On the point spectrum of Dirac operators

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Abstract. We study asymptotic properties of the discrete point spectrum of $H_0+V=\boldsymbol{\alpha}\mathbf{p}+\beta+V$ where V is a suitable potential. Firstly, replacing V by λV $(V\leqslant 0)$ we find that the number of eigenvalues which enter (resp. leave) the interval $(-1, 1)$ grows proportional to λ^3 as $\lambda\to\infty$. Secondly, if V $(V\leqslant 0)$ is long-range we determine the asymptotic behavior of the number of eigenvalues contained in $(0, E)$ as $E\uparrow 1$,

1. Introduction

Consider on $[L_2(\mathbb{R}^3)]^4$ the operator

$$H_0+V=\boldsymbol{\alpha}\mathbf{p}+\beta+V \tag{1.1}$$

where H_0 denotes the free Dirac Hamiltonian. Here $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3)$ and β are the usual 4×4 Dirac matrices satisfying the relations $\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}I$, $\alpha_i\beta+\beta\alpha_i=0$, $i=1, 2, 3$.

As is well known $\sigma(H_0)=\mathbb{R}\backslash(-1, 1)$ and if V is a suitable perturbation $\sigma_{\text{ess}}(H_0+V)=\sigma(H_0)$. In the gap $(-1, 1)$ H_0+V may have discrete eigenvalues. For nonpositive V we let $N^1(\lambda V)$ (resp. $N^{-1}(\lambda V)$) denote the total number of eigenvalues of $H_0+\lambda V$ that have entered (resp. left) the gap at 1 (resp. -1) as the coupling constant rose from 0 to λ. If V is a radially symmetric, rectangular potential well we know that $N^{\pm 1}(\lambda V)\to\infty$ as $\lambda\to\infty$, for the eigenvalue problem is explicitly solvable. The natural question arises whether this is true for more general potentials and, if yes, one may further be interested in the asymptotic behavior of $N^{\pm 1}(\lambda V)$ as $\lambda\to\infty$. In the Schrödinger case the number of negative eigenvalues grows like $\sim c\lambda^{3/2}$ where $c=(2\pi)^{-3}$ vol $\{(\mathbf{p}, \mathbf{x}) \mid p^2+V<0\}$ [1, p. 262]. So what is 'vol $\{(\mathbf{p}, \mathbf{x}) \mid \boldsymbol{\alpha}\mathbf{p}+\beta+V<0\}$'? I am indebted to B. Simon for once asking me this question. The answer will be given by Theorem (2.1). Moreover, we find that N^{+1} and N^{-1} are equal in leading order as $\lambda\to\infty$. This supports the intuitive idea that the eigenvalues move through the gap in an orderly fashion (one behind the other) and don't get stuck. In fact one may ask: can it happen that an eigenvalue $E(\lambda)$ obeys $E(\lambda)\downarrow a$ as $\lambda\to\infty$ where $a\in(-1, 1)$? We were unable to answer this question for general negative potentials, except in the radially symmetric case where the answer is negative (Remark 4 to Theorem (2.1)).

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In the proof of Theorem (2.1) we cannot use the well known Dirichlet–Neumann bracketing technique [1, p. 261]. We rather apply results of Birman and Solomyak on the eigenvalue asymptotics of certain integral kernels after we have found the right set-up for tackling the problem. The idea is to look at the Birman–Schwinger kernel $\lambda\,|V|^{1/2}(H_0-E)^{-1}\,|V|^{1/2}$ which has eigenvalue 1 if and only if E is eigenvalue of $H_0+\lambda V$. This kernel is a sum of two kernels which we refer to as the Dirac/Schrödinger part respectively (2.14) (2.15). It turns out that the Dirac part dominates over the Schrödinger part as far as the asymptotic behavior of the eigenvalues is concerned. More precisely, the Dirac part has eigenvalues which decrease as $0(n^{-1/3})$ whereas those of the Schrödinger part are $0(n^{-2/3})$. This implies that the eigenvalues of the full Birman–Schwinger kernel are also $0(n^{-1/3})$ which in turn yields the $0(\lambda^3)$ growth of $N(\lambda V)$.

