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Twistor geometry

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Abstract. The aim of this paper is to give a detailed exposition of the relation between the geometry of twistor space and the geometry of Minkowski space. The paper has a didactical purpose; no use has been made of differential geometry and cohomology.

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0. Introduction

Twistor space provides an alternative geometry for the geometry of Minkowski space. In order to give an idea of the physical motivation for Penrose's development of twistor theory I will quote the first three alineas of Penrose's contribution to the International Congress of Mathematicians held in 1978 (Penrose 1978):

"This century has seen two major revolutions in physical thought. The first of these, relativity, uprooted earlier ideas of the nature of time and space, and provided us with our present picture of the world as a real differential manifold of dimension four, possessing a pseudo-Riemannian metric with a (+−−−) signature. The second revolution, quantum theory, altered our picture of things yet more radically than did relativity – even to the extent that, as we were told, it became no longer appropriate to form pictures at all, in order to give accurate representations of physical processes on the quantum scale. And, for the first time, the complex field ℂ was brought into physics at a fundamental and universal level, not just as a useful or elegant device, as had often been the case earlier for many applications of complex numbers to physics, but at the very basis of physical law.

Thus, the allowable physical states were to form a complex vector space, in fact a Hilbert space. So, on the one hand, we had the real-manifold picture of space-time geometry, and on the other, the complex vector space view, according to which geometrical pictures were deemed inappropriate.

This conflict has remained with us since the conceptions of these great theories, to the extent that, even now, there is no satisfactory union between the two. Even at the most elementary level, there are still severe conceptual problems in providing a satisfactory interpretation of quantum mechanical observations in a way compatible with the tenets of special relativity. And quantum field theory, which represents the fully special-relativistic version of quantum theory, though it has had some very remarkable and significant successes, remains beset with inconsistencies and divergent integrals whose illeffects have been only partially circumvented. Moreover, the present status of the unification of general relativity with quantum mechanics remains merely a collection of hopes, ingenious ideas and massive but inconclusive calculations.

In view of this situation it is perhaps not unreasonable to search for a different viewpoint concerning the role of geometry in basic physics. Broadly speaking, "geometry", after all, means any branch of mathematics in which pictorial representations provide powerful aids to one's mathematical intuition. It is by no means necessary that these "pictures" should refer just to a spatio-temporal ordering of physical events in the familiar way. And since $\mathbb{C}$ plays such a basic universal role at the primitive levels of physics at which quantum phenomena are dominant, one is led to expect that the primitive geometry of physics might be complex rather than real. Moreover, the macroscopic geometry of relativity has many special features about it that are suggestive of a hidden complex manifold origin, and of certain underlying physical connections between the normal spatio-temporal relations between things and the complex linear superposition of quantum mechanics."

The idea behind twistor theory is to put the null lines of Minkowski space, being the world lines of non-interacting zero mass particles on the foreground instead of the space-time points. These space-time points then become derived objects; the basic quantities are the twistors. One of the advantages of this approach is that when the theory is quantized, the space-time points become fuzzy and the concept of null direction remains well-defined, in contrast to the conventional theory where the points remain well-defined and the null cones become fuzzy. Twistor space is the complex vector space $\mathbb{C}^4$ equipped with a Hermitian form of signature 0; the corresponding projective space $P\mathbb{C}^3$ is called projective twistor space. Null lines in Minkowski space correspond to null elements of $P\mathbb{C}^3$. This correspondence between Minkowski space and twistor space, called Penrose correspondence (Penrose 1967), will be the subject of this paper. By this correspondence physical problems in Minkowski space are transferred into problems of several complex variables on twistor space. Twistor theory provides a link between physics and complex manifold theory because in many cases the field equations of physics reduce to Cauchy Riemann equations, and therefore the solutions can be represented entirely in terms of complex manifolds, holomorphic vector bundles or cohomology classes on open complex manifolds with coefficients in certain holomorphic vector bundles. The most striking result of this approach was the solution of the Yang-Mills equations on S^4 by Atiyah, Hitchin, Drinfeld and Manin (1978). Twistor theory also appears to be the natural framework for the description of massless free fields (Eastwood, Penrose and Wells Jr 1981) and of self-dual Einstein manifolds (Penrose 1976).

In Section 1 we recall briefly the concepts of Minkowski space, Lorentz group, SL(2, $\mathbb{C}$), spinors and the interrelations between these. In Section 2 twistors are introduced, and a certain subset of them, the null twistors, are interpreted as null lines in compactified Minkowski space. In Section 3 we give interpretations of general twistors as Robinson congruences in compactified Minkowski space and as null planes in compactified complexified Minkowski space and show how the interpretations are related. In Section 4 we discuss how the conformal group acts on twistor space. In Section 5 massless free fields are briefly discussed.

I. Minkowski space, Lorentz group, SL(2, $\mathbb{C}$), spinors and all that

The real Minkowski space M is the real manifold $\mathbb{R}^4$ equipped with a scalar product

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu \tag{1.1}$$

where the metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.2}$$

and where the usual summation convention over repeated indices is used. If $\|x-y\|^2$ $(=(x-y)\cdot(x-y))$ is positive, zero, or negative then x and y are said to be timelike separated, null separated, and spacelike separated, respectively. If x and y are null separated then they can be joined by a light signal; note that the velocity of light is taken equal to 1. The Lorentz group L is the group of linear mappings Λ from M onto M which preserve the scalar product:

$$(\Lambda x)\cdot(\Lambda y) = x \cdot y \qquad \forall x, y \in M \tag{1.3}$$

If $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$ then $\Lambda \in L$ if and only if

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \tag{1.4}$$

Taking determinants of both sides of this equation gives

$$\det \Lambda = \pm 1 \qquad \forall \Lambda \in L \tag{1.5}$$

Taking $\rho = \sigma = 0$ in equation (1.4) gives

$$|\Lambda^0{}_0| \geqq 1 \qquad \forall \Lambda \in L \tag{1.6}$$

The elements $\Lambda \in L$ with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geqq 1$ form a subgroup L_0 of L of index 4 which is called the restricted Lorentz group.

The group SL(2, $\mathbb{C}$) is the group which consists of the 2×2 complex matrices with determinant equal to $+1$. This group is intimately related to L_0, as we will see.

Let $H(2)$ denote the group of complex 2×2 Hermitian matrices. Let

σ_μ ($\mu = 0, 1, 2, 3$) be the elements of $H(2)$ defined by

$$\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.7}$$

For each $x \in M$ we define $A_x \in H(2)$ by

$$A_x = x^\mu \sigma_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \tag{1.8}$$

This defines a linear 1-1 correspondence between M and $H(2)$. By inspection we find that

$$\|x\|^2 = 2 \det A_x \tag{1.9}$$

and

$$x \cdot y = \det(A_x + A_y) - \det A_x - \det A_y \tag{1.10}$$

Let $U \in \mathrm{SL}(2, \mathbb{C})$ and $A_x \in H(2)$. Then $UA_xU^\dagger \in H(2)$, where $U^\dagger$ is the Hermitian conjugate of U; so there exists a $x' \in M$ with $UA_xU^\dagger = A_{x'}$. So U defines a linear mapping $\Lambda(U)$ from M onto M by $x \mapsto x' = \Lambda(U)x$. This mapping is a Lorentz transformation; this follows immediately from the equations (1.3) and (1.10). By inspection we see that

$$\Lambda(U)\Lambda(U') = \Lambda(UU') \qquad \forall U, U' \in \mathrm{SL}(2, \mathbb{C}) \tag{1.11}$$

and

$$\Lambda(\mathbb{1}) = \tilde{\mathbb{1}} \tag{1.12}$$

where $\mathbb{1}$ and $\tilde{\mathbb{1}}$ are the unit elements of $\mathrm{SL}(2, \mathbb{C})$ and L respectively. It is easy to check that

$$\Lambda(U) = \Lambda(U') \leftrightarrow U = \pm U' \tag{1.13}$$

(see Appendix A).

If U is varied continuously until it reaches $\mathbb{1}$, $\Lambda(U)$ varies continuously to reach $\tilde{\mathbb{1}}$. From the equations (1.5) and (1.6) and the definition of L_0 it thus follows that $\Lambda(U) \in L_0$. The image of $\mathrm{SL}(2, \mathbb{C})$ under Λ is equal to L_0. This last statement is nontrivial; a proof can be found in Halpern (1968). So $U \mapsto \Lambda(U)$ is a homomorphism of $\mathrm{SL}(2, \mathbb{C})$ onto L_0 with kernel $\{\mathbb{1}, -\mathbb{1}\}$. This homomorphism is given explicitly by

$$[\Lambda(U)]^\mu{}_\nu = \mathrm{Tr}\,[\sigma_\mu U \sigma_\nu U^\dagger] \tag{1.14}$$

(see Appendix B).

It is assumed that the reader is familiar with theory of tensors. A tensor ψ of order $n + m$ has n contravariant (upper) indices and m covariant (lower) indices which take the values $0, 1, 2, 3$. Under a restricted Lorentztransformation ψ transforms into ψ' according to

$$\psi'^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_m} = \Lambda^{\alpha_1}{}_{\gamma_1}\Lambda^{\alpha_2}{}_{\gamma_2}\cdots\Lambda^{\alpha_n}{}_{\gamma_n}(\Lambda^{-1})^{\delta_1}{}_{\beta_1}(\Lambda^{-1})^{\delta_2}{}_{\beta_2}\cdots(\Lambda^{-1})^{\delta_m}{}_{\beta_m}\psi^{\gamma_1\gamma_2\cdots\gamma_n}_{\delta_1\delta_2\cdots\delta_m} \tag{1.15}$$

The raising and lowering of tensor indices is done as usual with the metric tensor, so we have e.g.

$$\psi_\mu = g_{\mu\nu}\psi^\nu \tag{1.16}$$

In the sequel we denote a tensor of order $n+m$ by $\psi^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}$, so this symbol stands for the whole tensor, not just for one particular component. The same remark will hold for spinors. Tensor indices will always be Greek letters, spinor indices will be Latin letters.

The concept of spinors is defined in analogy with the concept of tensors; the spinor indices will take only 2 values and there are four types of indices. A spinor ξ^a transforms under a restricted Lorentz transformation according to

$$\xi'^a = U^a{}_b\xi^b \tag{1.17}$$

where $U \in \mathrm{SL}(2, \mathbb{C})$ and is related to the restricted Lorentz transformation by the homomorphism discussed above. Since U is defined up to a sign we see that, strictly speaking, we should identify spinors which differ from each other by a sign.

A spinor $\xi^{\dot a}$, where a dot has been placed over the index, transforms according to

$$\xi'^{\dot a} = \bar U^{\dot a}{}_{\dot b}\xi^{\dot b} \tag{1.18}$$

where $\bar U$ is the complex conjugate of U. A spinor ξ_a transforms according to

$$\xi'_a = (U^{-1})^b{}_a\xi_b \tag{1.19}$$

and a spinor $\xi_{\dot a}$ according to

$$\xi'_{\dot a} = (\bar U^{-1})^{\dot b}{}_{\dot a}\xi_{\dot b} \tag{1.20}$$

Spinors with any number of indices, dotted or undotted, may be defined by the requirement that they transform in the same way as products of one-index spinors with the same indices.

One easily verifies that the operations of addition (of spinors with the same indices), multiplication and contraction (over a pair of one upper and one lower index, both dotted or both undotted) are spinor operations. The skew-symmetric Levi-Civita symbol

$$\varepsilon^{ab} = \varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.21}$$

is a spinor which is invariant under restricted Lorentz transformations and is used to raise and lower spinor indices:

$$\xi_a = \xi^b\varepsilon_{ba} \tag{1.22a}$$

$$\xi^a = \varepsilon^{ab}\xi_b \tag{1.22b}$$

We now introduce the "mixed quantity" $\sigma^{a\dot b}_\mu$, which is a spinor (with respect to the indices $a\dot b$) and a tensor (with respect to the index μ). In some particular frame the components of $\sigma^{a\dot b}_\mu$ are defined to be the matrix elements of the matrices σ_μ defined in equation (1.7).

