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Some nonstandard quantum electrodynamics

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Summary. Quantum electrodynamics in 3 + 1 dimensions is modified by means of non standard analysis in practical approximation to the classical development yielding a consistently defined mathematical theory, avoiding in particular the usual divergence problems. Detailed computations of vacuum polarization, including the Uehling term, display our methods explicitly.

0. Introduction

Quantum electrodynamics has been beset with the divergence problems from its very beginnings. Up to now no mathematical solution of these problems has been found, despite the fact that good numerical results have been given by using infinite renormalizations. In our mathematical theory we relax the principles of translation invariance, relativistic invariance, unbounded particle numbers and gauge invariance by applying a space cutoff, ultraviolet cutoff, particle number cutoff and using a nonvanishing photon mass, respectively. In distinction to the work done in constructive and axiomatic quantum field theory we do not remove the cutoffs in the end (cf. [1] part 6, [2]). In order to keep the damage low we apply nonstandard analysis, in particular we use an infinite space cutoff, an infinite particle number cutoff, and an infinitesimal photon mass. This allows us to closely follow some developments of standard quantum electrodynamics (in '3 + 1 dimensions') without running into the usual troubles. The heart of the problem lies in the initial values (value for time $t = 0$) of the free fields. That's where we apply cutoffs once and for all. The resulting initial cutoff fields have to be used not only for the interaction Hamiltonian (cf. 5.2) but also for the free Hamiltonians (cf. 2.15, 3.5, 4.8) and the other field functions (e.g. observables). Our modified Hamiltonians (free and interacting, resp.) in turn determine the modified time dependent fields, through Heisenberg's equation (cf. 2.20, 3.7, 4.11 and 5.4).

The use of A. Robinson's Nonstandard Analysis (cf. [3]) is not new in physics (cf. [4], [5], [6], [7], [8], [9], [10], [11], [12] and [22]). Since it does not seem necessary to give a full introduction into the subject, we give only a brief account (chapter I) of it in order to fix the main concepts. Furthermore we introduce the 'normapproximations' of certain distributions, in particular (the nonstandard) function $k(p-q)$ approximating $\delta^3(p-q)$, arising from a space cutoff by Fourier-transformation (cf. 1.8, 1.10 and 1.11).

In chapter 2 we first introduce the standard formalism for the Klein-Gordon field (cf. 2.1 through 2.4) of restmass m. The nonstandard modifications start with the introduction of nonstandard extensions $\mathbb{M}_{(1)} \supsetneqq \mathbb{M}_{(0)} \supsetneqq \mathbb{M}$ of the basic standard model $\mathbb{M}$ for analysis (cf. 2.5, 1.1 and 1.13). In view of the photon field (cf. chapter 2) we fix the restmass $m>0$ to belong to $\mathbb{M}_{(0)}$ and to be infinitesimal with respect to the standard model $\mathbb{M}$. The corresponding Fockspace then will be cut off at some nonstandard particle number ω belonging to $\mathbb{M}_0$ (cf. 2.6), yielding a corresponding cutoff-modification of the annihilation and creation operators $a^{(+)}(f)$. The resulting $a_\omega^{(+)}(f)$ thus become bounded operators on the cutoff Fockspace $\mathbb{F}^\omega$. By choosing ω to be a nonstandard natural number every state of the original Fockspace $\mathbb{F}$ can be approximated infinitesimally closely in $\mathbb{F}^\omega$ (norm topology).

In order to approximate the operator valued distributions $a_\omega^{(+)}(p)$ by operator valued functions we introduce a space cutoff Q in $\mathbb{M}_{(1)}$, which is infinite with respect to $\mathbb{M}_{(0)}$ (cf. 2.8). Then we introduce an UV cutoff $P>0$ which turns the annihilation and creation operators into uniformly bounded operator valued functions $b^{(+)}(p)$ (cf. 2.11 and 2.12). First and second order perturbation theory together suggest that P be standard (or at least finite). Some possible values are listed in 6.24.

The commutator $[b(p), b^+(q)]$ approximates $2\omega_p\delta^3(p-q)$ infinitesimally (cf. 2.10 and 2.13). The modified initial Klein-Gordon field $\chi(0, x)$ as well as the free Hamiltonian H_1^{KG} is defined by means of the cutoff annihilation and creation operators. The time dependent modified free Klein-Gordon field then has to be the corresponding Heisenberg field $\chi(t, x) = e^{itH_1^{\mathrm{KG}}}\chi(0, x)e^{-itH_1^{\mathrm{KG}}}$, an operator valued function in t, x over the particle number cutoff Fockspace $\mathbb{F}^\omega$ (cf. 2.20). The rest of chapter 2 is devoted to infinitesimally closely approximating the contractions of the modified free Klein-Gordon field.

It turns out that they agree almost completely with the usual contractions of standard quantum field theory as long as one restricts oneself to consider only those (0-quasistandard) particle states in $\mathbb{F}^{\omega-1}$ whose support is inside the cutoff P (cf. 2.29). The latter could be called the physical states (cf. also 5.13b).

In chapter 3 we develop the free photon field as a massive vector boson field with an infinitesimal restmass m. We start with the standard Stückelberg-Coester type formalism, which is based on the standard KG field. Then we replace the latter by the modified KG field of chapter 2. Thus the approximations of the contractions of chapter 2 carry over directly to the modified photon field.

In chapter 4 the standard Dirac field is modified along the lines of chapter 2 and 3. The developments so far are of interest only in view of interactions. Without such there would be no reason to modify the standard initial free fields.

Chapter 5 introduces electromagnetic interactions by means of the interaction Hamiltonian H_I. H_I arises from the standard interaction Hamiltonian by using the modified free fields instead of the standard free fields. Thus H_I becomes a bounded operator. One introduces the interaction picture in the usual way without the existence difficulties in connection with Haag's theorem (cf. 5.5 and 5.6). The Dyson expansion (from time s to time t) turns out to be a well defined infinite series which even converges (cf. 5.7 and 5.8). The theorem of Wick and Feynman's rules work in the usual way (cf. 5.9 and 5.10). It turns out that the modified first order theory of perturbation agrees very well with the standard one (cf. 5.13).

Chapter 6 deals with scattering of an electron in a (slightly modified) Coulomb field (cf. 6.1). The first order contribution ('Coulomb scattering', cf. 6.4) gives the expected result (cf. 6.6). Also the second order contribution ('vacuum polarization', cf. 6.7) agrees completely with the standard result (cf. 6.21). This is shown by explicit computations and approximations. We recover the standard results without any divergences, the charge renormalization is finite and the Uehling effect is secured (cf. 6.23). In 6.24 we list some possible values for the *UV* cutoff and the corresponding charge renormalization. A further development of the (renormalization-) theory should determine the *UV* cutoff as well as the other cut-offs much more precisely. It seems anyway that a lot more questions have to be raised than have been answered.

