Zeitschrift: Helvetica Physica Acta

Band: 55 (1982)

Heft: 3

Artikel: Evaluation of transient amplitudes between Dirac spinors

Autor: Caffo, M. / Remiddi, E.

DOI: https://doi.org/10.5169/seals-115286

Helvetica Physica Acta, Vol. **55** (1982) 339–347

0018–0238/82/030339–09$1.50 + 0.20/0

Evaluation of transient amplitudes between Dirac spinors[1]

By M. Caffo, CERN, 1211 Geneva 23 and
INFN, Sezione di Bologna
E. Remiddi, Département de Physique Théorique,
Université de Genève, 1211 Genève 4, Switzerland and
Istituto di Fisica, Università di Bologna and
INFN, Sezione di Bologna

(2. VII. 1982)

Abstract. We present a simple method for expressing transition amplitudes between fermion states of definite polarization in terms of momenta and polarization vectors only, rather than γ-matrices and spinors. As an illustration, we work out Coulomb and Bhabha scattering for polarized particles.

1. A transition amplitude A between two spinor states is written as

$$A = \bar{u}(j)Mu(i), \tag{1.1}$$

where M is a string or a sum of strings of γ-matrices and $\bar{u}(j), u(i)$ are the appropriate four component Dirac spinors corresponding to the sets of quantum numbers j, i (momentum, polarization and particle or antiparticle nature of the state). In textbooks the explicit values of components of free spinors (depending in particular on the representation used for the γ-matrices) are usually given, but very rarely used for the direct evaluation of equation (1.1). The standard treatment consists in considering the square

$$|A|^2 = \bar{u}(j)Mu(i)\bar{u}(i)\gamma_4 M^+ \gamma_4 u(j) \tag{1.2}$$

and in evaluating it as

$$|A|^2 = \mathrm{Tr}\,[MU(i, i)\gamma_4 M^+ \gamma_4 U(j, j)], \tag{1.3}$$

where one puts

$$U(i, i) = u(i)\bar{u}(i); \tag{1.4}$$

the explicit expression of $U(i, i)$ is then given. For a fixed polarization or the sum on the two spin states, it involves only γ-matrices in a way which is of course independent from their representation; it is familiar to everybody and will not be rewritten in this introduction.

The purpose of this note is rather to give the explicit expression of the

---

[1]) Partially supported by the Swiss National Science Foundation.

generalized operators

$$U(i, j) = u(i)\bar{u}(j), \tag{1.5}$$

again in terms of γ-matrices and of simple rational functions of momenta and spin-polarization vectors, to be used for rewriting equation (1.1) as

$$A = \mathrm{Tr}\,[MU(i, j)]. \tag{1.6}$$

After evaluation of the trace, A takes the explicit form of a suitable complex number, expressed in terms of momenta and polarization vectors, while everything else previously referring to spinors and γ-matrices has of course disappeared. Squaring A is then trivial.

The explicit form of equation (1.5) is given in equation (3.5) below. To our surprise, we could not find it and its use in textbooks, hence the present paper. Our result can be considered as an handy way of extracting helicity amplitudes; with respect to other methods for obtaining the same result, we observe that common helicity projectors apply usually to a given process only and are of no direct use for different processes, while our formulas do not suffer from that restriction.

For a simple M, the evaluation of equation (1.6) is as complicated as that of equation (1.3); yet, equation (1.6) provides another way of looking at equation (1.1) and one more way of studying its properties. In complicated cases, when M is the sum of several different contributions and interference effects are of interest equation (1.6) might prove to be more convenient than equation (1.3), as those equations are respectively linear and quadratic in the number of the contributions to M. We have in mind in particular the high energy e^+e^- scattering into various final channels, where the detailed study of the polarization of the concerned particles can provide a sensible way of investigating, for instance, parity violating interference effects and of testing the existing models of electroweak interactions.

The matters of this paper being anyhow simple, we were not afraid from being sometimes pedagogical: Section 2 contains a derivation of the well known equation (1.4); it is given for completeness, but also because obtained without any reference to the explicit values of the involved spinors, in a way which is slightly different from most textbooks; Section 3 establishes equation (1.5), which is the main result of this paper; Section 4 applies it explicitly to the very simple case of Coulomb potential scattering; Section 5 finally deals with the $e^+e^- \to e^+e^-$ process in tree approximation; we discuss the different features of the direct and annihilation channels, giving the explicit values of the transition amplitudes.

