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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **53 (1980)**

Heft 3

PDF erstellt am: **08.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-115127

Helvetica Physica Acta, Vol. 53 (1980), Birkhäuser Verlag, Basel

A simple model for irreversible dynamics from unitary time evolution

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(4. VIII. 1980)

Abstract. A Stark-type Hamiltonian describing the motion of a neutral spin-$\frac{1}{2}$ particle in a linear static magnetic field, is shown to give rise to a time evolution which, for every initial state of the total system and as time goes to ∞, drives the spin subsystem into a statistical state $\rho(\infty)$ with the following properties: (a) $\rho(\infty)$ has pure components corresponding to spin-up and spin-down states (with respect to the field direction). (b) $\rho(\infty)$ depends only on the initial state $\rho(0)$ of the spin subsystem. It is shown that this result is a consequence of a Stern–Gerlach-type quantum measurement process effected by the time evolution on suitable algebras of observables, and that irreversibility is due to the fact that the time evolution maps these algebras of observables onto a very small subalgebra in the infinite-time limit.

In an attempt to resolve what has become known as the 'problem of shape of molecules' [1], and in particular the 'paradox of optical isomers' [2], the following mechanism of blocking certain distinguished states of a system has been proposed [3]: Consider a two-level system which interacts with an external medium, called probe, in such a way that the time evolution of the combined system takes an initial product state $(\sum_{s=\pm 1} c_s\chi_s)\otimes\varphi$ (with distinguished orthogonal states $\chi_{\pm 1}\in\mathbb{C}^2$, arbitrary coefficients $c_{\pm 1}\in\mathbb{C}$, and probe state φ in some Hilbert space $\mathcal{H}$) into a final state $\sum_{s=\pm 1} c_s\chi_s\otimes\varphi_s$ where $\varphi_{\pm 1}\in\mathcal{H}$ are orthogonal. Since this final state leads to a reduced density operator for the two-level system that is diagonal in the basis $(\chi_s)_{s=\pm 1}$, it is concluded that the interaction with the probe forces the two-level system always to be eventually in either one of the states $\chi_{\pm 1}$ (in [3] they correspond to left/right-handed states of a molecule).

Although this conclusion is not without difficulty [4] (see also below), it is the purpose of this note to present a simple, yet physically rather suggestive Hamiltonian which effects such a 'reduction of the state vector' for *all* (normal) initial states of the total system if infinite time is allowed to elapse, and to analyze the mechanism responsible for this result:

Theorem 1. *Consider* $\sigma=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ *as operator acting in* $\mathbb{C}^2$. *Let* Q, P *be the (selfadjoint) position and momentum operators acting in* $L^2(\mathbb{R})$. *Define the Hamiltonian (acting in* $\mathbb{C}^2\otimes L^2(\mathbb{R})$*) by*

$$H=\omega\sigma\otimes 1+1\otimes\frac{1}{2m}P^2+\lambda\sigma\otimes Q \tag{1}$$

with constants $\omega, m, \lambda \in \mathbb{R}$ $(m, \lambda \neq 0)$. Then, for each density operator D on $\mathbb{C}^2 \otimes L^2(\mathbb{R})$, the reduced density operator

$$\rho(t) = tr_{L^2(\mathbb{R})}(e^{iHt} D e^{-iHt}) \qquad (t \in \mathbb{R})$$

satisfies

$$\lim_{|t|\to\infty} \rho(t) = \tfrac{1}{4} \sum_{s=\pm 1} (1 + s\sigma)\rho(0)(1 + s\sigma). \tag{2}$$

Proof. An easy consequence of the trace-class condition shows that it suffices to prove (2) for D's representing pure states. So let D be the orthogonal projector onto the vector $\sum_{s=\pm 1} \chi_s \otimes \varphi_s$ with $\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_{-1} = \begin{pmatrix}0\\1\end{pmatrix}$, and arbitrary $\varphi_{\pm 1} \in L^2(\mathbb{R})$ satisfying $\|\varphi_1\|^2 + \|\varphi_{-1}\|^2 = 1$. If follows that, for all $t \in \mathbb{R}$,

$$\rho(t) = \begin{pmatrix} \|\varphi_1\|^2 & \gamma(t) \\ \gamma(t)^* & \|\varphi_{-1}\|^2 \end{pmatrix}$$

where the Baker–Campbell–Hausdorff formula leads to

$$\gamma(t) e^{i(2\omega t + \lambda^2 t^3/m)} = \langle \varphi_{-1} \mid e^{-2i\lambda t Q} e^{-i\lambda t^2 P/m} \varphi_1 \rangle$$

$$= \int_{\mathbb{R}} e^{-2i\lambda t x} \varphi_{-1}(x)^* \varphi_1(x - \lambda t^2/m)\, dx. \tag{3}$$