In Appendix C we will show that $\sigma^{a\dot b}_\mu$ is invariant under restricted Lorentz transformations. Let ψ^{μ} be a tensor. Then a spinor $\zeta^{a\dot{b}}$ may be defined by contraction of the tensor ψ^{μ} and the mixed quantity $\sigma_{\mu}^{a\dot{b}}$:

$$\zeta^{a\dot{b}} = \sigma_{\mu}^{a\dot{b}}\psi^{\mu} \tag{1.23}$$

Raising and lowering indices gives

$$\zeta_{a\dot{b}} = \sigma^{\mu}_{a\dot{b}}\psi_{\mu} \tag{1.24}$$

In this way to each tensor there corresponds a spinor where for each tensor index there is a couple of one dotted and one undotted spinor index:

$$\zeta^{a\dot{b}\cdots c\dot{d}}_{e\dot{f}\cdots g\dot{h}} = \sigma_{\mu}^{a\dot{b}}\cdots\sigma_{\nu}^{c\dot{d}}\sigma^{\rho}_{e\dot{f}}\cdots\sigma^{\tau}_{g\dot{h}}\psi^{\mu\cdots\nu}_{\rho\cdots\tau} \tag{1.25}$$

The inverse of this equation is

$$\psi^{\mu\cdots\nu}_{\rho\cdots\tau} = \sigma^{\mu}_{a\dot{b}}\cdots\sigma^{\nu}_{c\dot{d}}\sigma_{\rho}^{e\dot{f}}\cdots\sigma_{\tau}^{g\dot{h}}\zeta^{a\dot{b}\cdots c\dot{d}}_{e\dot{f}\cdots g\dot{h}} \tag{1.26}$$

(see Appendix C).

In Appendix D we state and prove a number of useful properties of spinors and of the correspondence between spinors and tensors. See also Pirani (1965). A geometrical interpretation of a spinor is given in Appendix E.

II. Null twistors and null lines

Let $\mathscr{L}$ be a lightray in Minkowski space, i.e. a null straight line consisting of the points $\{x^{\mu} + \lambda y^{\mu} \mid \lambda \in \mathbb{R}\}$ where x^{μ} is a real vector and y^{μ} is a real future-pointing null vector.

According to the Theorems 3 and 9 of Appendix D the vector y^{μ} determines a spinor $\pi^{\dot{a}}$ up to a phase factor by

$$y^{\mu} = \sigma^{\mu}_{a\dot{b}}\bar{\pi}^{a}\pi^{\dot{b}} \tag{2.1}$$

Since $\mathscr{L}$ determines y^{μ} up to a positive constant, $\mathscr{L}$ determines $\pi^{\dot{b}}$ up to a complex constant. The vector x^{μ} determines the spinor $x^{a\dot{b}}$ by

$$x^{\mu} = \sigma^{\mu}_{a\dot{b}}x^{a\dot{b}} \tag{2.2}$$

Define the spinor ω^{a} by

$$\omega^{a} = ix^{a\dot{b}}\pi_{\dot{b}} \tag{2.3}$$

This spinor is independent of the choice of x^{μ} on $\mathscr{L}$, since if $\tilde{x}^{\mu} \in \mathscr{L}$ then $\tilde{x}^{\mu} = x^{\mu} + \lambda y^{\mu}$ for some $\lambda \in \mathbb{R}$, thus $\tilde{x}^{a\dot{b}} = x^{a\dot{b}} + \lambda\bar{\pi}^{a}\pi^{\dot{b}}$ and therefore $x^{a\dot{b}}\pi_{\dot{b}} = \tilde{x}^{a\dot{b}}\pi_{\dot{b}}$. So the null line $\mathscr{L}$ determines up to a common complex factor the two spinors ω^{a} and $\pi_{\dot{a}}$. The spinor pair

$$L = (\omega^{a}, \pi_{\dot{a}}) \tag{2.4}$$

is called a twistor. From this twistor the null line $\mathscr{L}$ is obtained as the set of solutions x^{μ} of the equations (2.2) and (2.3). This is proved in Lemma 1 of Appendix F. For any two twistors $L = (\omega^{a}, \pi_{\dot{a}})$ and $X = (\xi^{a}, \eta_{\dot{a}})$ we define their twistor product by

$$(L, X) = \omega^{a}\bar{\eta}_{a} + \pi_{\dot{a}}\bar{\xi}^{\dot{a}} \tag{2.5}$$

A null twistor is a twistor for which $(L, L) = 0$. The twistors form a four dimensional complex vector space which is called twistor space and is denoted by $\mathbb{T}$. The subset of null twistors is denoted by $\mathbb{T}^0$.

A projective twistor is an equivalence class of non-zero twistors which are all proportional to each other. The equivalence class which contains the twistor L will be denoted by $\mathbf{L}$. The set of projective twistors is called projective twistor space and is denoted by $\mathbb{PT}$. A projective null twistor is an equivalence class of null twistors. The subset of $\mathbb{PT}$ consisting of projective null twistors is denoted by $\mathbb{PT}^0$. Two projective twistors $\mathbf{L}$ and $\mathbf{L}'$ are said to be orthogonal (denoted by $\mathbf{L} \cdot \mathbf{L} = 0$) if $(L, L') = 0$ for each $L \in \mathbf{L}$ and each $L' \in \mathbf{L}'$. Since each null line in M determines a twistor up to a complex factor, it determines a projective twistor uniquely. In turn, this projective twistor determines the null line uniquely. However, not all projective twistors correspond in this way to null lines. We have already seen that the spinor $\pi_{\dot{a}}$ in equation (2.1) must be different from zero. In fact, we have the following theorem:

Theorem 2.1. *The equations* (2.1), (2.2) *and* (2.3) *provide a one-to-one correspondence between the null lines in M and the projective null twistors with* $\pi_{\dot{a}} \neq 0$.

So except for the demand that $\pi_{\dot{a}} \neq 0$ a projective twistor must be a projective null twistor if it corresponds to a null line in M. The proof of this theorem is an immediate consequence of the lemmas we prove in Appendix F: a null line determines a projective twistor which is a projective null twistor (Lemma 2) and has $\pi_{\dot{a}} \neq 0$; for each projective null twistor with $\pi_{\dot{a}} \neq 0$ the equations (2.2) and (2.3) have a real solution x^μ (Lemma 3) and so the complete solution of (2.2) and (2.3) is a null line (Lemma 1).

We continue with another important theorem:

Theorem 2.2. *Let $\mathscr{L}_1$ an $\mathscr{L}_2$ be two nonparallel null lines in M. Let the corresponding projective twistors be $\mathbf{L}_1$ and $\mathbf{L}_2$ respectively. $\mathscr{L}_1$ and $\mathscr{L}_2$ meet each other if and only if $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal.*

The proof is given in Appendix G.

A 2-dimensional complex subspace of $\mathbb{T}$ is called a plane. Two planes who have only the zero twistor in common are said not to intersect. Two nonequal intersecting planes have a line in common; a line is a 1-dimensional complex subspace of $\mathbb{T}$ and thus a projective twistor completed with the zero twistor. Now let x^μ be some fixed real vector and consider the set of twistors $(\omega^a, \pi_{\dot{a}})$ who satisfy equation (2.3). These are just the twistors which determine null lines through x^μ, and so they are null twistors. They form a plane in $\mathbb{T}^0$. Let this plane be denoted by $P(x^\mu) \cdot P(x^\mu)$ obviously does not intersect the plane $\pi_{\dot{a}} = 0$.

Theorem 2.3. *All planes in $\mathbb{T}^0$ who do not intersect the plane $\pi_{\dot{a}} = 0$ are equal to $P(x^\mu)$ for some $x^\mu \in M$.*

The proof is given in appendix H.

Theorem 2.4. *The planes $P(x^\mu)$ and $P(y^\mu)$ intersect if and only if x^μ and y^μ are null separated.*

Proof. $P(x^\mu) \cap P(y^\mu)$ consists of those twistors determining null lines through both x^μ and y^μ. If x^μ and y^μ are not null separated such twistors do not exist and $P(x^\mu)$ and $P(y^\mu)$ do not intersect. If x^μ and y^μ are null separated but not equal then there is a unique null line through both x^μ and y^μ and thus $P(x^\mu) \cap P(y^\mu)$ is a line.

Let $\mathbb{M}$ denote the set of all planes in $\mathbb{T}^0$; $\mathbb{M}$ is called compactified Minkowski space. M may be identified with the subset of planes $P(x^\mu)$ of $\mathbb{M}$. Note that $\mathbb{M}$ has a conformal structure, i.e. a null structure, but no metric: points of $\mathbb{M}$ have null separation if and only if the corresponding planes intersect. So $\mathbb{M}$ is obtained from M by adding to M one single point I (the plane $\pi_{\dot{a}} = 0$) and the set of points which have null separation with I. Geometrically this means that $\mathbb{M}$ consists of M and a lightcone added at infinity.

The concept of null lines may be extended from M to $\mathbb{M}$ by the definition that a null line in $\mathbb{M}$ is given by the set of all planes in $\mathbb{T}^0$ which pass through a given line; so the null lines in $\mathbb{M}$ are in one-to-one correspondence with the projective null twistors. Two null lines meet if there is a plane in $\mathbb{T}^0$ containing both the corresponding projective null twistors. Theorem 2.2 may now be extended as follows:

Theorem 2.5. *Two null lines in $\mathbb{M}$ meet if and only if the corresponding projective null twistors are orthogonal.*

The proof is given in Appendix I.

To sum up, the correspondence, called Penrose correspondence, between $\mathbb{M}$ and $\mathbb{T}^0$ is as follows: There is a one-to-one correspondence between the points x^μ of $\mathbb{M}$ and the planes $P(x^\mu)$ in $\mathbb{T}^0$. Null separated points correspond to intersecting planes. There is a one-to-one correspondence between the null lines $\mathscr{L}$ in $\mathbb{M}$ and the projective null twistors $\mathbf{L}(\mathscr{L})$. Intersecting null lines correspond to orthogonal projective null twistors. Finally

$$x^\mu \in \mathscr{L} \Leftrightarrow \mathbf{L}(\mathscr{L}) \subset P(x^\mu) \tag{2.6}$$

III. Non-null twistors

In this section we will give two geometrical interpretations of non-null twistors: as Robinson congruences and as null planes in compactified complexified Minkowski space, and examine the relationship of the two interpretations.

Robinson congruences

We start with a theorem:

Theorem 3.1. *Let* $\mathbf{L}$ *be a projective twistor and let* x^μ *be a point in* M*, with the restriction that if* $\mathbf{L}$ *corresponds to a null line in* M*,* x^μ *does not lie on this null line. Then there is exactly one null line through* x^μ *whose projective null twistor* $\mathbf{X}$ *satisfies* $\mathbf{L} \cdot \mathbf{X} = 0$.

The proof is given in Appendix J.

Let $\mathbf{L}$ be a projective null twistor corresponding to a null line $\mathscr{L}$ in M. $\mathscr{L}$ is determined completely by the system of all null lines which meet it. The system of null lines corresponds to the system of projective null twistors $\mathbf{X}$ which satisfy $\mathbf{L}\cdot\mathbf{X}=0$. This system of null lines is called a congruence since through any point $x^\mu \in M$ not lying on $\mathscr{L}$ there is exactly one null line of the system, according to Theorem 3.1. This concept may be generalized for non-null twistors.

The Robinson congruence for a non-null projective twistor $\mathbf{L}$ is the congruence of null lines corresponding to the projective null twistors $\mathbf{X}$ with $\mathbf{L}\cdot\mathbf{X}=0$. According to Theorem 3.1 there is exactly one null line of the Robinson congruence through any point x^μ of M.

The Robinson congruence can be described geometrically as follows. Consider a spacelike hyperplane S in M with x^0 constant. At each point of S the null line of the Robinson congruences is projected orthogonally into S. This gives us a vector field in S. This vector field in tangent to a system of circles on a nested system of coaxial circular tori. On each torus the circles link once through the torus and once about it. This system is completed by a circle which is the limiting member of the tori and a straight line through the centre of this circle and perpendicular to its plane. For a picture, see Penrose (1975), page 291.

Null planes in compactified complexified Minkowski space

Complexified Minkowski space M^c is the complex manifold $\mathbb{C}^4$ with scalar product

$$z_1\cdot z_2 = g_{\mu\nu} z_1^\mu z_2^\nu \tag{3.1}$$

where $g_{\mu\nu}$ is given by equation (1.2). Let $(\omega^a, \pi_{\dot a})$ be a twistor with $\pi_{\dot a}\neq 0$.

