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Lower bounds for zero energy eigenfunctions of Schrödinger operators[1])

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Abstract. Let g be a non-zero solution in $L^2(\mathbb{R}^n)$, $n \geq 2$, of $(-\Delta + V)g = 0$. If the potential V vanishes rapidly enough at infinity, then g cannot decay (in the L^2-sense) more rapidly than any power of $|x|$, i.e. $|x|^N g \notin L^2(\mathbb{R}^n)$ for some finite N.

1. Introduction

A non-relativistic quantum mechanical particle moving on a line in a potential V cannot be bound at zero energy if V is such that

$$\int_{-\infty}^{+\infty} (1+|x|)\,|V(x)|\,dx < \infty.$$

In other words the equation $-\psi'' + V\psi = 0$ has no non-zero solutions that are square-integrable over the real line $\mathbb{R}$. If $\mathbb{R}$ is replaced by $(0, \infty)$ for example, the same is true; more precisely, if $\int_0^\infty r|V(r)|\,dr < \infty$, there are no zero energy bound states in the $l = 0$ partial wave subspace of a three-dimensional quantum mechanical system in the spherically symmetric potential $V(r)$ (see e.g. [1], Chapters XVII.1 and II.1 respectively).

In the latter case one may however have zero energy bound states in the higher order partial wave subspaces ($l \geqq 1$), even if V has finite range (see [2], footnote on page 80 for a square well, [1] or [3], Remark 11.17(c) and Problem 11.11 for more general cases). The intuitive reason for this is roughly as follows: if $l > 0$, then the effective potential is $V(r) + l(l+1)r^{-2}$ which, at large r, is roughly $l(l+1)r^{-2}$ under the above assumptions on V; hence, if the particle has zero energy, it sees a wall of infinite extension of the form cr^{-2} $(c > 0)$ which can produce a bound state[2]) (no tunnelling is possible).

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[2]) Notice that $\int_1^\infty r \cdot cr^{-2}\,dr = \infty$, so that the centrifugal part of the effective potential does not satisfy the condition needed for proving the non-existence of zero energy bound states.

The zero energy bound state eigenfunctions in the l-th partial wave subspace of $L^2(\mathbb{R}^3)$ are known to behave like r^{-l-1} as $r \to \infty$. This is strikingly different from the exponential decay of eigenfunctions belonging to strictly negative eigenvalues: if $\lambda < 0$, $(-\Delta + V)\psi = \lambda\psi$ and $\psi \in L^2(\mathbb{R}^3)$ and if V decays sufficiently rapidly, then $\|e^{\kappa r}\psi\|_{L^2} < \infty$ for each $\kappa < |\lambda|^{1/2}$. The purpose of our paper is to prove quite generally (i.e. in $n \geq 2$ space dimensions and without assuming spherical symmetry) that zero energy bound states are weakly localized in the sense indicated above: if V satisfies suitable decay conditions and if $\psi \in L^2(\mathbb{R}^n)$ is such that $(-\Delta + V)\psi = 0$, then there is a number $N < \infty$ such that $\|(1+|x|)^N\psi\|_{L^2} = \infty$, i.e. ψ cannot decay faster (in the L^2-sense) than some negative power of $|x|$. This follows from a more general result which we state and prove in the form of a theorem in Section 3. The proof makes heavy use of an inequality involving the Laplacean that we established in a previous paper [4].

2. Notation and preliminary results

We use the following notation: the symbol x is used for vectors in $\mathbb{R}^n$, $n \geq 2$. We set $r = |x|$, $\partial_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, $\nabla \equiv \text{grad} = (\partial_1, \ldots, \partial_n)$, $\partial_r = \sum_{j=1}^n x_j r^{-1} \partial_j$ and $\Delta = \sum_{j=1}^n \partial_j^2$. We shall refer to the operator $(1-\Delta)^{-1}$ acting on functions defined on $\mathbb{R}^n$; it is given as the convolution operator by the Green's function of the negative Laplacean (one of the Bessel potentials in the terminology of [5]).

