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On quantization of the electromagnetic field

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Abstract. We first construct the Hilbert state space of the classical electromagnetic field. This construction is given in the helicity representation and uses the theory of the extensions of representations of the Poincaré group. Each state satisfies the Lorentz gauge condition; a particular rôle is played by the Lorentz radiation gauge (Coulomb condition). The results obtained show that the field operator splits into a classical scalar part and a quantum transverse one.

Introduction

The problem of the quantization of the electromagnetic field is a very old one [1]. During the twenties, Dirac studied the emission and the absorption of radiation [2]. In his model the relativistic invariance is completely destroyed, because the field is split into a radiation field and a static Coulomb field; then only the radiation field is quantized, the static field remains classical. Since then, the covariant treatment of this problem has been considered by many authors [3a].

In the standard procedure of quantization of the electromagnetic field, one difficulty is the Lorentz gauge condition [3b]: it is considered as a subsidiary condition acting on the state vectors which describe the electromagnetic field. However, the best known method is that due to Gupta and Bleuler [3c], [4], [5].

In their scheme, the state space is a vector space with indefinite metric. There are four kinds of photons: two kinds of transverse photons, the longitudinals photons and the scalar photons. The states describing the scalar photons have negative 'norm'. Longitudinal and scalar photon states are eliminated by the help of the weakened subsidiary condition: one must take the Lorentz gauge condition only for the annihilation part of the potential operators acting on the 'physically desirable' states. The 'physically undesirable' states are connected with gauge transformations.

Indeed, states with the same number of longitudinal and scalar photons have zero 'norm': one can then define different vacuum states with positive 'norm' using particular linear combinations of such longitudinal and scalar states. Moreover, for a given transverse state, the addition of such a linear combination corresponds to a description in terms of potentials related by a gauge transformation [6].

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In this paper, essentially based on the author's Ph.D. thesis presented to the Faculty of Sciences of the University of Geneva [7], we present a group theoretical approach of the problem with two main purposes: to construct the Hilbert state space of the electromagnetic field [8] and to obtain the decomposition into a classical part and a quantum one in a covariant manner. Moreover we analyse the rôle played by the invariance gauge and the gauge conditions, especially the Lorentz and Coulomb gauge conditions.

The main mathematical tools used are the extensions of zero mass representations of the restricted Poincaré group [9]. Indeed, the usual action of this group on four vectors satisfying the Lorentz condition is given by a decomposable representation [10]. The existence of an intertwining operator between this decomposable representation and the extension of zero mass representations with helicities 0, +1, −1 and 0 allows the passage from the canonical basis to the local helicity basis. It follows, in natural way, the decomposition of the field operator into a classical scalar part and a quantum transverse one. The first one is associated with the vacuum state and is gauge dependant. If one chooses the Coulomb gauge, this scalar part is identically zero. We will see that this choice is related to the direction of the time [11], if the radiation field interacts with a material system. The quantum transverse part of the field operator is composed by the two helicity components, which are gauge independant and associated with photon states. We call photons what Gupta and Bleuler call transverse photons; so we do not introduce the notions of longitudinal and scalar photons.

Thus we develop a formalism which, on the one hand, avoids the mathematical difficulties due to the indefinite metric and, on the other hand, splits the field as in Dirac's model. Moreover the gauge transformations are clearly related to the vacuum state representations. We point out that this approach does not follow the axiomatic point of view of the quantum field theory [12].

To conclude this short introduction, we briefly mention the contents of the following sections. In section I, we shall construct rigorously the Hilbert state space for the classical electromagnetic radiation field. In section II, we give the action of the restricted Poincaré group on this Hilbert space. The intertwining operator is explicitly built up. Then the extension of representations and its domain are discussed. In section III, we construct the Fock space and define the field operator and its components. Finally, section IV is devoted to the covariance and the gauge invariance of the field operator. We shall distinguish two cases: the free radiation field and the interaction of the radiation field with matter.

I. State space of the electromagnetic field

I.1. *Definitions*

Let M be the differential manifold $\mathbf{R}^3\backslash\{\vec{0}\}$, which is embedded in $\mathbf{R}^4$ by:

$$M \to C_+ \subset \mathbf{R}^4$$

$$\vec{p} = (p^1, p^2, p^3) \mapsto p^\mu = (p^1, p^2, p^3, |\vec{p}|/c) = (\vec{p}, |\vec{p}|/c),$$

where $|\vec{p}| = +((p^1)^2 + (p^2)^2 + (p^3)^2)^{1/2}$ and c is the light velocity. The range C_+ of this embedding can be explicitly written as

$$C_+ = \{p^\mu \in \mathbf{R}^4 \mid g_{\mu\nu}p^\mu p^\nu = 0 \quad \text{and} \quad p^4 > 0\},$$

where the $g_{\mu\nu}$'s are the coefficients of the following sesquilinear hermitian form:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}.$$

Let $H = L^2(M, dm(\vec{p}))$ be the Hilbert space whose elements are complex functions defined on M and square integrable with respect to the measure $dm(\vec{p}) = d^3p/c\,|\vec{p}|$.

The scalar product of two elements u and v of H is denoted by (u, v) and defined by:

$$(u, v) = \int dm(\vec{p})\overline{u(\vec{p})}v(\vec{p}), \tag{I.1}$$

where $\overline{u(\cdot)}$ is the complex conjugate of $u(\cdot)$.

Let $\overset{4}{\oplus} H$ be the direct sum of four copies of H: $\overset{4}{\oplus} H = H \oplus H \oplus H \oplus H$. An element of $\overset{4}{\oplus} H$ is denoted by f and its components by f^μ. We define in this space the canonical scalar product between two elements f and h of $\overset{4}{\oplus} H$:

$$(f, h) = (f^1, h^1) + (f^2, h^2) + (f^3, h^3) + c^2(f^4, h^4), \tag{I.2}$$

where (f^μ, h^μ) is defined by (I.1). The norm of f is given by $\|f\| = |(f, f)|^{1/2}$. Henceforth we will call this norm the euclidean norm. We also define a sesquilinear hermitian form:

$$B(f, h) = (f^1, h^1) + (f^2, h^2) + (f^3, h^3) - c^2(f^4, h^4). \tag{I.3}$$

Among the elements of $\overset{4}{\oplus} H$, we want to restrict our attention to those which satisfy the following condition:

$$g_{\mu\nu}p^\mu f^\nu(\vec{p}) = 0 \quad \text{for any} \quad p^\mu \in C_+,$$

which we will call the Lorentz condition. All the elements of $\overset{4}{\oplus} H$ obeying this condition form the subspace H_1 of $\overset{4}{\oplus} H$, i.e.

$$H_1 = \left\{ f \in \overset{4}{\oplus} H \mid f^4(\vec{p}) = \frac{\vec{p} \cdot \vec{f}(\vec{p})}{c\,|\vec{p}|}, \quad \text{where} \quad \vec{f}(\vec{p}) = (f^1(\vec{p}), f^2(\vec{p}), f^3(\vec{p})) \right\}.$$

We denote by b the restriction to H_1 of the form B (I.3). We have the following results:

b is a non negative form.