The application of the method to more complicated processes will be given elsewhere.

2. To introduce our notation we write Dirac's equation as

$$\begin{aligned} &\gamma_4(i\vec{p}\cdot\vec{\gamma} - p_0\gamma_4 + m)u(\vec{p}) = 0, \\ &u^+(\vec{p})(-i\vec{p}\vec{\gamma} - p_0\gamma_4 + m)\gamma_4 = 0, \end{aligned} \tag{2.1}$$

with hermitian γ-matrices. If $p_0 = E = +\sqrt{\vec{p}^2 + m^2}$ and $p = (\vec{p}, E)$, equation (2.1) becomes

$$\begin{aligned} &\gamma_4(i\not{p} + m)u_+(\vec{p}) = 0, \\ &u_+^+(\vec{p})\gamma_4(i\not{p} + m) = 0, \end{aligned} \tag{2.2}$$

while if $p_0 = E = -\sqrt{p^2+m^2}$ and $p = (\vec{p}, E)$, as in the previous case, one has

$$\begin{aligned}(-i\not{p}+m)\gamma_4 u_-(\vec{p}) &= 0,\\ u_-^+(\vec{p})(-i\not{p}+m)\gamma_4 &= 0.\end{aligned} \tag{2.3}$$

To write down explicitly the solutions u, an explicit representation of the γ-matrices has to be chosen. It is however possible to write the bilinear forms, such as equation (1.4) to which we are interested, without explicit reference to spinor components. To that purpose we define as usual the projectors $\Lambda_+(\vec{p})$ and $\Lambda_-(\vec{p})$

$$\begin{aligned}\Lambda_+(\vec{p}) &= \frac{1}{2E}[E+(m-i\vec{p}\vec{\gamma})\gamma_4] = \frac{1}{2E}(-i\not{p}+m)\gamma_4;\\ \Lambda_-(\vec{p}) &= \frac{1}{2E}[E-(m-i\vec{p}\vec{\gamma})\gamma_4] = \frac{1}{2E}\gamma_4(-i\not{p}-m)\end{aligned} \tag{2.4}$$

and the polarization vector $a = (\vec{a}, a_0)$ with the properties

$$(a\cdot p) = 0, \qquad (a\cdot a) = 1, \qquad [i\gamma_5\not{a}, i\not{p}] = 0, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{2.5}$$

We are then ready to introduce the 4 operators ($\lambda = \pm 1$)

$$\begin{aligned}P_+(p, a, \lambda) &= \Lambda_+(\vec{p})\tfrac{1}{2}(1+i\lambda\gamma_5\not{a})\Lambda_+(\vec{p})\\ &= \Lambda_+(\vec{p})\tfrac{1}{2}(1+i\lambda\gamma_5\vec{a}\cdot\vec{\gamma})\Lambda_+(\vec{p})\\ &= \tfrac{1}{2}(1+i\lambda\gamma_5\not{a})\frac{1}{2E}(-i\not{p}+m)\gamma_4,\\ P_-(p, a, \lambda) &= \Lambda_-(\vec{p})\tfrac{1}{2}(1+i\lambda\gamma_5\not{a})\Lambda_-(\vec{p})\\ &= \Lambda_-(\vec{p})\tfrac{1}{2}(1+i\lambda\gamma_5\vec{a}\cdot\vec{\gamma})\Lambda_-(\vec{p})\\ &= \gamma_4\frac{1}{2E}(-i\not{p}-m)\tfrac{1}{2}(1+i\lambda\gamma_5\not{a}).\end{aligned} \tag{2.6}$$

For given (p, a), call P_i, $i = 1, 2, 3, 4$ the four above operators. By using the appropriate lines of equations (2.6) one immediately verifies that they satisfy the relations

$$\begin{aligned}P_i^+ &= P_i,\\ P_i\cdot P_j &= \delta_{ij}P_j,\\ \sum_{i=1}^{4} P_i &= 1.\end{aligned} \tag{2.7}$$