As in the previous section we consider the equations

$$x^\mu = \sigma^\mu_{a\dot b} x^{a\dot b} \tag{3.2}$$

and

$$\omega^a = i x^{a\dot b}\pi_{\dot b} \tag{3.3}$$

with the difference that now x^μ is complex. These equations do have solutions; for instance, let $\xi^{\dot a}$ be a spinor with $\xi^{\dot a}\pi_{\dot a} = -i$ then $x_0^{a\dot b}=\omega^a\xi^{\dot b}$ is a solution. Suppose that $x_0^{a\dot b}+x_1^{a\dot b}$ is also a solution. Then $x_1^{a\dot b}\pi_{\dot b}=0$. From Theorem 4 of appendix D it follows that $x_1^{a\dot b}=\lambda^a\pi^{\dot b}$ for some spinor λ^a. We obtain:

Theorem 3.2. *The general solution of equation* (3.3) *has the form*

$$x^{a\dot b} = x_0^{a\dot b} + \lambda^a\pi^{\dot b},\ \lambda^a \text{ arbitrary}. \tag{3.4}$$

This solution corresponds to a plane in M^c. The difference of any two vectors of this plane is a null vector; such a plane is called a null plane. From the previous section we know that this null plane intersects M if and only if $(\omega^a, \pi_{\dot a})$ is a null twistor. A null plane which has the form of equation (3.4) is called an α-plane. A β-plane is a null plane of the form

$$x^{a\dot b} = x_0^{a\dot b} + \pi^a\lambda^{\dot b}, \qquad \lambda^{\dot b} \text{ arbitrary}. \tag{3.5}$$

Note that the complex conjugate of an α-plane is a β-plane and vice-versa. In Appendix K we will show that each null plane is either a α-plane or a β-plane.

Suppose the equation

$$\xi^a = ix^{a\dot{b}}\eta_{\dot{b}} \tag{3.6}$$

has the same α-plane as solution for $x^{a\dot{b}}$ as equation (3.3). Then

$$\xi^a = i[x_0^{a\dot{b}} + \lambda^a \pi^{\dot{b}}]\eta_{\dot{b}} \qquad \text{for all } \lambda^a \tag{3.7}$$

Then $\pi^{\dot{b}}\eta_{\dot{b}} = 0$, so $\eta_{\dot{b}} = c\pi_{\dot{b}}$ for some $c \in \mathbb{C}$.

It follows that $(\xi^a, \eta_{\dot{a}})$ and $(\omega^a, \pi_{\dot{a}})$ belong to the same projective twistor. Since to each α-plane corresponds a projective twistor (to the α-plane of equation (3.4) corresponds the twistor $(iz_0^{a\dot{b}}\pi_{\dot{b}}, \pi_{\dot{b}})$) we have established the following theorem:

Theorem 3.3. *The equations* (3.2) *and* (3.3) *provide a one-to-one correspondence between the α-planes in M^c and the projective twistors with $\pi_{\dot{a}} \neq 0$.*

The set of twistors $(\omega^a, \pi_{\dot{a}})$ who satisfy equation (3.3) for some fixed $x^\mu \in M^c$ form a plane (2-dimensional complex subspace) in $\mathbb{T}$, denoted by $P(x^\mu)$. Analogous to Theorem 2.3 we have

Theorem 3.4. *All planes in $\mathbb{T}$ who do not intersect the plane $\pi_{\dot{a}} = 0$ are equal to $P(x^\mu)$ for some $x^\mu \in M^c$.*

The proof is given in Appendix L.

Analogous to Theorem 2.4 is

Theorem 3.5. *The planes $P(x^\mu)$ and $P(y^\mu)$ intersect if and only if x^μ and y^μ are null separated.*

Proof. If x^μ and y^μ are null separated but not equal there is a unique α-plane through both x^μ and y^μ. This is proved in Theorem 2 of Appendix K. The rest of the proof is analogous to the proof of Theorem 2.4.

Let $\mathbb{M}^c$ denote the set of all planes in $\mathbb{T}$; $\mathbb{M}^c$ is called compactified complexified Minkowski space. M^c may be identified with the subset of planes $P(x^\mu)$ of $\mathbb{M}^c$.

Points of $\mathbb{M}^c$ have null separation if and only if the corresponding planes intersect in a line. The concept of α-plane may be extended to $\mathbb{M}^c$ by the definition that a α-plane in $\mathbb{M}^c$ is given by the set of all planes in $\mathbb{T}$ which pass through a given line; so there is a one-to-one correspondence between the α-planes and the projective twistors.

The Penrose correspondence between $\mathbb{M}^c$ and $\mathbb{T}$ is thus as follows: There is a one-to-one correspondence between the points x^μ of $\mathbb{M}^c$ and the planes $P(x^\mu)$ in $\mathbb{T}$. Null separated points correspond to intersecting planes. There is a one-to-one correspondence between the α-planes A in $\mathbb{M}^c$ and the projective twistors $\mathbf{L}(A)$. Finally,

$$x^\mu \in A \Leftrightarrow \mathbf{L}(A) \subset P(x^\mu) \tag{3.8}$$

Relationship between α-planes and Robinson congruences

For each projective twistor $\mathbf{L}$ we now have two geometrical interpretations: an interpretation as an α-plane A in $\mathbb{M}^c$ and an interpretation as a congruence R

of null lines in $\mathbb{M}$. The relationship between these two interpretations is as follows: each β-plane which intersects both A and $\mathbb{M}$ intersects $\mathbb{M}$ in a null line which belongs to R, and through each null line of R goes a β-plane which intersects A. This will be proved in Appendix M.

IV. Conformal transformations on twistor space

In the previous sections we introduced twistors as structures in compactified Minkowski space $\mathbb{M}$. We used a fixed origin and frame. The result however is to be interpreted as independent of this origin and frame. In this section we will examine what happens to twistor space when we apply a transformation to $\mathbb{M}$ which leaves its null structure invariant. The group of these transformations of $\mathbb{M}$ is called the conformal group and is denoted by C. It has the coset decomposition

$$C = C_0 + pC_0 + tC_0 + ptC_0 \tag{4.1}$$

where C_0 is the normal subgroup of C consisting of the conformal transformations which are continuously connected with the identity transformation, p denotes space inversion and t denotes time inversion. Since null lines are transformed into null lines by a conformal transformation, projective null twistors are transformed into projective null twistors. It is shown in Penrose (1967) that Robinson congruences are transformed into Robinson congruences. So to each conformal transformation there corresponds a transformation of $\mathbb{T}$ which leaves $\mathbb{T}^0$ invariant. The transform of $\mathbf{L}$ under a conformal transformation is denoted by $\tilde{\mathbf{L}}$. So we have

$$\mathbf{L}\cdot\mathbf{L}=0 \Leftrightarrow \tilde{\mathbf{L}}\cdot\tilde{\mathbf{L}}=0 \tag{4.2}$$

If $\mathbf{L}_1$ and $\mathbf{L}_2$ correspond to null lines which meet each other, then also the null lines corresponding to $\tilde{\mathbf{L}}_1$ and $\tilde{\mathbf{L}}_2$ meet each other. So

$$\mathbf{L}_1\cdot\mathbf{L}_2=0 \Leftrightarrow \tilde{\mathbf{L}}_1\cdot\tilde{\mathbf{L}}_2=0 \tag{4.3}$$

if $\mathbf{L}_1\cdot\mathbf{L}_1=\mathbf{L}_2\cdot\mathbf{L}_2=0$, according to Theorem 2.5. This implies, as was shown in Penrose (1967), that equation (4.3) holds for every $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{PT}$.

Let G be the group of transformations of $\mathbb{PT}$ preserving orthogonality. We have found a homomorphism from C into G. Actually, C and G are isomorphic, since for every $g \in G$ a corresponding conformal transformation can be constructed: let x^μ be a point in $\mathbb{M}$ and take two null lines through x^μ then g transforms these null lines into null lines which intersect in a point $\widetilde{x^\mu}$. This point $\widetilde{x^\mu}$ is independent of the choice of null lines through x^μ and the mapping $x^\mu \mapsto \widetilde{x^\mu}$ is a conformal transformation. So we have found that the group of transformations of $\mathbb{M}$ preserving its null structure is isomorphic to the group of transformations of $\mathbb{T}$ preserving orthogonality. The connection between these groups is given by equation (2.3), the equation which gives the connection between null lines in M and projective null twistors. For each $h \in C$ we have a transformation $T(h)$ of $\mathbb{PT}$:

$$\mathbf{L} \mapsto \tilde{\mathbf{L}} = T(h)\mathbf{L} \tag{4.4}$$

but since we prefer to work with $\mathbb{T}$ rather than with $\mathbb{PT}$ we would like to consider a corresponding transformation $U(h)$ of $\mathbb{T}$:

$$L \mapsto \tilde{L} = U(h)L \tag{4.5}$$

The condition $U(h)$ must satisfy is

$$\mathbf{U(h)L} = T(h)\mathbf{L} \qquad \forall L \in \mathbb{T} \tag{4.6}$$

but this equation does of course not determine $U(h)$ uniquely. So we have a freedom in our choice of $U(h)$. If $U(h)$ satisfies equation (4.6) it is said to induce $T(h)$.

A transformation U of $\mathbb{T}$ is called semilinear if

$$U(\lambda L + \mu L') = \zeta(\lambda)UL + \zeta(\mu)UL' \qquad \forall L, L' \in \mathbb{T}, \forall \lambda, \mu \in \mathbb{C} \tag{4.7}$$

where the mapping $\zeta : \mathbb{C} \to \mathbb{C}$ is given by

$$\zeta(\lambda) = \lambda \qquad \forall \lambda \in \mathbb{C} \tag{4.8}$$

or by

$$\zeta(\lambda) = \bar{\lambda} \qquad \forall \lambda \in \mathbb{C} \tag{4.9}$$

It is clear that each semilinear transformation U of $\mathbb{T}$ induces a transformation of $\mathbb{PT}$. We have the following theorem:

Theorem 4.1. *Each transformation T of $\mathbb{PT}$ which preserves orthogonality is induced by a semilinear transformation U of $\mathbb{T}$. If U_1 and U_2 are two semilinear transformations of $\mathbb{T}$ which both induce T then $U_1 = \lambda U_2$ for some $\lambda \in \mathbb{C}$. The semilinear transformations U of $\mathbb{T}$ which induce transformations of $\mathbb{PT}$ which preserve orthogonality are those which satisfy*

$$(UL, UL') = C\zeta((L, L')) \qquad \forall L, L' \in \mathbb{T}. \tag{4.10}$$

where C is a real constant.

This theorem is the application to twistor space of a general theorem first given in van den Broek (1983) which is a generalisation of a famous theorem of Wigner which is well known in quantum mechanics.