For $0 \leq a < b \leq \infty$ we set $\Omega(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$. Notice that $\Omega(0, \infty) = \mathbb{R}^n \setminus \{0\}$. The derivatives of locally integrable functions are understood to be in the sense of distributions. For $1 \leq q \leq \infty$, $k \geq 0$ and integer, $a \geq 0$ and $\Omega \equiv \Omega(a, \infty)$, $L^q(\Omega)$ denotes the Banach space of q-summable functions on Ω and $H^{k,q}(\Omega)$ the Sobolev space consisting of all $f \in L^q(\Omega)$ such that $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f \in L^q(\Omega)$ for all n-tuples $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers with $\sum_{j=1}^n \alpha_j \leq k$. We put

$$\|f\|_{H^{k,q}(\Omega)} = \sum_{\alpha_1 + \cdots + \alpha_n \leq k} \|\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f\|_{L^q(\Omega)}. \tag{1}$$

If $q = 2$, we use the simpler notation $H^k(\Omega) \equiv H^{k,2}(\Omega)$. Finally we write $\|\cdot\|_{L^q}$ for the norm in $L^q(\mathbb{R}^n)$ and $\|\cdot\|_{H^{k,q}}$ for that in $H^{k,q}(\mathbb{R}^n)$, and we denote by $H_c^{k,q}(\mathbb{R}^n \setminus \{0\})$ the set of functions $f \in H^{k,q}(\mathbb{R}^n)$ that have compact support in $\mathbb{R}^n \setminus \{0\}$.

The proof of our theorem is based on the Sobolev imbedding theorem and on the following known results that we announce as Propositions 1, 2 and 3.

Proposition 1. *If* $1 < q < \infty$, *then* $(1-\Delta)^{-1}$ *defines a bounded invertible operator from* $L^q(\mathbb{R}^n)$ *onto* $H^{2,q}(\mathbb{R}^n)$. *In particular, if* $f, \Delta f \in L^q(\mathbb{R}^n)$, *then*[3] $f \in H^{2,q}(\mathbb{R}^n)$ (see [5], Theorem V.3).

Proposition 2. *Let* $n \geq 2$, $p \in (n/2, \infty]$ *with* $p \geq n-2$. *Set* $\mu = 2 - n/p$. *Let* q *and* s *satisfy*

$$1 \leq q \leq 2 \leq s < \infty, \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{s}. \tag{2}$$

[3]) Write $f = (1-\Delta)^{-1}(f - \Delta f)$.

Let $\Gamma_{ns}=\{k+n-3/2-n/s \mid k=1,2,3,\ldots\}$. *Then there is a finite constant* C, *depending only on* n, p *and* s, *such that*

$$\|r^{\nu}f\|_{L^s}\leq C\,\|r^{\nu+\mu}\,\Delta f\|_{L^q} \tag{3}$$

for all $\nu\in\Gamma_{ns}$ *and all* $f\in H_c^{2,q}(\mathbb{R}^n\setminus\{0\})$. *If* $p=\infty$, *the inequality* (3) *holds with* C *replaced by* $2\nu^{-1}$. (See [4], Theorem 1 and proof of Theorem 2.)

Proposition 3. *Let* $R>0$ *and* $\Omega=\Omega(R,\infty)$, *and let* $\alpha\geq 0$. *Then* $f\in H^{k,q}(\Omega)\Rightarrow r^{-\alpha}f\in H^{k,q}(\Omega)$.

Proof. Clearly multiplication by $r^{-\alpha}$ defines a bounded operator in $L^q(\Omega)$, since $R>0$ and $\alpha\geq 0$. This proves the assertion for $k=0$. Next notice that

$$\partial_j r^{-\alpha}f=r^{-\alpha}\,\partial_j f-\alpha x_j r^{-\alpha-2}f. \tag{4}$$

Hence $f\in H^{1,q}(\Omega)\Rightarrow r^{-\alpha}f\in H^{1,q}(\Omega)$. The proof for $k>1$ is similar. ■

3. Lower bounds for zero energy eigenfunctions

We now state and prove our principal result.