To prove $\lim_{|t|\to\infty} \gamma(t) = 0$, suppose first that φ_{-1} has compact support and consider the bounded linear operator T from $L^2(\mathbb{R})$ to the space $C(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ (with supremum norm) such that, for every $\varphi_1 \in L^2(\mathbb{R})$ and all $t \in \mathbb{R}$, $(T\varphi_1)(t)$ is given by the right-hand side of (3). If φ_1 has compact support, too, then $T\varphi_1$ trivially is in the closed subspace of $C(\mathbb{R})$-functions vanishing at $\pm\infty$. By the boundedness of T, this extends to all $\varphi_1 \in L^2(\mathbb{R})$. Keeping now $\varphi_1 \in L^2(\mathbb{R})$ fixed and repeating the argument for the φ_{-1}-dependence of $\gamma(\cdot)$, we obtain $\lim_{|t|\to\infty} \gamma(t) = 0$ for all $\varphi_{\pm 1} \in L^2(\mathbb{R})$. q.e.d.

Remarks. 1) In an obvious interpretation of Hamiltonian (1), Theorem 1 says that, no matter what the initial state of the particle, all spin measurements performed after infinite time can be interpreted in terms of the proposition that the spin is exclusively in either one of the up/down states $\chi_{\pm 1}$ (with respective probabilities $tr_{\mathbb{C}^2}(\rho(0)(1 \pm \sigma)/2)$). And the proof gives the appealing detail that this 'state reduction' occurs already in finite time if the particle is initially localized in a finite region of space.

2) The irreversibility of the spin subdynamics as implied by Theorem 1, i.e. the absence of recurrence cycles and the approach of $\rho(t)$ to a persistent state $\rho(\infty)$ for every initial state, is in perfect formal analogy to that featured by Bloch-type semigroup evolutions (compare equation (2) vs. [5]). Equation (2) also entails that $\rho(\infty)$ is more mixed (in the sense of Uhlmann – see e.g. [6]) than $\rho(0)$, so the entropy of $\rho(\infty)$ is not less than that of $\rho(0)$.

3) Theorem 1 holds for still simpler, although hardly physical Hamiltonians obtained, for instance, by replacing P^2 in (1) by P or a function of Q. The latter variant is essentially the model analyzed along similar lines in [7].

Now for arbitrary $c_{r,s} \in \mathbb{C}$ $(r = 1, 2, \ldots; s = \pm 1)$ such that $\sum_{r=1}^{\infty} c_{r,s}^* c_{r,s'} = \langle \chi_s \mid \rho(\infty) \chi_{s'} \rangle$ $(s, s' = \pm 1)$, the statistical state $\rho(\infty)$ may be decomposed into pure states given by the (unnormalized) vectors $\sum_{s=\pm 1} c_{r,s}\chi_s$ $(r=1,2,\ldots)$. I.e. 'the knowledge of a non-pure statistical state does not imply the knowledge of the kind of ensemble to which it refers' [8] (the ensemble alluded to in Remark 1 is singled out, of course, in that it applies simultaneously to all states $\rho(\infty)$ as given by Theorem 1*)). This is the major reason why Theorem 1, as it stands, falls short of fully mimicking a quantum measurement process. In fact, any successful theory of measurement [7], [10] requires specification of a suitable algebra of observables which implies, at least for some non-pure states, a *unique* decomposition into pure states (corresponding to different 'pointer readings'). To be sure, such 'pointer positions' cannot be obtained from local position or momentum measurements of the particle:

Proposition 2. *Adopt the notations of Theorem* 1 *and of the proof thereof. Let* $\varphi_{\pm 1}\in L^2(\mathbb{R})$ *and define* $\varphi_{\pm 1}(t)\in L^2(\mathbb{R})$ by

$$e^{-iHt}\sum_{s=\pm 1}\chi_s\otimes\varphi_s=\sum_{s=\pm 1}\chi_s\otimes\varphi_s(t) \tag{4}$$

$(t\in\mathbb{R})$. *Furthermore, let the* C^**-algebra of quasilocal observables,* $\mathcal{A}$, *be the norm closure of*

$$\bigcup_{n=1}^{\infty}\{f_n(Q)Bf_n(Q)\mid B\in\mathcal{B}(L^2(\mathbb{R}))\}$$

where $\mathcal{B}(L^2(\mathbb{R}))$ *is the set of all bounded linear operators on* $L^2(\mathbb{R})$, *and* f_n *is the characteristic function of the interval* $[-n,n]$ $(n=1,2,\ldots)$. *Then*

$$\lim_{|t|\to\infty}\langle\varphi_s(t)\mid A\varphi_{s'}(t)\rangle=0\qquad(s,s'=\pm 1) \tag{5}$$

for all $A\in\mathcal{A}$. *The same holds if* Q *is replaced by* P *in the definition of* $\mathcal{A}$.

