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Exact Wigner functions of bicanonical unitary transformations

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Abstract. A class of unitary operators (called bicanonical) is defined. The Wigner functions describing these operators in phase-space induce *classical* canonical maps.

On the other hand, the Wigner function is completely given by the standard generating function of the classical map if this function exists globally and if the map is connected to the identity by a one-parameter group. Thus, a W.K.B. expansion yields the exact result by the lowest order in $\hbar$.

Information about topological limitations due to caustics is easily obtainable for these exact cases. They remain important when W.K.B. is only approximate.

1. Introduction

For quantum systems whose Hilbert-space if $\mathcal{H} = L^2(\mathbb{R}^n, d^nq)$ the Wigner isomorphism [1, 2, 3] is well defined and associates to each linear operator $\mathbf{F}$ on $\mathcal{H}$ a function $f(q, p)$, the Wigner-function of $\mathbf{F}$ (reference W.F.). The computation of $f = \Phi(\mathbf{F})$ is easy if $\mathbf{F}$ is a polynomial of the canonical operators $\mathbf{Q}^k$ and $\mathbf{P}_k$, representing linear coordinates and momenta. This is also the case for most Hamiltonians which are the sum of a kinetic and a potential energy. The arguments q and p of the W.F. are usually interpreted as the classical canonical variables of a classical system corresponding to the quantum one.

This idyllic situation changes badly in the case of projectors onto quantum states, and becomes still worse for unitary operators. Because unitary operators are important in many respects (change of coordinates, construction of projectors, S-matrix, evolution operators), it is worthwhile to study them in the language of W.F. Starting from a wellcome subset of unitary operators for which the W.F. can be computed exactly, we show in this paper:

1° That the W.F. of unitary operators form a complicated manifold subdivided into "botanic species" (C^∞ bounded functions, unbounded functions, functions with partly compact support, distribution kernels, . . .).

2° How these various species are due to geometrical properties of the corresponding classical systems.

For illustration, let us say that the W.F. $u(q, p)$ of an unitary operator $\mathbf{U}$ is comparable to the kernels $\langle q'|\,\mathbf{U}\,|q\rangle$, $\langle p'|\,\mathbf{U}\,|p\rangle$ or $\langle p'|\,\mathbf{U}\,|q\rangle$. In some cases, a sensible approximation for these functions has the form $N \exp{(i\hbar)^{-1}G}$ [4, 5], where N depends on the phase G. The phase is a geometric quantity, a generating function of the classical canonical transformation φ corresponding to $\mathbf{U}$. But it may also happen that no single G exists to generate φ. This failure, of pure geometric nature, invalidate the above form in a way that higher order terms of a W.K.B. expansion cannot improve anything.

The bicanonical unitary operators, defined in Section 2, represent essentially linear changes of canonical variables and coordinate transformations in configuration space. Their W.F. can be computed exactly, and by iterating non-commuting products of two such operators one generates a very large set of operators. This method is being used to produce chaos for instance (see M. Tabor [10] and references given therein). W.F. of bicanonical operators are closely related to standard generating functions, which we review in Section 3. They are a particular type of generating functions "discovered" independently by M. S. Marinov [6], who calls them phase action, and ourselves [7]. In fact, they were introduced by H. Poincaré [8], as was kindly told us by A. Weinstein who calls them Poincaré generating function [9].

The W.F. of the bicanonical maps representing finite transformations are roughly classified and then constructed in Section 4. One-parameter sub-groups are treated in Section 5. Typical examples of W.F. are given in Section 6.

The direct method used in this paper for the construction of W.F. of bicanonical operators can lead to valuable results in more complicated cases. It has the advantage of avoiding formal expansions and their convergence problems. The set of bicanonical unitary operators is not a group, but it contains essentially two groups which have been studied by several authors by means of various methods [11, 12, 13, 14, 15, 16]. The present paper gives a survey of this subject and fills some gaps. Our approach enhance the deep role played by the underlying symplectic geometry. Namely, the form of W.F. of unitary operators depends on global geometric properties (transversality, caustics, ...).

2. Wigner functions of operators – bicanonical maps

In order to fix the notations and to make this paper self-supporting we recall in this section the notions of linear polarization of phase space and of Wigner-function (W.F.) of operators. Afterwards, using W.F. of unitary operators, quantum and classical canonical maps are compared and bicanonical maps are defined.

Throughout this paper the phase-space E of the dynamical systems is supposed to be an affine symplectic manifold homeomorphic to $\mathbb{R}^{2n}$. The points x of E will usually be labelled by $2n$ linear canonical coordinates $X = (X^1, \ldots, X^{2n})$. Then X can be identified with a vector of the tangent space TE_0 at $X = 0$ and the symplectic 2-form l of E reads

$$l(X, Y) = X \cdot LY, \qquad L = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \tag{2.1}$$

We denote by $\Lambda = (\Lambda^{\mu\nu})$ the inverse matrix of $L = (L_{\mu\nu}) : \Lambda L = \mathbb{1}_{2n}$. A linear, anticanonical involution M of E,

$$M^2 = -\mathbb{1}, \qquad l(MX, MY) = -l(X, Y), \tag{2.2}$$

induces a linear polarization of E defined by

$$X_\pm = \tfrac{1}{2}(\mathbb{1} \pm M)X. \tag{2.3}$$

The sets $V^M_\pm = \{X_\pm \mid X \in E\}$ are Lagrangian submanifolds of E. By adapting the canonical coordinates to $V^M_\pm$, M can always be brought into the form

$$T = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} \tag{2.4}$$

in order that

$$X_+ = (q, 0), \qquad X_- = (0, p), \qquad q, p \in \mathbb{R}^n. \tag{2.5}$$

We shall conventionally call $E_q = V^M_+$ the configuration space of the system for any choice of M.

The Wigner map [1, 2, 3]

$$\Phi : \mathbf{F} \mapsto f \tag{2.6}$$

associates to each linear operator $\mathbf{F}$ defined on the Hilbert space $\mathcal{H} = L^2(E_q, d^n q)$ a function (or distribution kernel) f on E, called the Wigner function (W.F.) of $\mathbf{F}$.

Explicitly

$$f(X) = \int_{E_q} d^n q' \left(\exp \frac{i}{\hbar} q' \cdot p\right) \langle q - \tfrac{1}{2}q' |\mathbf{F}|\, q + \tfrac{1}{2}q' \rangle, \qquad X = (q, p). \tag{2.7}$$

Φ sends the product of two operators $\mathbf{F}$ and $\mathbf{G}$ onto the Moyal product of their images

$$\Phi : \mathbf{FG} \mapsto f \circ g,$$

where

$$(f \circ g)(X) = \int_{E \times E} \frac{d^{2n} Y d^{2n} Z}{(\pi\hbar)^{2n}} f(Y) g(Z) \exp 2(i\hbar)^{-1} l(X - Y, X - Z). \tag{2.8}$$

The Moyal bracket

$$\{f, g\}_M = \frac{1}{i\hbar}(f \circ g - g \circ f) = \Phi\left(\frac{1}{i\hbar}[\mathbf{F}, \mathbf{G}]\right) \tag{2.9}$$

is usually different from the Poisson bracket

$$\{f, g\}_P = \Lambda^{\mu\nu}\, \partial_\mu f\, \partial_\nu g. \tag{2.10}$$

But there are remarkable sets of functions (Section 4) on which these two brackets coincide. Another coincidence is

$$\{x^\mu, f\}_P = \{x^\mu, f\}_M = \Lambda^{\mu\nu}\, \partial_\nu f \tag{2.11}$$

where f is any C^1 function of X and x^μ the linear function

$$x^\mu(X) = X^\mu, \qquad \mu = 1 \cdots 2n. \tag{2.12}$$

x^μ is just the W.F. of the canonical operators $\mathbf{X}^\mu$ corresponding to the linear coordinate X^μ. From (2.11) follows

$$\{x^\mu, x^\nu\}_M = \{x^\mu, x^\nu\}_P = \Lambda^{\mu\nu}. \tag{2.13}$$

If u is the W.F. of an operator $\mathbf{U} \in \mathcal{U}$, the unitary group of $\mathcal{H}$, the quantities

$$\hat{x}^{\mu} = u^* \circ x^{\mu} \circ u \tag{2.14}$$

are W.F. of the transformed operators $\mathbf{U}^+\mathbf{X}^{\mu}\mathbf{U}$; the map $x \mapsto \hat{x}$ is canonical in the quantum sense because

$$\{\hat{x}^{\mu}, \hat{x}^{\nu}\}_M = \Lambda^{\mu\nu}. \tag{2.15}$$

But it is usually not canonical in classical sense, i.e.

$$\{\hat{x}^{\mu}, \hat{x}^{\nu}\}_P \neq \Lambda^{\mu\nu}.$$

In exceptional cases, the new set $\{\hat{x}^{\mu}\}$ satisfies both conditions (2.13), so that (2.14) defines a map $\varphi : x \mapsto \hat{x}$ which is canonical in the classical as well as in the quantum sense. In view of this property we propose the

Definition. An element $\mathbf{U} \in \mathcal{U}$ is said *bicanonical* if its W.F. u defines a map

$$\varphi : x \mapsto u^* \circ x \circ u \tag{2.16}$$

which verifies the classical conditions of canonicity

$$\{\varphi^{\mu}(X), \varphi^{\nu}(X)\} = \Lambda^{\mu\nu}, \tag{2.17}$$

for each set $X^{\mu} = x^{\mu}(X)$ of linear canonical coordinates of E.

Let $\mathbb{B}$ denote sub-set of bicanonical elements of $\mathcal{U}$. The combination of the Wigner map (2.6) and of the map $u \mapsto \varphi$ defined by (2.16) is a map $\beta : \mathbf{U} \mapsto \varphi$. Per definition, $\mathcal{B} = \beta(\mathbb{B})$ is a subset of classical canonical maps.

Theorem 1. *The kernel of*

$$\beta : \mathbb{B} \to \mathcal{B}$$

is homeomorphic to the unit circle S^1 of $\mathbb{C}$.

Proof. Let $\mathbf{V}, \mathbf{U} \in \mathbb{B}$, *such that* $\beta(\mathbf{V}) = \beta(\mathbf{U}) = \varphi$. Hence $u^* \circ x^{\mu} \circ u = v^* \circ x^{\mu} \circ v$. Using unitarity, $u \circ u^* = v \circ v^* = 1$, this equation becomes $x \circ v \circ u^* - v \circ u^* \circ x = 0$. We recognize a Moyal bracket. Using (2.11) we get

$$\{x^{\mu}, v \circ u^*\}_M = \Lambda^{\mu\nu}\, \partial_{\nu}(v \circ u^*) = 0.$$

Thus $v \circ u^*$ is constant; more precisely $v = e^{i\alpha} u$ with $\alpha \in \mathbb{R}$ to preserve unitarity. Since Φ (2.6) is linear and bijective, one has finally $\mathbf{V} = e^{i\alpha}\mathbf{U}$.