Theorem. *Let* $n\geq 2$, $R_0\in[0,\infty)$ *and set* $\Omega_0=\Omega(R_0,\infty)$. *Let* $V:\Omega_0\to\mathbb{C}$ *and assume that there is a number* $p\in[1,\infty]$ *such that* $p>n/2$ *and* $p\geq n-2$ *and such that* $r^{2-n/p}V\in L^p(\Omega_0)$. *Suppose* $g\in H^1(\Omega_0)$ *is such that* Δg *is a function and*

$$|(\Delta g)(x)|\leq|V(x)|\,|g(x)| \quad \text{a.e. on } \Omega_0. \tag{5}$$

Then, if $r^{\tau}g\in L^2(\Omega_0)$ *for each* $\tau<\infty$, *one must have* $g=0$ (*in the* L^2-*sense*).

Remark. (a) If $p=\infty$, the condition on the function V means that $|x|^2\,|V(x)|\leq\text{const}<\infty$, i.e. $V(x)$ should decay at least as rapidly as $|x|^{-2}$ for $|x|\to\infty$. If $p<\infty$, the condition on V means that

$$\int_{\Omega_0}|r^2V(x)|^p\frac{d^nx}{r^n}<\infty,$$

i.e. $r^2V(x)$ must tend to zero in an L^p-sense as $|x|\to\infty$. Of course local singularities of V are allowed, and for $n=2,3,4$ the result is very natural.

(b) Let $V:\mathbb{R}^n\to\mathbb{R}$ be such that $(1+r)^{2-n/p}V\in L^p(\mathbb{R}^n)$ for some $p\in(n/2,\infty]$ with $p\geq n-2$. Then $H=-\Delta+V$ is self-adjoint in $L^2(\mathbb{R}^n)$ on the domain $\{f\in H^1(\mathbb{R}^n)\mid Hf\in L^2(\mathbb{R}^n)\}$. If zero is an eigenvalue of H, then any associated eigenvector g has the following property: there is a number $N<\infty$ such that $\|r^Nf\|_{L^2}=\infty$. (To see this, it suffices to notice that an eigenvector g corresponding to the eigenvalue zero satisfies (5) with the equality sign.)

Proof. (i) We first fix s and q satisfying the hypotheses of Propositions 1 and 2. It suffices to choose the number s; q is then defined by $q^{-1}=p^{-1}+s^{-1}$.

If $p>2$, we take $s=2$. If $p\leq 2$ (which is possible only for $n=2,3$), we define s by $s^{-1}=3/4-(2p)^{-1}-(2n)^{-1}$. The assumptions made on p imply that $s\in[3,\infty)$ in the second case and that $1<q\leq\min\{2,p\}$ in both cases.

We set $\mu=2-n/p$ and choose a number $R\in(R_0,\infty)$ as follows. If $p=\infty$, we take $R=R_0+1$; if $p<\infty$, we let $C=C(n,p,s)$ be the constant appearing in Proposition 2 and take R so large that $C\|r^\mu V\|_{L^p(\Omega(R,\infty))}<\frac{1}{2}$, which is possible by the hypothesis made on V. We set $\Omega=\Omega(R,\infty)$ and $\lambda=\|r^\mu V\|_{L^p(\Omega)}$.

The Sobolev imbedding theorem ([6], Theorem 5.4 and Corollary 5.16) implies that, if $p>n/2$ and q and s are as above, one has the following imbeddings: $H^1(\Omega)\subset L^s(\Omega)$ and $H^{2,q}(\Omega)\subset L^s(\Omega)$; here $X\subset Y$ means that each $\xi\in X$ is also an element of Y and that there is a constant $\kappa=\kappa_{XY}$ such that $\|\xi\|_Y\leq\kappa\|\xi\|_X$ for each $\xi\in X$.