If U is a semilinear transformation of $\mathbb{T}$ the set $\{\lambda U \mid \lambda \in \mathbb{C}, \lambda \neq 0\}$, denoted by $\mathbf{U}$, is said to be a ray of semilinear transformations. So Theorem 4.1 says that for each $T \in G$ there is a unique ray of semilinear transformations of $\mathbb{T}$ which induce T and that there is a one-to-one correspondence between G and the rays which satisfy equation (4.10). Let $\mathbf{G}$ denote the group of these rays, then $\mathbf{G}$ is isomorphic to G and C. $\mathbf{G}$ has, analogous to equation (4.1), a coset decomposition

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{A}\mathbf{G}_0 + \mathbf{B}\mathbf{G}_0 + \mathbf{A}\mathbf{B}\mathbf{G}_0 \tag{4.11}$$

where $\mathbf{G}_0$ is the subgroup of $\mathbf{G}$ which is isomorphic to C_0 and consists of the rays whose elements satisfy equation (4.10) with C positive and ζ satisfying equation (4.8). The coset representatives $\mathbf{A}$ and $\mathbf{B}$ may be chosen such that they contain elements A and B respectively which satisfy

$$(AL, AL') = -(L, L') \qquad \forall L, L' \in \mathbb{T} \tag{4.12}$$

$$(BL, BL') = (L', L) \qquad \forall L, L' \in \mathbb{T} \tag{4.13}$$

The group of semilinear transformations U which satisfy

$$(UL, UL') = (L, L') \qquad \forall L, L' \in \mathbb{T} \tag{4.14}$$

and which have determinant equal to one is SU (2, 2). Each ray $\mathbf{U} \in \mathbf{G}_0$ contains 4 elements of SU (2, 2), say $U, iU, -U$ and $-iU$. So there is a 4:1 isomorphism

$$\mathrm{SU}\,(2,2) \xrightarrow{4:1} C_0 \tag{4.15}$$

In the literature this isomorphism is usually introduced via the two 2:1 isomorphisms

$$\mathrm{SU}\,(2,2) \xrightarrow{2:1} \mathrm{SO}\,(2,4) \xrightarrow{2:1} C_0 \tag{4.16}$$

Now we will derive explicitly for each conformal transformation the corresponding transformation of twistor space. For C_0 these results are first given by Klotz (1974), for pt by Penrose (1967) and for p and t by van den Broek (1983). Let a conformal transformation:

$$x^{\mu} \mapsto x^{\tilde{\mu}} \tag{4.17}$$

be given, then we have to find a semilinear transformation U of $\mathbb{T}$ such that

(i) $\mathbf{U} \in \mathbf{G}$, i.e. U satisfies equation (4.10)
(ii) if $L = (\omega^a, \pi_{\dot{a}})$ is a null twistor and if x^{μ} is a point of the corresponding null line:

$$\omega^a = ix^{a\dot{b}}\pi_{\dot{b}} \tag{4.18}$$

then, if $UL = (\widetilde{\omega}^a, \widetilde{\pi}_{\dot{a}})$, $\widetilde{x^{\mu}}$ is a point of the transformed null line:

$$\widetilde{\omega}^a = i\tilde{x}^{a\dot{b}}\tilde{\pi}_{\dot{b}}. \tag{4.19}$$

Note that (ii) determines U on $\mathbb{T}^0$ only; the extension to $\mathbb{T}$ then being determined by (i). Consider first restricted Lorentz transformations

$$\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \tag{4.20}$$

In this case the transformation of twistors follows immediately from the transformation of the spinor components (equations (1.17) and (1.20)):

$$\tilde{\omega}^a = Q^a{}_b\omega^b \tag{4.21a}$$

$$\tilde{\pi}_{\dot{a}} = (\bar{Q}^{-1})^{\dot{b}}{}_{\dot{a}}\pi_{\dot{b}} \tag{4.21}$$

where Q belongs to $\mathrm{SL}\,(2, \mathbb{C})$ and corresponds to Λ in the way discussed in Section 1.

Next consider the translations

$$\tilde{x}^{\mu} = x^{\mu} + a^{\mu} \qquad (a^{\mu} \text{ real}) \tag{4.22}$$

The corresponding twistor transformation is given by

$$\tilde{\omega}^a = \omega^a + ia^{a\dot{b}}\pi_{\dot{b}} \tag{4.23a}$$

$$\tilde{\pi}_{\dot{a}} = \pi_{\dot{a}} \tag{4.23b}$$

It is easily verified that this transformation is semilinear and satisfies equation (4.10), and that equation (4.19) follows from equation (4.18).

For the dilations

$$\tilde{x}^\mu = cx^\mu \qquad (c>0) \tag{4.24}$$

the corresponding twistor transformation is given by

$$\tilde{\omega}^a = \sqrt{c}\,\omega^a \tag{4.25a}$$

$$\tilde{\pi}_{\dot{a}} = \frac{1}{\sqrt{c}}\pi_{\dot{a}} \tag{4.25b}$$

as can also immediately be verified.

The accelerations

$$\tilde{x}^\mu = \frac{x^\mu - \frac{1}{2}a^\mu x_\nu x^\nu}{1 - a_\nu x^\nu + \frac{1}{4}(a_\nu a^\nu)(x_\rho x^\rho)} \quad (a^\mu \text{ real}) \tag{4.26}$$

correspond to the twistor transformation

$$\tilde{\omega}^a = \omega^a \tag{4.27a}$$

$$\tilde{\pi}_{\dot{a}} = \pi_{\dot{a}} + ia_{b\dot{a}}\omega^b \tag{4.27b}$$

One easily verifies that this transformation is semilinear and satisfies equation (4.10). To show that equation (4.19) is applied by equation (4.18) requires in this case some algebra, which is given in Appendix N.

Since each conformal transformation belonging to C_0 can be written as a product of restricted Lorentz transformations, translations, dilations and accelerations we have now established the correspondence between C_0 and $\mathbf{G}_0$.

Consider pt, the inversion of space and time:

$$\tilde{x}^\mu = -x^\mu \tag{4.28}$$

It is obvious that the corresponding twistor transformation is

$$\tilde{\omega}^a = -\omega^a \tag{4.29a}$$

$$\tilde{\pi}_{\dot{a}} = \pi_{\dot{a}} \tag{4.29b}$$

This transformation satisfies equation (4.12), so the coset $\mathbf{AG}_0$ of equation (4.11) corresponds to the coset ptC_0 of equation (4.1).

Consider finally the conformal transformation

$$\tilde{x}^\mu = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (-x^0, -x^1, x^2, -x^3) \tag{4.30}$$

This transformation is the time inversion t together with a rotation through π around x^2-axis. It is straightforward to verify that the corresponding twistor transformation is

$$\tilde{\omega}^a = \overline{\omega^a} \tag{4.31a}$$

$$\tilde{\pi}_{\dot{a}} = \overline{\pi_{\dot{a}}} \tag{4.31b}$$

This transformation satisfies equation (4.13), so the coset $\mathbf{BG}_0$ of equation (4.11) corresponds to the coset tC_0 of equation (4.1). Herewith the correspondence between C and $\mathbf{G}$ has been established.

V. Massless free fields

Now that we have finished our exposition on the geometries of twistor space and Minkowski space we will end this paper with a comparison of the description of massless free fields in both geometries. In the space time formalism the massless free fields are symmetric spinor fields satisfying a differential equation:

$$\nabla^{a\dot{a}}\nabla_{a\dot{a}}\phi = 0 \tag{5.1}$$

$$\nabla^{a\dot{a}}\phi_{\dot{a}\dot{b}\cdots\dot{c}} = 0 \tag{5.2}$$

$$\nabla^{a\dot{a}}\phi_{ab\cdots c} = 0 \tag{5.3}$$

Here $\nabla^{a\dot{a}}$ is the spinor which corresponds to the tensor ∂^{μ}.

A free massless field of positive frequency with helicity n is described by equation (5.1) if $n = 0$, by equation (5.2) if $n > 0$ and by equation (5.3) if $n < 0$. The number of indices of ϕ is equal to $2|n|$. The equations with $n = \pm\frac{1}{2}$ are the Dirac-Weyl neutrino equations, and the equations for $n = \pm 1$ are Maxwell's equations.

In the twistor formalism the massless free fields of helicity n are holomorphic functions on twistor space which are homogeneous of degree $-2n-2$. The differential equation has been absorbed into the geometry! The corresponding spinor field is obtained via a contour integral. If $f(L)$ is such a function, and if $n \geq 0$ then this contour integral is given by

$$\phi_{\dot{a}\dot{b}\cdots\dot{c}}(x^{\mu}) = \oint_{\Gamma} \pi_{\dot{a}}\pi_{\dot{b}} \cdots \pi_{\dot{c}} f(ix^{a\dot{b}}\pi_{\dot{b}}, \pi_{\dot{a}})\, \Delta\pi \tag{5.4}$$

where

$$\Delta\pi = \pi_{\dot{a}}\, d\pi^{\dot{a}} \tag{5.5}$$

and the contour Γ lies in the plane corresponding with x^{μ}, avoids singularities of f and varies continuously with x^{μ}.

Appendix A

Theorem. *Let U and U' be elements of* $\mathrm{SL}(2, \mathbb{C})$. *Then*

$$UAU^{\dagger} = U'AU'^{\dagger} \qquad \forall A \in H(2) \tag{A.1}$$

if and only if $U = \pm U'$

Proof. It is clear that equation (A.1) follows from $U = \pm U'$. Suppose U and U' satisfy equation (A.1). Let $V = U'^{-1}U$, then $VAV^{\dagger} = A$ for each $A \in H(2)$. Taking A equal to the unit matrix gives $VV^{\dagger} = \mathbb{1}$, so $V^{\dagger} = V^{-1}$ and we have $VA = AV$ for each $A \in H(2)$. Taking A equal to σ_3 gives

$$\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

or $v_{12}=v_{21}=0$. Taking A equal to σ_1 gives

$$\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$$

from which it follows that $v_{11}=v_{22}$.

$$V\in \mathrm{SL}\,(2,\mathbb{C})\to \det V=1\to v_{11}^2=1\to v_{11}=\pm 1\to V=\pm\mathbb{1}\to U=\pm U' \quad \text{QED}$$

Appendix B

Theorem. *Let U belong to* SL $(2,\mathbb{C})$ *and let the Lorentztransformation Λ be defined by $x'^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}$ where $UA_xU^{\dagger}=A_{x'}$ and $A_x=x^{\mu}\sigma_{\mu}$. Then*

$$\Lambda^{\mu}{}_{\rho}=\mathrm{Tr}\,(\sigma_{\mu}U\sigma_{\rho}U^{\dagger}). \tag{B.1}$$

Proof. From

$$\mathrm{Tr}\,(\sigma_{\mu}\sigma_{\nu})=\delta_{\mu\nu} \tag{B.2}$$

it follows that

$$\mathrm{Tr}\,(\sigma_{\mu}A_x)=\mathrm{Tr}\,(\sigma_{\mu}x^{\nu}\sigma_{\nu})=x^{\mu}. \tag{B.3}$$

Let x^{μ} be the vector whose ρ-th component equals 1 and whose other components are 0. Then

$$A_x=\sigma_{\rho} \tag{B.4}$$

$$x'^{\mu}=\Lambda^{\mu}{}_{\rho} \tag{B.5}$$

and thus

$$\Lambda^{\mu}{}_{\rho}=\mathrm{Tr}\,(\sigma_{\mu}A_{x'})=\mathrm{Tr}\,(\sigma_{\mu}UA_xU^{\dagger})=\mathrm{Tr}\,(\sigma_{\mu}U\sigma_{\rho}U^{\dagger}). \quad \text{QED}$$

Appendix C

Let the matrices σ^{μ} be defined by

$$\sigma^{\mu}_{ab}=g^{\mu\nu}\varepsilon_{ca}\varepsilon_{db}\sigma^{cd}_{\nu} \tag{C.1}$$

Note that this definition is consistent with the rising and lowering of the mixed quantity σ^{ab}_{μ}.

Lemma 1. $\sigma^{\mu}_{ab}=\sigma^{ab}_{\mu}$ (C.2)

Proof. Straightforward verification.

Lemma 2. $\sigma^{\mu}_{ab}\sigma^{ab}_{\nu}=\delta^{\mu}{}_{\nu}$ (C.3)

$$\sigma^{\mu}_{ab}\sigma^{cd}_{\mu}=\delta^{c}{}_{a}\delta^{d}{}_{b} \tag{C.4}$$

where $\delta^{\mu}{}_{\nu}$ and $\delta^{a}{}_{b}$ are the usual Kronecker delta symbols.

Proof. Straightforward verification.

We want to show that the mixed quantity $\sigma_\mu^{ab'}$, whose components are in some frame defined to be the elements of the matrices σ_μ, is invariant under restricted transformations. This will be achieved once we have proved the following theorem:

Theorem 1. *Let* Λ *be a restricted Lorentztransformation and let* U *be an element of* SL(2, $\mathbb{C}$) *corresponding to* Λ *via the* $2-1$ *homomorphism of* L_0 *and* SL(2, $\mathbb{C}$). *Then*

$$\sigma_\mu^{ab} = U^a{}_c \overline{U^b{}_d} \Lambda_\mu{}^\nu \sigma_\nu^{cd} \tag{C.5}$$

Proof. From Lemma 1 it follows that equation (1.14) can be written as

$$\Lambda^\mu{}_\nu = \sigma^\mu_{ba} U^b{}_c \sigma_\nu^{cd} \overline{U^a{}_d}$$

Multiplying both sides with $\Lambda_\rho{}^\nu \sigma_\mu^{ef}$ and using equation (C.4) gives

$$\Lambda^\mu{}_\nu \Lambda_\rho{}^\nu \sigma_\mu^{ef} = \Lambda_\rho{}^\nu U^e{}_c \sigma_\nu^{cd} \overline{U^f{}_d} \tag{C.6}$$

so equation (C.5) is proved if we show that

$$\Lambda^\mu{}_\nu \Lambda_\rho{}^\nu \sigma_\mu^{ef} = \sigma_\rho^{ef} \tag{C.7}$$

and this equation is an immediate consequence of equation (1.4).

Theorem 2. *Equation* (1.26) *follows from equation* (1.25).