1. Tools from nonstandard analysis

1.1. *The basic structure* $\mathbb{M}$

One usually does analysis inside a structure $\mathbb{M}$ which contains the set $\mathbb{R}$ of real numbers as subset, and possibly the Cartesian products $\mathbb{R}\times\mathbb{R}$, $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, etc. Furthermore it is convenient to have iterated powersets $\mathcal{P}^n(\mathbb{R})$ as subsets of $\mathbb{M}$ the n depending on the type of analysis in consideration. We do not want to be explicit about $\mathbb{M}$, just assuming it to be rich enough to carry what we need. For later convenience we shall also assume that subsets of elements of $\mathbb{M}$ are again elements of $\mathbb{M}$. Examples are the 'superstructures' of [14], pg. 23 or any model of Zermelo Fraenkel set theory (cf. [15], pg. 1).

In order to apply first order model theory we have to fix a first order language L with interpretation in $\mathbb{M}$. It is convenient to use the 'full language $L_{\mathbb{M}}$ over $\mathbb{M}$', having an individual constant for each element of $\mathbb{M}$, a predicate constant for each finitary relation over $\mathbb{M}$ and a functional constant for each function (with finitely many variables) over $\mathbb{M}$.

The interpretation of these 'constants' over $\mathbb{M}$ is the obvious one. Thus $\mathbb{M}$ becomes an $L_{\mathbb{M}}$-structure. Notice that quantified variables (in the $L_{\mathbb{M}}$-formulas) which always correspond to elements of $\mathbb{M}$ may actually cover subsets of $\mathbb{R}$ and more complicated things since $\mathcal{P}(\mathbb{R})\subseteq\mathbb{M}$, etc.

1.2. *A nonstandard model*

$\mathbb{M}'$ over $\mathbb{M}$ is (by definition) an $L_{\mathbb{M}}$-structure

$$\mathbb{M}' \supsetneq \mathbb{M}$$

such that

a) the embedding $\mathbb{M}\subset\mathbb{M}'$ preserves the validity of any $L_{\mathbb{M}}$-formula
b) there are elements s of $\mathbb{M}'$ belonging to the interpretation of $\mathbb{R}$ in $\mathbb{M}'$ which fulfill $s>r$ for all elements r belonging to (the interpretation of) $\mathbb{R}$ in $\mathbb{M}$. This means that there are nonstandard real numbers s beating all the standard reals r in magnitude.

The existence of such models is an immediate consequence of the 'compactness theorem' of first order model theory (cf. [3], pg. 21).

The elements of $\mathbb{M}'$ which belong to $\mathbb{M}$ are called standard elements.

1.3. *Internal and external sets*

A subset N of some $L_{\mathbb{M}}$-structure $\mathcal{S}$ is called internal if there is an element n in $\mathcal{S}$ such that $N=\{m \mid m \in n$ holds in $\mathcal{S}\}$. Then N can be represented by the element n. Subsets that are not internal are called external. Our basic $L_{\mathbb{M}}$-structure $\mathbb{M}$ is such that all subsets of elements are internal (cf. 1.1). This is not any more the case for the nonstandard models $\mathcal{S}=\mathbb{M}'$.

In order to apply (first order) theorems of $\mathbb{M}$ to subsets N of $\mathbb{M}'$ it is necessary to know that the latter are internal, because only those are covered by the variables in the theorems (through the resp. elements n which represent them). The subsets N one is particularly interested in are the 'definable' ones, i.e. those which are introduced through their definitions.

1.4. *Definable subsets*

A subset N of some element m of an $L_{\mathbb{M}}$-structure $\mathcal{S}$ is called definable if there exists an $L_{\mathbb{M}}$-formula $\varphi(x)$ having one free variable x (and possibly additional parameters from $\mathcal{S}$) such that $N=\{l \in m \mid \varphi(l)$ holds in $\mathcal{S}\}$.

1.5. Proposition. *A subset N of any nonstandard model $\mathbb{M}'$ is internal if and only if it is definable.*

Proof. If N is internal there is n in $\mathbb{M}'$ such that $N=\{l \in n \mid l \in n$ holds in $\mathbb{M}'\}$ hence it is definable (by the formula: $x \in n$). Conversely if N is definable such that $N=\{l \in m \mid \varphi(l)$ holds in $\mathbb{M}'\}$ where $\varphi(x)$ is the formula $\psi(x, m_1, \ldots, m_r)$, the $m_1, \ldots, m_r$ being all the parameters needed, then one knows that $\mathbb{M}$ fulfills the formula $\forall x_1 \cdots \forall x_r \forall z \exists y \forall x(x \in y \Leftrightarrow \psi(x, x_1, \ldots, x_r) \wedge x \in z)$, because in $\mathbb{M}$ every subset of any element z is internal, cf. 1.3. The formula then holds in $\mathbb{M}'$ too, since $\mathbb{M}'$ is a model for all theorems of $\mathbb{M}$. I.e. in $\mathbb{M}'$ we have a uniquely defined y_0 such that $\forall x(x \in y_0 \Leftrightarrow \varphi(x) \wedge x \in m)$. Or $y_0=\{x \in m \mid \varphi(x)$ holds in $\mathcal{S}\}=N$. qed.

1.6. *Remark.* Proposition 1.5 enables us to carry out nonstandard analysis by applying the theorems of standard analysis to things that are defined in the right way, namely according to 1.4.

1.7. *Finite, infinite, infinitesimal and quasistandard elements*

Any (nonstandard) real in $\mathbb{M}'$ whose (interpretation of) absolute value is bounded by some standard real (in $\mathbb{M}$!) is called finite. Otherwise it is called infinite.

Any nonstandard real $r \geq 0$ (≤ 0) which is sandwiched between every positive (negative) standard real and 0 is called infinitesimal. In this case one writes $r \sim 0$. More generally, $x \sim y$ means $|x-y| \sim 0$. This definition should be applied to complex x, y, too.

If some (nonstandard) complex number z fulfills $z \sim w$, where w is standard, then we write $\mathrm{st}(x)=w$. Notice that w is uniquely defined.

For any internal $f: \mathbb{R}^n \rightarrow C$ in $\mathbb{M}'$ belonging to $L^2_{\mathbb{C}}(\mathbb{R}^n)$ the Hilbertspace of square integrable functions in $\mathbb{M}'$, with respect to $d^n x$, we write $f \approx 0$ if the L^2-norm fulfills $\|f\| \sim 0$. More generally, $f \approx g$ if $f-g \approx 0$. Later on we will use the same notation $f \approx g$ for f, g belonging to some other Hilbertspaces e.g. $L^2_{\mathbb{C}}(\mathbb{R}^3, d^3p/2\omega_p)$ (cf. 2.1).