(ii) Let $\eta\in C^\infty(\mathbb{R}^n)$ be such that $0\leq\eta\leq1$, $\eta(x)=0$ if $|x|\leq R$ and $\eta(x)=1$ if $|x|\geq R+1$. Assume that g satisfies all the hypotheses stated in the theorem and set $g_0=\eta g$. We shall show that $r^\tau g_0$, $r^\tau\Delta g_0$ and each component of $r^\tau\nabla g_0$ belong to $L^q(\mathbb{R}^n)$ for each $\tau\in\mathbb{R}$.

The first assertion follows from the Hölder inequality and the hypothesis that $r^\tau g\in L^2(\Omega_0)$ for all τ: if $m\in(2,\infty]$ is defined by $m^{-1}=q^{-1}-\frac{1}{2}$, then

$$\|r^\tau g_c\|_{L^q}\leq\|r^\tau g\|_{L^q(\Omega)}\leq\|r^{-n}\|_{L^m(\Omega)}\|r^{\tau+n}g\|_{L^2(\Omega)}<\infty.$$

Next we observe that

$$r^\tau\Delta g_0=\eta r^\tau\Delta g+2r^\tau(\nabla\eta)\cdot\nabla g+r^\tau(\Delta\eta)g. \tag{6}$$

Since g and the components of ∇g are in $L^2(\Omega)$ and $\nabla\eta,\Delta\eta$ have compact support, the last two terms on the r.h.s. of (6) are in $L^q(\mathbb{R}^n)$ (remember that $q\leq2$). We denote by β_τ the sum of their L^q-norms and then have by the Hölder inequality:

$$\|r^\tau\Delta g_0\|_{L^q}\leq\|r^\tau\Delta g\|_{L^q(\Omega)}+\beta_\tau\leq\|r^\mu V\|_{L^p(\Omega)}\|r^{\tau-\mu}g\|_{L^s(\Omega)}+\beta_\tau. \tag{7}$$

In view of the last statement in (i), this leads to the following two inequalities, in which λ is the number defined in part (i) of the proof and κ_s, κ_{qs} are finite constants depending on the values of the subscript(s):

$$\|r^\tau\Delta g_0\|_{L^q}\leq\lambda\kappa_s\|r^{\tau-\mu}g\|_{H^1(\Omega)}+\beta_\tau, \tag{8}$$

$$\begin{aligned}\|r^\tau\Delta g_0\|_{L^q}&\leq\lambda\kappa_{qs}\|r^{\tau-\mu}g_0\|_{H^{2,q}}+\lambda\|r^{\tau-\mu}(1-\eta)g\|_{L^s(\Omega)}+\beta_\tau\\&\leq\lambda\kappa_{qs}\|r^{\tau-\mu}g_0\|_{H^{2,q}}+\lambda\gamma_\tau\kappa_s\|g\|_{H^1(\Omega)}+\beta_\tau,\end{aligned} \tag{9}$$

where $\gamma_\tau=\|r^{\tau-\mu}(1-\eta)\|_{L^\infty(\Omega)}<\infty$.

Since $g\in H^1(\Omega)$, the inequality (8) and Proposition 3 imply that $r^\tau\Delta g_0\in L^q(\mathbb{R}^n)$ for $\tau\leq\mu$; in particular $\Delta g_0\in L^q(\mathbb{R}^n)$. By Proposition 1, we then have $g_0\in H^{2,q}(\mathbb{R}^n)$.

Next we notice the identity

$$\Delta r^\tau g_0=r^\tau\Delta g_0+2\tau\,\partial_r(r^{\tau-1}g_0)+(n\tau-\tau^2)r^{\tau-2}g_0. \tag{10}$$

Since $\|\partial_r f\|_{L^q}\leq\|f\|_{H^{1,q}}\leq\|f\|_{H^{2,q}}$, (10) leads to

$$\|\Delta r^\tau g_c\|_{L^q}\leq\|r^\tau\Delta g_0\|_{L^q}+2|\tau|\,\|r^{\tau-1}g_0\|_{H^{2,q}}+(n|\tau|+\tau^2)\|r^{\tau-2}g_0\|_{L^q}. \tag{11}$$

Hence, if $\tau\leq\tau_0\equiv\min\{\mu,1\}$, we have $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$. Together with Proposition 1, this implies that $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for $\tau\leq\tau_0$.