Proof. Immediate consequence of Lemma 2.

Appendix D. Spinor algebra

Theorem 1. *Raising and lowering a pair of dummy spinor indices produces a sign change.*

Proof. It is sufficient to show that $\xi^a \eta_a = -\xi_a \eta^a$. From the equations (1.21) and (1.22) we obtain

$$\xi^a \eta_a = \varepsilon^{ab} \xi_b \eta^c \varepsilon_{ca} = -\xi_b \eta^c \delta^b{}_c = -\xi_b \eta^b \quad \text{QED}$$

Theorem 2. *Two spinors* ξ^a *and* η^a *are proportional if and only if* $\xi^a \eta_a = 0$.

Proof. It follows from theorem 1 that if ξ^a and η^a are proportional then $\xi^a \eta_a = 0$. It is straightforward to check that

$$\varepsilon^{ab} \varepsilon^{cd} + \varepsilon^{ac} \varepsilon^{db} + \varepsilon^{ad} \varepsilon^{bc} = 0. \tag{D.1}$$

Contracting with $\xi_c \eta_d$ gives

$$\varepsilon^{ab} \xi_c \eta^c - \xi^a \eta^b + \xi^b \eta^a = 0. \tag{D.2}$$

If $\xi_c \eta^c = 0$ then

$$\xi^a \eta^b = \xi^b \eta^a \tag{D.3}$$

from which it follows that ξ^a and η^a are proportional.

Theorem 3. *The spinors ξ^a and η^a have the property*

$$\xi^a \bar{\xi}^{\dot{b}} = \eta^a \bar{\eta}^{\dot{b}} \tag{D.4}$$

if and only if $\xi^a = e^{i\theta}\eta^a$ for some phase factor $e^{i\theta}$.

Proof. It is clear that (D.4) holds if $\xi^a = e^{i\theta}\eta^a$. Since $\eta_a\eta^a = 0$ (Theorem 2) contraction of equation (D.4) with η_a gives

$\eta_a\xi^a\bar{\xi}^{\dot{b}} = 0$ from which it follows that $\eta_a\xi^a = 0$.

Now theorem 2 says that η_a and ξ^a are proportional, and from equation (D.4) it follows that the factor of proportionality should have modulus unity.

Theorem 4. *If the spinors $\mu^{a\dot{b}}$ and $\pi_{\dot{b}}$ have the properties $\mu^{a\dot{b}}\pi_{\dot{b}} = 0$ and $\pi_{\dot{b}} \neq 0$ then $\mu^{a\dot{b}} = \lambda^a\pi^{\dot{b}}$ for some spinor λ^a.*

Proof. Let $\eta^{\dot{a}}$ be a spinor with the property $\eta_{\dot{a}}\pi^{\dot{a}} = 1$. Define λ^a by $\lambda^a = \mu^{a\dot{b}}\eta_{\dot{b}}$. Then

$$[\mu^{a\dot{b}} - \lambda^a\pi^{\dot{b}}]\pi_{\dot{b}} = 0 \tag{D.5}$$

and

$$[\mu^{a\dot{b}} - \lambda^a\pi^{\dot{b}}]\eta_{\dot{b}} = \mu^{a\dot{b}}\eta_{\dot{b}} - \mu^{a\dot{c}}\,\eta_{\dot{c}}\pi^{\dot{b}}\eta_{\dot{b}} = 0. \tag{D.6}$$

Since any spinor $\xi_{\dot{b}}$ is a linear combination of $\pi_{\dot{b}}$ and $\eta_{\dot{b}}$ we conclude that

$$\mu^{a\dot{b}} - \lambda^a\pi^{\dot{b}} = 0. \tag{D.7}$$

Theorem 5. *To a contraction of two tensors corresponds the spinor who is the contraction of the spinors who correspond to the original tensors.*

Proof. Straightforward consequence of lemma 2 of appendix C.

Theorem 6. *The spinor $g^{a\dot{b}c\dot{d}}$ corresponding to the metric tensor $g^{\mu\nu}$ is given by*

$$g^{a\dot{b}c\dot{d}} = \varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}}. \tag{D.8}$$

Proof. According to theorem 5 the spinor form of $x^\mu = g^{\mu\nu}x_\nu$ is

$$x^{a\dot{b}} = g^{a\dot{b}c\dot{d}}x_{c\dot{d}} \tag{D.9}$$

and from equation (1.22) it follows

$$x^{a\dot{b}} = \varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}}x_{c\dot{d}}. \tag{D.10}$$

This proves the theorem.

Theorem 7. *The vector x^μ is real if and only if the corresponding spinor $x^{a\dot{b}}$ is Hermitian, i.e. its components satisfy $x^{a\dot{b}} = \bar{x}^{\dot{b}a}$.*

Proof. If x^μ is real then

$$\bar{x}^{\dot{b}a} = \overline{x^{b\dot{a}}} = \overline{\sigma_\mu^{b\dot{a}}x^\mu} = \sigma_\mu^{a\dot{b}}x^\mu = x^{a\dot{b}}$$

If $x^{a\dot{b}} = \bar{x}^{\dot{b}a}$ then

$$\overline{x^\mu} = \overline{\sigma^\mu_{a\dot{b}}x^{a\dot{b}}} = \sigma^\mu_{\dot{b}\dot{a}}\bar{x}^{\dot{a}b} = \sigma^\mu_{b\dot{a}}x^{b\dot{a}} = x^\mu$$

Theorem 8. *The vector z^μ is a null vector* (i.e. $z_\mu z^\mu = 0$) *if and only if the corresponding spinor $z^{a\dot{b}}$ is equal to $\lambda^a \pi^{\dot{b}}$ for some spinors λ^a and $\pi^{\dot{b}}$.*

Proof. If $z^{a\dot{b}} = \lambda^a \pi^{\dot{b}}$ then $z_\mu z^\mu = z_{a\dot{b}} z^{a\dot{b}} = \lambda_a \pi_{\dot{b}} \lambda^a \pi^{\dot{b}} = 0$. If $z_\mu z^\mu = 0$ then

$z_{a\dot{b}} z^{a\dot{b}} = z^{c\dot{d}} \varepsilon_{ca} \varepsilon_{\dot{d}\dot{b}} z^{a\dot{b}} = 2(z^{0\dot{0}} z^{1\dot{1}} - z^{0\dot{1}} z^{1\dot{0}}) = 0.$

Let the spinor $\pi_{\dot{a}}$ be defined by its components in some particular frame according to

$$\pi_{\dot{0}} = -z^{0\dot{1}}; \qquad \pi_{\dot{1}} = z^{0\dot{0}}. \tag{D.11}$$

One verifies easily that $z^{a\dot{b}} \pi_{\dot{b}} = 0$. If $\pi_{\dot{b}}$ happens to be zero, one may take

$$\pi_{\dot{0}} = -z^{1\dot{1}}; \qquad \pi_{\dot{1}} = z^{1\dot{0}} \tag{D.12}$$

Except when $z^\mu = 0$, is which case the theorem obviously applies, we have found a spinor $\pi_{\dot{a}}$ which satisfies $z^{a\dot{b}} \pi_{\dot{b}} = 0$ and $\pi_{\dot{b}} \neq 0$. From theorem 4 it follows that $z^{a\dot{b}} = \lambda^a \pi^{\dot{b}}$ for some spinor λ^a.

Theorem 9. *The vector ψ^μ is real, null and futurepointing* ($\psi^0 > 0$) *if and only if the corresponding spinor $\psi^{a\dot{b}}$ can be written as*

$$\psi^{ab} = \xi^a \bar{\xi}^b \tag{D.13}$$

for some spinor ξ^a.

Proof. If equation (D.13) holds then ψ^μ is null (Theorem 8), real (Theorem 7) and futurepointing:

$$\psi^0 = \sigma^0_{ab} \xi^a \bar{\xi}^{\dot{b}} = \frac{1}{\sqrt{2}} (\xi^0 \bar{\xi}^{\dot{0}} + \xi^1 \bar{\xi}^{\dot{1}}) > 0.$$

Now suppose that ψ^μ is real, null and futurepointing. From Theorem 8 it follows that we may write

$$\psi^{a\dot{b}} = \lambda^a \pi^{\dot{b}} \tag{D.14}$$

From Theorem 7 we obtain

$$\lambda^a \pi^{\dot{b}} = \bar{\lambda}^{\dot{b}} \bar{\pi}^a \tag{D.15}$$

Contracting this with λ_a gives $\lambda_a \bar{\pi}^a = 0$, which means that λ^a and $\bar{\pi}^a$ are proportional (Theorem 2). So

$$\bar{\pi}^a = c\lambda^a \tag{D.16}$$

where c is a complex number. Substituting (D.16) into (D.15) shows that c should be a real number. Now

$$\psi^0 = \sigma^0_{a\dot{b}} \lambda^a \pi^{\dot{b}} = \sigma^0_{a\dot{b}} c\lambda^a \bar{\lambda}^{\dot{b}} = c(\lambda^0 \bar{\lambda}^{\dot{0}} + \lambda^1 \bar{\lambda}^{\dot{1}})$$

and therefore $c > 0$ since $\psi^0 > 0$. So we have

$$\psi^{a\dot{b}} = c\lambda^a \bar{\lambda}^{\dot{b}}, \qquad c > 0 \tag{D.17}$$

If we define $\xi^a = \sqrt{c}\lambda^a$ then equation (D.13) follows.

Theorem 10. *The tensor $\psi^{\mu\nu}$ is real and antisymmetric if and only if the corresponding spinor $\psi^{a\dot{b}c\dot{d}}$ can be written as*

$$\psi^{a\dot{b}c\dot{d}} = \varepsilon^{ac}\bar{\phi}^{\dot{b}\dot{d}} + \varepsilon^{\dot{b}\dot{d}}\phi^{ac} \tag{D.17}$$

for some symmetric spinor ϕ^{ab}.

Proof. Let $\psi^{\mu\nu}$ be given by

$$\psi^{\mu\nu} = \sigma^{\mu}_{a\dot{b}}\sigma^{\nu}_{c\dot{d}}[\varepsilon^{ac}\bar{\phi}^{\dot{b}\dot{d}} + \varepsilon^{\dot{b}\dot{d}}\phi^{ac}]. \tag{D.18}$$

It is clear that $\psi^{\mu\nu}$ is antisymmetric, and $\psi^{\mu\nu}$ is real since $\bar{\sigma}^{\mu}_{ab} = \sigma^{\mu}_{ba}$. Now let $\psi^{\mu\nu}$ be a real antisymmetric tensor, and let

$$\psi^{a\dot{b}c\dot{d}} = \sigma_{\mu}^{a\dot{b}}\sigma_{\nu}^{c\dot{d}}\psi^{\mu\nu} \tag{D.19}$$

Then $\psi^{a\dot{b}c\dot{d}} = -\psi^{c\dot{d}a\dot{b}}$ and so we have

$$\psi^{a\dot{b}c\dot{d}} = \tfrac{1}{2}[\psi^{a\dot{b}c\dot{d}} - \psi^{c\dot{b}a\dot{d}} + \psi^{c\dot{b}a\dot{d}} - \psi^{c\dot{d}a\dot{b}}] \tag{D.20}$$

From equation (D.1) it follows

$$0 = [\varepsilon^{ac}\varepsilon^{pq} + \varepsilon^{ap}\varepsilon^{qc} + \varepsilon^{aq}\varepsilon^{cp}]\psi_{p}{}^{\dot{b}}{}_{q}{}^{\dot{d}} = \varepsilon^{ac}\psi_{p}{}^{\dot{b}p\dot{d}} - \psi^{a\dot{b}c\dot{d}} + \psi^{c\dot{b}a\dot{d}} \tag{D.21}$$

and

$$0 = [\varepsilon^{\dot{b}\dot{d}}\varepsilon^{\dot{p}\dot{q}} + \varepsilon^{\dot{b}\dot{q}}\varepsilon^{\dot{d}\dot{p}} + \varepsilon^{\dot{b}\dot{p}}\varepsilon^{\dot{q}\dot{d}}]\psi^{c}{}_{\dot{p}}{}^{a}{}_{\dot{q}} + \varepsilon^{\dot{b}\dot{d}}\psi^{c}{}_{\dot{p}}{}^{a\dot{p}} + \psi^{c\dot{d}a\dot{b}} - \psi^{c\dot{b}a\dot{d}}. \tag{D.22}$$