The first equation says that the P_i's are hermitian; the second, that their eigenvalues are 0 or 1, hence they are projectors; the third that they form a complete set. One has further

$$\operatorname{Tr} P_i = 1, \qquad i = 1, \ldots 4,$$

showing that each of them projects in a single state. It is then possible to write

$$P_+(p, a, \lambda) = u(\vec{p}, a, \lambda)u^+(\vec{p}, a, \lambda), \tag{2.8}$$

where $\lambda = \pm 1$, $u(p, a, \lambda)$ is the eigenstate of $P_+(p, a, \lambda)$ to the eigenvalue 1, normalized to $u^+u=1$ and satisfying equation (2.2), as immediately seen. Similarly one can write

$$P_-(p, a, \lambda)=w(\vec{p}, a, \lambda)w^+(\vec{p}, a, \lambda), \tag{2.9}$$

where $\lambda=\pm 1$, $w^+w=1$ and w, w^+ satisfy equation (2.3). The more usual v's can be defined as

$$v(\vec{p}, a, \lambda)=Cw(\vec{p}^*, a^*, \lambda) \tag{2.10}$$

with $C^+C=1$, where $a^*=(\vec{a}, -a_0)$, $p^*=(-\vec{p}, E)$ as compared to $a=(\vec{a}, a_0)$, $p=(\vec{p}, E)$ and $(p\cdot a)=(p^*\cdot a^*)=0$, $(a\cdot a)=(a^*\cdot a^*)=1$.

The known standard formulas, to be used later, are then

$$\begin{aligned} u(\vec{p}, a, \lambda)\bar{u}(\vec{p}, a, \lambda)&=\frac{1}{2E}(-i\not{p}+m)\tfrac{1}{2}(1+i\lambda\gamma_5\not{a}),\\ v(\vec{p}, a, \lambda)\bar{v}(\vec{p}, a, \lambda)&=\frac{1}{2E}(-i\not{p}-m)\tfrac{1}{2}(1+i\lambda\gamma_5\not{a}), \end{aligned} \tag{2.11}$$

with the normalization $u^+u=v^+v=1$. They follow immediately from equations (2.9), (2.10) and the last of equations (2.4). Note that the products $u\bar{u}$ and $v\bar{v}$ in equations (2.11) strictly speaking are not projectors, as they do not satisfy equations (2.7).

3. We will now work out equation (1.5). For ease of notation, let us write

$$u(1)=u(\vec{p}_1, a_1, \lambda_1), \tag{3.1}$$

with the usual kinematics

$$p_1=(\vec{p}_1, E_1), \qquad p_1^2=-m_1^2, \qquad a_1^2=1, \qquad (p_1\cdot a_1)=0, \qquad \lambda=\pm 1 \tag{3.2}$$

and

$$U(1)=u(1)\bar{u}(1)=\frac{1}{2E_1}(-i\not{p}_1+m_1)\tfrac{1}{2}(1+i\lambda_1\gamma_5\not{a}_1). \tag{3.3}$$

Similarly

$$\begin{aligned} &u(2)=u(\vec{p}_2, a_2, \lambda_2),\\ &U(2)=u(2)\bar{u}(2)=\frac{1}{2E_2}(-i\not{p}_2+m_2)\tfrac{1}{2}(1+i\lambda_2\gamma_5\not{a}_2),\\ &p_2=(\vec{p}_2, E_2), \qquad p_2^2=-m_2^2, \qquad a_2^2=1, \qquad (p_2\cdot a_2)=0, \qquad \lambda=\pm 1. \end{aligned} \tag{3.4}$$

The masses m_1, m_2 can in general be different, as the concerned spinors are by no means restricted to refer to a single fermion line. Evidentiating for once spinor indices

$$U_{\alpha\beta}(1, 2)=u_\alpha(1)\bar{u}_\beta(2)=N_1^{-1}(1, 2)U_{\alpha\rho}(1)U_{\rho\beta}(2), \tag{3.5}$$

where $N_1(1, 2)$ is a complex constant given by

$$N_1(1, 2)=\bar{u}(1)u(2). \tag{3.6}$$

By using equations (3.3), (3.4) and taking the trace of the second and third term in equations (3.5), one obtains

$$|N_1(1,2)|^2 = \frac{1}{4E_1E_2}\{[m_1m_2-(p_1p_2)][1+\lambda_1\lambda_2(a_1a_2)]+\lambda_1\lambda_2(p_1a_2)(p_2a_1)\}. \tag{3.7}$$