We call 'kernel of b' the subspace $(H_1)_0$ of H_1 on which the quadratic form associated with b is identically zero:

$$\ker b = (H_1)_0 = \{g \in H_1 \mid b(g, g) \equiv 0\}.$$

$(H_1)_0$ is given by:

$$(H_1)_0 = \left\{ g \in H_1 \mid g^\mu(\vec{p}) = \chi(\vec{p})p^\mu \text{ and } \int dm(\vec{p})\,|\vec{p}|^2\,|\chi(\vec{p})|^2 < \infty \right\}. \tag{I.4}$$

I.2. *Quotient space*

Two elements f and h of H_1 are defined to be equivalent if they differ by an element g of $(H_1)_0$ i.e.

$$h = f + g \quad \text{and} \quad g^\mu(\vec{p}) = \chi(\vec{p})p^\mu, \qquad p^\mu \in C_+, \tag{I.5}$$

where the scalar function χ is finite in the sense given in (I.4). (I.5) is called a (Lorentz) gauge transformation, because it leaves invariant the Lorentz condition. The equivalence classes form the quotient space $\mathcal{H} = H_1/(H_1)_0$. The equivalence class of f is denoted by $\hat{f}$. The euclidean norm of $\hat{f}$ is defined as the quotient norm:

$$\|\hat{f}\| = \inf_{g \in (H_1)_0} \|f + g\|. \tag{I.6}$$

$\mathcal{H}$ equipped with this norm is a Banach space.

The scalar product between two classes $\hat{f}$ and $\hat{h}$ is defined as the scalar product between the representative elements f' and h' which are orthogonal to $(H_1)_0$:

$$\begin{aligned} &(\hat{f}, \hat{h}) = (f', h'), \\ &(f', g) = (h', g) = 0, \qquad \forall g \in (H_1)_0. \end{aligned} \tag{I.7}$$

(f', h'), (f', g) and (h', g) are given by (I.2). As $(H_1)_0$ is closed, this scalar product defines a Hilbert structure on $\mathcal{H}$. The norm $|(f', f')|^{1/2}$ of this hilbertian structure is exactly the quotient norm (I.6) [13].

On the other hand, the sesquilinear hermitian form is simply defined by $\hat{b}(\hat{f}, \hat{h}) = b(f, h)$ where f and h are any representative elements of the classes $\hat{f}$ and $\hat{h}$. $\hat{b}$ is a positive definite form. The quadratic form associated with it allows us to define another norm, called the $\hat{b}$-norm:

$$\|\hat{f}\|_{\hat{b}} = |\hat{b}(\hat{f}, \hat{f})|^{1/2}, \tag{I.8}$$

which is less than or equal to the euclidean norm:

$$\|\hat{f}\|_{\hat{b}} \leq \|\hat{f}\|.$$

I.3. *Coulomb condition*

Among all the representative elements of a class, there is a particular one which plays an important rôle in radiation theory. It is the one which obeys the Lorentz and Coulomb conditions, namely:

$$\vec{p} \cdot \vec{\tilde{f}}(\vec{p}) = 0.$$

This representative element $\tilde{f}$ is obtained by the means of a gauge transformation (I.5) acting on some other representative element f of the same class $\hat{f}$. Explicitly we have:

$$\tilde{f}^\mu(\vec{p}) = f^\mu(\vec{p}) + \chi(\vec{p})p^\mu \tag{I.9}$$

and χ is the following scalar function:

$$\chi(\vec{p}) = -\frac{c}{|\vec{p}|} f^4(\vec{p}) = -\frac{\vec{p} \cdot \vec{f}(\vec{p})}{|\vec{p}|^2}.$$

This particular representative element $\tilde{f}$ is orthogonal to $(H_1)_0$:

$$(\tilde{f}, g) = 0 \quad \text{for any} \quad g \in (H_1)_0.$$

Thus $\tilde{f}$ is the representative element f' of (I.7). Moreover $b(\tilde{f}, \tilde{h})$ is equal to $(\tilde{f}, \tilde{h})$ (because $\tilde{f}^4$ and $\tilde{h}^4$ are zero). Collecting these results, we obtain that the scalar product $(\hat{f}, \hat{h})$ of two classes $\hat{f}$ and $\hat{h}$ and the value $\hat{b}(\hat{f}, \hat{h})$ of the form $\hat{b}$ (evaluated on the same classes) are equal:

$$(\hat{f}, \hat{h}) = \hat{b}(\hat{f}, \hat{h}).$$

In particular, the euclidean norm $\|\cdot\|$ and the $\hat{b}$-norm $\|\cdot\|_{\hat{b}}$ of any class coincide.

From now on, we shall omit the symbol $\hat{}$ and denote by the same letter f an element of $\mathcal{H}$ (i.e. a class) and a representative element of this class.

A class f is called a state of the classical electromagnetic field: each class is indeed formed by elements which obey the Lorentz condition and two elements of the same class differ by a gauge transformation (I.5). We say that each class corresponds to an electric field and to a magnetic induction and each representative element of this class corresponds to a four potential obeying the Lorentz condition: we shall say that the field is described in the Lorentz gauge. If we represent this state by the element $\tilde{f}$ which obeys the Coulomb condition, we shall say that this state is given in the Lorentz radiation gauge.

II. Actions of the restricted Poincaré group $\mathcal{P}$

II.1. *Representations of* $\mathcal{P}$

We want to give the action of the restricted Poincaré group $\mathcal{P}$ in the various Hilbert spaces of the previous section. Let us recall that, in the Hilbert space $H = L^2(M, dm(\vec{p}))$, $\mathcal{P}$ acts by the unitary irreducible representations V_λ of zero mass, positive energy and helicity $\lambda(=-1, 0, +1)$. If (a, Λ) is any element of $\mathcal{P}$, where $a \in \mathbf{R}^4$ stands for a translation and Λ for a proper orthochronous Lorentz transformation, $V_\lambda(a, \Lambda)$ acts on an element u of H as follows:

$$(V_\lambda(a, \Lambda)u)(\vec{p}) = \exp(-i\hbar^{-1} g_{\alpha\beta} p^\alpha a^\beta) \cdot \exp(2i\lambda\theta(\Lambda, \vec{p})) \cdot u(\overrightarrow{\Lambda^{-1}p}) \tag{II.1}$$

where $\hbar$ is Planck's constant divided by 2π and $\theta(\Lambda, \vec{p})$ is a function of Λ and $\vec{p}$ with values in the interval $[0, 2\pi[$. If Λ is represented by the $SL(2, \mathbf{C})$-matrix

$$A(\Lambda) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{II.2}$$

then we have:

$$\exp(i\theta(\Lambda, \vec{p})) = \frac{\bar{\gamma}}{|\gamma|} \tag{II.3}$$

with $\gamma = a_{22}(|\vec{p}| + p^3) - a_{12}(p^1 + ip^2)$. The helicity λ takes the values -1, 0, $+1$.

