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On averaged angular time delay for two-body scattering

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Abstract. In this note, we introduce and discuss the concept of averaged angular time delay, in analogy with averaged total cross-sections.

I. Introduction

Recently, there has been a renewed interest in the study of time delay in scattering theory. (See Ref. [1–6], and the references in [6] for earlier results). These papers discuss the notion of global time delay and its relation to the scattering matrix, using a time-dependent approach.

In this note, we rigorously study the angular time delay from a stationary point of view. Heuristic discussions in this spirit can be found in Ref. [7, 8] and the references cited therein.

In particular, we introduce and discuss averaged angular time delay in analogy with recent treatments of averaged total cross-sections [9–11]. We show that it is equal to the trace of the on-shell global time-delay operator. In establishing this relation, we first have to prove the differentiability of the scattering matrix such that we know that time delay exists as a bounded operator. Furthermore, we have to show that the time-delay operator is trace class. This is done in Theorem 3 for the class of potentials $(1 + |\mathbf{x}|)V(\mathbf{x}) \in L^1 \cap R$. This condition roughly implies that $V(\mathbf{x}) = O(|\mathbf{x}|^{-4-\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \rightarrow \infty$. Note that the differentiability of the scattering matrix has also been proved in Ref. [1, 4] for a more general class of potentials, roughly allowing a $|\mathbf{x}|^{-1-\varepsilon}$ behavior at infinity. The proof we present here is extremely simple and, in addition, yields the trace class property of the time-delay operator. Finally, in Theorem 4, we establish the continuity of the averaged angular time delay (and averaged total cross-section) with respect to interactions.

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II. Some results on two-body scattering

Let $V: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real measurable function and assume $V \in R$, where R denotes the Rollnik class (i.e. $\|V\|_R^2 = \int d^3x d^3y |V(\mathbf{x})| |V(\mathbf{y})| |\mathbf{x} - \mathbf{y}|^{-2} < \infty$). If $H_0 = -\Delta$ denotes the usual self-adjoint realization of the kinetic energy operator, we define the Hamiltonian $H = H_0 + V$ through the method of forms [12]. Introducing

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}, \quad u(\mathbf{x}) = v(\mathbf{x}) \operatorname{sign} V(\mathbf{x}), \tag{2.1}$$

the symmetrized resolvent operator $uG_k v$ defined as the norm limit

$$uG_{\pm k} v = n - \lim_{\varepsilon \rightarrow 0^+} u(H_0 - k^2 \mp i\varepsilon)^{-1} v, \quad k \geq 0, \tag{2.2}$$

is Hilbert-Schmidt and satisfies

$$uG_{-k} v = (uG_k v)^*. \tag{2.3}$$

The scattering wave functions Φ^\pm obey the following inhomogeneous Lippmann-Schwinger equations

$$\Phi^\pm(k\boldsymbol{\omega}, \mathbf{x}) = \Phi_0^\pm(k\boldsymbol{\omega}, \mathbf{x}) - (uG_{\mp k} v \Phi^\pm)(k\boldsymbol{\omega}, \mathbf{x}), \quad k^2 \notin \mathcal{E}, \tag{2.4a}$$

which can also be written in the form

$$\Phi^\pm(k\boldsymbol{\omega}, \mathbf{x}) = ((1 + uG_{\mp k} v)^{-1} \Phi_0^\pm)(k\boldsymbol{\omega}, \mathbf{x}), \quad k^2 \notin \mathcal{E}, \tag{2.4b}$$

where the Φ_0^\pm are defined by

$$\Phi_0^-(k\boldsymbol{\omega}, \mathbf{x}) = u(\mathbf{x}) e^{ik\boldsymbol{\omega} \cdot \mathbf{x}}, \quad \Phi_0^+(k\boldsymbol{\omega}, \mathbf{x}) = v(\mathbf{x}) e^{ik\boldsymbol{\omega} \cdot \mathbf{x}}, \tag{2.5}$$

and where $\boldsymbol{\omega} \in S^2$, the unit sphere in \mathbb{R}^3 . The set \mathcal{E} is defined as

$$\mathcal{E} = \{k^2 \geq 0 \mid uG_k v \psi = -\psi \text{ for some } \psi \in L^2(\mathbb{R}^3), k \geq 0\}. \tag{2.6}$$