For any choice of λ_1, λ_2, equation (3.7) determines $N_1(1,2)$ up to an intrinsically undetermined phase. We find it useful to introduce the projectors

$$P_\pm(1,2) = \tfrac{1}{2}(1\pm\lambda_1\lambda_2); \tag{3.8}$$

as λ_1, λ_2 can take the values ± 1, $P_\pm(1,2)$ takes only the values 0 or 1. Equation (3.7) then gives

$$\begin{aligned} N_1^{-1}(1,2) &= 2\sqrt{E_1E_2} \\ &\cdot\{e^{-i\varphi_1^+}[(m_1m_2-p_1p_2)(1+a_1a_2)+(p_1a_2)(p_2a_1)]^{-1/2}P_+(1,2) \\ &+e^{-i\varphi_1^-}[(m_1m_2-p_1p_2)(1-a_1a_2)-(p_1a_2)(p_2a_1)]^{-1/2}P_-(1,2)\}. \end{aligned} \tag{3.9}$$

Equation (3.9) makes sense, of course, only if the values of the square brackets do not vanish. We will encounter and discuss in Section 5 a case of vanishing values. Note in equation (3.9) the presence of arbitrary, as yet unspecified, phases. Equations (3.5) and (3.9) are the required result.

So far for the particle-particle case. The extension to antiparticle-antiparticle or particle-antiparticle is straightforward. If

$$\begin{aligned} &v(i) = v(\vec{p}_i, b_i, \lambda_i) \\ &V(i) = v(i)\bar{v}(i) \\ &p_i = (\vec{p}_i, E_i), \qquad p_i^2 = -m_i^2, \qquad b_i^2 = 1, \qquad (p_ib_i) = 0, \qquad \lambda_i = \pm 1, \end{aligned} \tag{3.10}$$

we obtain correspondingly

$$\begin{aligned} V(1,2) &= v(1)\bar{v}(2) = N_2^{-1}(1,2)\,V(1)\,V(2), \\ W(1,2) &= v(1)\bar{u}(2) = N_3^{-1}(1,2)\,V(1)\,U(2), \\ X(1,2) &= u(1)\bar{v}(2) = N_4^{-1}(1,2)\,U(1)\,V(2), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &N_3^{-1}(1,2) = 2\sqrt{E_1E_2}. \\ &\{e^{-i\varphi_3^+}[(-m_1m_2-p_1p_2)(1+a_1a_2)+(p_1a_2)(p_2a_1)]^{-1/2}P_+(1,2) \\ &+e^{-i\varphi_3^-}[(-m_1m_2-p_1p_2)(1-a_1a_2)-(p_1a_2)(p_2a_1)]^{-1/2}P_-(1,2)\} \end{aligned} \tag{3.12}$$

and $N_2(1,2)$, $N_4(1,2)$ are identical to $N_1(1,2)$, $N_3(1,2)$ respectively (if allowance is made for the arbitrariness of the phases).

4. As a first application, we consider Coulomb potential scattering of an electron. If $\vec{k}$ is the momentum transfer, the amplitude is

$$A = -\frac{Ze^2}{\vec{k}^2}\,\bar{u}(2)\gamma_4 u(1). \tag{4.1}$$

According to the discussion of previous sections, we read A as

$$A = -\frac{Ze^2}{\vec{k}^2}\,\mathrm{Tr}\,[\gamma_4 U(1,2)], \tag{4.2}$$

where $U(1, 2)$ is given in equations (3.5). For the explicit calculation, we take the kinematics as

$$p_1 = (\vec{p}_1, E), \qquad p_2 = (\vec{p}_2, E), \qquad |\vec{p}_1| = |\vec{p}_2| = \sqrt{E^2 - m^2} = \beta E,$$

$$\vec{p}_2 = \vec{p}_1 + \vec{k}, \qquad \vec{p}_1 \cdot \vec{p}_2 = p^2 \cos\left(\frac{\theta}{2}\right),$$

where θ is the scattering angle and the vectors a_i, $i = 1, 2$ correspond to longitudinal polarization