Proof. By a simple continuity argument, we need to prove (5) only for all $A=f_n(Q)Bf_n(Q)$ with $B\in\mathcal{B}(L^2(\mathbb{R}))$ and $n=1,2,\ldots$. For any such A and $\varphi_s\in L^2(\mathbb{R})\cap L^1(\mathbb{R})$ it follows that

$$|\langle\varphi_s(t)\mid A\varphi_{s'}(t)\rangle|\leqslant\|B\|\cdot\|\varphi_{s'}\|\cdot\|f_n(Q)\varphi_s(t)\|$$

$$\begin{aligned}\|f_n(Q)\varphi_s(t)\|&=\|f_n(Q)e^{-is\lambda t^2P/(2m)}e^{-itP^2/(2m)}e^{-is\lambda tQ}\varphi_s\|\\&=\left\{\frac{m}{2\pi|t|}\int_{-n}^{n}\left|\int_{\mathbb{R}}\exp\left(\frac{im}{2t}\left(x-\frac{s\lambda t^2}{2m}-y\right)^2\right)\right.\right.\\&\qquad\left.\left.\times e^{-is\lambda ty}\varphi_s(y)\,dy\right|^2dx\right\}^{1/2}\\&\leqslant\left\{\frac{mn}{\pi|t|}\right\}^{1/2}\int_{\mathbb{R}}|\varphi_s(y)|\,dy\end{aligned}$$

*) Recall also the following characterizations of the decomposition of a statistical state into orthogonal vector states: (i) The underlying elementary events correspond to mutually compatible propositions. (ii) The resulting probability distribution is least mixed in the sense of Uhlmann (cf. [9]).

($t \in \mathbb{R}$), whence (5). The extension to arbitrary $\varphi_s \in L^2(\mathbb{R})$ proceeds similarly as in the proof of Theorem 1. The truncated-momentum version of (5) follows from $\|f_n(P)\varphi_s(t)\| = \|f_n(P)e^{-is\lambda tQ}\varphi_s\| = \|f_n(P - s\lambda t)\varphi_s\| \to 0$ as $|t| \to \infty$ for each $\varphi_s \in L^2(\mathbb{R})$. q.e.d.

Clearly it is the spreading of the wave packet and the uniform acceleration of the particle, respectively, which cause the spatial parts of every vector state of the particle to evolve so as to give asymptotically vanishing expectation values (and interference terms) for both classes of observables described in Proposition 2. So it is not surprising that we need macroscopic measuring devices like two semi-infinite screens, in order to detect whether the particle is deflected in the + or − direction, i.e. to get the desired 'pointer positions':

Proposition 3. *Assume the hypotheses of Proposition 2, and let f be the characteristic function of the interval* $[0, \infty[$. *Then, for* $u = \pm\infty$,

$$\lim_{t \to u} \left\langle \sum_{s=\pm 1} \chi_s \otimes \varphi_s(t) \,\middle|\, A \sum_{s=\pm 1} \chi_s \otimes \varphi_s(t) \right\rangle = \lim_{t \to u} \sum_{s=\pm 1} \langle \chi_s \otimes \varphi_s(t) \mid A(\chi_s \otimes \varphi_s(t)) \rangle \tag{6}$$

for all $A \in \mathcal{B}(\mathbb{C}^2) \otimes \{f(Q)\}''$ *(where* $''$ *denotes the double commutant, and where the limits in* (6) *exist). The same holds with* $f(Q)$ *replaced by* $f(P)$.

Proof. The claim is obvious if we can prove the more detailed statements

$$\lim_{|t| \to \infty} \langle \varphi_s(t) \mid \varphi_{s'}(t) \rangle = \delta_{s,s'} \|\varphi_s\|^2, \tag{7a}$$

$$\lim_{|t| \to \infty} \langle \varphi_s(t) \mid f(Q)\varphi_{s'}(t) \rangle = \delta_{s,s'}\, \delta_{s,-\mathrm{sig}(m\lambda)} \|\varphi_s\|^2, \tag{7b}$$

$$\lim_{t \to \pm\infty} \langle \varphi_s(t) \mid f(P)\varphi_{s'}(t) \rangle = \delta_{s,s'} \delta_{s,\mp \mathrm{sig}(\lambda)} \|\varphi_s\|^2 \tag{7c}$$

