta \mathbf{r}_K = S\mathbf{r}_K, \qquad S=-S^T, \tag{2.37}$$

yielding (see (2.36) and (2.22))

$$\delta \mathscr{L} = 0. \tag{2.38}$$

On the other hand, with (2.7) we have

$$\delta\mathscr{L} \doteq d/dt \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot S\mathbf{r}_K. \tag{2.39}$$

For arbitrary constant $S=-S^T$, we then have

$$d/dt \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot S\mathbf{r}_K \doteq 0. \tag{2.40}$$

In a three-dimensional Euclidean space, there are three linearly independent skew-symmetric 3×3 matrices. Thus, there are three linearly independent conservation equations of the form (2.40). Taken together, they can be written as one vectorial conservation equation

$$d/dt\, \mathbf{L} \doteq 0; \qquad \mathbf{L} = \sum_{K=1}^{N} m_K \mathbf{r}_K \wedge \dot{\mathbf{r}}_K, \tag{2.41}$$

where $\wedge$ stands for the usual vector (cross) product. To investigate the possibility that δU is a total time derivative, we now assume $S=S_K=0$. With (2.26), (2.27), we then have

$$B_K = B = R - 2Pt - Qt^2; \qquad A_K = A = -\tfrac{1}{2}\dot{B}, \tag{2.42}$$

and (2.33) reads

$$\delta U = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum \frac{\partial U}{\partial r_{KL}} r_{KL}^{-1}((\mathbf{r}_K-\mathbf{r}_L)\cdot B(\dot{\mathbf{r}}_K-\dot{\mathbf{r}}_L)-\tfrac{1}{2}(\mathbf{r}_K-\mathbf{r}_L)\cdot \dot{B}(\mathbf{r}_K-\mathbf{r}_L)) \tag{2.43}$$

For

$$A = 0, \qquad B = -\tau I \tag{2.44}$$

equation (2.43) reduces to

$$\delta U = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum -\tau \frac{\partial U}{\partial r_{KL}} r_{KL}^{-1}(\mathbf{r}_K - \mathbf{r}_L)\cdot(\dot{\mathbf{r}}_K - \dot{\mathbf{r}}_L)$$

$$= \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum -\tau \frac{\partial U}{\partial r_{KL}} \frac{d}{dt} r_{KL} = -\tau \frac{d}{dt} U. \tag{2.45}$$

With (2.22) we then have

$$\delta \mathscr{L} = -\tau \frac{d}{dt}\left(\sum_{K=1}^{N} \tfrac{1}{2} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K - U\right) \tag{2.46}$$

On the other hand, equation (2.7) yields

$$\delta \mathscr{L} \doteq -\tau \frac{d}{dt} \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K, \tag{2.47}$$

and we deduce the scalar conservation equation

$$\frac{d}{dt} H \doteq 0, \qquad H = \sum_{K=1}^{N} \tfrac{1}{2} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K + U. \tag{2.48}$$

In many physical applications the potential energy (2.2) can be given in the form

$$U(r_{KL}) = -\frac{1}{2} \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum C_{KL} r_{KL}^{-\beta}, \tag{2.49}$$

where the constants C_{KL} are arbitrary and the parameter β indicates the character of the internal two-body interaction, E.g. for Newtonian gravitational interaction one has $\beta = 1$. With (2.49) equation (2.43) reads

$$\delta U = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum -\tfrac{1}{2}\beta C_{KL} r_{KL}^{-\beta-2}((\mathbf{r}_K - \mathbf{r}_L)\cdot B(\dot{\mathbf{r}}_K - \dot{\mathbf{r}}_L)$$

$$-\tfrac{1}{2}(\mathbf{r}_K - \mathbf{r}_L)\cdot \dot{B}(\mathbf{r}_K - \mathbf{r}_L)). \tag{2.50}$$

This form for δU suggests the possibility that it can be written as

$$\delta U = d/dt \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum a C_{KL} r_{KL}^{-\beta}, \tag{2.51}$$

where a is some possibly time dependent scalar. A simple calculation gives

$$d/dt \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum a C_{KL} r_{KL}^{-\beta} = \sum_{\substack{K,L=1\\K\neq L}}^{N}\sum C_{KL} r_{KL}^{-\beta-2}(-\beta a(\mathbf{r}_K - \mathbf{r}_L)\cdot(\dot{\mathbf{r}}_K - \dot{\mathbf{r}}_L)$$

$$+\dot{a}(\mathbf{r}_K - \mathbf{r}_L)\cdot(\mathbf{r}_K - \mathbf{r}_L)). \tag{2.52}$$