\mathcal{E} is a closed subset of $[0, \infty)$ with Lebesgue measure zero containing the singular continuous spectrum and positive eigenvalues of H [12]. Then we have (for a proof see [12, 13]).

Theorem 1. *Let $V \in L^1(\mathbb{R}^3) \cap R$, then the scattering operator S , associated with the pair (H, H_0) is unitary, commutes with H_0 and in the spectral representation of H_0 , the corresponding on-shell operator $S(k)$ reads*

$$(S(k)\phi)(\boldsymbol{\omega}) = \phi(\boldsymbol{\omega}) - \frac{k}{2\pi i} \int_{S^2} d\boldsymbol{\omega}' f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') \phi(\boldsymbol{\omega}'), \quad k^2 \notin \mathcal{E}, \quad \phi \in L^2(S^2), \tag{2.7}$$

where $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ represents the on-shell scattering amplitude. For $k^2 \notin \mathcal{E}$, $S(k) - 1$ is a trace class-operator in $L^2(S^2)$, i.e. $[S(k) - 1] \in \mathcal{B}_1(L^2(S^2))$, $k^2 \notin \mathcal{E}$, and continuous in trace norm with respect to k . Furthermore, an explicit characterization of $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is obtained from

$$\begin{aligned} f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') &= -(4\pi)^{-1} (\Phi_0^+(k\boldsymbol{\omega}), \Phi^-(k\boldsymbol{\omega}')) \\ &= -(4\pi)^{-1} (\Phi^+(k\boldsymbol{\omega}), \Phi_0^-(k\boldsymbol{\omega}')), \quad k^2 \notin \mathcal{E}. \end{aligned} \tag{2.8}$$

Finally, $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is uniformly continuous in all variables if k^2 varies in compact intervals not intersecting \mathcal{E} .

Following [9–11], the averaged total cross-section $\bar{\sigma}(k)$ is defined as

$$\begin{aligned} \bar{\sigma}(k) &= (4\pi)^{-1} \int_{S^2} d\omega \int_{S^2} d\omega' |f(k, \omega, \omega')|^2 \\ &= \frac{\pi}{k^2} \|S(k) - 1\|_2^2 = -\frac{2\pi}{k} \operatorname{Re} (\operatorname{Tr} (S(k) - 1)). \end{aligned} \tag{2.9}$$

Theorem 1 then immediately leads to

Theorem 2. *Assume $V \in L^1(\mathbb{R}^3) \cap R$. Then for $k^2 \notin \mathcal{E}$, $\bar{\sigma}(k)$ is finite and continuous in k . In addition, the optical theorem*

$$\sigma(k, \omega) = \int_{S^2} d\omega' |f(k, \omega, \omega')|^2 = \frac{4\pi}{k} \operatorname{Im} f(k, \omega, \omega) \tag{2.10}$$

is valid.

3. Averaged angular time delay

In the following we discuss the necessary modifications of the above approach in order to study the concept of averaged angular time delay in a rigorous way. We start with

Lemma 1. *Let $(1 + |\mathbf{x}|)V \in L^1(\mathbb{R}^3) \cap R$ and $k^2 \notin \mathcal{E}$. Then $(1 + |\mathbf{x}|)^{1/2} \Phi^\pm(k\omega, \mathbf{x})$ is strongly continuous in $L^2(\mathbb{R}^3)$ with respect to k for all $\omega \in S^2$ and $f(k, \omega, \omega')$ is continuously differentiable with respect to k for all $\omega, \omega' \in S^2 \times S^2$. In particular, $S(k)$ is continuously differentiable in trace norm, i.e. $S'(k) \in \mathcal{B}_1(L^2(S^2))$ and $(-k(2\pi i)^{-1} f(k, \omega, \omega'))'$ is the kernel of $S'(k)$. (The ' denotes the derivative with respect to k).*