In our direct sum space $\overset{4}{\oplus} H$, the action of $\mathcal{P}$ is given by the following bounded operator:

$$(\mathcal{U}(a, \Lambda)f)^\mu(\vec{p}) = \exp(-i\hbar^{-1} g_{\alpha\beta} p^\alpha a^\beta) \Lambda^\mu_\nu f^\nu(\overrightarrow{\Lambda^{-1}p}). \tag{II.4}$$

The restriction to the subspace H_1 of $\overset{4}{\oplus} H$ and the quotient of H_1 by $(H_1)_0$ do not induce any formal modification to (II.4). However, we denote by $U(a, \Lambda)$ the action of $\mathcal{P}$ in the state space $\mathcal{H}$; $U(a, \Lambda)$ is unitary.

On the other hand, the operator which leaves invariant the choice of Coulomb representative element $\tilde{f}^\mu$ is given by:

$$(\tilde{U}(a, \Lambda)\tilde{f})^\mu(\vec{p}) = \exp(-i\hbar^{-1}g_{\alpha\beta}p^\alpha a^\beta)\left(\Lambda^\mu_\nu - c\frac{p^\mu}{|\vec{p}|}\Lambda^4_\nu\right)\tilde{f}^\nu(\overrightarrow{\Lambda^{-1}p}). \tag{II.5}$$

$\tilde{U}(a, \Lambda)$ is obtained by a gauge transformation (I.5) defined by the scalar function (cf. (I.9)).

$$\chi(\vec{p}) = -c\frac{(U(a, \Lambda)\tilde{f})^4(\vec{p})}{|\vec{p}|} = -\exp(-i\hbar^{-1}g_{\alpha\beta}p^\alpha a^\beta)c\frac{\Lambda^4_\nu \tilde{f}^\nu(\overrightarrow{\Lambda^{-1}p})}{|\vec{p}|}.$$

II.2. *Helicity representation*

The representation $\mathcal{U}(a, \Lambda)$ is decomposable. The problem is to find an intertwining operator $\Pi(\vec{p})$ such that:

$$\Pi(\vec{p})\mathcal{U}(a, \Lambda)\Pi^{-1}(\overrightarrow{\Lambda^{-1}p}) = \mathcal{W}(a, \Lambda) \tag{II.6}$$

where $\mathcal{W}(a, \Lambda)$ is an extension of the representations $V_\lambda(a, \Lambda)$ defined in [9]. The operator $\Pi(\vec{p})$ can be considered as a local change of basis in the space $\overset{4}{\oplus} H$.

In order to do this, we consider the underlying helicity representation. $\Pi(\vec{p})$ is constructed in the following way:

$$\Pi(\vec{p}) = \tau \cdot \alpha(\vec{p}),$$

where τ is defined by its action on the f^μ's:

$$(\tau f)^1 = f^3 + cf^4,$$
$$(\tau f)^2 = f^1 - if^2,$$
$$(\tau f)^3 = f^1 + if^2,$$
$$(\tau f)^4 = -f^3 + cf^4$$

and $\alpha(\vec{p})$ is composed of a rotation $\rho(\vec{p})$ and a boost $\beta(\vec{p})$ such that:

$$\alpha(\vec{p}) = \beta(\vec{p})\rho(\vec{p}) \quad \text{and} \quad \alpha(\vec{p})p^\mu = p^\mu_0 \tag{II.7}$$

with $p^\mu_0 = (0, 0, 1, c^{-1})$. $\Pi(\vec{p})$ takes the following explicit form:

$$(\Pi f)^1(\vec{p}) = \psi^1(\vec{p}) = -\hat{p}_\mu f^\mu(\vec{p}),$$
$$(\Pi f)^2(\vec{p}) = \psi^2(\vec{p}) = -X_\mu(\vec{p})f^\mu(\vec{p}),$$
$$(\Pi f)^3(\vec{p}) = \psi^3(\vec{p}) = -\overline{X_\mu(\vec{p})}f^\mu(\vec{p}),$$
$$(\Pi f)^4(\vec{p}) = \psi^4(\vec{p}) = -p_\mu f^\mu(\vec{p}),$$

with

$$p_\mu = (\vec{p}, -c\,|\vec{p}|), \qquad \hat{p}_\mu = \frac{-1}{|\vec{p}|^2}(\vec{p}, c\,|\vec{p}|), \tag{II.8}$$

$$X_1(\vec{p}) = -\frac{p^3}{|\vec{p}|} - \frac{p^2}{|\vec{p}|\,(|\vec{p}| + p^3)}(p^2 + ip^1),$$

$$X_2(\vec{p}) = i\frac{p^3}{|\vec{p}|} + \frac{p^1}{|\vec{p}|\,(|\vec{p}|+p^3)}(p^2+ip^1), \tag{II.9}$$
$$X_3(\vec{p}) = \frac{p^1-ip^2}{|\vec{p}|},$$
$$X_4(\vec{p}) = 0.$$

Then $\mathcal{W}(a, \Lambda)$ introduced in (II.6) can be written (in matrix notation) as follows:

$$\mathcal{W}(a,\Lambda) = \begin{pmatrix} V_0(a,\Lambda) & \overline{z(\Lambda,\vec{p})}V_+(a,\Lambda) & z(\Lambda,\vec{p})V_-(a,\Lambda) & |z(\Lambda,\vec{p})|^2 V_0(a,\Lambda) \\ 0 & V_+(a,\Lambda) & 0 & z(\Lambda,\vec{p})V_0(a,\Lambda) \\ 0 & 0 & V_-(a,\Lambda) & \overline{z(\Lambda,\vec{p})}V_0(a,\Lambda) \\ 0 & 0 & 0 & V_0(a,\Lambda) \end{pmatrix} \tag{II.10}$$

and its action is given formally by:

$$\psi'^{\mu}(\vec{p}) = (\mathcal{W}(a,\Lambda)\psi)^{\mu}(\vec{p}) = \mathcal{W}(a,\Lambda)^{\mu}_{\nu}\psi^{\nu}(\overrightarrow{\Lambda^{-1}p}).$$