($s, s' = \pm 1$) where $\varphi_{\pm 1} = \varphi_{\pm 1}(0)$ (recall (4)). (7a) is essentially the content of Theorem 1. To see (7b), note first that

$$\langle \varphi_s(t) \mid f(Q)\varphi_s(t) \rangle = \left\langle \varphi_s \,\middle|\, e^{itP^2/(2m)} f\!\left(Q - \frac{s\lambda t^2}{2m}\right) e^{-itP^2/(2m)} \varphi_s \right\rangle,$$

$$\begin{aligned} |\langle \varphi_s(t) \mid f(Q)\varphi_{-s}(t) \rangle| &= \left| \left\langle f\!\left(Q \mp \frac{s\lambda t^2}{2m}\right) e^{-itP^2/(2m)} \varphi_{\pm s} \,\right|\right. \\ &\qquad \left.\left. \times\, e^{\mp is\lambda t^2 P/m} e^{\pm 2is\lambda tQ} e^{-itP^2/(2m)} \varphi_{\mp s} \right\rangle \right| \\ &\leq \left\| f\!\left(Q \mp \frac{s\lambda t^2}{2m}\right) e^{-itP^2/(2m)} \varphi_{\pm s} \right\| \cdot \|\varphi_{\mp s}\|; \end{aligned}$$

then prove

$$w - \lim_{|t| \to \infty} e^{itP^2/(2m)} f\!\left(Q - \frac{s\lambda t^2}{2m}\right) e^{-itP^2/(2m)} = \delta_{s,-\mathrm{sig}(m\lambda)} 1 \tag{8}$$

by verifying the required matrix-element relations for the (total) set of functions $\mathbb{R} \ni x \mapsto e^{-(x-a)^2}$ ($a \in \mathbb{R}$); and finally use that (8) implies

$$s - \lim_{|t| \to \infty} f(Q - |\lambda t^2/(2m)|) \exp(-itP^2/(2m)) = 0.$$

Similarly, equation (7c) follows from

$$\langle \varphi_s(t) \mid f(P)\varphi_s(t)\rangle = \langle \varphi_s \mid f(P - s\lambda t)\varphi_s\rangle,$$

$$|\langle \varphi_s(t) \mid f(P)\varphi_{-s}(t)\rangle| = |\langle f(P \mp s\lambda t)\varphi_{\pm s} \mid e^{\pm is\lambda t^2 P/m} e^{\pm 2is\lambda tQ}\varphi_{\mp s}\rangle|,$$

and

$$s - \lim_{t \to \pm\infty} f(P - s\lambda t) = \delta_{s, \mp \mathrm{sig}(\lambda)} 1. \quad \text{q.e.d.}$$

Proposition 3 shows that, for every (pure) initial state $\sum_{s=\pm 1} \chi_s \otimes \varphi_s$ of the particle, the time-evolved state (4) for $t \to \infty$ induces a (mixed) state on $\mathcal{B}(\mathbb{C}^2) \otimes \{f(Q)\}''$ with *unique* decomposition into pure components corresponding to the alternatives

$$\left(\text{spin} \begin{array}{c}\text{up}\\ \text{down}\end{array}\right) \otimes (\text{particle at } \pm\infty)$$

(see [7] for a detailed discussion of this type of quantum measurement process). In addition, it exemplifies the typical situation where one and the same quantity (here the spin) can be measured by different classical measuring instruments (here represented by $\{f(Q)\}''$ and $\{f(P)\}''$) which are non-commutative among each other.

The often-asserted relationship between state reduction and irreversibility (cf. also Remarks 1 and 2), becomes now manifest from the following corollary of Proposition 3:

Theorem 4. *Keep notations as before. Then, for every* $A \in \mathcal{B}(\mathbb{C}^2) \otimes \{f(Q)\}''$, *the limits*

$$\alpha_\pm(A) = w - \lim_{t \to \pm\infty} e^{iHt} A e^{-iHt}$$

exist and are in $\{\sigma \otimes 1\}''$ *(the mappings* $\alpha_\pm(\cdot)$ *are endomorphisms if and only if restricted to the domain* $\{\sigma\}'' \otimes \{f(Q)\}''$*). The same holds if* $f(Q)$ *is replaced by* $f(P)$.

Proof. This is a straightforward result of (4), (6), (7) for all $\varphi_{\pm 1} \in L^2(\mathbb{R})$. Note that the weak limits in question can actually be computed very explicitly. q.e.d.

Indeed, for state reduction it is crucial that $\alpha_+(\cdot)$ maps the algebra of observables, $\mathcal{B}(\mathbb{C}^2) \otimes \{f(Q)\}''$, onto a *commutative* subalgebra, $\{\sigma \otimes 1\}''$; while for irreversibility it is essential that $\alpha_+(\cdot)$ is not invertible, i.e. that $\alpha_+(\cdot)$ maps the algebra of observables onto a *sufficiently small* subalgebra. But in the present model, the latter property is clearly a consequence of the former.

Acknowledgement

The author would like to thank G. A. Raggio and H. Primas for very helpful discussions.

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