The unitary transformation $\hat{\mathbf{F}} = \mathbf{U}^+\mathbf{F}\mathbf{U}$ of an operator $\mathbf{F}$ reads in phase space

$$\hat{f} = u^* \circ f \circ u. \tag{2.18}$$

The W.F. f and $\hat{f}$ are real if $\mathbf{F}$ is self-adjoint. Explicitly

$$\hat{f}(x) = \int_E d^{2n}x'\, k_u(X, X') f(X') \tag{2.19}$$

where

$$k_u(X, X') = \int_{\mathbb{R}^{2n}} \frac{d^{2n}v}{(2\pi)^{2n}}\, u^*\left(\frac{X+X'}{2} + \frac{\hbar}{2} V\right) u\left(\frac{X+X'}{2} - \frac{\hbar}{2} V\right) e^{-i\langle X-X', V\rangle}. \tag{2.20}$$

Note that u has an explicit dependence in $\hbar$ not indicated here. The property $(u^* \circ f \circ u)^* = u^* \circ f^* \circ u$ of the Moyal product ensures that k_u is real. Unitarity of u, $u^* \circ u = u \circ u^* = 1$, implies

$$\int_E d^{2n}X' k_u(X, X') = \int_E d^{2n}X' k_u(X', X) = 1 \tag{2.21}$$

From $k_{u^* \circ u} = k_{u^*} \cdot k_u = \mathbb{1}$ follows orthogonality

$$\int_E d^{2n}Y k_u(X, Y) k_u(X', Y) = \delta^{(2n)}(X - X') \tag{2.22}$$

and

$$k_u^{-1}(X, X') = k_{u^*}(X, X') = k_u(X', X).$$

In the particular case (2.16), (2.19) reads

$$\varphi^\mu(X) = \int_E d^{2n}X' k_u(X, X') X'^\mu \tag{2.16'}$$

and conversely

$$X^\mu = \int_E d^{2n}X' \varphi^\mu(X') k_u(X', X). \tag{2.23}$$

The suitable quantities for discussing limits $\hbar \to 0$ are the kernel k_u and its momenta in q and p. The function u itself has a highly singular behavior at $\hbar \sim 0$.

3. Standard generating functions

This section contains a brief review of the matter treated in previous paper [7], followed by additional developments about geometrical properties of canonical maps and their standard generating functions. These functions come later into play as phases of W.F. of unitary operators.

The standard description of canonical automorphisms of E, which is the natural geometrical substratum of W.F. of unitary operators, works as follows: A set of $4n$ parametric equations

$$\bar{X} = Y + \tfrac{1}{2}\Lambda \nabla g(Y) \stackrel{n}{=} \bar{\chi}(Y)$$

$$X = Y - \tfrac{1}{2}\Lambda \nabla g(Y) \stackrel{n}{=} \chi(Y), \qquad Y \in D \subset \mathbb{R}^{2,} \tag{3.1}$$

defines locally a canonical map

$$\varphi : E \ni X \mapsto \bar{X} = \varphi(X) \in E'. \tag{3.2}$$

We call g the *standard generating function* of φ because it belongs to a symplectic invariant procedure. Eliminating the variables Y^μ between the two sets (3.1) one obtains the equivalent equations

$$\bar{X} - X = \Lambda \nabla g\left(\frac{\bar{X} + X}{2}\right), \tag{3.3}$$

which define φ implicitly. The symplectic map Σ induced by φ in tangent space is given by

$$\Sigma(X) \equiv \left(\frac{\partial \varphi^\mu}{\partial X^\nu}(X)\right) = (\mathbb{1} + \tfrac{1}{2}\Lambda\Omega(Y))(\mathbb{1} - \tfrac{1}{2}\Lambda\Omega(Y))^{-1} \tag{3.4}$$

where

$$\Omega(Y) = \left(\frac{\partial^2 g}{\partial Y^\mu\, \partial Y^\nu}(Y)\right) \tag{3.5}$$

and $Y = \frac{1}{2}(\varphi(X) + X) = \frac{1}{2}(\bar{X} + X)$.

In view of later applications, it is essential to know:

1° If φ is symplectomorphic (canonical *and* diffeomorphic).

2° If g unfolds φ (generates it globally). These difficult analytical questions can be stated in a simple geometrical way. For this we introduce the product space $E \times E'$, the prime serving to distinguish the domain from the range of φ. Supplying $E \times E'$ with additional structures by means of the projectors

$$\begin{aligned} &\mathbb{P} : E \times E' \ni (X, X') \mapsto (X, 0) \in E \\ &\mathbb{P}' : E \times E' \ni (X, X') \mapsto (0, X') \in E' \end{aligned} \tag{3.6}$$

and of the symplectic 2-form $\mathscr{L}(X_1, X_2, X_1', X_2') = l(X_1, X_2) - l(X_1', X_2')$, we form the symplectic space $\mathscr{E} = (E \oplus E', \mathscr{L})$. The canonicity of the map $\varphi : E \to E'$ is then simply expressed by saying that its graph V is a Lagrangian submanifold of $\mathscr{E}$ [7]. Obviously, φ is symplectomorphic iff the restrictions to V of the above projections, $\mathbb{P}_V : V \to E$ and $\mathbb{P}'_V : V \to E'$, are diffeomorphic. Iff it is the case, no tanget vector of $V : (dx, d\bar{x}) \equiv (dx, \Sigma(x)\, dx)$, lays somewhere parallel to E or to E'. In other words, Σ has nowhere a vanishing or an infinite eigenvalue, and, since eigenvalues of $\Sigma(X)$ go by pairs $\sigma_k(X)$ and $1/\sigma_k(X)$, $k = 1 \cdots n$, the characteristic function

$$\Delta^2(X) = \det \tfrac{1}{2}(\Sigma(X) + \mathbb{1}) = \tfrac{1}{4} \prod_{k=1}^{n} (\sigma_k(X) + 1)\left(\frac{1}{\sigma_k(X)} + 1\right) \tag{3.7}$$

takes finite values. In conclusion, φ is diffeomorphic iff

$$|\Delta^2(X)| < \infty, \qquad X \in E, \qquad \|X\| < \infty, \tag{3.8}$$

where by definition

$$\|X\| = \sup |X^\mu|, \qquad \mu = 1 \cdots 2n. \tag{3.9}$$

[The restriction on $\|X\|$ is usual, E being regarded as an infinite open set].

To answer the question 2°, we first remark that putting g = constant in (3.1) yields the identity map $\bar{X} = X$, whose graph in $\mathscr{E}$ is the manifold

$$\mathcal{M} = \{(X, X') \mid X' = X\}. \tag{3.10}$$

Introducing the projector

$$\mathbb{P}_{\mathcal{M}} : \mathscr{E} \ni (X, X') \mapsto \left(\frac{X + X'}{2}, \frac{X + X'}{2}\right) \in \mathcal{M} \tag{3.11}$$

and the variable $Y = \frac{1}{2}(X + X')$, we can speak of points Y of $\mathcal{M}$. This convention allows one to say that $\mathbb{P}_{\mathcal{M}}$ projects the particular points $(X, \bar{X}) \in V$ of the graph of φ onto the "middle points" $Y = \frac{1}{2}(\bar{X} + X) \in \mathcal{M}$. Iff the restriction of $\mathbb{P}_{\mathcal{M}}$ to V is a bijective map $V \to D = \mathbb{P}_{\mathcal{M}}(V) \subset \mathcal{M}$, the standard description (3.1) of φ is global. One usually says there, that the projection $\mathbb{P}_{\mathcal{M}}$ of V into $\mathcal{M}$ makes no folds, and that g *unfolds* φ. By adapting to $\mathbb{P}_{\mathcal{M}}$ the above arguments (preceding (3.7)) one obtains easily the *unfolding condition*

$$|\Delta^2(X)| > 0, \qquad X \in E, \qquad \|X\| < \infty. \tag{3.12}$$

When (3.12) holds, the initial equation (3.3) has up to an additive constant a unique solution defined on a domain $D = \{Y \mid Y = \frac{1}{2}(\varphi(X) + X), X \in E\}$. The maps χ and $\bar{\chi}$ ((3.1)) define $\varphi = \bar{\chi} \cdot \chi^{-1}$ and from (3.4) follows for their Jacobian

$$\mathcal{N}^2(Y) = \det\left(\frac{\partial \chi}{\partial Y}\right) \equiv \det\left(\frac{\partial \bar{\chi}}{\partial Y}\right) = \det\left(\mathbb{1} \pm \tfrac{1}{2}\Lambda\Omega(Y)\right) \tag{3.13}$$

and

$$\mathcal{N}^2(Y) = \frac{1}{\Delta^2(\chi(Y))}, \qquad Y \in D. \tag{3.14}$$

To avoid later specification of integration domains we introduce the function

$$N(Y) = \theta_D(Y)\, |\mathcal{N}^2(Y)|^{1/2} \tag{3.15}$$

where θ_D is 1 over D and 0 elsewhere.

The fact that symplectomorphisms globally described in the standard way are given by a pair (g, D) makes the converse problem too intricate to be mastered by simple analytical conditions. As a guide for the reader's imagination we state important necessary conditions:

a) The inner $\mathring{D}$ of $D \subset \mathcal{M}$ (largest open subset of D) is homeomorphic to an open Cartesian subset of $\mathbb{R}^{2n}$.
b) g is at least C^1 on $\mathring{D}$. (3.16)
c) $0 < |\mathcal{N}^2(Y)| < \infty, \qquad Y \in \mathring{D},$
d) $\|\nabla g(Y)\| \to \infty, \qquad Y \to Y_L \in \partial D \quad \text{if} \quad \|Y_L\| < \infty.$

Conditions c) follow from (3.8) and (3.12). d) is necessary because sections of D may be compact and $\|\chi(Y_L)\| = \|\bar{\chi}(Y_L)\| = \infty$ for finite $\|Y_L\|$.

The next important question is to look for the generating function g of the product $\varphi = \varphi_2 \cdot \varphi_1$ of two symplectomorphisms unfolded by functions g_1 and g_2. In principle, g is the $\top$-product [7]

$$g = g_2 \top g_1 \tag{3.17}$$

defined explicitly in the following way: $g(Y)$ is equal to the value of the function

$$\psi(Y, Y', Y'') = 2(Y'' - Y) \cdot L(Y' - Y) + g_2(Y'') + g_1(Y') \tag{3.18}$$

taken at the stationary point with respect to variations of Y' and Y'' for fixed Y. Hence

$$g(Y) = \psi(Y, Y_1(Y), Y_2(Y))$$

where the pair Y_i, $i = 1, 2$, is solution of

$$Y_1 + \tfrac{1}{2}\Lambda\, \nabla g_2(Y_2) = Y_2 - \tfrac{1}{2}\Lambda\, \nabla g_1(Y_1) = Y. \tag{3.19}$$

These equations admit a unique solution for all $Y \subset D$ if $\varphi_2 \cdot \varphi_1$ is diffeomorphic and if the projection $\mathbb{P}_{\mathcal{M}}: V \to \mathcal{M}$ makes no fold. Otherwise, more than one solution may exist, the number of which depends on Y. There is a bifurcation set which can be studied using catastrophe theory [17]. In not too bad cases, the solutions are branches of a multivalued function. Each branch $g^{(k)}$ defines φ in a subset $D^{(k)} \subset \mathcal{M}$. The boundary $\partial D^{(k)}$ is a caustic on which $\mathcal{N}^2(Y)$ ((3.13), (3.23)) usually diverges. Multivalued generating functions also occur in the case of piecewise diffeomorphic canonical maps (see Example iii), Section 6).

The solutions Y_i of (3.19) are gometrically interpreted as "middle points", like Y. Putting $\bar{X} = \varphi_1(X)$, $\bar{\bar{X}} = \varphi_2(\bar{X}) = \varphi(X)$ for simplicity one sees easily that

$$Y_1 = \tfrac{1}{2}(\bar{X} + X), \qquad Y_2 = \tfrac{1}{2}(\bar{\bar{X}} + \bar{X}), \qquad Y = \tfrac{1}{2}(\bar{\bar{X}} + X). \tag{3.20}$$

Using this property, the composition law $\Sigma(X) = \Sigma_2(\bar{X})\Sigma_1(X)$ and the formula (3.4), one obtains the relation between the matrices of second derivatives (3.5) of g, g_1 and g_2:

$$\mathbb{1} - \tfrac{1}{2}\Lambda\Omega(Y) = (\mathbb{1} - \tfrac{1}{2}\Lambda\Omega_2(Y_2))(\mathbb{1} + \tfrac{1}{4}\Lambda\Omega_2(Y_2)\Lambda\Omega_1(Y_1))^{-1}(\mathbb{1} - \tfrac{1}{2}\Lambda\Omega_1(Y_1)), \tag{3.21}$$

and for the determinantal functions (3.13):

$$\mathcal{N}^2(Y) = \mathcal{N}_1^2(Y_1)\mathcal{N}_2^2(Y_2) \det(\mathbb{1} + \tfrac{1}{4}\Lambda\Omega_2(Y_2)\Lambda\Omega_1(Y_1))^{-1}. \tag{3.22}$$

This relation is important for the product of bicanonical maps because their amplitude is in many cases given by $|\mathcal{N}^2|^{1/2}$.