This last inclusion may now be combined with the inequality (9) to deduce that $r^\tau\Delta g_0\in L^q(\mathbb{R}^n)$ for $\tau\leq\tau_0+\mu$, and (11) then implies that $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$ if $\tau\leq2\tau_0$. Hence, by Proposition 1, $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for $\tau\leq2\tau_0$. By iterating this procedure one obtains that $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$ and $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for all $\tau\in\mathbb{R}$.

Finally we have for each $\tau \in \mathbb{R}$ (see (4)):

$$\|r^\tau \partial_j g_0\|_{L^q} \le \|\partial_j r^\tau g_0\|_{L^q} + |\tau| \, \|r^{\tau-1} g_0\|_{L^q}$$
$$\le \|r^\tau g_0\|_{H^{2,q}} + |\tau| \, \|r^{\tau-1} g_0\|_{L^q} < \infty.$$

(iii) We now show that $g(x)=0$ for $|x|>R+1$. For this, we let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(x)=1$ if $|x|\le 1$ and $\theta(x)=0$ if $|x|\ge 2$. For $a>0$ we define θ_a by $\theta_a(x)=\theta(x/a)$, and we set $\delta' = \| |\nabla\theta| \|_{L^\infty}$, $\delta'' = \|\Delta\theta\|_{L^\infty}$. We observe that

$$|(\nabla\theta_a)(x)| \le \frac{\delta'}{a}, \qquad |(\Delta\theta_a)(x)| \le \frac{\delta''}{a^2} \quad \forall x \in \mathbb{R}^n. \tag{12}$$

The identity

$$\Delta\theta_a g_0 = \theta_a \, \Delta g_0 + 2(\nabla\theta_a)\cdot\nabla g_0 + (\Delta\theta_a) g_0 \tag{13}$$

and a similar identity for $\partial_j \theta_a g_0$ imply that $\theta_a g_0 \in H_c^{2,q}(\mathbb{R}^n \setminus \{0\})$. By setting $f=\theta_a g_0$ in (3) and using (13) and (12) one finds that, for $\nu \in \Gamma_{ns}$:

$$\|r^\nu \theta_a g_0\|_{L^s} \le C \, \|r^{\nu+\mu}\theta_a \, \Delta g_0\|_{L^q} + \frac{2\delta'}{a} C \, \|r^{\nu+\mu} \, \nabla g_0\|_{L^q} + \frac{\delta''}{a^2} C \, \|r^{\nu+\mu} g_0\|_{L^q}. \tag{14}$$

Remembering that $r^\rho \, \Delta g_0$, $r^\rho \, \nabla g_0$ and $r^\rho g_0$ are in $L^q(\mathbb{R}^n)$ for each $\rho \in \mathbb{R}$, one may take the limit $a \to \infty$ in (14) (by using for instance the dominated convergence theorem) to obtain the inequality

$$\|r^\nu g_0\|_{L^s} \le C \, \|r^{\nu+\mu} \, \Delta g_0\|_{L^q}, \qquad \nu \in \Gamma_{ns}. \tag{15}$$

The r.h.s. of (15) may be majorized by using the inequality (7), with $\tau = \nu + \mu$. We note that β_τ satisfies $\beta_\tau \le (R+1)^\tau c(\eta, g)$, where $c(\eta, g)$ is a finite number that does not depend on τ. We also have, as in (9), that

$$\|r^\nu g\|_{L^s(\Omega)} \le \|r^\nu g_0\|_{L^s} + (R+1)^\nu \, \|g\|_{L^s(\Omega)} \le \|r^\nu g_0\|_{L^s} + \kappa_s (R+1)^\nu \, \|g\|_{H^1(\Omega)}.$$