$$a_i = \left(\frac{\vec{p}_i}{m\beta}, \frac{E}{m}\beta\right). \tag{4.3}$$

One then has

$$N_1^{-1}(1, 2) = \left[\frac{m}{E}\cos\left(\frac{\theta}{2}\right)\right]^{-1} P_+(1, 2) + \left[\sin\left(\frac{\theta}{2}\right)\right]^{-1} P_-(1, 2), \tag{4.4}$$

where $\lambda_i = \pm 1$, $P_\pm(1, 2)$ is given in equation (3.8) and phases have been dropped. After performing the trace, equation (4.2) reads

$$A = -\frac{Ze^2}{\vec{k}^2}\left[\cos\left(\frac{\theta}{2}\right)P_+(1, 2) + \frac{m}{E}\sin\left(\frac{\theta}{2}\right)P_-(1, 2)\right]. \tag{4.5}$$

Obtaining $|A|^2$ is straightforward, as the $P_\pm(1, 2)$ are projectors:

$$|A|^2 = \frac{(Ze^2)^2}{(\vec{k}^2)^2}\left[\cos^2\left(\frac{\theta}{2}\right)P_+(1, 2) + \left(\frac{m}{E}\right)^2 \sin^2\left(\frac{\theta}{2}\right)P_-(1, 2)\right]. \tag{4.6}$$

The Mott scattering cross section for fully polarized initial and final states is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{(4\pi)^2}(2E)^2 |A|^2 \\ &= \frac{(Z\alpha)^2}{4E^2} \frac{1}{\beta^4 \sin^4\left(\frac{\theta}{2}\right)}\left[\cos^2\left(\frac{\theta}{2}\right)P_+(1, 2) + \left(\frac{m}{E}\right)^2 \sin^2\left(\frac{\theta}{2}\right)P_-(1, 2)\right].\end{aligned} \tag{4.7}$$

When summing on the final polarization (which amounts here to averaging on the initial), it simplifies to the familiar expression

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4E^2} \frac{1}{\beta^4 \sin^4\left(\frac{\theta}{2}\right)}\left[1 - \beta^2 \sin^2\left(\frac{\theta}{2}\right)\right]. \tag{4.8}$$

A formula equivalent to equation (4.5) can be found in [1], where it is obtained by using the explicit representation dependent expression of the spinor components.

5. The t and s channel graphs contributing in lowest order to e^+e^- elastic

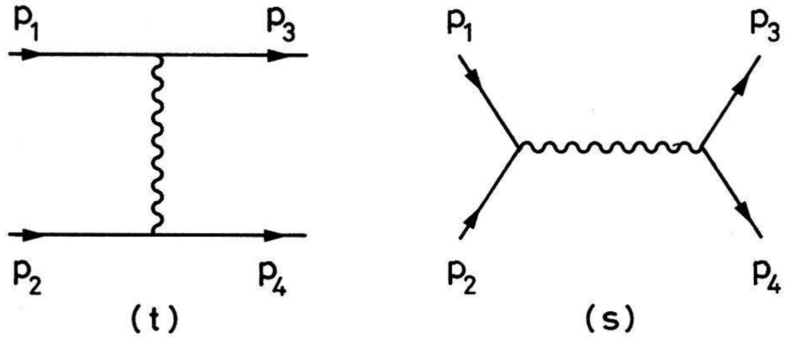

Figure 1
The t and s channel lowest order graphs for e^+e^- scattering.

scattering are shown in Fig. 1. The momenta of the in and out electrons are p_1, p_3, those of positrons p_2, p_4. The kinematics in the c.m. system is

$$\begin{aligned}
&p_1+p_2=p_3+p_4, \qquad s=-(p_1+p_2)^2=4E^2, \qquad p=\sqrt{E^2-m^2}=\beta E,\\
&\vec{p}_1=-\vec{p}_2=p(0,0,1), \qquad \vec{p}_3=-\vec{p}_4=p(\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta),\\
&t=-(p_3-p_1)^2=-4p^2\sin^2\frac{\theta}{2}.
\end{aligned} \tag{5.1}$$

The corresponding amplitude is

$$\begin{aligned}
&A(t,s)=A_t+A_s,\\
&A_t=\frac{e^2}{t}\bar{u}(3)\gamma_\mu u(1)\bar{v}(2)\gamma_\mu v(4),\\
&A_s=-\frac{e^2}{s}\bar{v}(2)\gamma_\mu u(1)\bar{u}(3)\gamma_\mu v(4).
\end{aligned} \tag{5.2}$$