The standard description of Hamiltonian flows succeeds using generating functions $g: \mathbb{R} \times \mathcal{M} \to \mathbb{R}$. If h denote the generator of a flow $\mathcal{G}^h$, and $t \in \mathbb{R}$ the group parameter, the solutions of the equations

$$\dot{X}_t = \{X_t, h\}_P = \Lambda\, \nabla h(X_t), \qquad X_0 = X \in E, \tag{3.23}$$

define the elements $\varphi_t: X \mapsto X_t$ of $\mathcal{G}^h$. The generating function of $\mathcal{G}^h$ satisfies the standard Hamilton–Jacobi equation [6, 7, 9].

$$\partial_t g_t(Y) = h(Y + \tfrac{1}{2}\Lambda\, \nabla g_t(Y)), \tag{3.24}$$

with initial condition

$$g_0(Y) = 0. \tag{3.25}$$

For each given t, g_t generates φ_t via (3.3). The group law $\varphi_{t'} \cdot \varphi_t = \varphi_{t'+t}$ of $\mathcal{G}^h$ implies

$$g_{t'} \top g_t = g_{t'+t} + c(t', t). \tag{3.26}$$

The cocycle c is a piecewise constant function of t', t only, equal to zero for sufficiently small t and t'. Equation (3.24) with initial condition (3.25) admits a unique solution g for some interval $[0, t_1]$ at least. But it is not true, as suggested in the paper by Marinov [6], that g_t unfolds φ_t in any case. The projection of the graph V_t into $\mathcal{M}$ can make folds for arbitrary small t already. The standard description of φ_t must be completed using solutions of (3.24) which fulfill boundary conditions differing from (3.25). A similar situation occurs when φ_t is only piecewise diffeomorphic (Example iii), Section 6).

In the next sections, we use mainly Hamiltonians of two particular forms. Results about the relation between h and the corresponding g_t are summarized for these cases in the next theorem.

Theorem 2. *Let $\mathcal{G}^h$ be a flow whose Hamiltonian is a function, of second degree in X,*

$$h(X)=\tfrac{1}{2}X\cdot\omega X+A\cdot X+h_0, \tag{3.27}$$

or of degree 1 in n commuting variables, for instance

$$h(X)=h_1(q)+p\cdot A(q). \tag{3.28}$$

Then the solution $g_t(Y)$ of (3.24–25) have the same functional form as the generator h. Moreover, the characteristic function (3.7) has the property

$$\infty>\Delta_t^2(X)\geq 0, \qquad \|X\|<\infty, \qquad X\in E, \qquad |t|<\infty. \tag{3.29}$$

Proof. The first part of Theorem 2 is proved in [7]. With h as in (3.27) the solution g_t is the second degree polynomial

$$g_t(Y)=Y\cdot LC_tY+Y\cdot L(\mathbb{1}-C_t)V_t+\gamma_t \tag{3.30}$$

where

$$\begin{aligned} C_t &= th\frac{t}{2}\Lambda\omega \\ V_t &= \int_0^t dt'(\exp t'\Lambda\omega)\Lambda A \\ \gamma_t &= h_0t+\tfrac{1}{4}V_t\cdot LV_t-\tfrac{1}{2}A\cdot\int_0^t dt'(\exp -t'\Lambda\omega)V_{t'}. \end{aligned} \tag{3.31}$$

The characteristic function is

$$\Delta_t^2=\prod_{k=1}^n\left(ch\frac{t}{2}\mu_k^*\right)\left(ch\frac{t}{2}\mu_k\right) \tag{3.32}$$

where $\pm\mu_k$, $k=1\cdots n$, are the eigenvalues of $\Lambda\omega$. It is ≥ 0 and finite for finite t.

With h of the form (3.28) one has

$$g_t(Y)=f_t(\xi)+\eta\cdot a_t(\xi), \qquad Y=(\xi,\eta)\in D=d\xi\times\mathbb{R}^n, \tag{3.33}$$

where f_t is a function and a_t a vector field satisfying

$$\begin{aligned} \partial_t a_t(\xi) &= \left(\mathbb{1}-\frac{1}{2}\frac{\partial a_t}{\partial\xi}\right)A(\xi+\tfrac{1}{2}a_t(\xi)), \qquad a_0(\xi)=0, \\ \partial_t f_t(\xi) &= h_1(\xi+\tfrac{1}{2}a_t(\xi))-\tfrac{1}{2}A(\xi+\tfrac{1}{2}a_t(\xi))\cdot\nabla f_t(\xi), \qquad f_0(\xi)=0. \end{aligned} \tag{3.34}$$

The characteristic function is independent of p:

$$\begin{aligned} \Delta_t^2(q) &= \det\left(\frac{\partial q_t}{\partial q}(q)\right)^{-1}\left[\det\frac{1}{2}\left(\mathbb{1}+\frac{\partial q_t}{\partial q}(q)\right)\right]^2 \\ &= \det\left[\mathbb{1}-\left(\frac{1}{2}\frac{\partial a_t}{\partial\xi}\left(\frac{q_t+q}{2}\right)\right)^2\right]^{-1}. \end{aligned} \tag{3.35}$$

The factor $\det(\ )^{-1}$ in the first equality is continuous, equal to 1 for $t=0$, and cannot change sign because all φ_t's are diffeomorphic by assumption. Thus Δ_t^2 fulfills (3.29).

$\mathscr{G}^h$ is automatically a flow if h has the form (3.27). Δ_t^2 is a finite constant for any finite t. Moreover, g_t unfolds $\varphi_t \in \mathscr{G}^h$ for all t iff $\Lambda\omega$ has no purely imarinary eigenvalues. If it has, $\mu_k = im_k$, $m_k \in \mathbb{R}$, $k = 1 \cdots n_0$, then Δ_t^2 vanishes at

$$t_{k,\nu} = (2\nu + 1)\frac{\pi}{m_k}, \qquad \nu \in Z. \tag{3.36}$$

The set $\{t_{k,\nu}\}$ is discrete; g_t is well defined and unfolds φ_t for each $t \notin \{t_{k,\nu}\}$.

When h has the form (3.28), $\mathscr{G}^h$ is a Hamiltonian flow iff the equations $\dot{q}_t = A(q_t)$, $q_0 = q$, define a n-dimensional flow. The solution (3.33) of (3.34) unfolds $\mathscr{G}^h$ iff $\Delta_t^2(q) > 0$ for all finite $|t|$ and $\|q\|$. If not, this solution unfolds $\mathscr{G}^h$ locally only, in a connected domain of $\mathbb{R} \times E_q$ containing the hyperplane $t = 0$ and delimited by two hypersurfaces $t_\pm(q)$ defined by $\Delta_{t_\pm}^2(q) = 0$. Additional solutions are necessary. In complicated cases, it is preferable to try first other descriptions [3, 5].

4. Wigner functions of bicanonical unitary operators

The set $\mathbb{B}$ of bicanonical operators (Section 2) cannot be characterized explicitly as a whole. The main difficulty is that $\mathbb{B}$ is not a subgroup of $\mathscr{U}$. However, it is possible to exhibit bicanonical subgroups of $\mathscr{U}$ which generate by repeated products a large part of $\mathbb{B}$ at least. One subgroup is homeomorphic to $ISp(E)$, the inhomogeneous symplectic group. The other ones, two by two isomorphic, form a continuous set containing the gauge transformations in particular. Let us first give a rough classification of bicanonical maps according to the analytical form of their W.F. This classification keeps sense for more general unitary operators.

There are "easy" $\mathbf{U} \in \mathbb{B}$. Their W.F. has exactly Van Vleck form [4, 5]

$$\Phi(\mathbf{U}) = u = Ne^{-(i/\hbar)g} \tag{4.1}$$

where N is the function (3.16) of second derivatives of g, and g is the standard generating function which unfolds the symplectomorphism $\varphi = \beta(\mathbf{U})$. u has no zero, is bounded in D ((3.16c)), and is continuous if g is C^2 (see theorem 3 and 4 below). There are "less easy" $\mathbf{U} \in \mathbb{B}$. Their W.F. is no longer completely specified by the generating function of $\varphi = \beta(\mathbf{U})$. The form (4.1) holds with a multiplicative correction which tends toward 1 when $\hbar \to 0$ (see Theorem 6). There is no "a priori" argument against an appropriate W.K.B. expansion of u.

In more "difficult" cases, u is a sum

$$u = \sum_k N_k e^{-(i/\hbar)g_k} \tag{4.2}$$

where N_k and g_k are related as in (4.1). The functions g_k are branches of a multivalued generating function which piece-wise unfolds φ. This geometrical complication occurs when φ is only piecewise bijective (Example iii), Section 6), or (and) when $\mathbb{P}_{\mathscr{M}}: V \to \mathscr{M}$ makes folds (Example iv)).

In "exceptional" cases u is a distribution of type δ. The corresponding classical map φ admits no standard generating function (Example i), Theorem 3).

This (non-exhaustive) list remains meaningful for non-bicanonical $\mathbf{U} \in \mathscr{U}$ in the following sense: Usually, the limit $(u^* \circ x^\mu \circ u)(X) = \int d^{2n}X' k_u(X, X')X'^\mu \to \varphi^\mu(X)$, $\hbar \to 0$, exists, and φ is canonical. If a single g unfolds φ, Van Vleck's form (4.1) is certainly a sensible first approximation for u. If more than one g is needed, it is necessary to begin with (4.2). No expansion in power of $\hbar$ can save the situation if an information contained in the classical underlying geometry has been overlooked in the first approximation.

The sets of functions considered in Theorem 2 play a central role in the generation of bicanonical maps [11, 18]; we need to define them properly:

i) The set of real polynomials of degree 2

$$\mathcal{A}_2 = \{g \mid g(X) = X \cdot \Omega X + A \cdot X + g_0, \Omega = \tilde{\Omega}\}. \tag{4.3}$$

The function (3.13) is constant here: $\mathcal{N}^2 = \det(\mathbb{1} + \Lambda\Omega)$. We shall denote by $\mathcal{A}_2'$ the subset of elements $g \in \mathcal{A}_2$ for which $\mathcal{N}^2 \neq 0$.

ii) The sets of real functions whose restrictions to a linear Lagrangian submanifold V_-^M (2.3) are polynomials of degree 1.

$$\mathcal{A}_M = \{g \mid g(X) = f(X_+) + X_- \cdot a(X_+), X_\pm \in V_\pm^M\}. \tag{4.4}$$

In the canonical chart (2.5) adapted to $V_\pm^M$ one has

$$g(X) = f(q) + p \cdot a(q). \tag{4.5}$$

The function f and the vector field a are supposed to be C^∞ on an open subset d_q of $\mathbb{R}^n$. Multivalued functions are not excluded. The function (3.13) depends on q only

$$\mathcal{N}^2(q) = \det\left(\mathbb{1} - \frac{1}{4}\left(\frac{\partial a}{\partial q}\right)^2\right). \tag{4.6}$$

$\mathcal{A}_M'$ will denote the subet of functions $g \in \mathcal{A}_M$ which unfold a symplectomorphism of E. The necessary conditions (3.16) give an idea of their properties.

We are now able to construct well behaved W.F. of bicanonical maps.