Consequently we obtain that

$$\|r^\nu g_0\|_{L^s} \le C\lambda \, \|r^\nu g_0\|_{L^s} + C\lambda\kappa_s (R+1)^\nu \, \|g\|_{H^1(\Omega)} + Cc(\eta, g)(R+1)^{\nu+\mu}. \tag{16}$$

If $p<\infty$, we have $C\lambda < \frac{1}{2}$, and (16) implies that, for $\nu \in \Gamma_{ns}$:

$$\|r^\nu g\|_{L^s(\Omega(R+1,\infty))} \le \|r^\nu g_0\|_{L^s(\mathbb{R}^n)} \le c_1(R, \eta, g)(R+1)^\nu, \tag{17}$$

where c_1 is a finite number independent of ν. If $p=\infty$, one may replace C by $2\nu^{-1}$ in (14)–(16) and obtains the validity of (17) for all $\nu \in \Gamma_{ns} \cap [4\lambda, \infty)$.

Now assume that $\|g\|_{L^2(\Omega(R+1,\infty))} \ne 0$. Then, as $\nu \to \infty$ ($\nu \in \Gamma_{ns}$), the l.h.s. of (17) grows faster than $(R+1)^\nu$, i.e. (17) is violated for ν large enough. Hence we must have $g=0$ on $\Omega(R+1,\infty)$.

(iv) To show that $g=0$ on $\Omega_0 = \Omega(R_0, \infty)$, it suffices to notice that $q \ge 2p/(p+2)$, so that one may apply the unique continuation theorem proved in [4] (see [4], Theorem 2). ■

Additional remarks

(a) It is interesting to point out that A. Hinz recently obtained *upper* bounds for zero energy eigenfunctions that have the form of a negative power of $|x|$, see [7].

(b) One may ask to what extent our condition $\|r^{2-n/p}V\|_{L^p} < \infty$ is optimal. For $p = \infty$, it requires that $|V(x)| \leq cr^{-2}$. The following example shows that one may have exponentially decreasing zero energy eigenfunctions for potentials V tending to zero at infinity but doing so more slowly than r^{-2}: if $-\Delta g + Vg = 0$, then $V = \Delta g/g$. By taking g of the form $g(x) = \exp[-\varphi(r)]$, one obtains

$$V(x) = |\varphi'(r)|^2 - \varphi''(r) - (n-1)r^{-1}\varphi'(r).$$

If for example φ is a smooth function that is constant near $r = 0$ and equal to $r^\alpha, 0 < \alpha < 1$, near infinity, then $g \in L^2(\mathbb{R}^n)$, hence it is a zero energy bound state eigenfunction, and $V(x)$ decays at infinity like $r^{-2+2\alpha}$. This gives a class of smooth potentials that decay like $r^{-\beta}, 0 < \beta < 2$, and give rise to zero energy eigenfunctions that decrease more rapidly than any negative power of $|x|$.

Note. This paper is an elaboration of one of the results announced in [8]. After submission of the paper for publication, our attention was drawn to Ref. [9] which contains various L^2 lower bounds for eigenfunctions of Schrödinger operators, in particular a theorem of the type of that given here.

REFERENCES

[1] K. Chadan and P. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer, New York, 1977.

[2] L. I. Schiff, *Quantum Mechanics*, McGraw-Hill, New York, 1968.

[3] W. O. Amrein, J. M. Jauch and K. B. Sinha, *Scattering Theory in Quantum Mechanics*, Benjamin, Reading, 1977.

[4] W. O. Amrein, A. M. Berthier and V. Georgescu, Ann. Inst. Fourier *31*, 153–168 (1981).

[5] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princton University Press, 1970.

[6] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[7] A. Hinz, Math. Zeitschr. *185*, 291–304 (1984).

[8] W. O. Amrein, A. M. Berthier and V. Georgescu, Comptes Rendus Acad. Sci. Paris I *295*, 575–578 (1982).

[9] R. Froese, I. Herbst, M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, Proc. Royal Soc. Edinburgh *95A*, 25–38 (1983).