Comparing (2.50) and (2.52), one sees that δU takes the form (2.51) if

$$B = 2aI \quad \text{and} \quad \dot{B} = 4/\beta\, \dot{a} I, \tag{2.53}$$

I being the unity matrix. Equation (2.53) is satisfied if either

$$B = 2aI \quad \text{and} \quad \beta = 2, \tag{2.54}$$

or

$$B = 2aI \quad \text{and} \quad \dot{a} = 0. \tag{2.55}$$

The last case is equivalent to (2.44) which we already have studied. In (2.54), the condition $\beta = 2$ represents a restriction to a special kind of non-Newtonian internal forces. In that case one generally has

$$B = (-2\gamma t - \lambda t^2)I; \qquad A = (\gamma + \lambda t)I. \tag{2.56}$$

Here we have left out the constant term in B, because this term is also contained in (2.44).

From (2.56) and (2.22), (2.51) one finds with $\beta = 2$

$$\delta\mathscr{L} = d/dt\left(\sum_{K=1}^{N} \tfrac{1}{2}\lambda m_K \mathbf{r}_K \cdot \mathbf{r}_K - \tfrac{1}{2} m_K (2\gamma t + \lambda t^2)\dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K \right.$$
$$\left. -\frac{1}{2} \sum_{\substack{K,L=1 \\ K\neq L}}^{N}\sum (2\gamma t + \lambda t^2) C_{KL} r_{KL}^{-2} \right). \tag{2.57}$$

On the other hand (2.7) yields

$$\delta\mathscr{L} \doteq d/dt \sum_{K=1}^{N} (m_K(\gamma + \lambda t)\mathbf{r}_K \cdot \dot{\mathbf{r}}_K - m_K(2\gamma t + \lambda t^2)\dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K). \tag{2.58}$$

The equations (2.57) and (2.58) together yield a linear combination (coefficients γ and λ) of two scalar conservation equations. These scalar conservation equations are found when we take in (2.57) and (2.58) either $\lambda = 0$ or $\gamma = 0$. In this way one finds

$$d/dtD \doteq 0, \tag{2.59}$$

$$D = 2t\left(\sum_{K=1}^{N} \frac{1}{2} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K - \frac{1}{2} \sum_{\substack{K,L=1 \\ K\neq L}}^{N}\sum C_{KL} r_{KL}^{-2} \right) - \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot \mathbf{r}_K,$$

or with (2.48)

$$D = 2tH - \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K \cdot \mathbf{r}_K. \tag{2.60}$$

Additionally one has

$$d/dtF \doteq 0 \tag{2.61}$$

$$F = t^2\left(\sum_{K=1}^{N} \frac{1}{2} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K - \frac{1}{2} \sum_{\substack{K,L=1 \\ K\neq L}}^{N}\sum C_{KL} r_{KL}^{-2} \right) - \sum_{K=1}^{N} (t m_K \mathbf{r}_K \cdot \dot{\mathbf{r}}_K$$
$$- \tfrac{1}{2} m_K \mathbf{r}_K \cdot \mathbf{r}_K), \tag{2.62}$$

or with (2.48)

$$F = t^2 H - \sum_{K=1}^{N} (t m_K \mathbf{r}_K \cdot \dot{\mathbf{r}}_K - \tfrac{1}{2} m_K \mathbf{r}_K \cdot \mathbf{r}_K) \tag{2.63}$$

Summarizing:

For the general case, we have the following conservation equations and corresponding Noetherian variations

$$(2.17)\quad d/dt\ \mathbf{P} \doteq 0, \qquad \delta \mathbf{r}_K = \boldsymbol{\alpha} \tag{2.64}$$

$$(2.18)\quad d/dt\ \mathbf{G} \doteq 0, \qquad \delta \mathbf{r}_K = t\boldsymbol{\varepsilon} \tag{2.65}$$

$$(2.41)\quad d/dt\ \mathbf{L} \doteq 0, \qquad \delta \mathbf{r}_K = S\mathbf{r}_K,\ S = -S^T \tag{2.66}$$

$$(2.48)\quad d/dt\ H \doteq 0, \qquad \delta \mathbf{r}_K = -\tau \dot{\mathbf{r}}_K, \tag{2.67}$$