In case $f_k, g_k : \mathbb{R}^n \to \mathbb{C}$ have an additional parameter $k \in \mathbb{R}^m$ we shall write $f_k \approxeq g_k$ to indicate that $f \approx g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}$ as functions whose variables include k. Any $f \in L^2_{\mathbb{C}}(\mathbb{R}^n)$ is called quasistandard if there exists a standard $f_1 \in L^2_{\mathbb{C}}(\mathbb{R}^n)$ such that $f \approx f_1$.

1.8. *Internal norm approximations of distributions*

We consider 'function valued distributions' $D: L^2_{\mathbb{C}}(\mathbb{R}^3) \to L^2_{\mathbb{C}}(\mathbb{R}^3)$ where $(D(f))(p) = \int d^3q D(p, q) f(q)$, e.g. $D = id$, $D(p, q) = \delta^3(p-q)$. An internal function $K(p, q)$ with the property

$$\int d^3q K(p, q) f(q) \approxeq \int d^3q D(p, q) f(q) \text{ for all quasistandard } f \in L^2_{\mathbb{C}}(\mathbb{R}^3)$$

is called an internal norm approximation of $D(p, q)$. Our next aim is to construct a handy internal norm approximation $k(p-q)$ of $\delta^3(p-q)$. For this we use some means from the theory of

1.9. *Fourier transformations*

By abuse of notation we describe Fourier transformations by

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p e^{ipx} f(p) \qquad (x \in \mathbb{R}^3 \text{ 'configuration space'})$$

$$f(p) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-ipx} f(x) \qquad (p \in \mathbb{R}^3 \text{ 'momentum space'})$$

'distinguishing' the transforms only by the names of the variables x, y, z and p, q, k respectively. So far we assumed f (and its transform) to belong to $\mathcal{S}(\mathbb{R}^3)$, the Schwartz space of C^∞ functions of rapid decrease. The two transformations are inverse to each other and can be uniquely extended to the whole of $L^2_{\mathbb{C}}(\mathbb{R}^3)$ (cf. [16] pg. 10 or [17] pg. 153 'Plancherels theorem'). The extensions are still unitary transformations.

The convolutions $f * k$ for $f(p), k(p)$ belonging to $L^2_{\mathbb{C}}(\mathbb{R}^3)$ is defined according to $(f * k)(p) = \int f(q) k(p-q)\, d^3q$ and has the Fourier transform $(2\pi)^{3/2} f(x) k(x)$, in case this product belongs to $L^2_{\mathbb{C}}(\mathbb{R}^3)$ (cf. [17] pg. 154/55).

1.10. *Space cutoff Q*

Let $k : \mathbb{R}^3 \to C$ be the function with

$$k(x) = \begin{cases} \dfrac{1}{(2\pi)^{3/2}} & \text{for } |x| \le Q \\ 0 & \text{for } |x| > Q \end{cases}$$

where Q is some positive infinite real in $\mathbb{M}'$ over $\mathbb{M}$. It follows that for any $f(x)$

belonging to $L^2_{\mathbb{C}}(\mathbb{R}^3)$,

$$(2\pi)^{3/2} f(x) \cdot k(x) = \begin{cases} f(x) & \text{for} \quad |x| \le Q \\ 0 & \text{for} \quad |x| > Q \end{cases}$$

belongs to $L^2_{\mathbb{C}}(\mathbb{R}^3)$, too.

Furthermore, for any standard $f(x)$ in $L^2_{\mathbb{C}}(\mathbb{R}^3)$ we have $(2\pi)^{3/2} fk \approx f$.

Proof. $\|f-(2\pi)^{3/2}fk\| = \int_{|x|>Q} |f(x)|^2 \, d^3x$ is infinitesimal, because for any standard $\varepsilon > 0$ there exists some standard $Q_\varepsilon > 0$, such that $\varepsilon > \int_{|x|>Q_\varepsilon} |f(x)|^2 \, d^3x \ge \int_{|x|>Q} |f(x)|^2 \, d^3x$.

1.11. Theorem. *The Fourier transform $k(p)$ of the space cutoff function $k(x)$ (cf. 1.10) fulfills $\int f(q)k(p-q)\, d^3q \approx f(p)$ for any quasistandard $f(q)$ from $L^2_{\mathbb{C}}(\mathbb{R}^3)$. I.e. $k(p-q)$ is a norm approximation of $\delta^3(p-q)$.*

Proof. First let $f(p)$ be a standard function from $L^2_{\mathbb{C}}(\mathbb{R}^3)$. Then $\int f(q)k(p-q)\, d^3q = (f * k)(p)$ has the Fourier transform $(2\pi)^{3/2} f(x)k(x)$ (cf. 1.0) fulfilling $(2\pi)^{3/2} f(x)k(x) \approx f(x)$. Since Fourier transformation is unitary, we get $(f*k)(p) \approx f(p)$.

For a quasistandard $f = f_1 + g$ where f_1 is atandard and $g \approx 0$ we have

$$\begin{aligned}(f*k)(p) &= ((f_1+g)*k)(p) = (f_1*k)(p) + (g*k)(p) \\ &\approx (f_1*k)(p) \quad \text{since } (g*k)(p) \text{ has the same norm as } g(x)k(x) \\ &\qquad\qquad \text{which is bounded by the infinitesimal norm of } g. \\ &\approx f_1(p) \quad \text{since } f_1 \text{ is standard.} \\ &\approx f(p) \quad \text{by definition of } f_1. \quad \text{qed.}\end{aligned}$$

For later use we will insert here a theorem from nonstandard complex analysis.

1.12. Theorem. *For any real infinitesimal $\varepsilon > 0$ we have*

$$\int_{-\infty}^{+\infty} \frac{e^{-ip_0(x_0-y_0)}}{p^2 - p_0^2 + m^2 - i\varepsilon} \, dp_0 \sim \pi i \frac{e^{-i\omega_p|x_0-y_0|}}{\omega_p}$$

where $p \in \mathbb{R}^3$, $\omega_p := \sqrt{m^2+p^2}$, $m > 0$ and standard, $p_0, x_0, y_0 \in \mathbb{R}$.

Proof. For real $m > 0$ the formula $\lim_{\varepsilon \to +0} l_\varepsilon = 0$ holds for

$$l_\varepsilon := \int_{-\infty}^{+\infty} \frac{e^{-ip_0(x_0-y_0)}}{p^2 - p_0^2 + m^2 - i\varepsilon} \, dp_0 - \pi i \frac{e^{-i\omega_p|x_0-y_0|}}{\omega_p}$$

in standard complex analysis.