These V_λ's, $\lambda = \pm 1, 0$, are given by (II.1) and $z(\Lambda, \vec{p})$ is the following complex valued function:

$$z(\Lambda,\vec{p}) = \frac{1}{|\vec{p}|}\frac{\bar{\alpha}\beta+\bar{\gamma}\delta}{|\alpha|^2+|\gamma|^2} \tag{II.11}$$

with

$$\alpha = a_{21}(|\vec{p}|+p^3) - a_{11}(p^1+ip^2),$$
$$\beta = a_{11}(|\vec{p}|+p^3) + a_{21}(p^1-ip^2),$$
$$\gamma = a_{22}(|\vec{p}|+p^3) - a_{12}(p^1+ip^2),$$
$$\delta = a_{12}(|\vec{p}|+p^3) + a_{22}(p^1-ip^2).$$

z appears (with θ) in the $SL(2, \mathbf{C})$-matrix representations of the little group S_{p_0} of p_0:

$$A(\theta, z) = \begin{pmatrix} e^{i\theta} & ze^{-i\theta} \\ 0 & e^{-i\theta} \end{pmatrix}.$$

$z(\Lambda, \vec{p})$ and $\theta(\Lambda, \vec{p})$ (see (II.3)) characterize the following element of S_{p_0}:

$$A(\theta(\Lambda,\vec{p}), z(\Lambda,\vec{p})) = A(\vec{p})A(\Lambda)A^{-1}(\overrightarrow{\Lambda^{-1}p}),$$

where $A(\vec{p})$ is the $SL(2, \mathbf{C})$-matrix associated with $\alpha(\vec{p})$ (II.7) and $A(\Lambda)$ with Λ (II.2).

Remark. Because of the little group multiplication law

$$A(\theta_1, z_1)A(\theta_2, z_2) = A(\theta_1+\theta_2, z_1+e^{2i\theta_1}z_2),$$

the function $z(\Lambda, \vec{p})$ obeys the 1-cocycle equation

$$z((\Lambda_1,\vec{p})(\Lambda_2,\vec{p})) = z(\Lambda_1,\vec{p}) + V_+(a_1,\Lambda_1)z(\Lambda_2,\overrightarrow{\Lambda_1^{-1}p})V_0^{-1}(a_1,\Lambda_1).$$

In the helicity basis, where the states are described by the components ψ^μ, the scalar product of $\overset{4}{\oplus} H$ is given by:

$$(\psi, \phi) = \int dm(\vec{p}) s'_{\mu\nu}(\vec{p}) \overline{\psi^\mu(\vec{p})} \phi^\nu(\vec{p}), \tag{II.12}$$

where the $s'_{\mu\nu}(\vec{p})$'s are the coefficients of the canonical scalar product (I.2) expressed in the new basis:

$$(s'_{\mu\nu}(\vec{p})) = \frac{1}{2}\begin{pmatrix} |\vec{p}|^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & |\vec{p}|^{-2} \end{pmatrix}.$$

The sesquilinear hermitian form corresponding to (I.3) is now denoted by $\beta(\psi, \phi)$ and is given by:

$$\beta(\psi, \phi) = \int dm(\vec{p}) g'_{\mu\nu} \overline{\psi^\mu(\vec{p})} \phi^\nu(\vec{p}), \tag{II.13}$$

where the $g'_{\mu\nu}$'s are the coefficients of the sesquilinear hermitian form expressed in the helicity basis:

$$(g'_{\mu\nu'}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

(II.13) becomes:

$$\beta(\psi, \phi) = \tfrac{1}{2}\{(\psi^2, \phi^2) + (\psi^3, \phi^3) - (\psi^1, \phi^4) - (\psi^4, \phi^1)\},$$

where (ψ^μ, ϕ^ν) is given by (I.1).

Remark. This sesquilinear hermitian form is invariant under the Poincaré group. The above expression of this form is one of the possible invariant sesquilinear forms defined by Rideau [9].

II.3. *Domain of the extension*

Consider the multiplication operator $z(\Lambda, \vec{p})$ (II.11). Because of the simple pole at the origin $\vec{p} = (0, 0, 0)$, it is an unbounded operator. The domain of the extension $\mathcal{W}(a, \Lambda)$ cannot be the space $\overset{4}{\oplus} H$. We restrict $\mathcal{W}(a, \Lambda)$ to the subspace $\overset{4}{\oplus} D = D \oplus D \oplus D \oplus D$, where D is the space of C^∞-functions of compact support on M; D is a dense subspace of H and $\overset{4}{\oplus} D$ is dense in $\overset{4}{\oplus} H$.

Following the same procedure as in section I, we define the subspace D_1 formed by the elements obeying the Lorentz condition, expressed now as follows:

$$D_1 = \{\psi \in \overset{4}{\oplus} D \mid \psi^4 \equiv 0\}.$$

So for every element of D_1, the first component ψ^1 is simply related to the fourth component f^4:

$$\psi^1(\vec{p}) = 2c\,|\vec{p}|^{-1} f^4(\vec{p}) = 2\frac{\vec{p}\cdot\vec{f}(\vec{p})}{|\vec{p}|^2}.$$

ψ^1 is called the scalar component.

Now the scalar product (II.12) restricted to D_1 is:

$$(\psi, \phi) = \tfrac{1}{2}\{(\psi^2, \phi^2) + (\psi^3, \phi^3)\} + \frac{1}{2}\int dm(\vec{p})\,|\vec{p}|^2\overline{\psi^1(\vec{p})}\phi^1(\vec{p}), \tag{II.14}$$

and the restriction to D_1 of the form β is given by (we still call it β):

$$\beta(\psi, \phi) = \tfrac{1}{2}\{(\psi^2, \phi^2) + (\psi^3, \phi^3)\}. \tag{II.15}$$

β is degenerate. Its kernel is the following:

$$(D_1)_0 = \ker\beta = \{\chi \in D_1 \mid \chi^2 \equiv \chi^3 \equiv 0\}. \tag{II.16}$$

The elements of $(D_1)_0$ give the gauge transformations (cf. (I.5)). We see then that these transformations do not affect the components ψ^2 and ψ^3; they modify only the scalar component ψ^1.

On the other hand, the action of $\mathcal{P}$ given by $\mathcal{W}(a, \Lambda)$ on ψ^2 is determined by $V_+(a, \Lambda)$ only: ψ^2 is the helicity $+1$ component, we call it ψ^+. The same is true for ψ^3 with the opposite sign of helicity: we call it ψ^-.