where

$$(2.17)\quad \mathbf{P} = \sum_{K=1}^{N} m_K \dot{\mathbf{r}}_K$$

$$(2.18)\quad \mathbf{G} = \sum_{K=1}^{N} m_K \mathbf{r}_K - t\mathbf{P}$$

$$(2.41)\quad \mathbf{L} = \sum_{K=1}^{N} m_K \mathbf{r}_K \wedge \dot{\mathbf{r}}_K$$

$$(2.48)\quad H = \sum_{K=1}^{N} \tfrac{1}{2} m_K \dot{\mathbf{r}}_K \cdot \dot{\mathbf{r}}_K + U$$

Additionally for $U(r_{KL}) = -\frac{1}{2}\sum C_{KL} r_{KL}^{-2}$, we have found

$$(2.59)\quad d/dt\, D \doteq 0, \qquad \delta \mathbf{r}_K = \gamma(\mathbf{r}_K - 2t\dot{\mathbf{r}}_K) \tag{2.68}$$

$$(2.61)\quad d/dt\, F \doteq 0, \qquad \delta \mathbf{r}_K = \lambda(t\mathbf{r}_K - t^2\dot{\mathbf{r}}_K) \tag{2.69}$$

where

$$(2.60)\quad D = 2tH - \sum_{K=1}^{N} m_K \mathbf{r}_K \cdot \dot{\mathbf{r}}_K$$

$$(2.63)\quad F = t^2 H - \sum_{K=1}^{N} (t m_K \mathbf{r}_K \cdot \dot{\mathbf{r}}_K - \tfrac{1}{2} m_K \mathbf{r}_K \cdot \mathbf{r}_K).$$

According to Steudel's equivalence rule (1.13) the Noetherian variation (2.67) corresponds to an infinitesimal translation of the time axes

$$\delta^* t = \tau. \tag{2.70}$$

The variations (2.64), (2.65), (2.66) and (2.70) together form a basis of the 10-*parametric group of infinitesimal Galilei transformations.*

The three vectorial conservation equations (2.17) (2.18), (2.41) and the scalar conservation eq. (2.48) yield the ten so-called *classical Eulerian integrals.*

Obviously, for U given by (2.49) with $\beta = 2$ the symmetry of the system is larger than the Galilei symmetry. We have the additional Noetherian variations (2.68) and (2.69), or equivalently

$$\delta^* \mathbf{r}_K = \gamma \mathbf{r}_K; \qquad \delta^* t = 2\gamma t, \tag{2.71}$$

and

$$\delta^* \mathbf{r}_K = \lambda t \mathbf{r}_K; \qquad \delta^* t = \lambda t^2, \tag{2.72}$$

respectively.

The variations (2.64), (2.65), (2.66), (2.70), (2.71) and (2.72) together form an infinitesimal basis of the *12-parametric symmetry group called Jacobi–Schrödinger group* (Havas 1978).

Usually, e.g. Havas 1978, one derives the conservation equations with Noether's theorem directly from the symmetry of the system. In this section, we have succeeded in the derivation of these conservation equations without the necessity of knowing the symmetry of the system at forehand.

3. The isotropic harmonic oscillator and Kepler's problem

3.1. *Introduction*

Let the state of a dynamical system be described by the coordinate vector $\mathbf{r} \in \mathbf{R}_n$ and the velocity vector $\dot{\mathbf{r}} = d/dt\,\mathbf{r}$. We assume the Lagrangian of the system is

$$\mathcal{L} = \tfrac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{1}{2\alpha}(\mathbf{r} \cdot \mathbf{r})^{\alpha}, \tag{3.1}$$

where α is a given real constant and the dot $\cdot$ stands for the usual inner product in $\mathbf{R}_n$.