Thus for each standard $\delta > 0$ there is a standard $\varepsilon(\delta) > 0$ such that $\forall \varepsilon (0 < \varepsilon < \varepsilon(\delta) \Rightarrow |l_\varepsilon| < \delta)$ holds. In $\mathbb{M}' \supset \mathbb{M}$ let $\varepsilon > 0$ be infinitesimal. Then $\varepsilon < \varepsilon(\delta)$ for each standard $\delta > 0$. Hence $|l_\varepsilon| < \delta$ for each standard $\delta > 0$, i.e. $|l_\varepsilon|$ is infinitesimal (cf. also [18]). qed.

1.13. *Iterated nonstandard models*

The nonstandard models $\mathbb{M}' \supseteqq \mathbb{M}'$ can be used again for extensions $\mathbb{M}'' \supsetneqq \mathbb{M}'$ subject to conditions 1.2a) und b) (with respect to the same first order language $L_{\mathbb{M}}$).

In the sequel we will use fixed iterated nonstandard models $\mathbb{M}_{(1)} \supsetneqq \mathbb{M}_{(0)} \supsetneqq \mathbb{M}$ in order to modify standard quantum field theory (belonging to $\mathbb{M}$).

We will distinguish between

a) standard and 0-standard elements belonging to $\mathbb{M}$ and $\mathbb{M}_{(0)}$ respectively
b) finite and 0-finite elements with respect to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$
c) infinitesimal ($r \sim 0$) and 0-infinitesimal ($r \underset{0}{\sim} 0$) with resp. to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$.
d) $f \approx 0$ and $f \underset{0}{\approx} 0$ with resp. to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$.
e) $f \approxeq 0$ and $f \underset{0}{\approxeq} 0$ with resp. to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$.
f) quasistandard f and 0-quasistandard f with respect to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$.
g) st x and 0-st x with resp. to $\mathbb{M}_{(1)} \supset \mathbb{M}$ and $\mathbb{M}_{(1)} \supset \mathbb{M}_{(0)}$ (cf. 1.7).

2. The Klein-Gordon field

First we consider the standard formulation (cf. [2]).

2.1. *Standard Fockspace*

The Fock space $\mathbb{F}$ is built up in the usual way by means of the 'one particle space' $\mathbb{F}_1 \cdot \mathbb{F}_1$ is the Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^3, d^3p/2\omega_p)$ where $\omega_p = \sqrt{m^2+p_1^2+p_2^2+p_3^2}$ ($m>0$ 'restmass'). The 'n-particle space' $\mathbb{F}_n$ is the symmetrical tensorproduct $\mathbb{F}_n := \mathbb{F}_1 \overset{n}{\underset{s}{\otimes}} = \mathbb{F}_1 \underset{s}{\otimes} \cdots \underset{s}{\otimes} \mathbb{F}_1$ of n factors $\mathbb{F}_1$. For $n=0$, the component $\mathbb{F}_0$ is set to be $\mathbb{F}_0 := \mathbb{C}$ (complex numbers).

Set $\mathbb{F} := \overline{\bigoplus_{n=0}^{\infty} \mathbb{F}_n}$, the topological completion of $\bigoplus_{n=0}^{\infty} \mathbb{F}_n$. The annihilation and creation operators a and a^+ are usually defined as operator valued distributions over $\mathbb{F}$ e.g.

$$(a^+(f)\Lambda)(p_1, \ldots, p_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{l=1}^{n+1} f(p_l)\Lambda(p_1, \ldots, \not{p}_l, \ldots p_{n+1})$$

2.2. *Remark.* The restrictions $a(f) \restriction \mathbb{F}_n$ and $a^+(f) \restriction \mathbb{F}_{n-1}$ both are bounded operators for each $n>1$ (and they are adjoint to each other). But $a(f)$ and $a^+(f)$ are unbounded for any $f \neq 0$.

2.3. *Commutators*

The definitions of a, a^+ yield the commutator relations, $[a(f), a(g)] = [a^+(f), a^+(g)] = 0$ and $[a(f), a^+(g)] = (f, g)$, (f, g) denoting the scalar product in the Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^3, dp^3/2\omega_p)$.

2.4. *Standard Klein-Gordon field*

The standard Klein-Gordon field $\phi(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ is given by

$$\phi(t, x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_p} (e^{-i(\omega_p t - px)} a(p) + e^{i(\omega_p t - px)} a^+(p))$$

which, for any fixed time t, is an operator valued distribution over the Fock space $\mathbb{F}$. Its initial value ($t=0$) reads

$$\phi(0, x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} (a(p) + a^+(-p))e^{ipx}.$$

The corresponding Hamiltonian H_0^{KG} is defined to be

$$H_0^{KG} = \int \frac{d^3p}{2\omega_p} \omega_p a^+(p)a(p).$$

2.5. *Modifications*

The standard Klein-Gordon field ϕ, belonging to the basic structure $\mathbb{M}$, will be viewed as belonging to the nonstandard models $\mathbb{M}_{(0)}$ and $\mathbb{M}_{(1)}$ of $\mathbb{M} \subsetneqq \mathbb{M}_{(0)} \subsetneqq \mathbb{M}_{(1)}$ (cf. 1.13). We will fix now an infinitesimal restmass $m > 0$, belonging to $\mathbb{M}_{(0)} \supset \mathbb{M}$. The further modifications of ϕ concern the Fock space as well as the annihilation and creation operators:

2.6. *Particle number cutoff*

Let ω be an infinite 0-standard natural number. We want to replace the Fockspace $\mathbb{F}$ (belonging to $\mathbb{M}_{(0)} \subset \mathbb{M}_{(1)}$) by $\mathbb{F}^\omega := \bigoplus_{n+1}^{\omega} \mathbb{F}_n$ (which belongs to $\mathbb{M}_{(0)} \subset \mathbb{M}_{(1)}$, too). Thus we have to replace

$$a(f) \quad \text{by} \quad a_\omega(f) := a(f) \restriction \mathbb{F}^\omega : \mathbb{F}^\omega \to \mathbb{F}^\omega$$

and

$$a^+(f) \quad \text{by} \quad a_\omega^+(f) := \begin{pmatrix} a^+(f) \restriction \mathbb{F}^{\omega-1} & 0 \\ 0 & 0 \end{pmatrix} : \mathbb{F}^\omega = \mathbb{F}^{\omega-1} \oplus \mathbb{F}_\omega \to \mathbb{F}^\omega.$$

Then a_ω, a_ω^+ become operator valued distributions over $\mathbb{F}^\omega$.

2.7. *Remark.* $a_\omega(f)$ and $a_\omega^+(f)$ are bounded operators which are adjoint to each other, for any $f \in L^2_{\mathbb{C}_1}(\mathbb{R}^3_1, d^3p/2\omega_p)$ from $\mathbb{M}_{(1)}$ (cf. 2.3).