The Dirac/Schrödinger part switch roles in Section 2. There we suppose V to be long-range so that the number of eigenvalues is infinite. We count the eigenvalues which are less than E and ask how this number grows as $E\uparrow 1$. Again, we consider the Birman–Schwinger kernel and observe that it is the Schrödinger part which dominates. One may say that the $E\uparrow 1$ limit is nonrelativistic. The relevant result is proved in Theorem (2.1). We also remark that the $E\uparrow 1$ limit has previously been investigated by H. Tamura [4] so that our result is merely a rediscovery of his in part. However, we feel that the approaches are sufficiently different and believe it is simpler *not* to square the Dirac operator.

In Remark 2 to Theorem (3.1) we comment on the relationship between the Dirac operator and the operator $\sqrt{p^2+1}+V$ and in Remark 4 we discuss the peculiarity of the potential $V(\mathbf{x})=-1/1+x^2$.

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2. The limit $\lambda\to\infty$

The main result of this section is

Theorem 2.1. *Suppose* $V\in L_3\cap L_{3/2}$, $V\leq 0$. *Then*

$$N^{\pm 1}(\lambda V)/\lambda^3\to\frac{1}{3\pi}\int|V|^3\,d^3x \quad as \quad \lambda\to\infty.$$

Remark. In the Schrödinger case the number of negative eigenvalues grows like $\sim\lambda^{3/2}\,\|V\|_{3/2}$. In the Dirac case, the property $V\in L_{3/2}$ is suppressed in the asymptotics, however, it is needed to ensure the finiteness of $N^{+1}(\lambda V)$.

We prove Theorem 2.1 in a sequence of four lemmas. For any compact, self-adjoint operator A, and any $t>0$, we define

$$N_t(A)=\dim P_{(t,\infty)}(A) \tag{2.1}$$

where $P_\Omega(A)$ denotes the projection-valued measure (of the Borel set Ω) associated with A. Let $\chi_M(\mathbf{x})$ denote the characteristic function of the set M.

Lemma 2.2. *Let $M=[-l, l]^3$. Then*

$$t^3 N_t\left(\chi_M \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} \chi_M\right) \to \frac{(2l)^3}{3\pi} \quad \textit{as} \quad t\downarrow 0. \tag{2.2}$$

Proof. The operator $\boldsymbol{\alpha}\mathbf{p}/p^2$ has kernel

$$\frac{\boldsymbol{\alpha}(\mathbf{x}-\mathbf{y})}{4\pi\,|x-y|^3} \tag{2.3}$$

This kernel is homogeneous of degree -2. So we can take advantage of the well developed theory for such operators. We follow closely the work of Birman and Solomyak [2], [3]. The method described in [2] (periodic continuation of the kernel and using a Fourier transform) carries over to our matrix kernel. For $l=\pi$ we find that the right side of (2.2) is ($|\ |$ denotes Lebesgue measure)

$$2\,|\{\mathbf{p}\in\mathbb{R}^3 \mid 1/p>1\}| = \frac{(2\pi)^3}{3\pi} \tag{2.4}$$

noting that $\boldsymbol{\alpha}\mathbf{p}/p^2$ has eigenvalues $\pm 1/p$ (each of multiplicity two). Upon scaling, the result (2.2) for arbitrary l follows.

Lemma 2.3. *Let $V\geq 0$ be continuous and have compact support. Then, as $t\downarrow 0$*

$$t^3 N_t\left(V^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} V^{1/2}\right) \to \frac{1}{3\pi}\int V^3\, d^3x \tag{2.5}$$

Proof. By approximating V by step functions and using Lemma 2.2. See [2] for how to deal with the cross terms.

Next we would like to get rid of the continuity restriction and merely assume $V\in L_3$. The corresponding problem in the Schrödinger case is nontrivial. To make the intended approximation argument work we need an upper bound on N_t in terms of $\|V\|_3$. Thanks to the work of Cwikel who solved the analogous problem for Schrödinger operators [5] we can state

Lemma 2.4. *Suppose $V\in L_3$, $V\geq 0$. Then*

$$t^3 N_t\left(V^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} V^{1/2}\right) \leq c\int V^3\, d^3x \tag{2.6}$$

Proof. Since $\boldsymbol{\alpha}\mathbf{p}/p^2 \leq 1/p$ (in the operator sense) it suffices to prove the estimate for the operator $V^{1/2}(1/p)V^{1/2}$. Now $V^{1/2}\in L_6$ and $1/p\in$ weak-L_6. So (2.6) follows from the result of [5] where c is a certain constant.