Remark. We have the well-known result that the gauge transformations do not modify the helicity components.

Going on with the procedure of section I, we define the quotient space $\mathcal{D} = D_1/(D_1)_0$; two equivalent elements ψ^μ and ϕ^μ differ only by their scalar component:

$$\phi^1(\vec{p}) = \psi^1(\vec{p}) + 2\chi(\vec{p}),$$

where $\chi(\vec{p})\,(2, 0, 0, 0)$ is an element of $(D_1)_0$ (it is easy to verify that the image by $\Pi(\vec{p})$ of an element g of $(H_1)_0$ (see (I.4)) is of this particular form).

The space $\mathcal{D}$ is provided with the quotient norm and the quotient form β (cf. §I.2, (I.6) and (I.8)).

The action of $\mathcal{P}$ on $\mathcal{D}$ is defined by a restriction of $\mathcal{W}(a, \Lambda)$ (owing to the Lorentz condition $\psi^4 \equiv 0$):

$$W(a, \Lambda) = \begin{pmatrix} V_0(a, \Lambda) & \overline{z(\Lambda, \vec{p})}V_+(a, \Lambda) & z(\Lambda, p)V_-(a, \Lambda) \\ 0 & V_+(a, \Lambda) & 0 \\ 0 & 0 & V_-(a, \Lambda) \end{pmatrix} \tag{II.17}$$

The representative element $\tilde{\psi}^\mu$ of the class ψ written in Lorentz radiation gauge has its scalar component equal to zero (Coulomb condition). It is obtained by means of the gauge transformation defined by $\chi(\vec{p}) = -\frac{1}{2}\psi^1(\vec{p})$ (see (I.9)):

$$\tilde{\psi}^1(\vec{p}) = \psi^1(\vec{p}) + 2\chi(\vec{p}) = 0.$$

This representation element is orthogonal to $(D_1)_0$: the quotient norm of the class ψ is given by the norm of $\tilde{\psi}^\mu$ (cf. (I.6, 7)). Then, by comparison of (II.14) and

(II.15), we see immediately that the scalar product (ψ, ϕ) of two classes ψ and ϕ and the value $\beta(\psi, \phi)$ of the form β (evaluated on the same classes) are equal:

$$(\psi, \phi) = \beta(\psi, \phi).$$

On the other hand, the action of $\mathscr{P}$ on $\mathscr{D}$ which leaves invariant the Lorentz radiation gauge is obtained by the intertwining operator Π acting on $\tilde{U}$ (see (II.5) and (II.6)) and then by restriction to $\mathscr{D}$:

$$\tilde{W}(a, \Lambda) = \Pi(\vec{p})\tilde{U}(a, \Lambda)\Pi^{-1}(\overrightarrow{\Lambda^{-1}p}), \tag{II.18}$$

$$\tilde{W}(a, \Lambda) = \begin{pmatrix} * & 0 & 0 \\ 0 & V_+(a, \Lambda) & 0 \\ 0 & 0 & V_-(a, \Lambda) \end{pmatrix}. \tag{II.19}$$

The asterisk stands for

$$V_0(a, \Lambda)\left(1 - c\frac{p^\mu}{|\vec{p}|}\Lambda^4_\mu\right).$$

This extension is reducible to the direct sum of $V_+(a, \Lambda)$ and $V_-(a, \Lambda)$:

$$\tilde{w}(a, \Lambda) = V_+(a, \Lambda) \oplus V_-(a, \Lambda),$$

$$\tilde{w}(a, \Lambda) = \begin{pmatrix} V_+(a, \Lambda) & 0 \\ 0 & V_-(a, \Lambda) \end{pmatrix}. \tag{II.20}$$

Its action is given by:

$$\psi'(\vec{p}) = (\tilde{w}(a, \Lambda)\psi)(\vec{p}) = V_+(a, \Lambda)\psi^+(\overrightarrow{\Lambda^{-1}p}) \oplus V_-(a, \Lambda)\psi^-(\overrightarrow{\Lambda^{-1}p}).$$

III. Quantization

III.1. *Fock space* [15]

We want to sketch the construction of the Fock space in this framework; it is not necessary for this construction to limit oneself to the subspaces D, D_1, $(D_1)_0$, $\mathscr{D}$.

(i) First we consider the tensorial product $H_n = H_1 \otimes \cdots \otimes H_1$ with $n \geq 1$ and $H_0 = \mathbf{C}$. The scalar product and the form β are simply defined by the product:

$$(\psi_{(n)}, \phi_{(n)}) = \prod_{j=1}^{n} (\psi_j, \phi_j),$$

$$\beta(\psi_{(n)}, \phi_{(n)}) = \prod_{j=1}^{n} \beta(\psi_j, \phi_j),$$

where $\psi_{(n)} = (\psi_1, \ldots, \psi_n)$ and $\phi_{(n)} = (\phi_1, \ldots, \phi_n)$ belong to H_n. (ψ_j, ϕ_j) is given by (II.14) and $\beta(\psi_j, \phi_j)$ by (II.15).

The kernel of β, $(H_n)_0$, is given by the direct sum of n subspaces of H_n and each subspace by the tensorial product of $(n-1)$ spaces H_1 and one space $(H_1)_0$:

$$(H_n)_0 = \{\psi_{(n)} = (\psi_1, \ldots, \psi_n) \mid \exists k,\ 1 \leq k \leq n,\ \psi_k \in (H_1)_0\}$$
$$= ((H_1)_0 \otimes H_1 \otimes \cdots \otimes H_1) \oplus \cdots \oplus (H_1 \otimes \cdots \otimes H_1 \otimes (H_1)_0).$$

The quotient space $H_n/(H_n)_0$ is isomorphic to the tensorial product $\mathcal{H}_n = \mathcal{H}\otimes\cdots\otimes\mathcal{H}$ of the quotient space $\mathcal{H} = H_1/(H_1)_0$:

$$H_n/(H_n)_0 \approx \mathcal{H}_n.$$

In $\mathcal{H}_n$ we define two (bounded) operators:

$$c(\eta)\psi_{(n)} = (\eta, \psi_1, \ldots, \psi_n), \qquad n \geq 0, \tag{III.1}$$

$$\begin{aligned} &a(\eta)\psi_{(n)} = \beta(\eta, \psi_1)(\psi_2, \ldots, \psi_n), \qquad n \geq 1, \\ &a(\eta)\psi_{(0)} \equiv 0 \quad \text{and} \quad \eta \text{ belongs to } \mathcal{H}. \end{aligned} \tag{III.2}$$