For an arbitrary variation $\delta\mathbf{r}$, the first order variation $\delta\mathcal{L}$ of $\mathcal{L}$ is

$$\delta\mathcal{L} = \dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}} - (\mathbf{r} \cdot \mathbf{r})^{\alpha-1}\mathbf{r} \cdot \delta\mathbf{r}, \tag{3.2}$$

or with the chain rule for differentiation

$$\delta\mathcal{L} = d/dt(\dot{\mathbf{r}} \cdot \delta\mathbf{r}) - \delta\mathbf{r} \cdot (\ddot{\mathbf{r}} + (\mathbf{r} \cdot \mathbf{r})^{\alpha-1}\mathbf{r}). \tag{3.3}$$

Consequently, the equation of motion is

$$\ddot{\mathbf{r}} + (\mathbf{r} \cdot \mathbf{r})^{\alpha-1}\mathbf{r} \doteq 0 \tag{3.4}$$

and from (3.3) remains

$$\delta\mathcal{L} \doteq d/dt(\dot{\mathbf{r}} \cdot \delta\mathbf{r}). \tag{3.5}$$

For $\alpha = 1$, equation (3.4) describes an n-dimensional isotropic harmonic oscillator with unit frequency. For $\alpha = -\frac{1}{2}$ we have Kepler's problem in n dimensions for a unit mass and a unit gravitational constant.

3.2. *Energy and angular momentum*

Let us now investigate a special variation $\delta\mathbf{r}$ of the form

$$\delta\mathbf{r} = \varepsilon(M\mathbf{r} + N\dot{\mathbf{r}}), \tag{3.6}$$

where ε is a vanishingly small constant and M and N constant $n \times n$ matrices. Substitution of (3.6) into (3.2) yields

$$\frac{1}{\varepsilon}\delta\mathcal{L} = \dot{\mathbf{r}} \cdot M\dot{\mathbf{r}} - (\mathbf{r} \cdot \mathbf{r})^{\alpha-1}\mathbf{r} \cdot M\mathbf{r} + \dot{\mathbf{r}} \cdot N\ddot{\mathbf{r}} - (\mathbf{r} \cdot \mathbf{r})^{\alpha-1}(\mathbf{r} \cdot N\dot{\mathbf{r}}). \tag{3.7}$$

The variation (3.6) is Noetherian if δL vanishes or can be written as a total time derivative. E.g. this is the case for

$$M=-M^T, \qquad N=\lambda I, \tag{3.8}$$

where I is the unit $n\times n$ matrix and λ some constant. One easily verifies

$$\frac{1}{\varepsilon}\,\delta\mathscr{L}=\lambda\, d/dt\left(\tfrac{1}{2}\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}-\frac{1}{2\alpha}\,(\mathbf{r}\cdot\mathbf{r})^{\alpha}\right). \tag{3.9}$$

On the other hand, from (3.5) and (3.8) one obtains

$$\frac{1}{\varepsilon}\,\delta\mathscr{L}\doteq d/dt(\dot{\mathbf{r}}\cdot M\mathbf{r}+\lambda\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}). \tag{3.10}$$

From (3.9) and (3.10) we then have

$$d/dt\left(\dot{\mathbf{r}}\cdot M\mathbf{r}+\lambda\left(\tfrac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}+\frac{1}{2\alpha}\,(\mathbf{r}\cdot\mathbf{r})^{\alpha}\right)\right)\doteq 0 \tag{3.11}$$

For $M=0$, $\lambda=-1$ we have the variation

$$\delta\mathbf{r}=-\varepsilon\dot{\mathbf{r}} \tag{3.12}$$

which corresponds to an infinitesimal time-shift $\delta^* t=\varepsilon$, and that yields the energy conservation law

$$d/dt\left(\tfrac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}+\frac{1}{2\alpha}\,(\mathbf{r}\cdot\mathbf{r})^{\alpha}\right)\doteq 0. \tag{3.13}$$

Furthermore, $\lambda=0$ yields the conservation equation

$$d/dt(\dot{\mathbf{r}}\cdot M\mathbf{r})\doteq 0, \qquad M=-M^T. \tag{3.14}$$

The underlying Noetherian variation is

$$\delta\mathbf{r}=\varepsilon M\mathbf{r}, \qquad M=-M^T, \tag{3.15}$$

which corresponds to an infinitesimal rotation in $\mathbf{r}$-spaces. There exist $\frac{1}{2}n(n-1)$ linearly independent skew-symmetric $n\times n$ matrices. Therefore, equation (3.14) represents a set of $\frac{1}{2}n(n-1)$ linearly independent integrals of the motion. In a 3-dimensional space, these conservation equations can be taken together in one vectorial equation:

$$d/dt(\mathbf{r}\wedge\dot{\mathbf{r}})\doteq 0, \tag{3.16}$$

where $\wedge$ stands for the usual vector cross product. Equation (3.16) is the familiar form for the angular momentum conservation. In the case $\alpha=1$ (harmonic oscillator) equation (3.7) reduces to

$$\frac{1}{\varepsilon}\,\delta\mathscr{L}=\dot{\mathbf{r}}\cdot M\dot{\mathbf{r}}-\mathbf{r}\cdot M\mathbf{r}+\dot{\mathbf{r}}\cdot N\ddot{\mathbf{r}}-\mathbf{r}\cdot N\dot{\mathbf{r}}. \tag{3.17}$$