Finally, we arrive at

Lemma 2.5. *Suppose $V\in L^3$, $V\geq 0$. Then (2.5) holds.*

Proof. Given $\varepsilon > 0$ pick $W_\varepsilon \in C_0^\infty$, $W_\varepsilon \geqslant 0$ such that $\|W_\varepsilon^{1/2} - V^{1/2}\|_6 < \varepsilon$. Write

$$V^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} V^{1/2} = W_\varepsilon^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} W_\varepsilon^{1/2} + (V^{1/2} - W_\varepsilon^{1/2}) \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} W_\varepsilon^{1/2}$$

$$+ W_\varepsilon^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} (V^{1/2} - W_\varepsilon^{1/2})$$

$$= W_\varepsilon^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} W_\varepsilon^{1/2} + \Delta_\varepsilon. \tag{2.7}$$

For any self-adjoint, compact operator A we let $s_n(A)(\lambda_n^\pm(A))$ denote its singular values (positive/negative eigenvalues). s_n and $\lambda_n^+(\lambda_n^-)$ are arranged in descending (ascending) order. The operators appearing in (2.7) have the useful property that their spectra are located symmetrically with respect to 0. For the self-adjoint members this means $s_1 = s_2 = \pm\lambda_1^\pm$, etc. To see this note that $\boldsymbol{\alpha}\mathbf{p}$ is unitarily equivalent to $-\boldsymbol{\alpha}\mathbf{p}$ via a transformation which commutes with the potentials [8]. Moreover, we remark that $t^\sigma N_t(A) \to c$ as $t \downarrow 0$ if and only if $n^\tau \lambda_n^+(A) \to c^\tau$ as $n \to \infty$ where $\tau = 1/\sigma$. If A is the first operator on the r.h.s. of (2.7) this applies with $\sigma = 3$.

Next we shall show that $\sup_n n^{1/3} s_n(\Delta_\varepsilon)$ tends to zero as $\varepsilon \downarrow 0$. By the above remarks this carries over to the eigenvalues and Lemma 1 of [2] applies with the result that Δ_ε drops out in the $t \downarrow 0$ asymptotics. On account of the inequalities [7]

$$s_{n+m-1}(A+B) \leqslant s_n(A) + s_m(B) \tag{2.8}$$

$$s_{n+m-1}(AB) \leqslant s_n(A) s_m(B) \tag{2.9}$$

we obtain with $Z_\varepsilon = V^{1/2} - W_\varepsilon^{1/2}$

$$s_{n+m-1}\left(Z_\varepsilon \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} W_\varepsilon^{1/2}\right) \leqslant s_n\left(Z_\varepsilon \frac{1}{\sqrt{p}}\right) s_m\left(\frac{\boldsymbol{\alpha}\mathbf{p}}{p} \frac{1}{\sqrt{p}} W_\varepsilon^{1/2}\right)$$

$$\leqslant s_n\left(Z_\varepsilon \frac{1}{\sqrt{p}}\right) s_m\left(\frac{1}{\sqrt{p}} W_\varepsilon^{1/2}\right) \tag{2.10}$$

where we used $s_1(A) = \|A\|$ in the second step.

Applying [6, Theorem XI.22] we obtain

$$s_n\left(Z_\varepsilon \frac{1}{\sqrt{p}}\right) \leqslant \text{const.}\, \|Z_\varepsilon\|_6\, n^{-1/6} \tag{2.11}$$

$$s_m\left(\frac{1}{\sqrt{p}} W_\varepsilon^{1/2}\right) \leqslant \text{const.}\, \|W_\varepsilon^{1/2}\|_6\, m^{-1/6} \tag{2.12}$$

By assumption $\|Z_\varepsilon\|_6 \leqslant \varepsilon$ showing that $\sup_n n^{1/3} s_n(\Delta_\varepsilon) \leqslant 0(\varepsilon)$. Therefore (using Lemma 2.3 and [2, Lemma 1])

$$\lim_{t \downarrow 0} t^3 N_t\left(V^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} V^{1/2}\right) = \lim_{\varepsilon \downarrow 0} \lim_{t \downarrow 0} t^3 N_t\left(W_\varepsilon^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2} W_\varepsilon^{1/2}\right)$$

$$= (\tfrac{1}{3}\pi)\, \|V\|_3^3,$$

completing the proof of Lemma 2.4.