So equation (D.20) becomes

$$\psi^{a\dot{b}c\dot{d}} = \tfrac{1}{2}[\varepsilon^{ac}\psi_{p}{}^{\dot{b}p\dot{d}} + \varepsilon^{\dot{b}\dot{d}}\psi^{c}{}_{\dot{p}}{}^{a\dot{p}}] \tag{D.23}$$

Let the spinor ϕ^{ab} be defined by

$$\phi^{ab} = \frac{1}{2}\psi^{b}{}_{\dot{p}}{}^{a\dot{p}} \tag{D.24}$$

Then

$$\phi^{ba} = \tfrac{1}{2}\psi^{a}{}_{\dot{p}}{}^{b\dot{p}} = -\tfrac{1}{2}\psi^{b\dot{p}a}{}_{\dot{p}} = \tfrac{1}{2}\psi^{b}{}_{\dot{p}}{}^{a\dot{p}} = \phi^{ab} \tag{D.25}$$

so ϕ^{ab} is symmetric. Since $\psi^{\mu\nu}$ is real we have

$$\bar{\psi}^{\dot{a}b\dot{c}d} = \overline{\psi^{a\dot{b}c\dot{d}}} = \overline{\sigma_{\mu}^{a\dot{b}}}\,\overline{\sigma_{\nu}^{c\dot{d}}}\psi^{\mu\nu} = \sigma_{\mu}^{b\dot{a}}\sigma_{\nu}^{d\dot{c}}\psi^{\mu\nu} = \psi^{b\dot{a}d\dot{c}} \tag{D.26}$$

from which it follows that

$$\bar{\phi}^{\dot{a}\dot{b}} = \frac{1}{2}\bar{\psi}^{\dot{b}}{}_{p}{}^{\dot{a}p} = \tfrac{1}{2}\psi_{p}{}^{\dot{b}p\dot{a}} \tag{D.27}$$

With (D.24) and (D.27) we can write (D.23) as

$$\psi^{a\dot{b}c\dot{d}} = \varepsilon^{ac}\bar{\phi}^{\dot{b}\dot{d}} + \varepsilon^{\dot{b}\dot{d}}\phi^{ac} \quad \text{QED}$$

Appendix E. Geometrical interpretation of a spinor

A spinor ξ^{a} (not equal to zero) determines a real futurepointing null vector

$$\psi^{\mu} = \sigma^{\mu}_{a\dot{b}}\xi^{a}\bar{\xi}^{\dot{b}} \tag{E.1}$$

according to Theorem 9 of Appendix D. All spinors which determine this vector are equal to ξ^a up to a phase factor; this is an immediate consequence of Theorem 3 of Appendix D. Let η^a be a spinor with

$$\xi^a \eta_a = 1 \tag{E.2}$$

and define the vector ω^μ by

$$\omega^\mu = \sigma^\mu_{a\dot{b}}[\xi^a \bar{\eta}^{\dot{b}} + \eta^a \bar{\xi}^{\dot{b}}] \tag{E.3}$$

Then ω^μ is real (Theorem 7 of Appendix D) and $\omega^\mu \omega_\mu = -2$, so ω^μ and ψ^μ are linearly independent. Let $\hat{\eta}^a$ also be a spinor with

$$\xi^a \hat{\eta}_a = 1 \tag{E.4}$$

and define

$$\hat{\omega}^\mu = \sigma^\mu_{a\dot{b}}[\xi^a \bar{\hat{\eta}}^{\dot{b}} + \hat{\eta}^a \bar{\xi}^{\dot{b}}] \tag{E.5}$$

Then

$$\xi^a(\hat{\eta}_a - \eta_a) = 0 \tag{E.6}$$

and thus, according to Theorem 2 of Appendix D,

$$\hat{\eta}^a = \eta^a + \lambda \xi^a, \qquad \lambda \in \mathbb{C} \tag{E.7}$$

It follows that

$$\hat{\omega}^\mu = \omega^\mu + (\lambda + \bar{\lambda})\psi^\mu. \tag{E.8}$$

The plane spanned by ω^μ and ψ^μ is determined uniquely *bij* ξ^a and is called a "flag" with ψ^μ as "flagpole". Let the vector χ^μ be defined by

$$\chi^\mu = i\sigma^\mu_{a\dot{b}}[\xi^a \bar{\eta}^{\dot{b}} - \eta^a \bar{\xi}^{\dot{b}}]. \tag{E.9}$$

χ^μ is real and $\chi_\mu \chi^\mu = -2$. From $\psi_\mu \omega^\mu = \psi_\mu \chi^\mu = \omega_\mu \chi^\mu = 0$ one easily deduces that ψ^μ, χ^μ and ω^μ are linearly independent. If the phase of ξ^a is changed:

$$\xi^a \to \xi^a e^{i\theta}, \qquad \theta \in \mathbb{R} \tag{E.10}$$

then

$$\eta^a \to \eta^a e^{-i\theta} \tag{E.11}$$

and

$$\omega^\mu \to \sigma^\mu_{a\dot{b}}(\xi^a \bar{\eta}^{\dot{b}} \varepsilon^{2i\theta} + \eta^a \bar{\xi}^{\dot{b}} e^{-2i\theta}) = \omega^\mu \cos 2\theta + \chi^\mu \sin 2\theta. \tag{E.12}$$

So by this change of phase the flag is rotated round the flagpole through an angle 2θ. By a change of sign ($\theta = \pi$) the flag remains unchanged. So the geometrical interpretation as flags of the spinors ξ^a and $-\xi^a$ is the same.

Appendix F

Lemma 1. *Let a twistor $L = (\omega^a, \pi_{\dot{a}})$ be given and suppose that the equations (2.2) and (2.3) have a real solution x_0^μ. Then the complete real solution of these equations is a null line. The direction of this null line is given by equation (2.1).*

Proof. Suppose that x^μ is also a real solution of (2.2) and (2.3). Then

$$x^{a\dot{b}}\pi_{\dot{b}} = x_0^{a\dot{b}}\pi_{\dot{b}} \tag{F.1}$$

from which it follows, according to theorem 4 of appendix D that

$$x^{a\dot{b}} - x_0^{a\dot{b}} = \lambda^a\pi^{\dot{b}} \tag{F.2}$$

for some spinor λ^a. Since x^μ and x_0^μ are real, theorem 7 of appendix D gives

$$\lambda^a\pi^{\dot{b}} = \bar{\lambda}^{\dot{b}}\bar{\pi}^a \tag{F.3}$$

Contracting with λ_a gives

$$\lambda_a\bar{\pi}^a = 0 \tag{F.4}$$

so

$$\lambda^a = c\bar{\pi}^a \tag{F.5}$$

for some $c \in \mathbb{C}$, according to theorem 2 of appendix D. It follows that

$$x^\mu - x_0^\mu = cy^\mu \tag{F.6}$$

where y^μ is given by equation (2.1).

From equation (F.3) it follows that c is real. Since y^μ is real and null (theorem 9 of appendix D) the vector x^μ lies on the null line through x_0^μ with direction y^μ. On the other hand, it is obvious that each point of this line indeed satisfies the equations (2.2) and (2.3).

Lemma 2. *If the equations* (2.2) *and* (2.3) *have a real solution x^μ, then $L = (\omega^a, \pi_{\dot{a}})$ is a null twistor.*

Proof. Since x^μ is real we have $x^{a\dot{b}} = \bar{x}^{\dot{b}a}$. If equation (2.3) holds then

$$(L, L) = \omega^a\bar{\pi}_a + \pi_{\dot{a}}\bar{\omega}^{\dot{a}} = ix^{a\dot{b}}\pi_{\dot{b}}\bar{\pi}_a - i\bar{x}^{\dot{a}b}\bar{\pi}_b\pi_{\dot{a}} = ix^{a\dot{b}}\pi_{\dot{b}}\bar{\pi}_a - i\bar{x}^{\dot{b}a}\bar{\pi}_a\pi_{\dot{b}} = 0$$

Lemma 3. *Let $L = (\omega^\alpha, \pi_{\dot{a}})$ be a null twistor and let $\pi_{\dot{a}} \neq 0$. Then the equations* (2.2) *and* (2.3) *have a real solution x^μ.*

Proof. If L is a null twistor then

$$\omega^a\bar{\pi}_a + \bar{\omega}^{\dot{a}}\pi_{\dot{a}} = 0. \tag{F.7}$$

Suppose $\omega^a\bar{\pi}_a \neq 0$. Let $x^{a\dot{b}}$ be given by

$$x^{a\dot{b}} = \frac{i\omega^a\bar{\omega}^{\dot{b}}}{\omega^c\bar{\pi}_c}. \tag{F.8}$$

Then

$$ix^{a\dot{b}}\pi_{\dot{b}} = \frac{-\omega^a\bar{\omega}^{\dot{b}}\pi_{\dot{b}}}{\omega^c\bar{\pi}_c} = \omega^a$$

and

$$\bar{x}^{\dot{a}b} = \frac{-i\bar{\omega}^{\dot{a}}\omega^b}{\bar{\omega}^{\dot{c}}\pi_{\dot{c}}} = \frac{i\bar{\omega}^{\dot{a}}\omega^b}{\omega^c\bar{\pi}_c} = x^{b\dot{a}}.$$

So $x^{a\dot{b}}$ satisfies equation (2.3) and x^{μ} is real according to Theorem 7 of Appendix D. Now suppose $\omega^a\bar{\pi}_a = 0$. Choose a spinor ξ^a such that

$$\xi^a\bar{\pi}_a = i. \tag{F.9}$$

Let $x^{a\dot{b}}$ be given by

$$x^{a\dot{b}} = \omega^a\bar{\xi}^{\dot{b}} + \bar{\omega}^{\dot{b}}\xi^a. \tag{F.10}$$

Then

$$ix^{a\dot{b}}\pi_{\dot{b}} = i\omega^a\bar{\xi}^{\dot{b}}\pi_{\dot{b}} + i\bar{\omega}^{\dot{b}}\xi^a\pi_{\dot{b}} = \omega^a$$

and

$$\bar{x}^{\dot{a}b} = \bar{\omega}^{\dot{a}}\xi^b + \omega^b\bar{\xi}^{\dot{a}} = x^{b\dot{a}}$$

which proves the lemma.

Appendix G. Proof of Theorem 2.2

Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ meet. Let x^{μ} be the point of intersection. Suppose $L_1 = (\omega^a, \pi_{\dot{a}}) \in \mathbf{L}_1$ and $L_2 = (\xi^a, \eta_{\dot{a}}) \in \mathbf{L}_2$. Then

$$\omega^a = ix^{a\dot{b}}\pi_{\dot{b}} \tag{G.1}$$

$$\xi^a = ix^{a\dot{b}}\eta_{\dot{b}} \tag{G.2}$$

and

$$x^{a\dot{b}} = \bar{x}^{\dot{b}a}. \tag{G.3}$$

It follows that

$$(L_1, L_2) = \omega^a\bar{\eta}_a + \pi_{\dot{a}}\bar{\xi}^{\dot{a}} = ix^{a\dot{b}}\pi_{\dot{b}}\bar{\eta}_a - i\bar{x}^{\dot{a}b}\bar{\eta}_b\pi_{\dot{a}} = 0$$

and so $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal.