(ii) Next we define the tensor space $\mathcal{T}(\mathcal{H})$ as the direct sum of the $\mathcal{H}_n$'s:

$$\mathcal{T}(\mathcal{H}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j;$$

an element Ψ of $\mathcal{T}(\mathcal{H})$ is a sequence of elements $\psi_{(n)}$ of $\mathcal{H}_n$ such that:

$$\Psi = \{\psi_{(0)}, \psi_{(1)}, \psi_{(2)}, \ldots\} \quad \text{and} \quad \|\Psi\|^2 = \sum_{n=0}^{\infty} \|\psi_{(n)}\|^2 < \infty.$$

Obviously, the scalar product and the form β are defined in $\mathcal{T}(\mathcal{H})$ by the sum:

$$(\Psi, \Phi) = \sum_{n=0}^{\infty} (\psi_{(n)}, \phi_{(n)}),$$

$$\beta(\Psi, \Phi) = \sum_{n=0}^{\infty} \beta(\psi_{(n)}, \phi_{(n)}).$$

The operators $c(\cdot)$ and $a(\cdot)$ extend (canonically) to $\mathcal{T}(\mathcal{H})$:

$$c(\eta)\Psi = \{0, c(\eta)\psi_{(0)}, c(\eta)\psi_{(1)}, \ldots\},$$

$$a(\eta)\Psi = \{a(\eta)\psi_{(1)}, a(\eta)\psi_{(2)}, a(\eta)\psi_{(3)}, \ldots\}.$$

Moreover we define two other operators:

(1) the symmetrisation operator S:

$$S\Psi = \{S\psi_{(0)}, S\psi_{(1)}, S\psi_{(2)}, \ldots\}$$

with $S\psi_{(n)} = (n!)^{-1}\sum_\sigma (\psi_{\sigma(1)}, \ldots, \psi_{\sigma(n)})$ and the sum runs over the $n!$ permutations of $(1, \ldots, n)$;

(2) the number operator N:

$$N\Psi = \{0\psi_{(0)}, 1\psi_{(1)}, 2\psi_{(2)}, \ldots\}.$$

(iii) Now we consider the subspace $\mathcal{G}$ of $\mathcal{T}(\mathcal{H})$ generated by the finite linear combinations of elements of the form $\{0, \ldots, 0, \psi_{(k)}, 0, \ldots\}$. In other words, an element Ψ of $\mathcal{G}$ is such that it exists a subset $\mathcal{I}$ of the set $\mathbf{N}$ and $\psi_{(k)} \equiv 0$ for $k = \mathbf{N}\backslash\mathcal{I}$. $\mathcal{G}$ is dense in $\mathcal{T}(\mathcal{H})$.

The symmetrised space $S\mathcal{T}(\mathcal{H})$ is the Fock space $\mathcal{F}$. We call $\mathcal{F}_0$ the symmetrised subspace $S\mathcal{G}$; $\mathcal{F}_0$ is a dense subspace of $\mathcal{F}$.

III.2. *Field operator*

We have seen in section II that the euclidean norm $\|\cdot\|$ and the β-norm $\|\cdot\|_\beta$ are the same on the quotient space $\mathcal{H}$. But to emphasize the invariant character under the Poincaré group we will use the form β instead of the scalar product, in particular for the adjointness: Let $\mathcal{O}$ be a linear operator defined on $\mathcal{T}(\mathcal{H})$; we call adjoint of $\mathcal{O}$ an operator $\mathcal{O}\dagger$ such that

$$\beta(\Phi, \mathcal{O}\dagger\Psi) = \beta(\mathcal{O}\Phi, \Psi), \qquad \Phi, \Psi \in \mathcal{T}(\mathcal{H}).$$

We define now the annihilation operator with domain $\mathcal{G}$:

$$a(\cdot) = S\mathscr{a}(\cdot)\sqrt{N}S$$

and its adjoint, the creation operator:

$$a\dagger(\cdot) = S\sqrt{N}^{c}(\cdot)S.$$

They obey the following commutation rules:

$$[a(\eta), a(\xi)] = [a\dagger(\eta), a\dagger(\xi)] = 0,$$
$$[a(\eta), a\dagger(\xi)] = \beta(\eta, \xi)S, \qquad \eta, \xi \in \mathcal{H}.$$

The field operator $\mathscr{A}(\cdot)$, with domain $\mathcal{F}_0$, is defined by

$$\mathscr{A}(\cdot) = \tfrac{1}{2}(a(\cdot) + a\dagger(\cdot)).$$

$\mathscr{A}(\cdot)$ is essentially selfadjoint on $\mathcal{F}_0$. the commutation rules are:

$$[\mathscr{A}(\psi), \mathscr{A}(\phi)] = \frac{i}{2}\,\mathrm{Im}\,\beta(\psi, \phi), \qquad \psi, \phi \in \mathcal{H}.$$

III.3. *Components of the field operator*

Until now we have associated a field operator $\mathscr{A}(\psi)$ to each state ψ of the electromagnetic field. To write down explicitly this state ψ, we have to choose a representative element ψ^μ. Similarly, for the field operator $\mathscr{A}(\psi)$ we must define its component $\mathscr{A}_\mu(\psi)$.

(i) Let us consider the state ψ. We choose the Coulomb representative element $\tilde{\psi}^\mu$:

$$\tilde{\psi}^\mu = (0;\ \psi^+;\ \psi^-;\ 0). \tag{III.3}$$

The components $\tilde{\mathscr{A}}_\mu(\psi)$ of the field operator $\mathscr{A}(\psi)$ in Lorentz radiation gauge are defined as:

$$\tilde{\mathscr{A}}_\mu(\psi) = (0;\ \mathscr{A}_+(\psi);\ \mathscr{A}_-(\psi);\ 0), \tag{III.4}$$

where $\mathscr{A}_\pm(\psi)$ acts only on the λ-helicity components $(\phi_n^k)^\pm$ of a state Φ of $\mathcal{F}_0$, $\Phi = \{\phi_{(0)}, \phi_{(1)}, \phi_{(2)}, \ldots\}$, where $\phi_{(n)} = (\phi_n^1, \ldots, \phi_n^n)$ belongs to $\mathcal{H}_n$ and $\phi_n^k = (0; (\phi_n^k)^+; (\phi_n^k)^-; 0)$ to $\mathcal{H}$. This $\mathscr{A}_\pm(\psi)$ may be written as $\mathscr{A}(\psi^\pm)$.