Next we want to prove Theorem 2.1 using Lemma 2.5. To this end note that

$$(H_0-E)^{-1}=\frac{\boldsymbol{\alpha}\mathbf{p}+\beta+E}{p^2+1-E^2} \tag{2.13}$$

Thus the Birman–Schwinger kernel $|V|^{1/2}(H_0-E)^{-1}|V|^{1/2}$ is a sum of a 'Dirac part'

$$|V|^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}}{p^2+1-E^2}|V|^{1/2} \tag{2.14}$$

and a 'Schrödinger part'

$$|V|^{1/2}\frac{\beta+E}{p^2+1-E^2}|V|^{1/2} \tag{2.15}$$

It is useful if both parts have norm limits as $E\uparrow 1$ or $E\downarrow -1$. The natural condition for this is that $V\in L_3\cap L_{3/2}$. Then the limits are

$$|V|^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}}{p^2}|V|^{1/2}, \qquad E=\pm 1 \tag{2.16}$$

for the Dirac part and

$$|V|^{1/2}\frac{\beta\pm 1}{p^2}|V|^{1/2}, \qquad E=\pm 1 \tag{2.17}$$

for the Schrödinger part.

Proof of Theorem 2.1. By the Birman–Schwinger principle $N^{\pm 1}(\lambda V)=\dim P_{(1/\lambda,\infty)}(|V|^{1/2}(H_0\mp 1)^{-1}|V|^{1/2})$. From the proof of Lemma 2.5 we know that

$$n^{1/3}\lambda_n^+\left(|V|^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}}{p^2}|V|^{1/2}\right)\to c>0 \quad \text{as} \quad n\to\infty. \tag{2.18}$$

From [6, Theorem XI.22] we conclude that the singular values of (2.17) obey

$$s_n\leq \text{const.}\,\|\,|V|^{1/2}\|_3^2\, n^{-2/3} \tag{2.19}$$

Thus $n^{1/3}s_n\to 0$ as $n\to\infty$. So the Dirac part determines the asymptotics since, by Lemma 2.3,

$$\begin{aligned}\lim_{\lambda\uparrow\infty} N^{\pm 1}(\lambda V)/\lambda^3 &= \lim_{\lambda\uparrow\infty} N_{1/\lambda}\left(|V|^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}}{p^2}|V|^{1/2}\right)\Big/\lambda^3\\ &=\text{r.h.s. of (2.5)},\end{aligned}$$

completing the proof of Theorem (2.1).

Remarks. 1. If $V\geq 0$ all goes through with $E=1$ and $E=-1$ switched.

2. That the eigenvalues move from the right to the left follows immediately from the fact that the eigenvalues of the Birman–Schwinger kernel *decrease* monotonically as $E\downarrow -1$ [8].

3. In order to see that as $\lambda\to\infty$ infinitely many eigenvalues pass through any given point $E\in(-1,1)$ we need only show that $\dim P_{(0,\infty)}$

$(|V|^{1/2}(H_0-E)^{-1}|V|^{1/2})=\infty$. On account of estimate (3.7) this is true provided

$$\dim P_{(0,\infty)}\left(|V|^{1/2}\frac{1}{p^2+1-E^2}|V|^{1/2}\right)=\infty,$$

which is obvious since on $L_2(\text{supp } V)$ the operator has trivial kernel.

4. If V is radially symmetric the preceding remark applies to each reducing subspace (two component spinors). An eigenvalue which is supposed to get stuck (as described in the Introduction) would necessarily be passed by infinitely many others. Thus degenerate eigenvalues would occur which, however, is impossible for the separated equations.

5. Let $V(x)=-1/1+x^2$. Then the operator (2.17) at $E=1$ $(E=-1)$ has spectrum $[0, 8]$ $([-8, 0])$ (see e.g. the Appendix to [10]). Since the operator (2.16) is compact we conclude that $H_0+\lambda V$ has infinitely (finitely) many bound states if $\lambda>\frac{1}{8}$ $(0<\lambda<\frac{1}{8})$. The special case $\lambda=\frac{1}{8}$ remains undecided.