Then to make (3.6) a Noetherian variation, equation (3.8) is a too strong condition. We also may set

$$M=-M^T, \qquad N=N^T. \tag{3.18}$$

Then, equation (3.17) yields

$$\frac{1}{\varepsilon}\,\delta\mathcal{L} = d/dt(\tfrac{1}{2}\dot{\mathbf{r}}\cdot N\dot{\mathbf{r}} - \tfrac{1}{2}\mathbf{r}\cdot N\mathbf{r}). \tag{3.19}$$

On the other hand, from (3.18) and (3.5) one obtains

$$\frac{1}{\varepsilon}\,\delta\mathcal{L} \doteq d/dt(\dot{\mathbf{r}}\cdot M\mathbf{r} + \dot{\mathbf{r}}\cdot N\dot{\mathbf{r}}). \tag{3.20}$$

From (3.19) and (3.20) we now get

$$d/dt(\dot{\mathbf{r}}\cdot M\mathbf{r} + \tfrac{1}{2}\dot{\mathbf{r}}\cdot N\dot{\mathbf{r}} + \tfrac{1}{2}\mathbf{r}\cdot N\mathbf{r}) \doteq 0 \tag{3.21}$$

Thus, besides the conservation equations (3.13) and (3.14), for the isotropic harmonic oscillator we have additionally

$$d/dt(\tfrac{1}{2}\dot{\mathbf{r}}\cdot N\dot{\mathbf{r}} + \tfrac{1}{2}\mathbf{r}\cdot N\mathbf{r}) \doteq 0, \qquad N = N^T, \tag{3.22}$$

The underlying Noetherian variation for (3.22) is

$$\delta\mathbf{r} = \varepsilon N\dot{\mathbf{r}}, \qquad N = N^T. \tag{2.23}$$

With the canonical momentum $\mathbf{p} = \dot{\mathbf{r}}$, one can write

$$\delta\mathbf{r} = \varepsilon N\mathbf{p}; \qquad \delta\mathbf{p} = \varepsilon N\ddot{\mathbf{r}} = -\varepsilon N\mathbf{r}$$

Thus in phase space, the Noetherian variation (3.23) can be interpreted as a rotation of the coordinate and momentum subspaces as a whole around an axis perpendicular to the phase space.

3.3. *The Runge–Lenz vector*

With the ansatz (3.6), it is only possible to find integrals of the motion of which the main term is quadratic in the coordinates. To make the finding of more complicated integrals of the motion possible, one has to investigate variations which are more complicated functions of $\mathbf{r}$ and of $\dot{\mathbf{r}}$. For instance, as an alternative ansatz one could also take

$$\delta x_i = A_{ijk}x_jx_k + B_{ijk}\dot{x}_jx_k + C_{ijk}\dot{x}_j\dot{x}_k,$$

or

$$\delta x_i = A_{ijkl}x_jx_kx_l + B_{ijkl}\dot{x}_jx_kx_l + C_{ijkl}\dot{x}_j\dot{x}_kx_l + D_{ijkl}\dot{x}_j\dot{x}_k\dot{x}_l,$$

etc.

Here, x_i is the ith component of the vector $\mathbf{r}$, and a summation over repeated indices from 1 through n is implied. In the following, we only investigate a subset of these possible variations. We set

$$\delta x_i = A_{ijk}\dot{x}_jx_k \tag{3.24}$$

and will try to find conditions for the coefficients A_{ijk} that make the variation (3.24) a Noetherian variation.

In terms of the components, the equations (3.2) and (3.5) read

$$\delta\mathcal{L} = \dot{x}_i\,\delta\dot{x}_i - (x_jx_j)^{\alpha-1}x_i\,\delta x_i, \tag{3.25}$$

and

$$\delta\mathscr{L} \doteq d/dt(\dot{x}_i\,\delta x_i), \tag{3.26}$$

respectively.