Proof. The strong continuity of $(1 + |\mathbf{x}|)^{1/2} \Phi^\pm(k\omega, \mathbf{x})$ is clear from (2.4) and the condition on V . (cf. [12], Section IV.5). To prove the rest of the lemma, we first note that, using (2.4) and (2.8)

$$\begin{aligned} f'(k, \omega, \omega') &= i(4\pi)^{-1} (|\omega \cdot \mathbf{x}|^{1/2} \Phi_0^+(k\omega), \operatorname{sign}(\omega \cdot \mathbf{x}) |\omega \cdot \mathbf{x}|^{1/2} \Phi_0^-(k\omega')) \\ &\quad + (4\pi)^{-1} (\Phi^+(k\omega), (uG'_k v \Phi^-)(k\omega')) \\ &\quad - i(4\pi)^{-1} (|\omega' \cdot \mathbf{x}|^{1/2} \Phi_0^+(k\omega), \operatorname{sign}(\omega' \cdot \mathbf{x}) |\omega' \cdot \mathbf{x}|^{1/2} \Phi_0^-(k\omega')), \end{aligned} \tag{3.1}$$

$k^2 \notin \mathcal{E}$,

where $uG'_k v$ denotes the Hilbert–Schmidt operator represented by the kernel

$$(uG'_k v)(\mathbf{x}, \mathbf{y}) = i(4\pi)^{-1} u(\mathbf{x}) e^{ik|\mathbf{x}-\mathbf{y}|} v(\mathbf{y}). \tag{3.2}$$

Equation (3.1) proves the assertions for the on-shell scattering amplitude $f(k, \omega, \omega')$. Next, following Ref. [13], we introduce the maps

$$A_V(k): \begin{cases} L^2(\mathbb{R}^3) \rightarrow L^2(S^2) \\ g(\mathbf{x}) \rightarrow (A_V(k)g)(\omega) = \int_{\mathbb{R}^3} d^3x u(\mathbf{x}) e^{-ik\omega \cdot \mathbf{x}} g(\mathbf{x}) \end{cases} \tag{3.3}$$

and

$$A_V(k)^* : \begin{cases} L^2(S^2) \rightarrow L^2(\mathbb{R}^3) \\ \phi(\boldsymbol{\omega}) \rightarrow (A_V(k)^* \phi)(\mathbf{x}) = \int_{S^2} d\boldsymbol{\omega} u(\mathbf{x}) e^{ik\boldsymbol{\omega} \cdot \mathbf{x}} \phi(\boldsymbol{\omega}). \end{cases} \quad (3.4)$$

Since $A_{|V|}(k) \in \mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ we have that

$$S(k) - 1 = \frac{-k}{2\pi i} A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \in \mathcal{B}_1(L^2(S^2)), \quad k^2 \notin \mathcal{E}. \quad (3.5)$$

For $k_0^2 \notin \mathcal{E}$ and $|k - k_0|$ small enough such that also $k^2 \notin \mathcal{E}$, we consider, using equations (2.4), (2.7) and (2.8),

$$\begin{aligned} [S(k) - S(k_0)] / (k - k_0) = & -\frac{1}{2\pi i} \left\{ A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \right. \\ & + \left[\frac{k_0}{k - k_0} (A_{|V|}(k) - A_{|V|}(k_0)) (1 + |\mathbf{x}|)^{-1/2} \right] \\ & \cdot [(1 + |\mathbf{x}|)^{1/2} (1 + uG_k v)^{-1} A_V(k)^*] \\ & + k_0 A_{|V|}(k_0) \frac{1}{k - k_0} [(1 + uG_k v)^{-1} - (1 + uG_{k_0} v)^{-1}] A_V(k)^* \\ & + [k_0 A_{|V|}(k_0) (1 + uG_{k_0} v)^{-1} (1 + |\mathbf{x}|)^{1/2}] \\ & \left. \cdot \left[\frac{1}{k - k_0} (1 + |\mathbf{x}|)^{-1/2} (A_V(k)^* - A_V(k_0)^*) \right] \right\} \end{aligned} \quad (3.6)$$