Now suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal projective null twistors. If $(\omega^a, \pi_{\dot{a}}) \in \mathbf{L}_1$ and $(\xi^a, \eta_{\dot{a}}) \in \mathbf{L}_2$ then

$$\omega^a\bar{\pi}_a + \bar{\omega}^{\dot{a}}\pi_{\dot{a}} = 0 \tag{G.4}$$

$$\xi^a\bar{\eta}_a + \bar{\xi}^{\dot{a}}\eta_{\dot{a}} = 0 \tag{G.5}$$

and

$$\omega^a\bar{\eta}_a + \xi^{\dot{a}}\pi_{\dot{a}} = 0. \tag{G.6}$$

Since the corresponding null lines are supposed to be nonparallel we also have

$$\bar{\pi}^a\bar{\eta}_a \neq 0. \tag{G.7}$$

Consider first the case that $\omega^a\bar{\pi}_a \neq 0$. Then a point on $\mathscr{L}_1$ is given by equation (F.8), and so all points of $\mathscr{L}_1$ are given by

$$x^{a\dot{b}} = \frac{i\omega^a\bar{\omega}^{\dot{b}}}{\omega^c\bar{\pi}_c} + A\bar{\pi}^a\pi^{\dot{b}} \qquad A \in \mathbb{R} \tag{G.8}$$

Take

$$A = \frac{\omega^c \bar{\pi}_c \xi^d \bar{\eta}_d + \omega^c \bar{\eta}_c \bar{\omega}^{\dot{d}} \eta_{\dot{d}}}{i\bar{\pi}^c \bar{\eta}_c \pi^{\dot{d}} \eta_{\dot{d}} \omega^e \bar{\pi}_e}. \tag{G.9}$$

From the equations (G.4) and (G.5) it follows that A is real. For this particular point x^μ on $\mathscr{L}_1$ define the spinor χ^a by

$$\chi^a = ix^{a\dot{b}}\eta_{\dot{b}}. \tag{G.10}$$

It is now a trivial exercise to show that

$$\chi^a \bar{\pi}_a = \xi^a \bar{\pi}_a \tag{G.11}$$

and

$$\chi^a \bar{\eta}_a = \xi^a \bar{\eta}_a. \tag{G.12}$$

Since $\bar{\pi}_a$ and $\bar{\eta}_a$ are independent, due to equation (G.7), we have

$$\chi^a = \xi^a. \tag{G.13}$$

So x^μ also belongs to $\mathscr{L}_2$, and thus $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect.

Now consider the case where $\omega^a \bar{\pi}_a = 0$. We may choose $(\xi^a, \eta_{\dot{a}})$ from $\mathbf{L}_2$ such that

$$\bar{\eta}^a \bar{\pi}_a = i \tag{G.14}$$

due to equation (G.7). From equation (F.10) it follows that $\mathscr{L}_1$ consists of the points

$$x^{a\dot{b}} = \omega^a \eta^{\dot{b}} + \bar{\omega}^{\dot{b}} \bar{\eta}^a + A\bar{\pi}^a \pi^{\dot{b}}. \tag{G.15}$$

Take

$$A = -i\xi^a \bar{\eta}_a. \tag{G.16}$$

A is real according to equation (G.5). As in the previous case one shows that

$$\xi^a = ix^{a\dot{b}}\eta_{\dot{b}} \tag{G.17}$$

for this particular x^μ on $\mathscr{L}_1$, and so $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect.

Appendix H. Proof of Theorem 2.3

Lemma 1. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two different projective null twistors. The plane they span lies entirely in $\mathbb{T}^0$ if and only if $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal.*

Proof. Let this plane be denoted by P. Each element of P has the form $L_1 + L_2$ where $L_1 \in \mathbf{L}_1$ and $L_2 \in \mathbf{L}_2$. If $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal then $(L_1 + L_2, L_1 + L_2) = 0$ and thus P lies entirely in $\mathbb{T}^0$. Now suppose that P lies in $\mathbb{T}^0$ and let $L_1 \in \mathbf{L}_1$ and $L_2 \in \mathbf{L}_2$. Then $(L_1 + L_2, L_1 + L_2) = (L_1, L_2) + (L_2, L_1) = 0$ and $i(L_1 + iL_2, L_1 + iL_2) = (L_1, L_2) - (L_2, L_1) = 0$. So $(L_1, L_2) = 0$ and thus $\mathbf{L}_1$ and $\mathbf{L}_2$ are orthogonal.

Lemma 2. *Let $L_1 = (\omega^a, \pi_{\dot{a}})$ and $L_2 = (\xi^a, \eta_{\dot{a}})$ be two linearly independent twistors. If the plane they span does not intersect the plane $\chi_{\dot{a}} = 0$ then $\pi^{\dot{a}}\eta_{\dot{a}} \neq 0$.*

Proof. Let this plane be denoted by P. Elements of P are given by $(\lambda\omega^a + \mu\xi^a, \lambda\pi_{\dot{a}} + \mu\eta_{\dot{a}})$ where $\lambda, \mu \in \mathbb{C}$. If $\lambda\pi_{\dot{a}} + \mu\eta_{\dot{a}} = 0$ for some $\lambda, \mu \in \mathbb{C}$ then we must have $\lambda\omega^a + \mu\xi^a = 0$ since P does not intersect the plane $\chi_{\dot{a}} = 0$. But then $\lambda = \mu = 0$ since L_1 and L_2 are linearly independent. So $\pi_{\dot{a}}$ and $\eta_{\dot{a}}$ are linearly independent. From Theorem 2 of Appendix D it now follows that $\pi^{\dot{a}}\eta_{\dot{a}} \neq 0$.

We are now able to prove Theorem 2.3.

Let P be a plane in $\mathbb{T}^0$ which does intersect the plane $\chi_{\dot{a}} = 0$. Let $L_1 = (\omega^a, \pi_{\dot{a}})$ and $L_2 = (\xi^a, \eta_{\dot{a}})$ be two linearly independent null twistors from P. Then $\mathbf{L}_1$ and $\mathbf{L}_2$ are different and, according to Lemma 1, orthogonal. Lemma 2 says that $\pi^{\dot{a}}\eta_{\dot{a}} \neq 0$ which means that the null lines corresponding to $\mathbf{L}_1$ and $\mathbf{L}_2$ are nonparallel. According to Theorem 2.2 these null lines meet. Let x^μ be the intersection point. Then $P(x^\mu)$ contains both L_1 and L_2, and thus $P(x^\mu) = P$.

Appendix I. Proof of Theorem 2.5

Suppose the null lines $\mathscr{L}_1$ and $\mathscr{L}_2$ correspond to the projective null twistors $\mathbf{L}_1$ and $\mathbf{L}_2$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ meet then there is a plane in $\mathbb{T}^0$ which contains both $\mathbf{L}_1$ and $\mathbf{L}_2$. If $\mathscr{L}_1$ is equal to $\mathscr{L}_2$ then $\mathbf{L}_1$ is equal to $\mathbf{L}_2$ and thus $\mathbf{L}_1 \cdot \mathbf{L}_2 = 0$. If $\mathscr{L}_1$ is not equal to $\mathscr{L}_2$ then $\mathbf{L}_1$ is not equal to $\mathbf{L}_2$. Now $\mathbf{L}_1 \cdot \mathbf{L}_2 = 0$ according to lemma 1 of Appendix H. If $\mathbf{L}_1 \cdot \mathbf{L}_2 = 0$ then the plane spanned by $\mathbf{L}_1$ and $\mathbf{L}_2$ lies in $\mathbb{T}^0$ according to Lemma 1 of Appendix H. So this plane is both an element of $\mathscr{L}_1$ and of $\mathscr{L}_2$. So $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect.

Appendix J. Proof of Theorem 3.1

Let $\mathbf{L}$ be a projective twistor and let x^μ be a point of M, with the restriction that if $\mathbf{L}$ corresponds to a null line in M, x^μ does not lie on this null line. Let $(\omega^a, \pi_{\dot{a}}) \in \mathbf{L}$ and define

$$\bar{\eta}^a = \omega^a - ix^{a\dot{b}}\pi_{\dot{b}} \tag{J.1}$$

$$\xi^a = ix^{a\dot{b}}\eta_{\dot{b}} \tag{J.2}$$

The righthandside of equation (J.1) is non-zero, due to the restriction made above. The projective twistor $\mathbf{X}$ containing $(\xi^a, \eta_{\dot{a}})$ is a projective null twistor corresponding to a null line through x^μ, according to equation (J.2). Further we have

$$\omega^a\bar{\eta}_a + \pi_{\dot{a}}\bar{\xi}^{\dot{a}} = \omega^a\bar{\eta}_a - i\pi_{\dot{a}}\bar{x}^{\dot{a}b}\bar{\eta}_b = \omega^a\bar{\eta}_a - i\pi_{\dot{b}}x^{a\dot{b}}\bar{\eta}_a = \bar{\eta}^a\bar{\eta}_a = 0.$$

So $\mathbf{L} \cdot \mathbf{X} = 0$, and we have shown that at least one projective null twistor $\mathbf{X}$ exists which determines a null line through x^μ and satisfies $\mathbf{L} \cdot \mathbf{X} = 0$. Now we show that $\mathbf{X}$ is uniquely determined. Let $\mathbf{X}$ be any projective null twistor with the requested properties, and let $(\xi^a, \eta_{\dot{a}}) \in \mathbf{X}$. Then equation (J.2) holds and since $\mathbf{L} \cdot \mathbf{X} = 0$ we have

$$\omega^a\bar{\eta}_a + \pi_{\dot{a}}\bar{\xi}^{\dot{a}} = 0 \tag{J.3}$$

Substituting equation (J.2) into equation (J.3) gives

$$\omega^a \bar{\eta}_a - i\pi_{\dot{a}} \bar{x}^{\dot{a}b} \bar{\eta}_b = (\omega^a - ix^{a\dot{b}} \pi_{\dot{b}}) \bar{\eta}_a = 0. \tag{J.4}$$

It follows that

$$\omega^a - ix^{a\dot{b}} \pi_{\dot{b}} = \lambda \bar{\eta}^a \tag{J.5}$$

for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. From the equations (J.5) and (J.2) it follows that **X** is uniquely determined.

Appendix K

Theorem 1. *Any null plane in* $\mathbb{M}^c$ *is either a* α*-plane or a* β*-plane.*

Proof. It is clear from the definitions that a null plane cannot be a α-plane and a β-plane at the same time. Let x_0^μ, x_1^μ and x_2^μ be three non collinear points of a null plane in M^c. The null plane then consists of the points

$$x^\mu = x_0^\mu + \lambda(x_1^\mu - x_0^\mu) + \mu(x_2^\mu - x_0^\mu) \qquad \lambda, \mu \in \mathbb{C} \tag{K.1}$$

From Theorem 8 of Appendix D it follows that

$$x_1^{a\dot{b}} = x_0^{a\dot{b}} + \alpha^a \beta^{\dot{b}} \tag{K.2}$$

$$x_2^{a\dot{b}} = x_0^{a\dot{b}} + \gamma^a \delta^{\dot{b}} \tag{K.3}$$

$$\gamma^a \delta^{\dot{b}} - \alpha^\alpha \beta^{\dot{b}} = \chi^a \phi^{\dot{b}} \tag{K.4}$$

where none of the spinors α^a, $\beta^{\dot{b}}$, γ^a, $\delta^{\dot{b}}$, χ^a and ϕ^b is equal to zero. Contracting equation (K.4) with $\phi_{\dot{b}}$ gives

$$\gamma^a \delta^{\dot{b}} \phi_{\dot{b}} = \alpha^a \beta^{\dot{b}} \phi_{\dot{b}} \tag{K.5}$$

If both sides of this equation are zero, then $\delta^{\dot{b}}\phi_{\dot{b}} = \beta^{\dot{b}}\phi_{\dot{b}} = 0$, so $\delta^{\dot{b}}$ and $\beta^{\dot{b}}$ are both proportional to $\phi^{\dot{b}}$, and thus proportional to each other:

$$\delta^{\dot{b}} = \nu \beta^{\dot{b}} \quad \text{for some} \quad \nu \in \mathbb{C} \tag{K.6}$$

Equation (K.1) may now be written as

$$x^{a\dot{b}} = x_0^{a\dot{b}} + [\lambda \alpha^a + \mu\nu\gamma^a] \beta^{\dot{b}} \tag{K.7}$$

It follows that the null plane is a α-plane. If both sides of equation (K.5) are nonzero then

$$\gamma^a = \nu \alpha^a \quad \text{for some} \quad \nu \in \mathbb{C} \tag{K.8}$$

Equation (K.1) may now be written as

$$x^{a\dot{b}} + x_0^{a\dot{b}} + \alpha^a[\lambda\beta^{\dot{b}} + \mu\nu\delta^{\dot{b}}]. \tag{K.9}$$

It follows that the null plane is a β-plane. This proves the theorem.

Theorem 2. *If* $x^\mu \in M^c$ *and* $y^\mu \in M^c$ *are different and null separated there is a unique* α*-plane which contains both* x^μ *and* y^μ.