Theorem 3. *The function* (4.1) $u = N \exp(i\hbar)^{-1}g$ *is for any* $g \in \mathcal{A}_2'$ *the* W.F. *of a bicanonical map* $\mathbf{U}$ *element of a sub-set* $\mathbb{B}_2' \subset \mathbb{B}$. *The corresponding map* $\varphi = \beta(\mathbf{U})$ ((2.16)) *belongs to* $ISp(E)$. *The kernel* (2.20) *is local and reads*

$$k_u(X, X') = \delta^{(2n)}(\varphi(X) - X'). \tag{4.7}$$

Proof. Let $g(X)$ be as in (4.3). We have from (2.20) and (4.1)

$$\begin{aligned} k_u(X, X') &= |\det(\mathbb{1} - \Lambda\Omega)| \int_{\mathbb{R}^{2n}} \frac{d^{2n}V}{(2\pi)^{2n}} e^{iV \cdot L(X - X' + \Lambda\Omega(X + X') + \Lambda A)} \\ &= |\det(\mathbb{1} - \Lambda\Omega)|\, \delta^{(2n)}((\mathbb{1} + \Lambda\Omega)X + \Lambda A - (\mathbb{1} - \Lambda\Omega)X'). \end{aligned}$$

Since $g \in \mathcal{A}_2'$, $\mathbb{1} - \Lambda\Omega$ is regular and k_u is given by (4.7) with

$$\varphi(X) = (\mathbb{1} - \Lambda\Omega)^{-1}(\mathbb{1} + \Lambda\Omega)X + (\mathbb{1} - \Lambda\Omega)A \equiv \Sigma X + a. \tag{4.8}$$

Obviously, Σ is symplectic and $\varphi \in ISp(E)$. $\mathbf{U}$ is unitary because k_u fulfills

(2.21–2); it is bicanonical since

$$(u^* \circ x^\mu \circ u)(X) = \int d^{2n}X' \, \delta^{(2n)}(\varphi(X - X'))X'^\mu = \varphi^\mu(X).$$

The vector space $\mathcal{A}_2$ is a Lie algebra for Poisson and Moyal brackets (they coincide on $\mathcal{A}_2$). Moreover, $\mathcal{A}_2$ is an (incomplete) associative algebra for the $\top$-product: $g_1, g_2 \in \mathcal{A}_2 \Rightarrow g_1 \top g_2 \in \mathcal{A}_2$ if the product exists. Assuming that these three functions belong to $\mathcal{A}_2'$, one must have in corollary to Theorem 1 and Theorem 3

$$N_2 e^{-(i/\hbar)g_2} \circ N_1 e^{-(i/\hbar)g_1} = N e^{-(i/\hbar)g_2 \top g_1 - i\alpha}, \tag{4.9}$$

where α is a real constant and N is given by (3.22). Thus, the map β preserves the product law. The closure $\mathbb{B}_2$ of $\mathbb{B}_2'$ is a group and $\beta : \mathbb{B}_2 \to ISp(E)$ is a group homomorphism, which leads to the metaplectic representation of $ISp(E)$ [12, 13, 14, 19]. When $g_2 \top g_1$ does not exist, the left handside of (4.9) still makes sense. The corresponding classical map is an exceptional element $\varphi = (\Sigma_2\Sigma_1, a_2 + \Sigma_2 a_1)$ of $ISp(E)$ characterized by the fact that $\Sigma = \Sigma_2\Sigma_1$ has eigenvalues equal to -1. The W.F. $u = u_2 \circ u_1$ is then a distribution for which the exponential form does not exist, but which can be factorized according to the

Theorem 3′. *Let* $\mathbf{U}$ *be any element of* $\mathbb{B}_2$ *and* $\varphi = (\Sigma, a) = \beta(\mathbf{U})$ *the corresponding classical map. The* W.F. $u = \Phi(\mathbf{U})$ *can be written*

$$u = u_0 \circ \delta_I^*$$

where u_0 *has the form* (4.1) *and* δ_I^* *is the distribution*

$$\delta_I^*(X) = (\pi\hbar)^{2m} \, \delta^{(2m)}(\tfrac{1}{2}(\mathbb{1} - I)X).$$

$2m$ *is the number of eigenvalues* -1 *of* Σ, *and* $\frac{1}{2}(\mathbb{1} - I)$ *the projector onto the corresponding symplectic sub-space* E_1. (*By convention* $\delta^{(0)}$ *is the unit function.*)

Proof. If $\Delta^2 = \det \frac{1}{2}(\Sigma + \mathbb{1}) \neq 0$, one has $m = 0$ and Theorem 3 applies. If $\Delta^2 = 0$, Σ has $2m$ eigenvalues -1 and the corresponding symplectic sub-space E_1 of E is such that $(\Sigma + \mathbb{1})^{2m} E_1 = 0$. A unique exceptional element $I \in Sp(E)$ exists having the properties $I^2 = \mathbb{1}$, $\tilde{I} = I$, $I\Sigma = \Sigma I$, $\frac{1}{2}(\mathbb{1} - I)E = E_1$. The symplectic map $\Sigma_0 = \Sigma I$ has no eigenvalue $-1 : \Delta_0^2 = \det(\Sigma_0 + \mathbb{1}) \neq 0$. The distribution δ_I^* represents the parity in E_1 (Section 6, Example i)) and the identity in $E - E_1$. Up to an arbitrary phase, u_0 is given by (4.1) knowing Σ_0. In conclusion, $u_0 \circ \delta_I^*$ verifies

$$(u_0 \circ \delta_I^*)^* \circ \times \circ (u_0 \circ \delta_I^*) = \Sigma X + a,$$

and by virtue of Theorem 1 it must be equal to u assuming the arbitrary constant phase of u_0 is correctly chosen.

Theorem 4. *To any* $g \in \mathcal{A}_M'$ *the relation* (4.1) *associated the* W.F. *of a bicanonical map* $\mathbf{U} \in \mathbb{B}_M' \subset \mathbb{B}$. *The corresponding symplectomorphism* $\varphi = \beta(\mathbf{U})$ *leaves the Lagrangian sub-manifold* E_q ((2.5)) *invariant:*

$$\begin{aligned} \varphi^k(X) &= \bar{q}^k(q), \qquad k = 1 \cdots n, \\ \varphi^{k+n}(X) &= \bar{p}_k(q, p) = \frac{\partial q^l}{\partial \bar{q}^k}(q) p_l. \end{aligned} \tag{4.10}$$

The kernel k_u has the properties

$$\int d^n p' k_u(q, p \mid q', p') = \delta^{(n)}(q' - \bar{q}(q)) \tag{4.11}$$

$$\int d^n p k_u(q, p \mid q', p') = \delta^{(n)}(q - \bar{q}^{-1}(q')) \tag{4.12}$$

$$\int d^n p' k_u(q, p \mid q', p') p_i' = \delta^{(n)}(q' - \bar{q}(q)) \bar{p}_j(q, p). \tag{4.13}$$

Proof. Calling $Y = (\xi, \eta)$ the variables of g, (4.5) reads $g(\xi, \eta) = f(\xi) + \eta \cdot a(\xi) \cdot g$ generates a map $\varphi: X = (q, p) \mapsto \bar{X} = (\bar{q}, \bar{p})$ given parametrically by

$$\begin{aligned} \bar{q} &= \xi + \tfrac{1}{2} a(\xi) \overset{n}{=} \bar{Q}(\xi) \\ q &= \xi - \tfrac{1}{2} a(\xi) \overset{n}{=} Q(\xi) \end{aligned} \qquad \xi \in d_q \subset \mathbb{R}^n \tag{4.14}$$

$$\bar{p} = \left(\mathbb{1} + \frac{1}{2} \frac{\partial a}{\partial \xi}\right)^{-1} \left[\left(\mathbb{1} - \frac{1}{2} \frac{\partial a}{\partial \xi}\right) p - \nabla f(\xi)\right]_{\xi = \frac{1}{2}(\bar{q} + q)}. \tag{4.15}$$

g being of degree 1 in η a trivial integration in (2.20) yields

$$\begin{aligned} k_u(q, p \mid q', p') = \int \frac{d^n z}{(2\pi)^n} & N\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right) N\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right) \\ & \times \delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right) + \bar{Q}\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right) - 2q'\right) \\ & \times \exp \frac{i}{\hbar} p' \cdot \left(\bar{Q}\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right) - \bar{Q}\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right)\right) \\ & \times \exp \frac{i}{\hbar} p \cdot \left(Q\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right) - Q\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right)\right) \\ & \times \exp \frac{i}{\hbar} \left(f\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right) - f\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right)\right) \end{aligned} \tag{4.16}$$

This kernel is non-local and depends on $\hbar$. By integrating over p' the first exponential gives

$$(2\pi\hbar)^n \delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2} + \frac{\hbar}{2} z\right) - \bar{Q}\left(\frac{q+q'}{2} - \frac{\hbar}{2} z\right)\right).$$

The argument vanishes for $z = 0$ only because $\bar{Q}: \xi \mapsto \bar{q}$ is bijective by assumption. Performing the z integration one gets

$$\int d^n p' k_n(q, p \mid q', p') = \left|\det \frac{\partial Q}{\partial \xi} \left(\frac{q+q'}{2}\right)\right| \delta^{(n)}\left(2\bar{Q}\left(\frac{q+q'}{2}\right) - 2q'\right). \tag{4.17}$$

The argument of $\delta^{(n)}$ has again a single zero at $q' = \bar{q}(q)$, and (4.17) is identical to (4.11). Using the bijectivity of $Q: \xi \mapsto q$ one proves (4.12) in the same way. To prove (4.13) we calculate the moment of k_u (4.16) with respect to p_j'. Taking the

$\delta^{(n)}$ in (4.16) into account the p' dependent term is

$$p'_j \exp\frac{2i}{\hbar} p' \cdot \left(\bar{Q}\left(\frac{q+q'}{2}+\frac{\hbar}{2}z\right)-q'\right)$$

$$= -i\left(\frac{\partial \bar{Q}}{\partial \xi}\left(\frac{q+q'}{2}+\frac{\hbar}{2}z\right)^{-1}\right)^k_j \frac{\partial}{\partial z^k}\exp\frac{2i}{\hbar}p'\cdot(\bar{Q}-q').$$

The matrix $\partial\bar{Q}/\partial\xi$ is regular by assumption. The p' integration yields

$$\int dp' k_u(q,p\mid q',p')p' = \frac{\hbar}{i}\int d^n V \delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2}-V\right)\right.$$
$$\left.+\bar{Q}\left(\frac{q+q'}{2}+V\right)-2q'\right) N\left(\frac{q+q'}{2}+V\right) N\left(\frac{q+q'}{2}-V\right)$$
$$\times \exp\frac{i}{\hbar}p\cdot\left(Q\left(\frac{q+q'}{2}-V\right)-Q\left(\frac{q+q'}{2}+V\right)\right)$$
$$\times \exp\frac{i}{\hbar}\left(f\left(\frac{q+q'}{2}+V\right)-f\left(\frac{q+q'}{2}-V\right)\right)$$
$$\times\left(\frac{\partial\bar{Q}}{\partial\xi}\left(\frac{q+q'}{2}+V\right)\right)^{-1}\nabla_V \delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2}+V\right)-q'\right).$$

The integrand contributing for $V=0$ only, even functions of V can be permutated with ∇_V. The integral is equal to

$$-\tfrac{1}{2}N^2\left(\frac{q+q'}{2}\right)\int d^n V \delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2}-V\right)-\bar{Q}\left(\frac{q+q'}{2}+V\right)\right)\delta^{(n)}\left(\bar{Q}\left(\frac{q+q'}{2}+V\right)-q'\right)$$
$$\times\left(\frac{\partial\bar{Q}}{\partial\xi}\left(\frac{q+q'}{2}\right)\right)^{-1}\nabla_V\left[p\cdot\left(Q\left(\frac{q+q'}{2}-V\right)-Q\left(\frac{q+q'}{2}+V\right)\right)\right.$$
$$\left.+f\left(\frac{q+q'}{2}+V\right)-f\left(\frac{q+q'}{2}-V\right)\right]$$

Taking the form (4.6) of N^2 into account, an easy computation leads to

$$\int d^n p' k_u(q,p\mid q',p')p' = \left|\det\frac{\partial Q}{\partial\xi}\left(\frac{q+q'}{2}\right)\right|\delta^{(n)}\left(2\bar{Q}\left(\frac{q+q'}{2}\right)-2q'\right)$$
$$\times\left(\frac{\partial\bar{Q}}{\partial\xi}\left(\frac{q+q'}{2}\right)\right)^{-1}\left[\widetilde{\left(\frac{\partial Q}{\partial\xi}\left(\frac{q+q'}{2}\right)\right)}p-\nabla f\left(\frac{q+q'}{2}\right)\right]. \tag{4.18}$$

One recognizes (4.17) in the first two factors and (4.15) in the last ones. This achieves the proof of (4.13).