2.8. *Space cutoff*

Let $k : \mathbb{R}^3_{(1)} \to \mathbb{R}^3_{(1)}$ be the 1-internalnorm approximation (in momentum space) of the Dirac δ-function evolving from a 0-infinite space cutoff $Q \in \mathbb{M}_{(1)}$ (in configuration space) according to 1.10 and 1.11 (cf. also 1.13). We set

$$a_{k,\omega}^{(+)}(p) := 2\omega_p \int \frac{d^3q}{2\omega_q} k(p-q) a_\omega^{(+)}(q) : \mathbb{F}^\omega \to \mathbb{F}^\omega.$$

It follows immediately:

2.9. *Remark.* $a_{k,\omega}(p)$ and $a_{k,\omega}^+(p)$ are bounded operators which are adjoint to each other (cf. 2.7) and belong to $\mathbb{M}_{(1)}$.

2.10. *Commutators*

The following relations hold

a) $[a_{k,\omega}((p), a_{k,\omega}(q)] = [a^+_{k,\omega}(p), a^+_{k,\omega}(q)] = 0$
b) $[a_{k,\omega}(p), a^+_{k,\omega}(q)] \upharpoonright \mathbb{F}^{\omega-1} = 2\omega_p 2\omega_q \cdot K_m(p, q)$

where $K_m(p, q) := \int d^3u(1/2\omega_u)k(p-u)k(q-u)$ is a 1-internal norm approximation of $\delta^3(p-q)/2\omega_q$.

Proof. a) is obvious.

b) $[a_{k\omega}(p), a^+_{k\omega}(q)] \upharpoonright \mathbb{F}^{\omega-1} = 2\omega_p 2\omega_q \int_{\mathbb{R}^3} d^3u \frac{1}{2\omega_u} k(p-u)k(q-u).$

For any 0-quasistandard $f \in L^2_{\mathbb{C}_1}(\mathbb{R}^3, d^3p)$ we find

$$\int d^3pK_m(p, q)f(p) = \int d^3u \int d^3pf(p)k(p-u)\frac{1}{2\omega_u} k(q-u)$$

$$\underset{0}{\approxeq} \int d^3u \frac{f(u)}{2\omega_u} k(q-u) \underset{0}{\approxeq} \frac{f(q)}{2\omega_q} \text{ qed.}$$

2.11. *UV cutoff*

Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the characteristic function of the ball $\mathbb{K}_P$ around $0 \in \mathbb{R}^3$ with large standard radius P (ultra violet cutoff). I.e.

$$h(p) = \begin{cases} 1 & \text{for } |p| \leq P \\ 0 & \text{for } |p| > P \end{cases}.$$

Possible explicit values of P will be discussed in 6.23 and 6.24. We set $a^{(+)}_{h,k,\omega}(p) := h(p)a^+_{k\omega}(p)$. For notational convenience we will write $b^{(+)}(p)$ instead of $a^{(+)}_{h,k,\omega}(p)$.

2.12. *Remark*

a) $p \mapsto b^{(+)}(p) : \mathbb{F}^\omega \to \mathbb{F}^\omega$ is an operator valued function belonging to $\mathbb{M}_{(1)}$.
It is uniformly bounded.
$b(p)$ is adjoint to $b^+(p)$ (cf. 2.9).
b) The standard and the modified annihilation operator resp. are very close in the sense that $a^{(+)}(p)\Lambda \underset{0}{\approxeq} b^{(+)}(p)\Lambda$ for any 0-quasistandard $\Lambda \in \mathbb{F}_n$, $n \leq \omega$, which has support in $\mathbb{K}^n_P$.

2.13. *The modified commutators*

The following relations hold, because of 2.10

a) $[b(p), b(q)] = [b^+(p), b^+(q)] = 0$
b) $[b(p), b^+q)] \upharpoonright \mathbb{F}^{\omega-1} = h(p)h(q)2\omega_p 2\omega_q K_m(p, q)$

2.14. *The initial cutoff Klein-Gordon field* $\chi(0, x)$

Now we are going to replace a and a^+ in the initial value of the Klein-Gordon field (cf. 2.4) by the modified annihilation and creation operators $b(p)$

and $b^+(p)$ respectively. Thus, the initial cutoff Klein-Gordon field $\chi(0, x)$ is defined to be the 1-internal uniformly bounded operator valued function

$$x \mapsto \chi(0, x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_p} e^{ipx}(b(p) + b^+(-p)).$$

2.15. *Modified free Hamiltonian*

The modified free Hamiltonian H_1^{KG} is constructed likewise from the standard free Hamiltonian H_0, by inserting the cutoff creation and annihilation operators into H_0^{KG}, yielding

$$H_1^{KG} = \int \frac{d^3p}{2\omega_p} \omega_p b^+(p) b(p)$$

2.16. *Remark.* H_1^{KG} is a 1-internal bounded operator over the cutoff Fock space $\mathbb{F}^\omega$.

2.17. *Remark.* Since $b(p)$ and $b^+(p)$ are adjoints it follows that H_1^{KG} is selfadjoint.

2.18. Definition. Let $\mathbb{G} \subset \mathbb{F}^{\omega-1}$ be the (external) subset consisting of all 0-quasistandard states of $\mathbb{F}^{\omega-1}$ having the property that for all $n \leq \omega - 1$ their $\mathbb{F}_n$-components have support in $\mathbb{K}_p^n$ (cf. 2.11). Furthermore let $\mathbb{G}_n := \mathbb{G} \cap \mathbb{F}_n$. One sees immediately that $\mathbb{G} \subset \mathbb{M}_{(1)}$ is a subset of the domain of $H_0^{KG} \in \mathbb{M}_{(1)}$.

2.19. Theorem. *Any state* $\Lambda \in \mathbb{G}$ *fulfills* $H_1^{KG}\Lambda \underset{0}{\approx} \Lambda'$ *where*

$$\Lambda'(p_1, \ldots, p_n) := \sum_{i=1}^{n} \omega_{p_i} \Lambda(p_1, \ldots, p_n).$$

Proof. For such a Λ we get

$$(H_1^{KG}\Lambda)(p_1 \cdots p_n)$$
$$= \left(\int \frac{d^3q}{\omega_q} a^+(q) \int \frac{d^3p}{2} h(p) 2\omega_p k(p-q) h(p) 2\omega_p \right.$$
$$\left. \times \int \frac{d^3r}{2\omega_r} k(p-r) a(r) \Lambda \right)(p_1 \cdots p_n)$$
$$\underset{0}{\approx} \sum_{l=1}^{n} \frac{1}{\sqrt{n}} \int \frac{d^3p}{2} h(p) 2\omega_p k(p-p_l) h(p) \sqrt{n} \Lambda(p, p_1 \cdots \not{p_l} \cdots p_n)$$
$$\underset{0}{\approx} \sum_{l=1}^{n} \omega_{p_l} \Lambda(p_1 \cdots p_l \cdots p_n)$$

(For more details cf. [23]). qed.