Proof. Since $x^\mu - y^\mu$ is a null vector we have from theorem 8 of appendix D

$$x^{a\dot{b}} - y^{a\dot{b}} = \lambda^a \pi^{\dot{b}}. \tag{K.10}$$

Since x^μ and y^μ are different $\pi^{\dot{b}} \neq 0$. The α-plane given by the points

$$\{x^{a\dot{b}} + \mu^a \pi^{\dot{b}} \mid \mu^a \text{ arbitrary}\} \tag{K.11}$$

contains x^μ $(\mu^a = 0)$ and y^μ $(\mu^a = -\lambda^a)$. Suppose there is a second α-plane containing x^μ and y^μ consisting of the points

$$\{x^{a\dot{b}} + \xi^a \eta^{\dot{b}} \mid \xi^a \text{ arbitrary}\} \tag{K.12}$$

Then

$$\xi^a \eta^{\dot{b}} = -\lambda^a \pi^{\dot{b}} \tag{K.13}$$

for some spinor ξ^a, since the α-plane contains y^μ. Contracting this equation with $\eta^{\dot{b}}$ gives, since $\lambda^a \neq 0$, $\pi^{\dot{b}} \eta_{\dot{b}} = 0$, so $\pi^{\dot{b}}$ and $\eta^{\dot{b}}$ are proportional, which means that the α-planes are the same.

Appendix L. Proof of Theorem 3.4

Let P be a plane in $\mathbb{T}$ which does not intersect the plane $\chi_{\dot{a}} = 0$. Let $L_1 = (\omega^a, \pi_{\dot{a}})$ and $L_2 = (\xi^a, \eta_{\dot{a}})$ be two linearly independent twistors from P. Then $\pi^{\dot{a}} \eta_{\dot{a}} \neq 0$ according to Lemma 2 of Appendix H. Define x^μ by

$$x^{a\dot{b}} = \frac{i}{\pi^{\dot{c}} \eta_{\dot{c}}} (\omega^a \eta^{\dot{b}} - \xi^a \pi^{\dot{b}}) \tag{L.1}$$

Then $\omega^a = i x^{a\dot{b}} \pi_{\dot{b}}$ and $\xi^a = i x^{a\dot{b}} \eta_{\dot{b}}$, so $L_1 \in P(x^\mu)$ and $L_2 \in P(x^\mu)$. It follows that $P(x^\mu) = P$.

Appendix M. Relationship between α-planes and Robinson congruences

Lemma 1. *Let $L = (\omega^a, \pi_{\dot{a}})$ be a non-null twistor with $\pi_{\dot{a}} \neq 0$. Let A be the corresponding α-plane in M^c and let R denote the corresponding Robinson congruence of null lines in M. Then a β-plane B which intersects both A and M intersects M in a null line which belongs to R.*

Proof. If $x_0^\mu \in A \cap B$ then

$$\omega^a = i x_0^{a\dot{b}} \pi_{\dot{b}} \tag{M.1}$$

and B consists of the points

$$\{x_0^{a\dot{b}} + \eta^a \chi^{\dot{b}} \mid \chi^{\dot{b}} \text{ arbitrary}\} \tag{M.2}$$

for some fixed η^a. Since B intersects M there is a spinor $\xi^{\dot{b}}$ such that

$$x_0^{a\dot{b}} + \eta^a \xi^{\dot{b}} = \bar{x}_0^{\dot{b}a} + \bar{\eta}^{\dot{b}} \bar{\xi}^a \tag{M.3}$$

according to Theorem 7 of Appendix D. Now $B \cap M$ consists of the points

$$\{x_0^{a\dot{b}} + \eta^a \xi^{\dot{b}} + \lambda \eta^a \bar{\eta}^{\dot{b}} \mid \lambda \in \mathbb{R}\}. \tag{M.4}$$

This is a null line, and the corresponding projective null twistor **X** has a representative $X=(\phi^a, \bar{\eta}_{\dot{a}})$ where

$$\phi^a = i(x_0^{a\dot{b}} + \eta^a \xi^{\dot{b}})\bar{\eta}_{\dot{b}} \tag{M.5}$$

We have to show that this null line belongs to R, i.e. that $\mathbf{L}\cdot\mathbf{X}=0$. This will be achieved if we show that

$$\omega^a \eta_a + \pi_{\dot{a}} \bar{\phi}^{\dot{a}} = 0 \tag{M.6}$$

The equations (M.5) and (M.3) give

$$\bar{\phi}^{\dot{a}} = -i(\bar{x}_0^{\dot{a}b} + \bar{\eta}^{\dot{a}}\bar{\xi}^b)\eta_b = -i(x_0^{b\dot{a}} + \eta^b \xi^{\dot{a}})\eta_b = -ix_0^{b\dot{a}}\eta_b.$$

From equation (M.1) it now follows that

$$\pi_{\dot{a}}\bar{\phi}^{\dot{a}} = -ix_0^{b\dot{a}}\eta_b\pi_{\dot{a}} = -\omega^b\eta_b$$

and this proves equation (M.6) and thus the lemma.

Lemma 2. Let $L=(\omega^a, \pi_{\dot{a}})$ be a non-null twistor with $\pi_{\dot{a}} \neq 0$. Let A be the corresponding α-plane in M^c and let R be the corresponding Robinson congruence. For any null line $\mathscr{L} \in R$ the (unique) β-plane B through $\mathscr{L}$ intersects A.

Proof. Suppose $\mathscr{L}$ is a null line of R corresponding to the projective twistor **X**. Let x^μ be a point of $\mathscr{L}$ and let $(\xi^a, \eta_{\dot{a}})$ belong to **X**. Then $\mathscr{L}$ is given by the points

$$\{x^{a\dot{b}} + \lambda\bar{\eta}^a\eta^{\dot{b}} \mid \lambda \in \mathbb{R}\}. \tag{M.7}$$

The unique β-plane B through $\mathscr{L}$ is given by the points

$$\{x^{a\dot{b}} + \bar{\eta}^a\rho^{\dot{b}} \mid \rho^{\dot{b}} \text{ arbitrary}\}. \tag{M.8}$$

B intersects A if and only if there exists a spinor $\rho^{\dot{b}}$ such that

$$\omega^a = i(x^{a\dot{b}} + \bar{\eta}^a\rho^{\dot{b}})\pi_{\dot{b}}. \tag{M.9}$$

Since $\mathscr{L}$ belongs to R we have

$$\omega^a\bar{\eta}_a + \pi_{\dot{a}}\bar{\xi}^{\dot{a}} = 0 \tag{M.10}$$

and since $x^\mu \in \mathscr{L}$ we have

$$\xi^a = ix^{a\dot{b}}\eta_{\dot{b}}. \tag{M.11}$$

Substituting equation (M.11) into equation (M.10) gives, using $x^{a\dot{b}} = \bar{x}^{\dot{b}a}$

$$(\omega^a - ix^{a\dot{b}}\pi_{\dot{b}})\bar{\eta}_a = 0. \tag{M.12}$$

It follows that

$$\omega^a - ix^{a\dot{b}}\pi_{\dot{b}} = \lambda\bar{\eta}^a \qquad \text{for some } \lambda \in \mathbb{C}. \tag{M.13}$$

Choose $\rho^{\dot{b}}$ such that

$$\rho^{\dot{b}}\pi_{\dot{b}} = -i\lambda. \tag{M.14}$$

Then $i(x^{a\dot{b}} + \bar{\eta}^a\rho^{\dot{b}})\pi_{\dot{b}} = ix^{a\dot{b}}\pi_{\dot{b}} + \lambda\bar{\eta}^a = \omega^a$, due to equation (M.13). So equation (M.9) holds for this choice of $\rho^{\dot{b}}$, so A and B intersect.

Appendix N

In this appendix we will show that equation (4.19) follows from the equations (4.18), (4.26) and (4.27). Substituting equations (4.26) and (4.27) into equation (4.19) gives

$$\omega^{a}=i\frac{x^{a\dot b}-\frac{1}{2}a^{a\dot b}x_{c\dot d}x^{c\dot d}}{1-a_{c\dot d}x^{c\dot d}+\frac{1}{4}a_{c\dot d}a^{c\dot d}x_{e\dot f}x^{e\dot f}}[\pi_{\dot b}+ia_{g\dot b}\omega^{g}]. \tag{N.1}$$

Multiplying this expression with the denominator of the righthandside and using equation (4.18) gives

$$\begin{aligned}-a_{c\dot d}x^{c\dot d}x^{a\dot b}\pi_{\dot b}+\tfrac{1}{4}a_{c\dot d}a^{c\dot d}x_{e\dot f}x^{e\dot f}x^{a\dot b}\pi_{\dot b}&\\=-x^{a\dot h}a_{g\dot h}x^{g\dot b}\pi_{\dot b}-\tfrac{1}{2}a^{a\dot b}x_{c\dot d}x^{c\dot d}\pi_{\dot b}+\tfrac{1}{2}a^{a\dot h}x_{c\dot d}x^{c\dot d}a_{g\dot h}x^{g\dot b}\pi_{\dot b}.&\end{aligned} \tag{N.2}$$

This expression is proved if we prove the two following expressions:

$$a_{c\dot d}a^{c\dot d}x_{e\dot f}x^{e\dot f}x^{a\dot b}=2a^{a\dot h}x_{c\dot d}x^{c\dot d}a_{g\dot h}x^{g\dot b} \tag{N.3}$$

$$2a_{c\dot d}x^{c\dot d}x^{a\dot b}=2x^{a\dot h}a_{g\dot h}x^{g\dot b}+a^{a\dot b}x_{c\dot d}x^{c\dot d} \tag{N.4}$$

It is straightforward to verify that

$$a^{a\dot b}a_{g\dot h}+\tfrac{1}{2}a^{k\dot h}a_{k\dot h}\delta^{a}{}_{g} \tag{N.5}$$

and with this equation (N.3) is proved immediately.

To prove equation (N.4) we start from the real antisymmetric tensor $a^{\mu}x^{\nu}-a^{\nu}x^{\mu}$. According to Theorem 10 of Appendix D we can write the corresponding spinor as

$$a^{a\dot b}x^{c\dot d}-a^{c\dot d}x^{a\dot b}=\varepsilon^{ac}\bar\phi^{\dot b\dot d}+\varepsilon^{\dot b\dot d}\phi^{ac} \tag{N.6}$$

Here ϕ^{ac} is given by

$$\phi^{ac}=\tfrac{1}{2}[a^{c}{}_{\dot p}x^{a\dot p}-a^{a\dot p}x^{c}{}_{\dot p}] \tag{N.7}$$

according to the proof of this theorem. From Theorem 7 of Appendix D it follows that

$$\bar\phi^{\dot b\dot d}=\tfrac{1}{2}[a_{p}{}^{\dot d}x^{p\dot b}-a^{p\dot b}x_{p}{}^{\dot d}]. \tag{N.8}$$

Combining the equations (N.6), (N.7) and (N.8) and contracting with $x_{c\dot d}$ gives

$$\begin{aligned}x_{c\dot d}x^{c\dot d}a^{a\dot b}-x_{c\dot d}a^{c\dot d}x^{a\dot b}&=\tfrac{1}{2}\varepsilon^{ac}x_{c\dot d}\,a_{p}{}^{\dot d}x^{p\dot b}-\tfrac{1}{2}\varepsilon^{ac}x_{c\dot d}a^{p\dot b}x_{p}{}^{\dot d}\\&\quad+\tfrac{1}{2}\varepsilon^{\dot b\dot d}x_{c\dot d}a^{c}{}_{\dot p}x^{a\dot p}-\tfrac{1}{2}\varepsilon^{\dot b\dot d}x_{c\dot d}a^{a\dot p}x^{c}{}_{\dot p}.\end{aligned} \tag{N.9}$$

Raising and lowering indices gives

$$\begin{aligned}x_{c\dot d}x^{c\dot d}a^{a\dot b}-x_{c\dot d}a^{c\dot d}x^{a\dot b}&=-\tfrac{1}{2}x^{a\dot d}a_{p\dot d}x^{p\dot b}+\tfrac{1}{2}x^{a\dot d}a^{p\dot b}x_{p\dot d}\\&\quad-\tfrac{1}{2}x^{c\dot b}a_{c\dot p}x^{a\dot p}+\tfrac{1}{2}x^{c\dot b}a^{a\dot p}x_{c\dot p}.\end{aligned} \tag{N.10}$$

Noting that because of equation (N.5) the second and the fourth term of the righthandside are both equal to $\frac{1}{4}x_{c\dot d}x^{c\dot d}a^{a\dot b}$ and that the first and the third term of the righthandside are equal we obtain

$$\tfrac{1}{2}x_{c\dot d}x^{c\dot d}x^{a\dot b}-x_{c\dot d}a^{c\dot d}x^{a\dot b}=-x^{a\dot d}a_{p\dot d}x^{p\dot b} \tag{N.11}$$

which is identical to equation (N.4)

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