3. The limit $E\uparrow 1$

Define

$$\tilde{n}(E, V)=\dim P_{(0,E)}(H_0+V), \qquad E\in(0, 1) \tag{3.1}$$

and

$$n(E, V)=\dim P_{(-\infty,E)}(-\Delta+V), \qquad E<0. \tag{3.2}$$

We make the same assumptions on V as in [1, Thm. XIII.82], namely

$$-c_1(x+1)^{-\beta}\leq V(\mathbf{x})\leq -c_2(x+1)^{-\beta} \tag{3.3}$$

$$|V(\mathbf{x})-V(\mathbf{y})|\leq c_3[\min\{x, y\}+1]^{-\beta-1}|\mathbf{x}-\mathbf{y}| \tag{3.4}$$

for some $0<\beta<2$, $c_1, c_2, c_3>0$.

Let

$$g(E, V)=\frac{1}{6\pi^2}\int_{V\leq E}(E-V(\mathbf{x}))^{3/2}\,d^3x \tag{3.5}$$

Then one can prove

$$n(E, V)/g(E, V)\to 1 \quad \text{as} \quad E\uparrow 0. \tag{3.6}$$

The analog for Dirac operators is contained in

Theorem 3.1.

$\tilde{n}(E, V)/\tilde{g}(E, V)\to 1$ as $E\uparrow 1$

where $\tilde{g}(E, V)=2^{5/2}g(E-1, V)$.

In a remark below we shall explain how one can relax the conditions on V and allow local singularities.

Proof. We introduce $P=(\beta+1)/2$ which projects onto the upper two spinor components. Then (recalling (2.1))

$$N_1(|V|^{1/2}(H_0-E)^{-1}|V|^{1/2}) \geq N_1\left(\frac{1+E}{2}|V|^{1/2}\frac{\beta+1}{p^2+1-E^2}|V|^{1/2}\right) \tag{3.7}$$

$$= 2n(E^2-1,(1+E)V) \tag{3.8}$$

where we used $P(\boldsymbol{\alpha}\mathbf{p})P=0$ and $N_1(\cdot)\geq N_1(P\cdot P)$ in the first step and the Birman–Schwinger principle in the second. Again by the Birman–Schwinger principle and (3.8)

$$\tilde{n}(E,V) = N_1(|V|^{1/2}(H_0-E)^{-1}|V|^{1/2}) - N_1(|V|^{1/2}H_0^{-1}|V|^{1/2}) \tag{3.9}$$

$$\geq 2n(E^2-1,(1+E)V) - N_1(|V|^{1/2}H_0^{-1}|V|^{1/2}) \tag{3.10}$$

Pick $\delta>0$ and suppose $1-\delta<E<1$. Then

$$\tilde{n}(E,V) \geq 2n(E^2-1,(2-\delta)V) - N_1(|V|^{1/2}H_0^{-1}|V|^{1/2}). \tag{3.11}$$

Now we remark that

$$\lim_{\delta\downarrow 0}\lim_{E\uparrow 1} g(E^2-1,(2-\delta)V)/\tilde{g}(E,V) = 1 \tag{3.12}$$

This follows from (3.3) by means of some straightforward estimates which we skip here.

Upon dividing (3.11) by $\tilde{g}(E,V)$ and letting first $E\uparrow 1$ and then $\delta\downarrow 0$ (using (3.6) and (3.12)) we arrive at

$$\lim_{E\uparrow 1} \tilde{n}(E,V)/\tilde{g}(E,V) \geq 1. \tag{3.13}$$

Next we prove the converse inequality by separating small and large p. Let $\chi_\delta(p)$ denote the characteristic function of $[0,\delta]$. Using $\boldsymbol{\alpha}\mathbf{p}\leq p$, $\chi_\delta(\mathbf{p})p\leq\delta$, $E\leq 1$, one gets

$$\frac{\boldsymbol{\alpha}\mathbf{p}+\beta+E}{p^2+1-E^2} \leq \frac{\delta+\beta+1}{p^2+1-E^2} + \frac{(1-\chi_\delta(\mathbf{p}))}{p} \tag{3.14}$$