Looking e.g. at the kernel of the operator $(1 + |\mathbf{x}|)^{1/2} (1 + uG_k v)^{-1} A_V(k)^*$, we infer from Lemma 1 that this operator is in $\mathcal{B}_2(L^2(S^2), L^2(\mathbb{R}^3))$ and continuous with respect to k in Hilbert-Schmidt norm. Similarly $[k_0 / (k - k_0)] [A_{|V|}(k) - A_{|V|}(k_0)] (1 + |\mathbf{x}|)^{-1/2}$ converges in Hilbert-Schmidt norm to a $\mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ -valued operator with kernel $-ik_0(\boldsymbol{\omega} \cdot \mathbf{x})(1 + |\mathbf{x}|)^{-1/2} u(\mathbf{x}) \exp(-ik_0 \boldsymbol{\omega} \cdot \mathbf{x})$. Thus recalling also (3.5) we have that $S'(k_0) \in \mathcal{B}_1(L^2(S^2))$ with kernel $(-k_0(2\pi i)^{-1} f(k_0, \boldsymbol{\omega}, \boldsymbol{\omega}'))'$. ■

In analogy with eq. (2.9) we now introduce the averaged angular time delay $\bar{\tau}(k)$ by the following definition

$$\bar{\tau}(k) = \int_{S^2} d\boldsymbol{\omega} \tau_a(k, \boldsymbol{\omega}), \quad k > 0, \quad k^2 \notin \mathcal{E}, \quad (3.7)$$

where the angular time delay τ_a is given by [7, 8]

$$\begin{aligned} \tau_a(k, \boldsymbol{\omega}) = & \frac{1}{4\pi k} \frac{\partial}{\partial k} (\text{Re } kf(k, \boldsymbol{\omega}, \boldsymbol{\omega})) \\ & + \frac{k}{8\pi^2} \int_{S^2} d\boldsymbol{\omega}' |f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')|^2 \frac{\partial}{\partial k} (\arg f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')), \quad k > 0, \quad k^2 \notin \mathcal{E}. \end{aligned} \quad (3.8)$$

We then have

Theorem 3. *Let $(1 + |\mathbf{x}|)V \in L^1(\mathbb{R}^3) \cap R$. Then, for $k > 0$ and $k^2 \notin \mathcal{E}$, the averaged angular time delay $\bar{\tau}(k)$ is finite and continuous in k . Furthermore*

$$\bar{\tau}(k) = \text{Tr}(\tau(k)), \quad k > 0, \quad k^2 \notin \mathcal{E}, \quad (3.9)$$

where $\tau(k)$ is the on-shell time delay operator defined by

$$\tau(k) = -\frac{i}{2k} S^*(k)S'(k), \quad k > 0, \quad k^2 \notin \mathcal{E}. \tag{3.10}$$

Proof. the continuity of $\bar{\tau}(k)$ and the fact that $\tau(k) \in \mathcal{B}_1(L^2(S^2))$ follow from Lemma 1. Equation (3.9) is directly obtained from equations (2.7), (3.7)–(3.8) and unitarity of S which allows (3.10) to be written in the form

$$\tau(k) = \frac{1}{2} \left[-\frac{i}{2k} S^*(k)S'(k) + \frac{i}{2k} S^{*'}(k)S(k) \right], \quad k > 0, \quad k^2 \notin \mathcal{E}. \quad \blacksquare \tag{3.11}$$

Remark 1. a) If V is spherically symmetric, then

$$\bar{\tau}(k) = \sum_l (2l+1) \frac{1}{k} \frac{\partial}{\partial k} \delta_l(k),$$

where $\delta_l(k)$ is the partial wave phase shift.

b) For a connection between $\bar{\tau}(k)$ and two-body Levinson’s theorem in a generalised form we refer to [14–16] and [6].

c) Definition (3.7) and Theorem 3 easily extend to relative time delay (as discussed e.g. in [3]).