The unitarity of $u = N\exp(i\hbar)^{-1}g$ is immediately verified using the properties (4.11–12) of k_u. The bicanonicity follows from (4.12–13):

$$(u^* \circ q^k \circ u)(q, p) = \int d^n q' q'^k \delta^{(n)}(q' - \bar{q}(q)) = \bar{q}^k(q) = \varphi^k(X)$$
$$(u^* \circ p_k \circ u)(q, p) = \int d^n q' \delta^{(n)}(q' - \bar{q}(q)) \bar{p}_k(q, p) = \bar{p}_k(q, p) = \varphi^{k+n}(X). \tag{4.19}$$

Since φ ((4.10)) is canonical, $\mathbf{U}$ is bicanonical.

The set $\mathscr{A}_M$ has the same properties as $\mathscr{A}_2$ with respect to Poisson and Moyal brackets, and to the $\top$-product. The law (4.9) holds with the same restriction for the bicanonical elements of $\mathbb{B}'_M$. The closure of $\mathbb{B}'_M$ is a subgroup $\mathbb{B}_M$ of $\mathscr{U}$ which corresponds to the group $\mathscr{G}_M$ of canonical maps leaving V_+^M invariant. Pairs $\mathscr{A}_M$ and $\mathscr{A}_{M'}$ as well as $\mathbb{B}_M$ and $\mathbb{B}_{M'}$ are identical up to an equivalence:

Theorem 5. *Given $u \in \Phi(\mathbb{B}_2)$ inducing a map $\varphi = (\Sigma, q) \in ISp(E)$, and a pair M, $M' = \Sigma M \Sigma^{-1}$, of polarizations, the map*

$$u : f \mapsto \hat{f} = u^* \circ f \circ u \tag{4.20}$$

defines an isomorphism between the groups $\Phi(\mathbb{B}_M)$ and $\Phi(\mathbb{B}_{M'})$, and a bijection between $\mathscr{A}_M$ and $\mathscr{A}_{M'}$ which preserves the Lie algebra structure and the associative $\top$-product. The following diagram is commutative

$$\begin{array}{ccc} \mathscr{A}'_M & \xrightarrow{m} & \mathscr{A}'_{M'} \\ {\scriptstyle (4.1)}\downarrow & & \downarrow{\scriptstyle (4.1)} \\ \Phi(\mathbb{B}_M) & \xrightarrow{u} & \Phi(\mathbb{B}_{M'}) \end{array} \tag{4.21}$$

Proof. Let $u_M \in \Phi(\mathbb{B}_M)$. From (4.7–8) follows

$$(u^* \circ u_M \circ u)(X) = u_M(\Sigma X + a) \stackrel{n}{=} \hat{u}_M(X).$$

u_M leaves invariant the submanifold V_+^M of E. Thus $\hat{u}_M$ leaves $V_+^{M'}$ invariant, $M' = \Sigma M \Sigma^{-1}$, and consequently $\hat{u}_M = u_{M'} \in \Phi(\mathbb{B}_{M'})$. The group structure is obviously preserved. If now $f_M \in \mathscr{A}_M$, it is clear from the Definition (4.4) that $\hat{f}_M \in \mathscr{A}_{M'}$. The Lie structure is preserved because (4.20) is linear and $u^* \circ \{f_1, f_2\}_M \circ u = \{\hat{f}_1, \hat{f}_2\}_M$. The diagram (4.21) is commutative because

$$(u^* \circ u_M \circ u)(X) = u_M(\Sigma X + a) \stackrel{(4.1)}{=} N(\Sigma X + a) e^{-(i/\hbar) g_M(\Sigma X + a)}$$
$$= (u^* \circ N \circ u)(X) e^{-(i/\hbar)(u^* \circ g_M \circ u)(X)}.$$

This property and (4.9) imply that

$$u^* \circ (g_2 \top g_1) \circ u = (u^* \circ g_2 \circ u) \top (u^* \circ g_1 \circ u). \tag{4.22}$$

A corollary of (4.7) (Theorem 3) and (4.11–13) (Theorem 4) is the following local property:

$$(u^* \circ f \circ u)(X) = f(\varphi(X)) \tag{4.23}$$

for any f if $u \in \Phi(\mathbb{B}_2)$ and for $f \in \mathscr{A}_M$ if $u \in \Phi(\mathbb{B}_M)$.

Products of bicanonical maps belonging to different sub-groups may be bicanonical or not.

Theorem 6. *If $u_2 \in \Phi(\mathbb{B}_2)$ and $u_M \in \Phi(\mathbb{B}_M)$, then the products $u_2 \circ u_M$ and $u_M \circ u_2$ are bicanonical. Assuming further that u_2 and u_M are of the form (4.1) with $g_2 \in \mathscr{A}_2'$, $g_M \in \mathscr{A}_M'$, and that a unique $g = g_2 \top g_1$ exists, then*

$$u_2 \circ u_M = Ne^{-(i/\hbar)g} e^{i\alpha} F \tag{4.24}$$

with

$$F(X, \hbar) \to 1, \qquad \hbar \to 0.$$

F is generically $\neq 1$ ($\hbar \neq 0$) unless either u_2 and u_M belong to a same subgroup, or commute $u_2 \circ u_M = u_M \circ u_2$.

Proof. From (4.7–8) follows

$$\begin{aligned}(u_2 \circ u_M)^* \circ x^\mu \circ (u_2 \circ u_M) &= u_M^* \circ (\Sigma_\nu^\mu x^\nu + a^\mu) \circ u_M \\ &= \Sigma_\nu^\mu u_M^* \circ x^\nu \circ u_M + u_M^* \circ u_M \circ a^\mu \\ &= \Sigma_\nu^\mu \varphi_M^\nu(x) + a^\mu = \varphi_2^\mu(\varphi_M(x)) = (\varphi_2 \cdot \varphi_M)(x)\end{aligned}$$

and

$$\begin{aligned}(u_M \circ u_2)^* \circ x^\mu \circ (u_M \circ u_2) &= u_2^* \circ \varphi_M^\mu(x) \circ u_2 \\ &= \varphi_M^\mu(\varphi_2(x)) = (\varphi_M \cdot \varphi_2)(x).\end{aligned}$$

Hence, both products satisfy the definition (2.16–17) of bicanonicity. If u_2 or u_M belongs to $\Phi(\mathbb{B}_2) \cap \Phi(\mathbb{B}_M)$, both belong to one of the subgroups and $F = 1$ because (4.9) holds.

If $u_2, u_M \notin \Phi(\mathbb{B}_2) \cap \Phi(\mathbb{B}_M)$ but commute, they induce commuting symplectomorphisms $\varphi_2 = (\Sigma, a)$ and φ_M:

$$\Sigma_\nu^\mu \varphi_M^\nu(X) + a^\mu = \varphi_M^\mu(\Sigma X + a).$$

Because φ_M is non-linear, this relation implies that φ_2 and φ_M act non-trivially on two non-intersecting symplectic subspaces of E. The result is trivially $u_2 \circ u_M = u_2 u_M$, $g = g_2 + g_M$, $F = 1$. In the generic case

$$(u_2 \circ u_M)(X) = (\pi\hbar)^{-2n} \int d^{2n}z \, d^{2n}z' N_2 N_M(z') e^{-(i/\hbar)\psi(X,z',z)}$$

where ψ is the function (3.18). By assumption ψ has a unique stationary point $(z_0, z_0') = (z_2(X), z_M(X))$ for each X, and $g(X) = (g_2 \top g_M)(X) = \psi(X, z_2(X), z_M(X))$ exists. By choosing a new origin, $z = z_2 + Y$, $z' = z_M + Y'$, one obtains

$$\begin{aligned}\psi(X, z', z) &= g(X) + \chi(X, Y, Y') \\ \chi(X, Y, Y') &= 2Y \cdot LY' + \tfrac{1}{2} Y \cdot \Omega_2 Y + \tfrac{1}{2} Y' \cdot \Omega_M Y' \\ &\quad + g_M(z_M + Y') - Y' \circ \nabla g_M(z_M) - \tfrac{1}{2} Y' \cdot \Omega_M(z_M) Y_M' \\ &= (Y, Y') \cdot B(z_2) \begin{pmatrix} Y \\ Y' \end{pmatrix} + r(z_2, Y');\end{aligned}$$

there, Ω_2 and Ω_M are the matrices of second derivatives of g_2 and g_M respectively. The function r vanishes like the third power of Y' at the origin. On the other hand, the relation (3.22) yields

$$\det B(z_2) = \det \begin{pmatrix} \frac{1}{2}\Omega_2 & -\Lambda \\ \Lambda & \frac{1}{2}\Omega_M \end{pmatrix} = \det (\mathbb{1} + \tfrac{1}{4}\Lambda\Omega_2\Lambda\Omega_M) = \mathcal{N}_2^2 \mathcal{N}_M^2(z_2)\mathcal{N}^{-2}(X),$$

where the $\mathcal{N}$'s are the norm functions (3.13) associated to the g's. Eventually,

$$(u_2 \circ u_M)(X) = N(X)e^{-(i/\hbar)g(X)}$$
$$\times \left[\frac{|\det B|^{1/2}}{\pi^{2n}} \int d^nV\, d^nV' \,\frac{N_M(z_2+\sqrt{\hbar}V')}{N_M(z_2)}\, e^{-(i/\hbar)r(z_2,\sqrt{\hbar}V')} e^{-i(V,V')\cdot B\binom{V}{V'}}\right.$$

$\hbar^{-1}r(z_2, \sqrt{\hbar}V')$ vanishes at the limit $\hbar = 0$ and the factor in brackets tends toward

$$e^{i\alpha} = \prod_{\mu=1}^{4n} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \exp\left[-i\frac{b_\mu}{|b_\mu|}t^2\right] = \exp \frac{\pi}{4i} \sum_\mu \frac{b_\mu}{|b_\mu|},$$

where b_μ denote the eigenvalues of the real symmetric matrix B. $B(z_2(x))$ is by assumption continuous and regular for all X (unicity of g). Its eigenvalues are continuous and cannot change sign without vanishing. Therefore α is independent of X. The function F of the statement is $e^{-i\alpha}$ times the above bracket. It depends on X and $\hbar$ in generic cases.

This theorem shows in particular that a bicanonical u is not always completely specified by the generating function of the corresponding classical map φ. The function $N \exp (i\hbar)^{-1} g_2 \top g_M$ is not exactly unitary. A correction F depending on $\hbar$ is necessary.

It is not difficult but teadious to see that products $u_M \circ u_{M'}$ are generically no longer bicanonical. The residual property is

$$(u_{M'} \circ u_M)^* \circ x^\mu \circ (u_{M'} \circ u_M) \xrightarrow[\hbar \to 0]{} \varphi^\mu(X) \tag{4.25}$$

where φ is generated by $g_{M'} \top g_M$.