Since $n(A+B)\leq n(A)+n(B)$ for any two self-adjoint operators A, B where $n(\cdot)=\dim P_{(-\infty,0)}(\cdot)$ [1, p. 274] a little agrument shows that for any $\varepsilon\in(0,1)$

$$N_1\left(|V|^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta+E}{p^2+1-E^2}|V|^{1/2}\right) \leq N_{1-\varepsilon}\left(|V|^{1/2}\frac{\delta+\beta+1}{p^2+1-E^2}|V|^{1/2}\right) + N_\varepsilon\left(|V|^{1/2}\frac{1-\chi_\delta(\mathbf{p})}{p}|V|^{1/2}\right) \tag{3.15}$$

$$= 2n\left(E^2-1,\frac{2+\delta}{1-\varepsilon}V\right) + N_\varepsilon\left(|V|^{1/2}\frac{1-\chi_\delta(\mathbf{p})}{p}|V|^{1/2}\right) \tag{3.16}$$

Since the last member on the right does not depend on E and is finite (the operator is compact) it tends to zero as $E\uparrow 1$ upon division by $\tilde{g}(E,V)$. Combining

(3.9) and (3.16) yields

$$\lim_{E\uparrow 1} \tilde{n}(E, V)/\tilde{g}(E, V) \leq 2 \lim_{E\uparrow 1} \frac{n(E^2-1, (2+\delta)/(1-\varepsilon)V)}{\tilde{g}(E, V)} \tag{3.17}$$

Letting $\delta\downarrow 0$ and $\varepsilon\downarrow 0$ we obtain

$$\lim_{E\uparrow 1} \tilde{n}(E, V)/\tilde{g}(E, V) \leq 1. \tag{3.18}$$

This finishes the proof of the theorem.

Next we briefly describe how one can include local singularities in the potential. If the local singularities are still in L_3 there is no problem, for in proving Theorem 3.1 we have reduced the Dirac problem to the Schrödinger case and for the latter there is a remark in [1, p. 277] about how to cope with local singularities. We recall that one can prove (3.6) for $-\Delta+V+W$ where V obeys (3.3) and (3.4) and W is such that $n(0, \lambda W)$ is *finite* for *all* λ. Here one may ask what happens if $W=\chi_1(\mathbf{x})/x^2$? Then $-\Delta+\lambda W$ is unbounded from below if $\lambda<-\frac{1}{4}$ (i.e. this is true for any self-adjoint extension). However if $\lambda>-\frac{1}{4}$ the operator has no negative bound state and one would expect (3.6) to be true for $-\Delta+V+\lambda W$, $\lambda>-\frac{1}{4}$. This is exactly the situation one is confronted with in the Dirac case if a local Coulomb singularity is present. Let us consider

$$H_0+V-\lambda W, \qquad \lambda\in(0, 1) \tag{3.19}$$

where

$$W(\mathbf{x})=\frac{1}{x}\chi_1(\mathbf{x}) \tag{3.20}$$

and V obeys the assumptions (3.3) and (3.4).

Then our proof of Theorem 3.1 breaks down completely for parts of the Birman–Schwinger kernel become *noncompact.* We even know that [9, Lemma 5.5]

$$\sigma_{\text{ess}}\left(W^{1/2}\frac{\boldsymbol{\alpha}\mathbf{p}}{p^2}W^{1/2}\right)=[-1, 1] \tag{3.21}$$

Therefore the estimate (3.15) becomes useless since the second term on the r.h.s. is infinite (in fact $\sigma_{\text{ess}}(W^{1/2}[1-\chi_\delta(\mathbf{p}))/p]W^{1/2}]=[-\pi/2, \pi/2]$ for any δ by arguments of [9]).

To overcome these difficulties we look at the modified Birman–Schwinger kernel

$$|V|^{1/2}(H_0-\lambda W-E)^{-1}|V|^{1/2}. \tag{3.22}$$

We recall from [8] that for any $E\in\rho(H_0-\lambda W)$ we have

$$(H_0-\lambda W-E)^{-1}=R_0(E)+\lambda R_0(E)W^{1/2}(1-\lambda W^{1/2}R_0(E)W^{1/2})^{-1}W^{1/2}R_0(E) \tag{3.23}$$