2.10. *Modified Klein-Gordon field*

The modified Klein-Gordon field $\chi(t, x)$ is defined according to

$$\chi(t, x) := e^{itH_1^{KG}} \chi(0, x) e^{-itH_1^{KG}}$$

where $\chi(0, x)$ is the initial cutoff Klein-Gordon field of 2.14 and H_1^{KG} the modified free Hamiltonian of 2.15. (This definition is obviously consistent for the value $t = 0$).

Thus $\chi(t, x)$ is for each $t \in \mathbb{R}^3_{(1)}$ an 1-internal uniformly bounded operator valued function in x over the modified Fock space $\mathbb{F}^\omega$.

In order to be able to compute contractions of χ (cf. 2.26) we need the following result.

2.21. Lemma. *For any $\Lambda \in \mathbb{G}_n$, $n \leq \omega - 1$, and any $g \in \mathbb{G}_1$ the following terms belong to $\mathbb{G}$ and fulfill*

$$\int_{\mathbb{R}^3} \frac{d^3p}{2\omega_p} g(p) b^{(+)}(p) \Lambda \underset{0}{\approx} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_p} g(p) a^{(+)}(p) \Lambda$$

Proof. Use 2.12b.

2.22. Theorem. *For any 0-finite t and any $\Lambda \in \mathbb{G}$*
a) $e^{itH_1^{KG}}\Lambda$ *belongs to $\mathbb{G}$ and fulfills*
b) $e^{itH_0^{KG}}\Lambda \underset{0}{\approx} e^{itH_1^{KG}}\Lambda$.

The following is an immediate

2.23. Corollary. *For any 0-finite t, any $g \in L^2_{\mathbb{C}}(\mathbb{R}^3, d^3p/2\omega_p)$ which is 0-quasi-standard and for any $\Lambda \in \mathbb{G}$ the following formulas hold*

$$\text{a)}\quad e^{itH_1^{KG}} \int \frac{d^3p}{2\omega_p} g(p) b(p) e^{-itH_1^{KG}} \Lambda \underset{0}{\approx} \int \frac{d^3p}{2\omega_p} e^{-i\omega_p t} g(p) b(p) \Lambda$$

$$\text{b)}\quad e^{itH_1^{KG}} \int \frac{d^3p}{2\omega_p} g(p) b^+(-p) e^{-itH_1^{KG}} \Lambda \underset{0}{\approx} \int \frac{d^3p}{2\omega_p} e^{+i\omega_p t} g(p) b^+(-p) \Lambda$$

Proof of 2.23 *from* 2.22 straightforward, using 2.21. For the proof of Theorem 2.22 we use

2.24. Definition. $U(t) := e^{itH_1^{KG}} e^{-itH_0^{KG}}$

2.25. Lemma. $\dot{U}(t)\Lambda \underset{0}{\approx} 0$ *for* $\Lambda \in \mathbb{G}$.

Proof. By applying 2.19.

Proof of 2.22

a) $e^{itH_1^{KG}}\Lambda \in \mathbb{G}$ follows from b) since $e^{itH_0^{KG}}\Lambda \in \mathbb{G}$
b) We are going to show that

$$W(t) := \|U(t)\Lambda - \Lambda\|^2 \underset{0}{\approx} 0 \text{ for } \Lambda \in \mathbb{G} \text{ and any 0-finite } t,$$

which yields

$$e^{itH_1^{KG}}\Lambda - e^{itH_0^{KG}}\Lambda = (e^{itH_1^{KG}} e^{-itH_0^{KG}} - id) e^{itH_0^{KG}}\Lambda \underset{0}{\approx} 0.$$

We have

$$\left|\frac{dW(t)}{dt}\right| = |(\dot{U}(t)\Lambda, (U(t)-id)\Lambda) + ((U(t)-id)\Lambda, \dot{U}(t)\Lambda)$$

$$\leq 2\,\|\dot{U}(t)\Lambda\| \cdot \|(U(t)-id)\Lambda\| \leq 2\,\|\dot{U}(t)\Lambda\| \cdot 2\,\|\Lambda\| \underset{0}{\sim} 0.$$

$W(t) \underset{0}{\sim} 0$ (for t 0-finite) then follows from the mean value theorem of differential calculus. qed.

2.26. *Contractions*

Contractions of operators are defined in the following way. Let $A_i(x_0)$, $i=1,2$ be time dependent operators in the following sense $A_i(x_0) = e^{ix_0H_1^{KG}}A_ie^{-ix_0H_1^{KG}}$ where A_i is a (smeared) polynomial in $b(p)$, $b^+(q)$. The contraction $\overline{A_1(x_0)A_2(y_0)}$ is set to be $\overline{A_1(x_0)A_2(y_0)} := T(A_1(x_0)A_2(y_0)) - :A_1(x_0)A_2(y_0):$ (cf. also [19] p. 83) where the 'time ordered product' equals

$$T(A_1(x_0)A_2(y_0)) := \begin{cases} A_1(x_0)A_2(y_0) & \text{for} \quad x_0 > y_0 \\ A_2(y_0)A_1(x_0) & \text{for} \quad x_0 < y_0 \end{cases}$$

and the 'normal product': $A_1(x_0)A_2(y_0)$: arises from $A_1(x_0)A_2(y_0)$ by transposing 'annihilation parts' $e^{ix_0H_1^{KG}}b(r)e^{-ix_0H_1^{KG}}$ and $e^{iy_0H_1^{KG}}b(s)e^{-iy_0H_1^{KG}}$ with 'creation parts' $e^{ix_0H_1^{KG}}b^+(u)e^{-x_0H_1^{KG}}$ and $e^{iy_0H_1^{KG}}b^+(v)e^{-y_0H_1^{KG}}$ until the latter occur only as left side factors of the former.

2.27. Definition. By abuse of notation we will write

$$\chi(t, x) = \frac{1}{(2\pi)^{3/2}} \int d^3p\chi(t, p)e^{ipx}.$$

For $t=0$ this implies $\chi(0, p) = b(p) + b^+(-p)/2\omega_p$ (cf. 2.14). It follows easily

2.28. Lemma.

$$\overline{\chi(x_0, p)\chi(y_0, q)} = \begin{cases} \left[e^{ix_0H_1^{KG}}\dfrac{b(p)}{2\omega_p}e^{-ix_0H_1^{KG}}, e^{iy_0H_1^{KG}}\dfrac{b^+(-q)}{2\omega_q}e^{-iy_0H_1^{KG}}\right] & \text{for} \quad x_0 > y_0 \\ \left[e^{iy_0H_1^{KG}}\dfrac{b(q)}{2\omega_q}e^{-iy_0H_1^{KG}}, e^{ix_0H_1^{KG}}\dfrac{b^+(-p)}{2\omega_p}e^{ix_0H_1^{KG}}\right] & \text{for} \quad x_0 < y_0 \end{cases}$$