The groups $\mathbb{B}_M$ present all the difficulties listed at the beginning of this section. There are elements $\mathbf{U} \in \mathbb{B}_M$ which induce non diffeomorphic maps φ, although they are unitary (Example iii)). Other ones induce symplectomorphisms whose graphs make folds on $\mathcal{M}$ (Example iv)). In both cases, it may happen that a multivalued function g exists, whose branches $g^{(1)} \cdots g^{(m)}$ unfold φ piecewise, and the W.F. u takes the form (4.2). These difficulties are in principle solved by the theory of Maslov [20]. Integral representations may also be helpful:

Theorem 4′. *Let* $\mathbf{U} \in \mathbb{B}$, *such that* $\varphi = \beta(\mathbf{U})$ *is a symplectomorphism. Then its W.F.* u *is given in integral form by*

$$u(q, p) = \int d^nq' e^{-(i/\hbar)(2p\cdot(q-q')+\chi(q'))} |\det S(q')|^{1/2} \delta^{(n)}\left(q - \frac{q'+\bar{q}(q')}{2}\right), \tag{4.26}$$

with

$$\bar{q}^k(q') = \varphi^k(q'), \qquad S = \left(\frac{\partial \bar{q}}{\partial q}\right),$$

$$\bar{p}_k(q', p') = \varphi^{k+n}(q', p') = (S^{-1}(q'))^l_k(p'_l - \partial_l\chi(q')).$$

u has the form (4.1) iff $q' \mapsto \xi(q') = \frac{1}{2}(\bar{q}(q') + q')$ *is a diffeomorphism, and the form (4.2) if* $\xi(q') = q$ *has the same number m of solution for each q.*

Proof. Supposing first that $\xi(q') = \frac{1}{2}(\bar{q}(q') + q')$ is a bijection $q' \leftrightarrow \xi$ of $\mathbb{R}^n$, the Jacobian matrix $(\partial\xi/\partial q')$ is everywhere regular and the inverse map $q' = Q(\xi)$ is well defined. Choosing ξ as a new integration variable and defining $\bar{Q}(\xi) = \bar{q}(Q(\xi))$ we have

$$d^n q' \, |\det S(q')|^{1/2} = d^n \xi \det\left(\frac{\partial Q}{\partial \xi}\right) \left|\det \frac{\partial \bar{Q}}{\partial \xi}\right|^{1/2} \left|\det \frac{\partial Q}{\partial \xi}\right|^{-1/2},$$

and

$$u(q, p) = \left|\det\left(\frac{\partial \bar{Q}}{\partial \xi}\right)\left(\frac{\partial Q}{\partial \xi}\right)\right|^{1/2} e^{-(i/\hbar)(2p \cdot (q - Q(q)) + \chi(Q(q)))}.$$

Defining $a(\xi) = \bar{Q}(\xi) - Q(\xi)$ and using the property $\bar{Q}(\xi) + Q(\xi) = 2\xi$, we have $(\partial\bar{Q}/\partial\xi)(\partial Q/\partial\xi) = \mathbb{1} - (\partial q/\partial\xi)^2$ and $\xi - Q(\xi) = \bar{Q}(\xi) - \xi = \frac{1}{2}a(\xi)$. Thus

$$u(q, p) = \left|\det\left(\mathbb{1} - \left(\frac{\partial a}{\partial \xi}\right)^2\right)\right|^{1/2} e^{-(i/\hbar)(p \cdot a(q) + f(q))} = N(q) e^{-(i/\hbar) g(q,p)},$$

where $f(q) = \chi(Q(q))$. This is exactly the form prescribed in (4.1).

When $q' \mapsto \xi = \bar{q}(q') + q'$ is not bijective, the matrix

$$\left(\frac{\partial \xi}{\partial q'}\right) = \frac{1}{2}\left(\mathbb{1} + \frac{\partial \bar{q}}{\partial q}(q')\right) = \left(\frac{\partial Q}{\partial \xi}\right)^{-1}$$

is not everywhere regular and more than one solution $Q(\xi)$ exist. u is a sum of terms like (4.2) with each $g^{(k)} \in \mathcal{A}_M$.

5. One-parameter groups of bicanonical maps

One-parameter subgroups of $\mathcal{U}$ are given in Wigner form by

$$u_t \circ = e^{-(it/\hbar)} h \circ \tag{5.1}$$

where h is the W.F. of the self-adjoint group generator **H**. The W.F. u_t, $t \in \mathbb{R}$, satisfy the equations

$$i\hbar \, \partial_t u_t = h \circ u_t \tag{5.2}$$

$$u_0 = 1 \qquad \text{(unit function)} \tag{5.3}$$

and have the properties

$$u_{t'} \circ u_t = u_{t'+t} \qquad \text{(group law)} \tag{5.4}$$

$$u_t^* \circ u_t = u_t \circ u_t^* = 1 \qquad \text{(unitarity)} \tag{5.5}$$

On the basis of Sections 3 and 4, it is not difficult to construct one-parameter sub-groups of bicanonical maps:

Theorem 7. *Let* $h \in \mathcal{A}_M$ *(respectively* $\mathcal{A}_2$*) be the generator of a flow* $\mathcal{G}^h$. *Being assumed that the solution* g *of (3.24–25) unfolds* $\mathcal{G}^h$ *for* $t \in \tau = [-t_1, t_1]$, *the* *following relation holds:*

$$e^{-(it/\hbar)}h\circ = (N_t e^{-(i/\hbar)g_t})\circ \quad , \qquad t \in \tau, \tag{5.6}$$

where g_t *is equal to* g *at "time"* t *and*

$$N_t(Y) = +\sqrt{\det\left(\mathbb{1} + \tfrac{1}{2}\Lambda\left(\frac{\partial^2 g_t}{\partial Y\,\partial Y}(Y)\right)\right)}. \tag{5.7}$$

Proof. We know from Theorem 2 that $h \in \mathscr{A}_M$ (respectively $\mathscr{A}_2$) implies $g_t \in \mathscr{A}_m$ (respectively $\mathscr{A}_2$) and $N_t^2 = 1/\Delta_t^2 \geqslant 0$. By assumption g_t unfolds φ_t, $t \in \tau$. Thus $g_t \in \mathscr{A}'_M$ (respectively $\mathscr{A}'_2$) and $\infty > N_t^2 > 0$, $t \in \tau$; $\hat{u}_t = N_t \exp(i\hbar)^{-1} g_t$ is bicanonical (Theorem 4 (respectively 3)). The law (4.9) is valid and yields using (3.26)

$$\hat{u}_{t'} \circ \hat{u}_t = N_{t'+t} e^{-(i/\hbar)(g_{t'+t}+\alpha(t',t))}, \qquad t, t', t'+t \in \tau.$$

The real function α is at least C^2 in the limited domain of $t, t' \in \tau$. Differentiating with respect to t' at $t'=0$ gives

$$\partial_{t'}\hat{u}_{t'} \circ \hat{u}_t|_{t'=0} = \partial_t \hat{u}_t - \frac{i}{\hbar}\,\hat{u}_t\,\partial_{t'}\alpha(t', t)|_{t'=0}.$$

But

$$\partial_{t'}\hat{u}_{t'}|_{t'=0} = -\frac{i}{\hbar}\,\partial_{t'} g_{t'}|_{t'=0} + \partial_{t'} N_{t'}|_{t'=0} = -\frac{i}{\hbar}\,h$$

by virtue of (3.24–25) and because N_t^2 is even in g_t ((3.13)). Thus

$$i\hbar\,\partial_t \hat{u}_t = h \circ \hat{u}_t - \alpha'(0, t)\hat{u}_t,$$

and

$$\alpha'(0, t) = \frac{1}{\hat{u}_t}(i\hbar\,\partial_t\hat{u}_t - h\circ\hat{u}_t) = \mathrm{Re}\,\frac{1}{\hat{u}_t}(i\hbar\,\partial_t\hat{u}_t - h\circ\hat{u}_t) = \partial_t g_t - \mathrm{Re}\,\frac{h\circ\hat{u}_t}{\hat{u}_t}.$$

With $h(X)\circ = h(X\circ) = h(X + \tfrac{1}{2}i\hbar\Lambda\nabla)$, $h \in \mathscr{A}_2$, or $h(X)\circ = h_1(q\circ) + \tfrac{1}{2}(p\circ A(q\circ)\circ p\circ)$, $h \in \mathscr{A}_M$, it is easy to verify that

$$\mathrm{Re}\,\frac{h\circ\hat{u}_t}{\hat{u}_t}(X) = h(X + \tfrac{1}{2}\Lambda\nabla g_t(X)).$$

Hence $\alpha'(0, t) = \partial_t g_t(X) - h(X + \tfrac{1}{2}\Lambda\nabla g_t(X))$, and $\alpha' \equiv 0$ since g_t satisfies (3.24) by assumption. Therefore $\hat{u}_t$ fulfills (5.2–3) like u_t. (5.6) follows from the unicity of the solution of (5.2–3).

In the particular case $h \in \mathscr{A}_2$, this theorem can be easily extended to all $t \in \mathbb{R}$. Using the notations of Theorem 2, we first remark that the root

$$\Delta_t = \prod_{k=1}^{n} ch\frac{t}{2}\,\mu_k \tag{5.8}$$

of Δ_t^2 is an entire function of $t \in \mathbb{C}$, whose zeros form a discrete set $t_{k,\nu} = (1+2\nu)i\pi\mu_k^{-1}$, $\nu \in Z$, $(\mu_k \neq 0)$. Thus $1/\Delta_t$ is meromorphic in the complex t-plane (in contradistinction to $N_t = |\Delta_t|^{-1}$), and $N_t = \Delta_t^{-1}$ for $t \in \tau$. Similarly, the function g_t defined in (3.30) is meromorphic in t and its poles lay in $\{t_{k,\nu}\}$ like those of Δ_t.

Defining for $t \in \mathbb{C}$

$$W_t = \frac{1}{\Delta_t} e^{-(i/\hbar)g_t}, \tag{5.9}$$

we have a meromorphic function of t which coincides with u_t for $t \in \tau$. Since (5.4) holds for $t, t', t+t' \in \tau$, and since $W_{t'} \circ W_t$ and $W_{t'+t}$ are meromorphic in t' (t fixed), these functions are everywhere equal and

$$W_{t'} \circ W_t = W_{t'+t}, \qquad t, t', t'+t \in \mathbb{C}/\{t_{k,\nu}\}. \tag{5.10}$$

Thus $W_t = u_t$, $t \in \mathbb{R}$; the continuity must be understood in a distribution sense if some μ_k is purely imaginary (Theorem 3′).

No such general statement is possible in the case of $h \in \mathcal{A}_M$. However, formula (4.28) holds for any t and renders possible the discussion of u_t outside τ. $\Delta_t^2(q) = 0$ is a hypersurface in $\mathbb{R} \times E_q$ instead of a hyperplane in $\mathbb{R} \times E$ as above.

6. Examples of bicanonical maps

The support of the W.F. u of a bicanonical map may be punctual (Example i)) or a subset of E (Example ii)). As mentioned at the beginning of Section 4, a bicanonical u may be only piece-wise one-to-one (Example iii)) or equal to a sum of exponentials (Example iv)).

Symplectic sub-groups with purely imaginary time lead to Gibbs states of thermal equilibrium (Example v)).