Substitution of (3.24) into (3.25) gives

$$\delta\mathscr{L} = A_{ijk}\dot{x}_i\ddot{x}_j x_k + A_{ijk}\dot{x}_i\dot{x}_j\dot{x}_k - (x_m x_m)^{\alpha-1}A_{ijk}x_i\dot{x}_j x_k \tag{3.27}$$

Let us now look at the last term of (3.27) first. The coefficients have to be chosen such that either this term vanishes or is a total time derivative. The term vanishes if one takes $A_{ijk} = -A_{kji}$. Then $\delta\mathscr{L} = A_{ijk}\dot{x}_i\ddot{x}_j x_k$, which will generally neither vanish nor be a total time derivative.

When the last term of (3.27) should be a total time-derivative, a candidate is

$$d/dt[(x_m x_m)^{\alpha}(\mathbf{b}\cdot\mathbf{r})], \tag{3.28}$$

where $\mathbf{b}$ is some constant vector with components $B_1, B_2, \ldots, B_n$. The question now is whether conditions for B_i and A_{ijk} can be found such that

$$-(x_m x_m)^{\alpha-1}A_{ijk}x_i\dot{x}_j x_k = 2\alpha(x_m x_m)^{\alpha-1}B_i x_i x_j\dot{x}_j + (x_m x_m)^{\alpha}B_i\dot{x}_i, \tag{3.29}$$

or

$$-A_{ijk}x_i\dot{x}_j x_k = 2\alpha B_i x_i\dot{x}_j x_j + B_i x_j\dot{x}_i x_j. \tag{3.30}$$

This condition is:

$$A_{ijk} = -2\alpha B_i\delta_{jk} - B_j\delta_{ik} + C_{ijk}, \tag{3.31}$$

where

$$C_{ijk} = -C_{kji} \tag{3.32}$$

and δ_{ij} stands for the usual Kronecker symbol. With (3.31) and (3.32) the second term of (3.27) generally becomes a complicated expression. Only in the special case $\alpha = -\frac{1}{2}$ this term vanishes. Therefore in the following we restrict ourselves to this case (Kepler's problem).

With (3.31) and

$$\alpha = -\tfrac{1}{2}, \tag{3.33}$$

the first term of (3.27) becomes

$$A_{ijk}\dot{x}_i\ddot{x}_j x_k = B_i\dot{x}_i\ddot{x}_j x_j - B_j\dot{x}_i\ddot{x}_j x_i + C_{ijk}\dot{x}_i\ddot{x}_j x_k \tag{3.34}$$

$$= (\mathbf{b}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\mathbf{r}) - (\mathbf{b}\cdot\ddot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r}) + C_{ijk}\dot{x}_i\ddot{x}_j x_k. \tag{3.35}$$

A simple calculation now gives

$$(\mathbf{b}\cdot\ddot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r}) = d/dt[(\mathbf{b}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r})] - (\mathbf{b}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\mathbf{r}) - (\mathbf{b}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) \tag{3.36}$$

$$= d/dt[(\mathbf{b}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r}) - (\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})] - (\mathbf{b}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\mathbf{r}) + 2(\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}) \tag{3.37}$$

Thus,

$$A_{ijk}\dot{x}_i\ddot{x}_j x_k = 2(\mathbf{b}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\mathbf{r}) - 2(\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}) + C_{ijk}\dot{x}_i\ddot{x}_j x_k - d/dt[(\mathbf{b}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r}) - (\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})]. \tag{3.38}$$

Because of (3.32), the third term on the right-hand side of (3.38) is skew-symmetric for the exchange of $\mathbf{r}$ and $\dot{\mathbf{r}}$. The first two terms together have the same property. Then C_{ijk} can be chosen such that the first three terms vanish together:

$$C_{ijk}=2B_k\delta_{ij}-2B_i\delta_{jk}. \tag{3.39}$$

With (3.31) and (3.33) we then have

$$A_{ijk}=-B_i\delta_{jk}-B_j\delta_{ik}+2B_k\delta_{ij}, \tag{3.40}$$

which with (3.24) yields the Noetherian variation

$$\delta\mathbf{r}=2(\mathbf{b}\cdot\mathbf{r})\dot{\mathbf{r}}-(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{b}-(\mathbf{b}\cdot\dot{\mathbf{r}})\mathbf{r}. \tag{3.41}$$

This Noetherian variation has no obvious geometrical interpretation. As a consequence, this symmetry has been unknown for a long time. Our method has enabled us to find this symmetry by a systematic search.