As in previous sections, we restrict ourselves to longitudinal polarizations. We put

$$\begin{aligned}
&U(1,3)=u(1)\bar{u}(3)=N_1^{-1}(1,3)U(1)U(3),\\
&V(4,2)=v(4)\bar{v}(2)=N_2^{-1}(4,2)V(4)V(2),
\end{aligned} \tag{5.3}$$

so obtaining

$$\begin{aligned}
N_1^{-1}(1,3)&=\{\bar{u}(1)u(3)\}^{-1}\\
&=\lambda_1\left[\frac{m}{E}\cos\left(\frac{\theta}{2}\right)\right]^{-1}P_+(1,3)+e^{-i\lambda_1\varphi}\left[\sin\left(\frac{\theta}{2}\right)\right]^{-1}P_-(1,3),\\
N_2^{-1}(4,2)&=\{\bar{v}(4)v(2)\}^{-1}\\
&=-\lambda_2\left[\frac{m}{E}\cos\left(\frac{\theta}{2}\right)\right]^{-1}P_+(4,2)+e^{i\lambda_2\varphi}\left[\sin\left(\frac{\theta}{2}\right)\right]^{-1}P_-(4,2),
\end{aligned} \tag{5.4}$$

where $U(1,3)$, $N(1,3)$ etc. are as in previous sections, a part the trivial change of notation; the otherwise arbitrary phases of the coefficients of the various projectors $P_\pm(1,2)$ have been specified here for convenience of later use. Equations

(5.2) then become

$$\begin{aligned} A_t &= \frac{e^2}{t} \mathrm{Tr}\,[\gamma_\mu U(1,3)] \cdot \mathrm{Tr}\,[\gamma_\mu V(4,2)] \\ &= \frac{e^2}{t} \frac{1}{2E^2} \{\lambda_1\lambda_2[p^2(2+\lambda_1\lambda_2(1-\cos\theta)) + E^2(1+\cos\theta)]P_+(1,3)P_+(4,2) \\ &\quad + \lambda_2 e^{-i\lambda_1\varphi} mE \sin\theta P_-(1,3)P_+(4,2) \\ &\quad - \lambda_1 e^{i\lambda_2\varphi} mE \sin\theta P_+(1,3)P_-(4,2) \\ &\quad - e^{-i\lambda_1\varphi} e^{i\lambda_2\varphi} m^2(1-\cos\theta)P_-(1,3)P_-(4,2)\}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} A_s &= -\frac{e^2}{s} \mathrm{Tr}\,[\gamma_\mu U(1,3)\gamma_\mu V(4,2)] \\ &= -\frac{e^2}{s} \frac{1}{2E^2} \{[E^2(1+\cos\theta)(1-\lambda_1\lambda_2) - m^2\cos\theta(1+\lambda_1\lambda_2)]P_+(1,3)P_+(4,2) \\ &\quad - \lambda_2 e^{-i\lambda_1\varphi} 2mE \sin\theta P_-(1,3)P_+(4,2) \\ &\quad + \lambda_1 e^{i\lambda_2\varphi} 2mE \sin\theta P_+(1,3)P_-(4,2) \\ &\quad + [e^{-i\lambda_1\varphi} e^{i\lambda_2\varphi} E^2(1-\cos\theta)(1-\lambda_1\lambda_2) \\ &\quad - m^2\cos\theta(1+\lambda_1\lambda_2)]P_-(1,3)P_-(4,2)\}. \end{aligned} \tag{5.6}$$

Note that A_t is a product of traces, while A_s consists of a unique trace; but one could also introduce the operators

$$\begin{aligned} W(4,3) &= v(4)\bar{u}(3) = N_3^{-1}(4,3)V(4)U(3), \\ X(1,2) &= u(1)\bar{v}(2) = N_4^{-1}(1,2)U(1)V(2), \end{aligned} \tag{5.7}$$

so that the amplitudes can be evaluated as

$$\begin{aligned} A_t &= \frac{e^2}{t} \mathrm{Tr}\,[\gamma_\mu X(1,2)\gamma_\mu W(4,3)], \\ A_s &= -\frac{e^2}{s} \mathrm{Tr}\,[\gamma_\mu X(1,2)] \cdot \mathrm{Tr}\,[\gamma_\mu W(4,3)]. \end{aligned} \tag{5.8}$$