Remark 2. Under appropriate additional assumptions on V , the singular continuous spectrum of H is empty (cf. [13] and references therein). In that case \mathcal{E} consists of the positive point spectrum of H . It has been shown in [15] that under suitable conditions on V , the positive eigenvalues of H decouple from the scattering phenomena in the sense that the scattering amplitude remains continuous at these points. However, the threshold point $k = 0$ needs a separate discussion. If $0 \in \mathcal{E}$, one has to distinguish whether it is a bound state of H , or a resonance, or both. (cf. e.g. Ref. [17]). Only if $0 \notin \mathcal{E}$ or if 0 is a bound state of H , the averaged total cross-section $\bar{\sigma}(0_+)$ remains finite. On the contrary, the limit of the averaged time delay, $\lim_{k \rightarrow 0_+} \bar{\tau}(k)$, never exists, irrespective whether $0 \in \mathcal{E}$ or not.

Finally, we state a continuity result for the averaged angular time delay and the averaged total cross-section with respect to the interactions. Assume $V, V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$ to be real valued, $n = 1, 2, \dots$ and denote by \mathcal{E}_n the exceptional sets corresponding to $H_n = H_0 + V_n$ (cf. equation (2.6)). Furthermore, let $\bar{\sigma}_n(k)$ and $\bar{\tau}_n(k)$ be the averaged total cross-section respectively the averaged angular time delay for the interaction V_n . Then we have

Theorem 4. a) Suppose $V, V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$, $n = 1, 2, \dots$ and

$$\lim_{n \rightarrow \infty} \|V_n - V\|_1 = 0, \quad \lim_{n \rightarrow \infty} \|V_n - V\|_{\mathcal{R}} = 0. \quad \text{Then}$$

$$\lim_{n \rightarrow \infty} \bar{\sigma}_n(k) = \bar{\sigma}(k), \quad k^2 \notin \mathcal{E}. \tag{3.12}$$

b) Assume $(1 + |\mathbf{x}|)V, (1 + |\mathbf{x}|)V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$, $n = 1, 2, \dots$ and

$$\lim_{n \rightarrow \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_1 = 0, \quad \lim_{n \rightarrow \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_{\mathcal{R}} = 0. \quad \text{Then}$$

$$\lim_{n \rightarrow \infty} \bar{\tau}_n(k) = \bar{\tau}(k), \quad k^2 \notin \mathcal{E}. \tag{3.13}$$

Proof. Continuity of the scattering amplitude $f_n(k, \omega, \omega')$ and the scattering matrix $S_n(k)$ associated with the pair of Hamiltonians (H_n, H_0) under the hypothesis a) has been proved in Ref. [18]. By exactly the same methods, assumption b) implies continuity of $f'_n(k, \omega, \omega')$ and $S'_n(k)$, completing the proof. (We remark that by the joint continuity of $uG_k v$ in k and V , $k^2 \in \mathcal{E}$ implies $k^2 \notin \mathcal{E}_n$ for n large enough). ■

Of course, the concept of time delay is not restricted to the simple case of two-body potential scattering. In the literature, one finds discussion of time delay for multiparticle systems ([7], [19] and [6]). Also nonlocal [20], dissipative [6, 21] and Coulomb interactions [3] were investigated. Finally, we also mention a recent treatment of the Lax–Philips scattering theory [5]. It is clear that our approach presented above could be extended to all these situations.

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