With (3.28) and (3.33), equation (3.27) yields

$$\delta\mathcal{L}=d/dt[(\mathbf{r}\cdot\mathbf{r})^{-1/2}(\mathbf{b}\cdot\mathbf{r})-(\mathbf{b}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r})+(\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})] \tag{3.42}$$

On the other hand, with (3.41) equation (3.5) yields

$$\delta\mathcal{L}\doteq d/dt[2(\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})-2(\mathbf{b}\cdot\dot{\mathbf{r}})(\mathbf{r}\cdot\dot{\mathbf{r}})]. \tag{3.43}$$

Now, equations (3.42) and (3.43) together yield

$$d/dt[(\mathbf{b}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})-(\mathbf{b}\cdot\dot{\mathbf{r}})(\mathbf{r}\cdot\dot{\mathbf{r}})-(\mathbf{r}\cdot\mathbf{r})^{-1/2}(\mathbf{b}\cdot\mathbf{r})]\doteq 0 \tag{3.44}$$

or

$$\mathbf{b}\cdot d/dt[(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\mathbf{r}-(\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}-(\mathbf{r}\cdot\mathbf{r})^{-1/2}\mathbf{r}]\doteq 0. \tag{3.45}$$

In the case $\mathbf{R}_n=\mathbf{R}_3$ we also may write

$$\mathbf{b}\cdot d/dt[\dot{\mathbf{r}}\wedge(\mathbf{r}\wedge\dot{\mathbf{r}})-(\mathbf{r}\cdot\mathbf{r})^{-1/2}\mathbf{r}]\doteq 0, \tag{3.46}$$

which is the usual formulation of the conservation of the so-called *Runge–Lunz vector.*

Summarizing:

For the system (3.1) we have derived the Noetherian variation

(3.12) $\delta\mathbf{r}=-\varepsilon\dot{\mathbf{r}}$

yielding the energy conservation

(3.13) $d/dt\left(\tfrac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}+\dfrac{1}{2\alpha}(\mathbf{r}\cdot\mathbf{r})^{\alpha}\right)\doteq 0$

and the Noetherian variation

(3.15) $\delta\mathbf{r}=\varepsilon M\mathbf{r}, \qquad M=-M^T$

yielding the conservation of angular momentum

(3.14) $d/dt(\dot{\mathbf{r}}\cdot M\mathbf{r})\doteq 0, \qquad M=-M^T$

or for $\mathbf{R}_n=\mathbf{R}_3$

(3.16) $d/dt(\mathbf{r}\wedge\dot{\mathbf{r}})\doteq 0.$

For the isotropic harmonic oscillator ($\alpha = 1$) we have found additionally the Noetherian variation

(3.23) $\delta\mathbf{r} = \varepsilon N\dot{\mathbf{r}}, \qquad N = N^T$

which yields

(3.22) $d/dt(\frac{1}{2}\dot{\mathbf{r}} \cdot N\dot{\mathbf{r}} + \frac{1}{2}\mathbf{r} \cdot N\mathbf{r}) \doteq 0, \qquad N = N^T.$

For Kepler's problem ($\alpha = -\frac{1}{2}$) additionally to (3.12) and (3.15), we have found the Noetherian variation

(3.41) $\delta\mathbf{r} = 2(\mathbf{b} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{b} - (\mathbf{b} \cdot \dot{\mathbf{r}})\mathbf{r},$

with **b** an arbitrary constant infinitesimal vector, yielding the conservation of the Runge-Lenz vector:

(3.45) $d/dt((\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\mathbf{r} - (\mathbf{r} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}} - (\mathbf{r} \cdot \mathbf{r})^{-1/2}\mathbf{r}) \doteq 0.$

or for $\mathbf{R}_n = \mathbf{R}_3$

(3.46) $d/dt(\dot{\mathbf{r}} \wedge (\mathbf{r} \wedge \dot{\mathbf{r}}) - (\mathbf{r} \cdot \mathbf{r})^{-1/2}\mathbf{r}) \doteq 0.$

All these results have been obtained by systematic investigation of possible Noetherian variations and without an *a priori* knowledge of the symmetry of the system.

Acknowledgements

The idea for this paper had been born during the years of pleasant and instructive cooperation with L. J. F. Broer. The paper was actuated by discussions with L. A. Turski. During the finalization of this paper the author received many valuable remarks from A. Thellung, R. M. Santilli and from G. Wanders. To all of them I want to express my sincere gratitude.

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