2.29. Theorem. *For any 0-finite time values $x_0 \neq y_0$, any 0-quasistandard $f: \mathbb{R}^3 \to \mathbb{C}$ and for any $\Lambda \in \mathbb{G}$ we have*

a) $$\int d^3qf(q)\overline{\chi(x_0, p)\chi(y_0, q)}\Lambda \underset{0}{\approxeq} f(-p)\frac{e^{-i\omega_p|x_0-y_0|}}{2\omega_p}h(p)\Lambda$$

b) $$\int d^3pf(p)\overline{\chi(x_0, p)\chi(y_0, q)}\Lambda \underset{0}{\approxeq} f(-q)\frac{e^{-i\omega_q|x_0-y_0|}}{2\omega_q}h(q)\Lambda$$

(*cf.* 1.7 *and* 1.13 *for the meaning of* $\underset{0}{\approxeq}$)

Proof. a) For $x_0 > y_0$ it follows

$$\int d^3q f(q) \overline{\chi(x_0, p)\chi}(y_0, q)\Lambda$$

$$= e^{ix_0 H_I^{KG}}\left(\frac{b(p)}{2\omega_p} e^{-i(x_0-y_0)H_I^{KG}} \int d^3q f(q) \frac{b^+(-q)}{2\omega_q} e^{i(x_0-y_0)H_I^{KG}}\right.$$

$$\left. - e^{-i(x_0-y_0)H_I^{KG}} \int d^3q f(q) \frac{b^+(-q)}{2\omega_q} e^{i(x_0-y_0)H_I^{KG}} \frac{b(p)}{2\omega_p}\right) e^{-ix_0 H_I^{KG}}\Lambda \qquad \text{(cf. 2.28)}$$

$$\underset{0}{\approx} e^{ix_0 H_I^{KG}} \int \frac{1}{2\omega_p} \frac{d^3q}{2\omega_q} f(q) e^{-i\omega_q(x_0-y_0)}[b(p), b^+(-q)] e^{-ix_0 H_I^{KG}}\Lambda$$

(cf. 2.12b and 2.23b)

$$= f(-p) \frac{e^{-i\omega_p(x_0-y_0)}}{2\omega_p} h(p)\Lambda \qquad \text{(cf. 2.10b, 2.22a)}$$

For $x_0 < y_0$ we compute the 'adjoint' $(\int d^3q f(q)\chi(x_0, p)\chi(y_0, q))^*\Lambda$ along the same lines. b) is proved similarly. qed.

2.30. Definition. Let

$$\Delta_F(p_0, p, q_0, q) := -i \frac{\delta^1(p_0+q_0)\delta^3(p+q)h(p)}{p^2 - p_0^2 + m^2 - i\varepsilon}$$

where $\varepsilon \in \mathbb{R}_{(1)}$ is 0-infinitesimal, $\varepsilon > 0$.

2.31. Lemma. *For any 0-finite times $x_0 \neq y_0$, any 0-quasistandard $f: \mathbb{R}^3 \to \mathbb{C}$, and any $\Lambda \in \mathbb{G}$ we have*

a) $$\int d^3q f(q) \overline{\chi(x_0, p)\chi}(y_0, q)\Lambda$$

$$\underset{0}{\approx} \frac{1}{2\pi} \int d^3q\, dq_0\, dp_0 f(q)\Delta_F(p_0, p, q_0, q) e^{-i(p_0x_0+q_0y_0)}\Lambda$$

b) $$\int d^3p f(p) \overline{\chi(x_0, p)\chi}(y_0, q)\Lambda$$

$$\underset{0}{\approx} \frac{1}{2\pi} \int d^3p\, dp_0\, dq_0 f(p)\Delta_F(p_0, p, q_0, q) e^{-i(p_0x_0+q_0y_0)}\Lambda$$

Proof of b)

$$\frac{1}{2\pi} \int d^3p\, dp_0\, dq_0 f(q)\Delta_F(p_0, p, q_0, q) e^{-i(p_0x_0+q_0y_0)}$$

$$= \frac{-i}{2\pi} \int dq_0 h(-q) \frac{e^{-iq_0(y_0-x_0)}}{q^2 - q_0^2 + m^2 - i\varepsilon} f(-q)$$

$$\underset{0}{\simeq} f(-q) \frac{e^{-i\omega_q|x_0-y_0|}}{2\omega_q} h(q) \qquad \text{(cf. 1.12)}$$

which implies the desired result (cf. 2.29b). qed.

3. The free photon field

We are going to start with the standard Stückelberg-Coester type formalism along the lines of [20] pg. 136–137 for photons of 'small' mass $m>0$. The 'gauge' λ will be fixed at $\lambda=1$.

3.1. Definition. The Fock space $\mathbb{B}$ for the standard photon field will be the symmetrical tensorproduct $\mathbb{B}:={}^0\mathbb{F}\otimes_s{}^1\mathbb{F}\otimes_s{}^2\mathbb{F}\otimes_s{}^3\mathbb{F}$ of the Fockspaces ${}^l\mathbb{F}=\mathbb{F}$, $l=0\cdots3$ for the K.G. field of mass m (cf. 2.1) with the usual 'indefinite metric'.

Let ${}^la(p)$, ${}^la^+(q)$ be the canonical annihilation and creation operators resp. over $\mathbb{B}$ (cf. [20]). Let $\varepsilon^0(k):=1/m(\omega_k,k)\in\mathbb{R}^4$, $k\in\mathbb{R}^3$ and let $\varepsilon^1(k)$, $\varepsilon^2(k)$, $\varepsilon^3(k)$ be space like real vectors of $\mathbb{R}^4$ which together form an orthonormal and complete system of $\mathbb{R}^4$ in the Lorentz metric. The standard free photon field is now defined by

$$A_\rho(t,x):=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega_k}\sum_{\lambda=0}^{3}\varepsilon_\rho^\lambda(k)\{e^{-i(\omega_kt-kx)}\,{}^\lambda a(k)+e^{i(\omega_kt-kx)}\,{}^\lambda a^+(k)\}.$$

Its initial value is

$$A_\rho(0,x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega_k}\sum_{\lambda=0}^{3}e^{ikx}\{\varepsilon_\rho^\lambda(k)^\lambda a(k)+\varepsilon_\rho^\lambda(-k)^\lambda a^+(-k)\}.$$

The free photon Hamiltonian is given by

$$H_0^{\text{ph}}:={}^0H_0+{}^1H+{}^2H_0+{}^3H_0$$

where

$${}^lH_0:=\int\frac{d^3p}{2\omega_p}\,\omega_p{}^la^+(p){}^la(p).$$

In order to modify the standard $A_\rho(t,x)$ we view it as belonging to $\mathbb{M}_{(1)}$, with the mass m belonging to $\mathbb{M}_{(0)}\subset\mathbb{M}_{(1)}$. Then we proceed as in chapter 2.