where $R_0(E)=(H_0-E)^{-1}$. This means we consider the physically distinguished realization of $H_0-\lambda W$. The kernel $K(E, W)=W^{1/2}R_0(E)W^{1/2}$ has its essential spectrum on $[-1, 1]$ (cf. [9]) and possible discrete eigenvalues outside this interval. Thus under the assumption $\lambda\in(0, 1)$ the kernel $\lambda K(E, W)$ can at most have eigenvalue 1 for a set of discrete values of the coupling constant λ. Henceforth we assume that λ is not such a value (otherwise we perturb it a little and all goes through we minor modifications). Since $K(E, W)$ is norm continuous in E we may assume that $1 \in \rho(K(E, W))$ for all E sufficiently close to 1. By the way this means that $H_0 - \lambda W$ has only a finite number of bound states.

Now insert (3.23) in (3.22). From Theorem 3.1 we conclude that

$$N_1(|V|^{1/2}(H_0 - E)^{-1}|V|^{1/2}) = 0[(1-E)^{3/2-3/\beta}] \tag{3.24}$$

as $E \uparrow 1$. We have to estimate the same quantity for the correction which arises from the second term on the r.h.s. of (3.23). It turns out to be of a lower order than (3.24) which, by now familiar arguments, implies that the overwhelming majority of eigenvalues above 1 stems from the operator $|V|^{1/2} R_0(E) |V|^{1/2}$.

Now

$$s_{n+m-1}(|V|^{1/2} R_0(E) W^{1/2}(1-K(E, W))^{-1} W^{1/2} R_0(E) |V|^{1/2}) \le C s_n(A) s_m(A^*) \tag{3.24}$$

where $A = |V|^{1/2} R_0(E) W^{1/2}$ and $\|(1-K(E, W))^{-1}\| \le C$ for E near 1. Now, using (2.8) and (2.13),

$$s_n(A) \le s_n\left(|V|^{1/2} \frac{\beta + E}{p^2 + 1 - E^2} W^{1/2}\right) + s_1\left(|V|^{1/2} \frac{\boldsymbol{\alpha}\mathbf{p}}{p^2 + 1 - E^2} W^{1/2}\right) \tag{3.25}$$

The last term stays bounded as $E \uparrow 1$ for the kernel $\sim |V(\mathbf{x})|^{1/2} |\mathbf{x} - \mathbf{y}|^{-2} W(y)^{1/2}$ is compact. The other one is bounded by

$$s_n\left(|V|^{1/2} \frac{(\beta + E)p^\gamma}{p^2 + 1 - E^2}\right) \|p^{-\gamma} W^{1/2}\| \tag{3.26}$$

where we set $\gamma = \frac{5}{4}$. The boundedness of the last factor is clear since it has kernel $\sim |\mathbf{x} - \mathbf{y}|^{-7/4} W(\mathbf{y})^{1/2}$. The first factor can be estimated by means of [6, Theorem XI.22] which yields

$$s_n \le \frac{(1-E)^{3/2q - 3/8}}{n^{1/q}}. \tag{3.27}$$

Finally we see that for N_1 of the operator on the left of (3.24) we get the bound $\le c(1-E)^{3/2 - 3q/8}$ which is of lower order than (3.24) if $8/\beta > q > \max[6/\beta, 4]$. Since $0 < \beta < 2$, such a choice is always possible. This proves that Theorem 3.1 is also valid for the operator (3.19).

Remarks. 1. Theorem 3.1 is, of course, still true if we count the eigenvalues contained in $(-1, E)$ rather than $(0, E)$.

2. We should also point out the close relationship between the Dirac Hamiltonian and the operator $\sqrt{p^2+1} + V(\mathbf{x})$ acting on $L_2(\mathbb{R}^3)$. Theorems 2.1 (for $N^{+1}(\lambda V)$) and (3.1) could be proved for the latter operator as well, with the only difference that, due to spin degeneracy, the number of eigenvalues is doubled in the Dirac case. The relevant property for the asymptotics of $N(\lambda V)$ is that the relativistic kinetic energy is proportional to p for $p \to \infty$ (which enters via (2.4)). The $E \uparrow 1$ limit depends on the fact that $\sqrt{1+p^2} \approx 1 + p^2/2$ for small p and, indeed, Theorem 3.1 also describes the $E \uparrow 1$ limit of $p^2/2 + 1 + V$ (up to the spin factor). We have exploited this connections in our proofs.

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