(i) *The Wigner function of parity*

The parity operation $\mathbf{\Pi}$ is defined by the relations

$$\mathbf{\Pi} \mathbf{X}^\mu \mathbf{\Pi} = -\mathbf{X}^\mu, \qquad \mu = 1 \cdots n, \tag{6.1}$$

$$\mathbf{\Pi}^+ = \mathbf{\Pi} = \mathbf{\Pi}^{-1}, \tag{6.2}$$

up to a sign (the intrinsic parity of the system). For one choice of the sign, the W.F. of $\mathbf{\Pi}$ is

$$u(X) = (\pi\hbar)^n \, \delta^{(2n)}(X), \tag{6.3}$$

which satisfies the relations $u \circ X \circ u = -X$ and $u \circ u = u^* \circ u = 1$, equivalent to (6.1–2). u represents the classical symplectic map $\varphi(X) = -X$. The support of u is punctual, indicating that no standard generating function unfolds φ. This map is an exceptional element of $Sp(E)$ forming together with the identity the center of this group. φ lies at the intersection of many sub-groups of $Sp(E)$, as for instance

$$\Sigma_t = e^{tI}, \qquad I^2 = -\mathbb{1}, \qquad |t| \leq \pi, \tag{6.4}$$

with

$$LI = \Omega = \tilde{\Omega} > 0. \tag{6.5}$$

Σ_t is equal to φ for $t = \pm\pi$. According to Section 5, this sub-group is represented by

$$u_t(X) = \left(\cos\frac{t}{2}\right)^{-n} \exp\left[-\frac{i}{\hbar} X \cdot \Omega X tg \frac{t}{2}\right]. \tag{6.6}$$

This simple form is due to $I^2=-\mathbb{1}$: the eigenvalues of I are ± 1, and

$$g_t(X)=X\cdot\left(Lth\frac{t}{2}I\right)X=X\cdot LIXtg\frac{t}{2}.$$

Accordingly, $\det\Omega=1$ and $X\cdot\Omega X>0$ ((6.5)). Therefore

$$u_t(X)\xrightarrow[t\to\pm\pi]{}\lim_{\varepsilon=+0}(\sin\varepsilon)^{-n}\exp\left[\mp\frac{i}{\hbar}X\cdot\Omega X\operatorname{ctg}\varepsilon\right]=(\mp i\pi\hbar)^n\delta^{(2n)}(X) \tag{6.7}$$

or

$$u_t|_{t=\pm\pi}=(\mp i)^n u. \tag{6.8}$$

(ii) *A bicanonical map whose* W.F. *vanishes in a part of E*

Taking $n=1$ for simplicity, the generating function

$$g(\xi,\eta)=2\eta\lambda\operatorname{tg}k\xi,\qquad \lambda k=\text{const.}>1,\quad k>0, \tag{6.9}$$

defines a map $\varphi\in\mathscr{G}^M$. Equations (4.15) and (4.14) give

$$\bar p=\frac{\cos^2 k\xi-\lambda k}{\cos^2 k\xi+\lambda k}\,p=\frac{\cos^2 k\dfrac{q+\bar q}{2}-\lambda k}{\cos^2 k\dfrac{q+\bar q}{2}+\lambda k}\,p \tag{6.10}$$

$$\bar q=\xi+\lambda\operatorname{tg}k\xi\overset{n}{=}\bar Q(\xi)$$

$$q=\xi-\lambda\operatorname{tg}k\xi\overset{n}{=}Q(\xi),\qquad |\xi|\leqslant\frac{\pi}{2k}. \tag{6.11}$$

The functions $\bar Q$ and Q map $D=[-\pi/2k;\pi/2k]$ onto $\mathbb{R}$. Moreover

$$0\geqslant\Delta^2=\frac{\cos^4 k\xi}{\cos^4 k\xi-\lambda^2k^2}\geqslant-\frac{1}{\lambda^2k^2-1},\qquad k\lambda>1,\quad \xi\in D. \tag{6.12}$$

Δ^2 reaches zero for $\xi=\pm\pi/2k$ only. For these values, φ sends $q=\mp\infty$ onto $\bar q=\pm\infty$. Therefore, φ is symplectomorphic and is unfolded by the function (6.9) defined on $D\times\mathbb{R}$. The W.F. of the bicanonical map which represents φ is

$$u(\xi,\eta)=N(\xi)e^{-(i/\hbar)2\eta\lambda\operatorname{tg}k\xi} \tag{6.13}$$

where, according to (3.15),

$$N(\xi)=\theta\left(\frac{\pi}{2k}-|\xi|\right)\left|\left(\frac{\lambda k}{\cos^2 k\xi}\right)^2-1\right|^{1/2} \tag{6.14}$$

The kernel (4.16) of u is

$$\begin{aligned}k_u(q,p\mid q',p')=&\int\frac{dt}{2\pi}N\left(\frac{q+q'}{2}-\frac{\hbar}{2}t\right)N\left(\frac{q+q'}{2}+\frac{\hbar}{2}t\right)\\&\times\exp\left\{i\left[(p'-p)t+(p+p')\frac{\lambda}{\hbar}\left(\operatorname{tg}k\frac{q+q'+\hbar t}{2}-\operatorname{tg}k\frac{q+q'-\hbar t}{2}\right)\right]\right\}\\&\times\delta\left(q-q'+\lambda\operatorname{tg}k\frac{q+q'+\hbar t}{2}+\lambda\operatorname{tg}k\frac{q+q'-\hbar t}{2}\right).\end{aligned} \tag{6.15}$$

The θ in N limitates the integration variable to the domain

$$\frac{\pi}{2k}-\left|\frac{q+q'}{2}\right|\geqslant\left|\frac{\hbar t}{2}\right|. \tag{6.16}$$

If $\hbar\to 0$, the norm factors and the arguments of δ no longer depend on t, and

$$\frac{1}{\hbar}\left(\operatorname{tg} k\frac{q+q'+\hbar t}{2}-\operatorname{tg} k\frac{q+q'-\hbar t}{2}\right)\xrightarrow[\hbar\to 0]{}\frac{kt}{\cos^2 k\dfrac{q+q'}{2}}. \tag{6.17}$$

Integrating (6.15) over t yields

$$\begin{aligned}\lim_{\hbar=0} k_u(q,p\mid q',p') &= \theta(\pi-k\mid q+q'|)\left[\lambda^2k^2\left(1+\operatorname{tg}^2 k\frac{q+q'}{2}\right)-1\right]\\ &\quad\times\delta\left(q-q'-2\lambda\operatorname{tg} k\frac{q+q'}{2}\right)\\ &\quad\times\delta\left(\left(1+\frac{\lambda k}{\cos^2 k\dfrac{q+q'}{2}}\right)p'-\left(1-\frac{\lambda k}{\cos^2 k\dfrac{q+q'}{2}}\right)p\right).\end{aligned} \tag{6.18}$$

The first δ contributes for

$$q'-q=2\lambda\operatorname{tg} k\frac{q+q'}{2}. \tag{6.19}$$

Taking (6.16) into account we have the unique solution

$$q'=\bar{q}(q),$$

and the second δ contributes exactly for

$$p'=\bar{p}(q,p),$$

where $\bar{q}$ and $\bar{p}$ are the classical transformations defined in (6.10–11). The norm factor in (6.18) is just convenient to give

$$\lim_{\hbar=0} k_u(q,p\mid q',p')=\delta(q'-\bar{q}(q))\,\delta(p'-\bar{p}(q,p)). \tag{6.20}$$

When $\hbar\neq 0$, the argument of δ in (6.15) vanishes at values $k\hbar t/2=\pm\tau$, given by

$$\sin^2\tau=\left(1-\frac{2\lambda}{q'-q}\operatorname{tg} k\frac{q+q'}{2}\right)\cos^2 k\frac{q+q'}{2},\qquad 0\leqslant\tau\leqslant\frac{\pi}{2}-\frac{k}{2}|q+q'|. \tag{6.21}$$

A solution $\tau(q,q')$ exists in the domain of (q,q')

$$0\leqslant\sigma(q,q')=\frac{2\lambda}{q'-q}\operatorname{tg} k\frac{q'+q}{2}<1. \tag{6.22}$$

σ is 1 if q and q' are classically related by (6.19), and decreases smoothly. k_u is zero for $\sigma>1$ or $\sigma<0$. $\tau(q,q')=0$ on the classical curve $\sigma(q,q')=1$ and increases smoothly to $\pi/2$ on the straight line $q'=-q$ $(\sigma=0)$.

Explicitly

$$k_u(q, p \mid q', p') = \theta(\pi - k \ |q+q'|)\theta(\sigma)\theta(1-\sigma)\frac{2}{\pi k}$$
$$\times \frac{\left[\lambda^2 k^2 - \cos^4\left(k\frac{q+q'}{2} - \tau\right)\right]^{1/2}\left[\lambda^2 k^2 - \cos^4\left(k\frac{q+q'}{2} + \tau\right)\right]^{1/2}}{\left|\cos^2\left(k\frac{q+q'}{2} - \tau\right) - \cos^2\left(k\frac{q+q'}{2} + \tau\right)\right|}$$
$$\times \frac{1}{\hbar}\cos\frac{1}{\hbar}\left[\frac{2\tau}{k}(p'-p) + (p'+p)(q'-q)\frac{\sin 2\tau}{\sin k(q'+q)}\right]. \tag{6.23}$$

Here, σ and τ depend on q and q' only, as given in (6.21–2). The support of k_u is that part of $E \times E$ defined by (see Fig. 1)

$$p, p' \in \mathbb{R} \tag{6.24}$$
$$q + q' \in [-\pi/k, +\pi/k]$$
$$q' - q \geqslant 2\lambda \text{ tg } k\frac{q+q'}{2} \geqslant 0, \qquad q' - q \leqslant 2\lambda \text{ tg } k\frac{q+q'}{2} \leqslant 0. \tag{6.25}$$

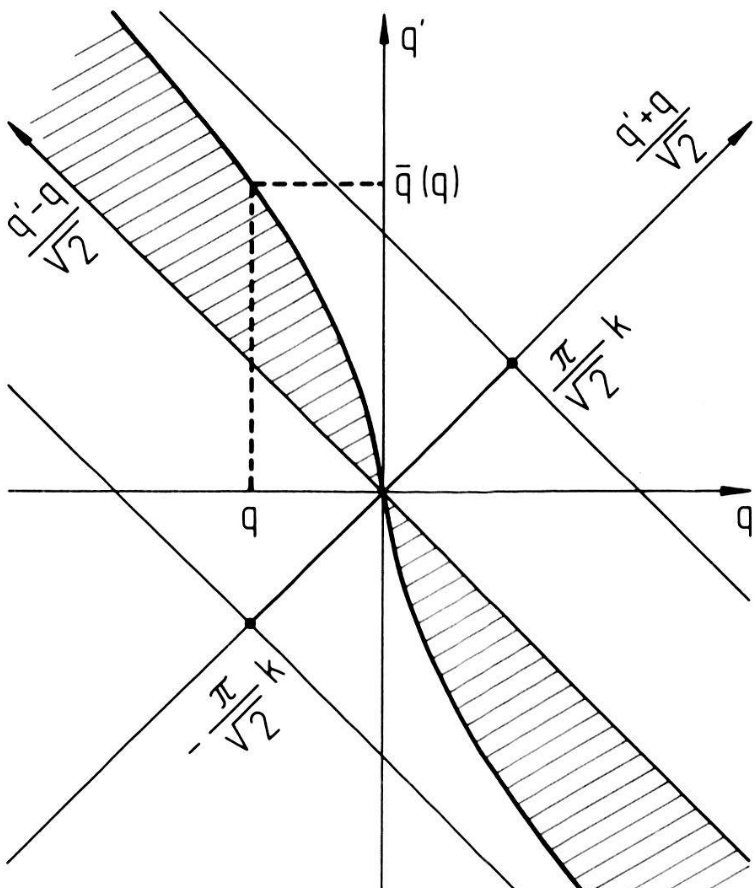

Figure 1
The shaded area is the support of k_u in (q, q')-space. The thick line is the graph of the classical canonical map induced by k_u. It is also the support of k_u at the limit $\hbar = 0$.