(ii) A particular attention must be paid to 'zero' class of $\mathcal{H}$(i.e. the kernel of β, $(H_1)_0$). The representative elements of this class are of the type (cf. (II.16)):

$$\chi^\mu = (\chi, 0, 0, 0).$$

As this class is the kernel of the form β, the operator $\mathscr{a}(\chi)$: $\mathcal{H}_n \to \mathcal{H}_{n-1}$ defined by (III.2) is identically zero:

$$\mathscr{a}(\chi)\psi_{(n)} = \beta(\chi, \psi_1)(\psi_2, \ldots, \psi_n) \equiv 0,$$

for every $\psi_{(n)} = (\psi_1, \ldots, \psi_n)$ belonging to $\mathcal{H}_n$. Thus the annihilation operator $a(\chi)$ associated with this class is identically zero.
On the other hand, the operator $c(\chi): \mathcal{H}_n \to \mathcal{H}_{n+1}$ defined by (III.1) applies the 'zero' class of $\mathcal{H}_n$ onto the 'zero' class of $\mathcal{H}_{n+1}$:

$$c(\chi)\psi_{(n)} = (\chi, \psi_1, \ldots, \psi_n) \in (H_{n+1})_0,$$

for every $\psi_{(n)}$ of $\bar{\mathcal{H}}_n$. The creation operator $a\dagger(\chi)$ commutes with every annihilation operator $a(\psi)$:

$$[a(\psi), a\dagger(\chi)] = \beta(\psi, \chi)S \equiv 0, \qquad \forall \psi \in \mathcal{H}.$$

Under these conditions, the field operator $\mathcal{A}(\chi)$ reduces to $\frac{1}{2}a\dagger(\chi)$ and commutes with every field operator $\mathcal{A}(\psi)$ associated with any state ψ of $\mathcal{H}$:

$$\mathcal{A}(\chi) = \tfrac{1}{2}a\dagger(\chi),$$
$$[\mathcal{A}(\chi), \mathcal{A}(\psi)] = 0, \qquad \forall \psi \in \mathcal{H}.$$

We will say that the field operator $\mathcal{A}(\chi)$ is *classical*. The components of the field operator $\mathcal{A}(\chi)$ associated with this particular state are

$$\mathcal{A}_\mu(\chi) = (\mathcal{A}_0(\chi); 0; 0; 0).$$

$\mathcal{A}_0(\chi)$ is called the scalar (classical) part (we put the index '$_0$' instead of '$_1$' because this part has no helicity). Note that if we choose the Coulomb representative element $\tilde{\chi}^\mu = (0; 0; 0; 0)$, $\tilde{\mathcal{A}}_\mu(\chi)$ is identically zero.

(iii) Let us consider a symmetrised element $S\psi_{(n)}$ of $S\mathcal{H}_n$. We call it *a state with n photons*. The Fock space $\mathcal{F}$ is the space of states with an undetermined number of photons.

The operators $a(\psi)$ and $a\dagger(\psi)$ are the annihilation and creation operators of a photon in the state ψ.

The 'zero' class of $\mathcal{F}$ (which is the direct sum of the 'zero' classes of each subspace $S\mathcal{H}_n$) is the state without any photon: we call it *the vacuum.* It is represented by the following element of $\mathcal{F}$: $\eta = \{\eta_{(0)}, 0, \ldots\}$ and $\eta_{(0)} = 1 \in \mathcal{H}_0$.

Remark. In the canonical basis, we can define in the same way the field operator and its components:

$$\tilde{f}^\mu = (\tilde{f}^1, \tilde{f}^2, \tilde{f}^3, 0), \tag{III.3 bis}$$

$$\tilde{A}_\mu(f) = (\tilde{A}_1(f), \tilde{A}_2(f), \tilde{A}_3(f), 0). \tag{III.4 bis}$$

We use a different symbol to distinguish between the canonical and helicity bases (see §IV.2). But, in the canonical basis, the rôle played by the scalar classical component $\tilde{A}_4(f) \equiv 0$ is less transparent.

IV. Covariance and gauge invariance

We want to discuss the covariance of the field operator under the action of the restricted Poincaré group $\mathcal{P}$. Contrary to section III, we have to consider the subspace $\mathcal{D}$ of $\mathcal{H}$, which is the domain of the extension of representations $W(a, \Lambda)$ (II.17). We start with the electromagnetic radiation field alone. Next we will look at the interaction of this field with matter.

IV.1. *Radiation field (free propagation)*

Let us consider the field operator $\mathscr{A}(\psi)$ associated with the state ψ. Both are given in the Lorentz radiation gauge (III.3, 4). The restricted extension $\tilde{w}(a, \Lambda)$ given by (II.20) leaves invariant the Lorentz radiation gauge: it applies the Coulomb representative element $\tilde{\psi}^{\mu}$ of the state ψ onto the Coulomb representative element $\tilde{\psi}'^{\mu}$ of the state ψ'. The state ψ' is obtained by the action of the Poincaré transformation (a, Λ) on the state ψ.

It is the same for the field operator:

We recall that the action of any unitary operator $\mathscr{U}$ on the field operator is given by:

$$\mathscr{U}\mathscr{A}(\psi)\mathscr{U}^{-1} = \mathscr{A}(\mathscr{U}\psi).$$

Thus the action of a Poincaré transformation (a, Λ) on the field operator $\mathscr{A}(\psi)$ in Lorentz radiation gauge, action which preserves this gauge, is given by the restricted extension $\tilde{\omega}(a, \Lambda)$:

$$\tilde{w}(a, \Lambda)\mathscr{A}(\psi)\tilde{w}^{-1}(a, \Lambda) = \mathscr{A}(\tilde{w}(a, \Lambda)\psi) = \mathscr{A}(\psi'). \tag{IV.1}$$

By considering (II.5) and (II.18), we see that $\tilde{W}(a, \Lambda)$ (and also $\tilde{w}(a, \Lambda)$) is compound of the extension $W(a, \Lambda)$ followed by a gauge transformation. But we have shown that a gauge transformation affects only the scalar component of a state and leaves invariant the helicity components. For a fixed state, the choice of a particular gauge corresponds to the choice of a particular representative element (because a state is an equivalence class). Thus, in this framework, this choice influences only the scalar component. As the gauge transformations are all in the same equivalence class, i.e. the 'zero' class $(D_1)_0$, those transformations are representative elements of this class. In the Fock space $\mathscr{F}$, the direct sum of the 'zero' classes $S(\mathscr{H}_n)_0$ of each subspace $S\mathscr{H}_n$ defines the vacuum. Then we can say that a gauge transformation changes only the representation of the vacuum [6].