The last way of writing A_s is surely the natural one for studying processes like $e^+e^- \to \mu^+\mu^-$, in which the in and out particles are different. A minor technical problem arises however if we restrict ourselves, as previously, to longitudinal polarizations; one has in fact

$$N_4(1,2) = \beta P_+(1,2). \tag{5.9}$$

The coefficient of $P_-(1,2)$ being equal to zero, equation (5.9) cannot be inverted for its use in equation (5.7); similarly for $N_3(4,3)$. To overcome this point, it is sufficient to give to the polarization vectors a small transversal component, for instance

$$a_i = \frac{1}{m\sqrt{1+\varepsilon^2}} \left(\frac{1}{\beta} \vec{p}_i + m\varepsilon\vec{n}_i,\ \beta E_i \right), \tag{5.10}$$

with

$$\vec{n}_i \cdot \vec{p}_i = 0, \qquad \varepsilon \ll 1.$$

One then obtains

$$|N_4(1,2)|^2 = \beta^2 P_+(1,2) + (\varepsilon\beta)^2 P_-(1,2) \tag{5.11}$$

and

$$N_4^{-1}(1,2) = \beta^{-1} P_+(1,2) - i(\varepsilon\beta)^{-1} P_-(1,2); \tag{5.12}$$

similarly

$$N_3^{-1}(4,3) = \beta^{-1} P_+(4,3) - ie^{-i\lambda_4\varphi}(\varepsilon\beta)^{-1} P_-(4,3) \tag{5.13}$$

(the choice of the phases will be commented in a moment).

After carrying out explicitly the traces and some straightforward algebra, the result reads

$$\begin{aligned}
A_t = \frac{e^2}{t}\frac{1}{2E^2}\{&[2p^2(1+\lambda_2\lambda_4)+m^2(\lambda_2\lambda_4+\cos\theta)]P_+(1,2)P_+(4,3)\\
&+\lambda_2 e^{i\lambda_2\varphi} mE\sin\theta P_-(1,2)P_+(4,3)\\
&+\lambda_4 e^{-i\lambda_4\varphi} mE\sin\theta P_+(1,2)P_-(4,3)\\
&-[E^2(1+\cos\theta)(1+\lambda_2\lambda_4)-e^{i\lambda_2\varphi}e^{-i\lambda_4\varphi}m^2(\lambda_2\lambda_4+\cos\theta)P_-(1,2)P_-(4,3)]\},\\
A_s = -\frac{e^2}{s}\frac{1}{E^2}\{&-m^2\cos\theta P_+(1,2)P_+(4,3)\\
&-\lambda_2 e^{i\lambda_2\varphi} mE\sin\theta P_-(1,2)P_+(4,3)\\
&-\lambda_4 e^{-i\lambda_4\varphi} mE\sin\theta P_+(1,2)P_-(4,3)\\
&+e^{i\lambda_2\varphi}e^{-i\lambda_4\varphi}E^2(1+\lambda_2\lambda_4\cos\theta)P_-(1,2)P_-(4,3)\}.
\end{aligned} \tag{5.14}$$

They look different from equations (5.5), (5.6), but they are in fact identical, thanks to the particular phases choosen in equations (5.4), (5.12) and (5.13), when expressed in terms of the same set of projectors $P_\pm(1,3)$, $P_\pm(4,2)$ used there. A different choice of phases in equations (5.12), (5.13) would give equality up to phases, not identity between the two expressions. To obtain the cross section is now trivial: one adds the amplitudes as given in equations (5.5), (5.6) or in equations (5.14), squares them recalling that different projectors do not interfere and, when needed, sums or averages over polarizations, which amounts to replace by 1 or $\frac{1}{2}$ the concerned projectors.

Acknowledgements

One of the authors (E.R.) acknowledges the kind hospitality extended to him in various occasions at the Department of Theoretical Physics of the University of Geneva, where this work was initiated and completed.

REFERENCES

[1] J. J. Sakurai, *Advanced Quantum Mechanics*, p. 314 (Addison-Wesley, Reading Mass. 1967).