(iii) *A one-parameter group of bicanonical maps inducing a non-diffeomorphic classical map*

The function

$$h = pq^2 \in \mathcal{A}_M \tag{6.26}$$

is the W.F. of the non essentially self-adjoint operator

$$\mathbf{H} = \mathbf{QPQ} \tag{6.27}$$

Both defect indices of $\mathbf{H}$ are equal to one. h is the generator of the one-parameter group [3]

$$\varphi_t: \begin{array}{l} q \mapsto q_t(q) = \dfrac{q}{1-tq} \\ p \mapsto p_t(q,p) = (1-tq)^2 p \end{array} \tag{6.28}$$

φ_t is canonical, but not symplectomorphic. It is only piece-wise diffeomorphic [3], namely in the domains $\mathcal{D}_t = \{p \in \mathbb{R}, q < 1/t\}$ and $\mathcal{D}_t^c = \{p \in \mathbb{R}, q > 1/t\}$. The functions

$$g_t^{(\pm)}(\xi, \eta) = -\eta \frac{2}{t}(1 \pm \sqrt{1+(t\xi)^2}) \tag{6.29}$$

are two branches of an algebraic function. They unfold φ_t locally, $g_t^{(-)}$ in $\mathcal{D}_t$ and $g_t^{(+)}$ in $\mathcal{D}_t^c$. They satisfy the standard Hamilton–Jacobi equation (3.26) with initial conditions

$$\begin{array}{ll} g_t^{(-)}(\xi,\eta) \sim h(\xi,\eta)t \to 0, & t \to 0, \\ g_t^{(+)}(\xi,\eta) \to -2\xi\eta, & t \to +\infty. \end{array} \tag{6.30}$$

The operator $\exp -(i/\hbar)t\mathbf{H}$ admits a continuous one-parameter set of unitary extensions:

$$(\mathbf{U}_t^{(\alpha)}\psi)(q) = \frac{e^{i\alpha\theta(tq-1)}}{1-tq}\,\psi\left(\frac{q}{1-tq}\right). \tag{6.31}$$

But the group law $\mathbf{U}_{t'}\mathbf{U}_t = \mathbf{U}_{t'+t}$ is verified for $e^{i(\alpha/\hbar)} = \pm 1$ only. The W.F. of $\mathbf{U}_t^{(\alpha)}$ has not the form (4.1). It is a sum

$$u_t^{(\alpha)}(X) = u_t^{(-)}(X) + e^{i\alpha}u_t^{(+)}(X), \tag{6.32}$$

where

$$u_t^{(\pm)}(X) = (1+(tq)^2)^{-1/2}e^{-(i/\hbar)g_t^{\pm}(X)}. \tag{6.33}$$

The components are isometric,

$$(\overset{*}{u}{}_t^{(-)} \circ u_t^{(-)})(q,p) = \theta(1-tq) = 1 - (\overset{*}{u}{}_t^{(+)} \circ u_t^{(+)})(q,p) \tag{6.34}$$

and orthogonal in distribution sense

$$\overset{*}{u}{}_t^{(+)} \circ u_t^{(-)} = \overset{*}{u}{}_t^{(-)} \circ u_t^{(+)} = 0. \tag{6.35}$$

The canonical map (6.28) is reproduced for q,

$$(\overset{*}{u}{}_t^{(\alpha)} \circ q \circ u_t^{(\alpha)})(X) = q_t(q) \tag{6.36}$$

whereas one obtains for p

$$(\overset{*}{u}{}_t^{(\alpha)} \circ p \circ u_t^{(\alpha)})(X) = p_t(q, p) - \alpha(1-tq)^2\,\delta(1-tq). \tag{6.37}$$

The correction term is zero in distribution sense like $X^2\,\delta(X)$, but the classical quantity $dq_t = (1-tq)^{-2}\,dq$ has a pole in $q = 1/t$. Thus, the action element transforms according to

$$p\,dq \mapsto p_t\,dq_t - \alpha\,\delta(1-tq)\,dq. \tag{6.38}$$

Strictly speaking, the map φ_t (6.28) is not defined in $tq = 1$. The singular term in (6.37) introduces a distinction between the classical maps induced by the various extensions $\mathbf{U}_t^{(\alpha)}$, $\alpha \in \mathbb{R}$, without which the present example would contradict Theorem 1.

(iv) *A bicanonical map near the parity operation*

The generating function

$$g(\xi, \eta) = \frac{2\alpha^3}{\xi^2+\alpha^2}\,\eta \equiv 2\eta a(\xi), \tag{6.39}$$

where α is an arbitrary positive constant, unfolds the symplectomorphism $\bar{X} = \varphi(X)$ given by

$$\begin{aligned} \bar{q} &= \xi + \frac{\alpha^3}{\xi^2+\alpha^2} \qquad & \bar{p} &= \left(1 + \frac{2\alpha^3\xi}{(\xi^2+\alpha^2)^2}\right)\eta \\ q &= \xi - \frac{\alpha^3}{\xi^2+\alpha^2} \qquad & p &= \left(1 - \frac{2\alpha^3\xi}{(\xi^2+\alpha^2)^2}\right)\eta. \end{aligned} \tag{6.40}$$

The corresponding bicanonical map is given by the W.F.

$$u(q, p) = (1 - a'(q)^2)^{1/2} \exp\frac{2}{i\hbar}\,pa(q), \tag{6.41}$$

which has the exponential form (4.1). But the product of u with the parity $\Pi(X) = \pi\hbar\,\delta(X)$ no longer has this property. One has

$$\begin{aligned} (u \circ \Pi)(X) &= \frac{1}{\pi\hbar}\int d^2Y\,d^2z e^{-(2i/\hbar)I(Y,z)}u(X+Y)\,\delta(X+z) \\ &= \int_{-\infty}^{\infty} d\xi\,\delta(q - a(\xi))(1 - a'(\xi)^2)^{1/2} \exp\left[-\frac{2i}{\hbar}\right]p(\xi + a(\xi) - q) \end{aligned} \tag{6.42}$$

The equation $a(\xi) = q$ has two solutions

$$\xi_\pm(q) = \pm\alpha\sqrt{\frac{\alpha}{q} - 1} \tag{6.43}$$

and q is limited to the domain $[0, \alpha]$. One finds

$$(u \circ \Pi)(q, p) = N(q)\left[\exp\left[\frac{2i\alpha}{\hbar} p\sqrt{\frac{\alpha}{q}-1}\right] + \exp\left[-\frac{2i\alpha}{\hbar} p\sqrt{\frac{\alpha}{q}-1}\right]\right] \tag{6.44}$$
$$= 2N(q)\cos\left[\frac{2\alpha}{\hbar} p\sqrt{\frac{\alpha}{q}-1}\right]$$

where

$$N(q) = \theta(q)\theta(\alpha - q)\left(\frac{\alpha^4}{4q^3(\alpha-q)} - 1\right)^{1/2} \tag{6.45}$$

(v) *The* W.F. *of the density operator of a gibbs state*

The identity (5.6)

$$e^{-(it/\hbar)}h \circ = (N_t e^{-(i/\hbar)g_t})\circ$$

holds for complex values of t with the same restriction (3.8). Putting

$$t = -i\hbar\beta, \qquad \beta > 0, \tag{6.46}$$

and defining

$$G(\beta, X) = \frac{i}{\hbar} g_{-i\hbar\beta}(X) \tag{6.47}$$

$$n(\beta, X) = N_{-i\hbar\beta}(X), \tag{6.48}$$

we have the W.F. of the operator $\exp -\beta\mathbf{H}$:

$$u(\beta, X) = n(\beta, X)e^{-G(\beta, X)} \tag{6.49}$$

G is real because g_t is odd in t, and

$$(n(\beta, X))^2 = \det\left(\mathbb{1} - \frac{i\hbar}{2}\Lambda\frac{\partial^2 G}{\partial X\,\partial X}(\beta, X)\right) \tag{6.50}$$

is real because N_t^2 is even in t.

A case interesting statistics is

$$h(X) = \tfrac{1}{2}X \cdot \omega X, \qquad \tilde{\omega} = \omega > 0. \tag{6.51}$$

h is the W.F. of the Hamiltonian of coupled oscillators with a positive spectrum. Because $\omega > 0$, it can be diagonalized in a symplectic basis in which its eigenvalues are ω_k, $k = 1 \cdots n$, with multiplicity 2 [3].

From Section 5 we get

$$n(\beta, X) = \prod_{k=1}^{n} \frac{1}{ch\frac{\hbar\beta}{2}\omega_k} \tag{6.52}$$

$$G(\beta, X) = \frac{1}{\hbar}\sum_{k=1}^{n}(q_k^2 + p_k^2)\,th\frac{\hbar\beta\omega_k}{2} \tag{6.53}$$

and

$$Trne^{-G} = \int \frac{d^{2n}X}{(2\pi\hbar)^n} \prod_{k=1}^{n} \frac{\exp\left[-(q_k^2+p_k^2)\frac{1}{\hbar}\, th\frac{\hbar\beta\omega_k}{2}\right]}{ch\frac{\hbar\beta\omega_k}{2}} = \prod_{k=1}^{n} \frac{1}{sh\frac{\hbar\beta\omega_k}{2}}. \tag{6.54}$$

By normalizing u to unit trace one obtains

$$\rho(\beta, X) = \prod_{k=1}^{n} \left(th\frac{\hbar\beta\omega_k}{2}\right) \exp\left[-\frac{1}{\hbar}\, th\frac{\hbar\beta\omega_k}{2}(q_k^2+p_k^2)\right]. \tag{6.55}$$

This is the exact W.F. of the density operator $\exp(\Omega-\beta\mathbf{H})$ of this Gibbs ensemble, if $\beta=1/kT$. For high temperature, $\hbar^{-1}th\frac{1}{2}\hbar\beta\omega_k \sim \omega_k/2kT$, ρ becomes

$$\rho(\beta, X) \simeq \frac{e^{-(h(X)/kT)}}{\prod_{j=1}^{n}\left(\frac{2kT}{\omega_j}\right)}. \tag{6.56}$$

This is the classical approximation for $\hbar \to 0$. At vanishing temperature, $th\frac{1}{2}\hbar\beta\omega_k \simeq 1$, ρ tends toward the ground state of the system

$$\rho(\beta, X) \propto e^{-(1/\hbar)X\cdot X}.$$

7. Conclusion

The algebraic content of this paper can be summarized as follows: The bicanonical operators studied here form two distinct species of groups which belong to a ray representation of the groups of classical canonical maps $ISp(E)$ and $\mathscr{G}^M$ respectively. The classical maps φ are represented by W.F. u which are built up from a pure geometrical quantity, the standard generating function g of φ: The phase of u is $-\hbar^{-1}g$ and its amplitude is the square root of a Jacobi determinant containing the second derivatives of g. The standard picture of canonical maps comes thus naturally into play together with the description of unitary operators by means of their Wigner function.

This ideal picture is altered to some extent by complications of geometrical nature. If the transversality condition $\infty > N > 0$ does not hold, g fails to unfold the classical map φ or does not exist at all. In some cases, a way-out is to work with a multivalued function; u becomes a sum of exponentials, one for each branch of g.

Bicanonical maps have more kinematical than dynamical applications. But, in any case, they form a useful investigation tool because we possess exact results on them. It is not difficult to convince oneself that Van Vleck's formula (4.1) can still make sense for more general unitary operators. It is the first meaningful approximation in an asymptotic expansion in $\hbar$, under the condition that the phase unfolds the classical canonical map induced by the quantum map at the limit $\hbar=0$. A $\hbar-$ expansion is meaningless if one does not take care to add the appropriate number of exponentials in the first approximation step. This is necessary whenever the classical problem reveals caustics. Interference phenomena play then a leading role in the quantum problem.

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