On the other hand, the field operator attached to the scalar component is classical and gauge dependant. We have choosen the Lorentz radiation gauge in which this operator is identically zero. Thus the radiation field is divided into a classical and gauge dependant part and a quantum and gauge independant part. The latter part is described by states of photons and quantum field operators acting on them. Moreover the action of every Poincaré transformation (a, Λ) is mixed with a gauge transformation (which depends on (a, Λ)) such that the choice of the Lorentz radiation gauge is left invariant: the classical part of the field is still zero.

We insist that this situation is only possible for a radiation field which propagates freely. We will see now that the situation is different if the radiation field interacts with matter.

IV.2. *Interaction with matter*

Let us consider a material system (e.g. an atom) which interacts with an electromagnetic radiation field. The evolution of this system is governed by a Schrödinger equation. By virtue of Galilei principle, the field which appears in the hamiltonian of a massive particle must be a fourvector [8].

Now the action of the Poincaré group given in canonical representation by

$\tilde{U}(a, \Lambda)$ and in helicity representation by $\tilde{W}(a, \Lambda)$ and $\tilde{w}(a, \Lambda)$ (II.5, 19, 20) is not the one which must act on fourvectors. Thus we have to work with $U(a, \Lambda)$ and $W(a, \Lambda)$ (see (II.4, 17)). Therefore the choice of the Lorentz radiation gauge is no more left invariant by the Poincaré group. One possibility would be to abandon the Lorentz radiation gauge. But, to our knowledge, the experimentalists always use that particular gauge (without justifying their choice). How to conciliate then the relativistic covariance of the field and the use of that particular gauge?

We suggest that the choice of the gauge is connected with the direction n of time, where time is the parameter of the evolution. The space-time symmetry is broken by this direction n (even if no electromagnetic radiation is present). Furthermore this direction determines the representation of the fourmomentum and spin operators of a relativistic particle [11]. The fourpotential appears with the fourmomentum in the hamiltonian and, from a passive point of view, every gauge transformation implies a change of the fourmomentum operator. Thus it is not surprising that this direction n determines the choice of the gauge [14].

Remark. For the forthcoming discussion, we use the canonical representation again to avoid the local dependance on $\vec{p}$.

As the Coulomb condition, which determines the Lorentz radiation gauge, is a transversality condition, we postulate that the gauge is fixed by the following relation:

$$g_{\mu\nu}n^{\mu}f^{\nu}=0, \qquad f\in\mathcal{H}, \tag{IV.2}$$

where the n^{μ}'s are the coordinates of the time-like fourvector n and the f^{ν}'s are the components of a representative element of the state f. (IV.2) appears as a generalised transversality condition which fixes the representative element.

The Coulomb condition corresponds to the choice of a frame such that n has the following coordinates:

$$\tilde{n}^{\mu}=(0,0,0,1). \tag{IV.3}$$

Under a Poincaré transformation (a, Λ), the coordinates n^{μ} change as usual for a fourvector:

$$n'^{\mu}=\Lambda^{\mu}_{\nu}n^{\nu}+a^{\mu}. \tag{IV.4}$$

Starting with $\tilde{n}^{\mu}$, the field operator $A(f)$ is given in the Lorentz radiation gauge. After a Poincaré transformation (a, Λ), n has the components given by (IV.4) and the field operator $A(f')$ is obtained with the help of $U(a, \Lambda)$ given by (II.4):

$$U(a,\Lambda)A(f)U^{-1}(a,\Lambda)=A(U(a,\Lambda)f)=A(f'). \tag{IV.5}$$

Thus, as $U(a, \Lambda)$ does not preserve the Lorentz radiation gauge, the field operator is no more written in that gauge, but in a gauge fixed by (IV.2), i.e.

$$g_{\mu\nu}n'^{\mu}f'^{\nu}=0.$$

This dependance on the components n^{μ} suggests to use them as a superselection rule and to introduce a family of state spaces $\{\mathcal{H}_{n^{\mu}}\}$ and a family of Fock spaces $\{\mathcal{F}_{n^{\mu}}\}$; the construction of the previous sections corresponds to $\mathcal{H}_{\tilde{n}^{\mu}}$ and $\mathcal{F}_{\tilde{n}^{\mu}}$. We lose in this manner the ambiguity due to the quotient structure: for every state f of $\mathcal{H}_{n^{\mu}}$, the condition (IV.2) fixes unequivocally how to write it (i.e. the gauge is fixed).

Example. Suppose that for n given by $\tilde{n}^\mu$, we first pick up a representative element f^μ of a state f such that (IV.2) is not satisfied. By means of a gauge transformation (I.5), we may obtain the right representative element $\tilde{f}^\mu$:

$$\tilde{f}^\mu(\vec{p}) = f^\mu(\vec{p}) + \chi(\vec{p})p^\mu.$$

The condition (IV.2) gives the function χ:

$$g_{\mu\nu}\tilde{n}\tilde{f}^\nu(\vec{p}) = 0 \Rightarrow \chi(\vec{p}) = -\frac{\vec{p}\cdot\vec{f}(\vec{p})}{|\vec{p}|^2},$$

which is an already known result (see (I.9)). $\tilde{f}^\mu$ will be written with the index $n^\mu : f^\mu_{n^\mu}$.

Remark. In helicity representation, the components of n depend on $\vec{p}$. For example, the components $\tilde{n}^\mu$ become:

$$\tilde{n}^\mu(\vec{p}) = \Pi(\vec{p})^\mu_\nu n^\nu = \left(\frac{c}{|\vec{p}|}, 0, 0, c\,|\vec{p}|\right). \tag{IV.3 bis}$$

The condition (IV.2) is written as (cf. (II.13)):

$$g'_{\mu\nu}n^\mu(\vec{p})\psi^\nu(\vec{p}) = n^2(\vec{p})\psi^+(\vec{p}) + n^3(\vec{p})\psi^-(\vec{p}) - n^4(\vec{p})\psi^1(\vec{p}) = 0 \tag{IV.2 bis}$$

for every $\vec{p}$ belonging to C_+.

The transformation law (IV.4) becomes:

$$n'^\mu(\vec{p}) = \Lambda(\vec{p})^\mu_\nu(\vec{p}) + a(\vec{p})^\mu, \tag{IV.4 bis}$$

where

$$\Lambda(\vec{p}) = \Pi(\vec{p})\Lambda\Pi^{-1}(\vec{p})$$

and

$$a(\vec{p}) = \Pi(\vec{p})a, \qquad a \in \mathbf{R}^4.$$

Finally, the law (IV.5) of the field operator is given by:

$$W(a, \Lambda)\mathcal{A}(\psi)W^{-1}(a, \Lambda) = \mathcal{A}(W(a, \Lambda)\psi) = \mathcal{A}(\psi'), \tag{IV.5 bis}$$

where $W(a, \Lambda)$ is defined by (II.17).

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