3.2. *The cutoffs*

We apply first the particle number cutoff (cf. 2.6) to each tensor factor ${}^0\mathbb{F}$, ${}^1\mathbb{F}$, ${}^2\mathbb{F}$, ${}^3\mathbb{F}$ of $\mathbb{B}$, yielding

$$\mathbb{B}^{4\omega}:={}^0\mathbb{F}^\omega\bigotimes_s{}^1\mathbb{F}^\omega\bigotimes_s{}^2\mathbb{F}^\omega\bigotimes_s{}^3\mathbb{F}^\omega\qquad(\omega\in\mathbb{N}_{(0)}).$$

Then we define ${}^\lambda a_\omega^{(+)}(p)$ according to 2.6, for $\lambda=0,1,2,3$. The space cutoff is applied to each ${}^\lambda a^{(+)}(p)$, according to 2.8, for $\lambda=0,1,2,3$, yielding $a_{k\omega}^{(+)}(p)$. The UV cutoff yields ${}^\lambda b^{(+)}(p)={}^\lambda a_{h,k,\omega}^{(+)}(p)$, according to 2.11, for $\lambda=0,1,2,3$.

3.3. *Commutators*

The following holds

$$[{}^\lambda b(p),{}^{\lambda'}b(q)]=[{}^\lambda b^+(p),{}^{\lambda'}b^+(q)]=0$$

$$[{}^\lambda b(p),{}^{\lambda'}b^+(q)]\restriction\mathbb{B}^{\omega-1}=-g_{\lambda\lambda'}h(p)h(q)2\omega_p2\omega_qK_m(p,q)\qquad\text{(cf. 2.10, 2.13).}$$

The restriction $\upharpoonright \mathbb{B}^{\omega-1}$ in the second equation could obviously be weakened to $\upharpoonright {}^0\mathbb{F}^{\omega-1} \otimes_s {}^1\mathbb{F}^{\omega-1} \otimes_s {}^2\mathbb{F}^{\omega-1} \otimes_s {}^3\mathbb{F}^{\omega-1}$.

3.4. *Initial cutoff photon field*

The initial cutoff photon field $B(0, x)$ over $\mathbb{B}^{4\omega}$ is now defined according to

$$B_\rho(0, x) := \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega_k} \sum_{\lambda=0}^{3} e^{ikx}\{\varepsilon_\rho^\lambda(k)^\lambda b(k) + \varepsilon_\rho^\lambda(-k)^\lambda b^+(-k)\}$$

which is a 1-internal uniformly bounded operator valued function in x (cf. 2.14).

3.5. *Modified free photon Hamiltonian*

Definition of the modified free photon Hamiltonian $\mathrm{H}_1^{\mathrm{ph}}$ according to

$$H_1^{\mathrm{ph}} := {}^0H_1^{\mathrm{KG}} + {}^1H_1^{\mathrm{KG}} + {}^2H_1^{\mathrm{KG}} + {}^3H_1^{\mathrm{KG}}$$

where

$${}^lH_1^{\mathrm{KG}} = \int \frac{d^3p}{2\omega_p} \omega_p^l {}^lb^+(p)^l b(p) \qquad (l = 0, 1, 2, 3) \qquad \text{(cf. 3.2, 2.16).}$$

3.6. Definition. Let $\mathbb{D} \subset \mathbb{B}^{\omega-1}$ be the (external) subset consisting of all 0-quasistandard photon states Λ whose components $\Lambda_{n_0,n_1,n_2,n_3}$, $n_0+n_1+n_2+n_3 \leq \omega-1$ have support in $\mathbb{K}_P^{n_0} \times \mathbb{K}_P^{n_1} \times \mathbb{K}_P^{n_2} \times \mathbb{K}_P^{n_3}$ (cf. 2.11 for $\mathbb{K}_P$).

3.7

The modified free photon field $B_\rho(t, x)$ is defined according to

$$B_\rho(t, x) := e^{itH_1^{\mathrm{ph}}} \qquad B_\rho(0, x)e^{-itH_1^{\mathrm{ph}}}$$

(cf. also 2.20).

3.8. Theorem. *Any state* $\Lambda = {}^0\Lambda_{n_0} \otimes_s {}^1\Lambda_{n_1} \otimes_s {}^2\Lambda_{n_2} \otimes_s {}^3\Lambda_{n_3} \in \mathbb{D}$ *fulfills*

$${}^lH_1^{\mathrm{ph}}\Lambda \underset{0}{\approx} -g_{ll}\left(\sum_{k=1}^{n_l} \omega_{l_{p_k}}\right) \cdot \Lambda$$

where ${}^l\Lambda_{n_l}$ *has the variables* ${}^lp_1, \ldots, {}^lp_{n_l}$

Proof. Apply 2.19 to Definition 3.5.

3.9. Lemma. *For any* ${}^l\Lambda_{n_l} \in \mathbb{D}_n \subset \mathbb{B}_{n_l}$ *and any* $g \in \mathbb{G}_1$ *(cf. 2.16) the following terms belong to* $\mathbb{D}$ *and fulfill*

$$\int \frac{d^3p}{2\omega_p} g(p)^l b^{(+)}(p)^l \Lambda_{n_l} \underset{0}{\approx} \int \frac{d^3p}{2\omega_p} g(p)^l a^{(+)}(p)^l \Lambda_{n_l}$$

Proof. Apply Lemma 2.21.

3.10. Theorem. *For any 0-finite time t and any* $\Lambda \in \mathbb{D}$

a) $e^{itH_0^{ph}}\Lambda$ *and* $e^{itH_1^{ph}}\Lambda$ *belong to* $\mathbb{D}$
b) $e^{itH_0^{ph}}\Lambda \underset{0}{\approx} e^{itH_1^{ph}}\Lambda$.

This follows immediately from 2.22. In the same way one generalizes the remaining definitions and assertions of chapter 2. Then Lemma 2.31 carries over to

3.11. Lemma. *For any 0-finite times* x_0 *and* y_0, *any 0-quasistandard* $f:\mathbb{R}^3 \to \mathbb{C}$ *and any* $\Lambda \in \mathbb{D}$ *we have*

a) $$\int d^3q f(q)\overline{B_\mu(x_0,p)B}_\nu(y_0,q)\Lambda$$

$$\underset{0}{\approx} \frac{1}{2\pi}\int d^3q\,dp_0\,dq_0 f(q)(-g_{\mu\nu})\Delta_F(p_0,p,q_0,q)e^{-i(p_0x_0+q_0y_0)}\Lambda$$

b) $$\int d^3p f(p)\overline{B_\mu(x_0,p)B}_\nu(y_0,q)\Lambda$$

$$\underset{0}{\approx} \frac{1}{2\pi}\int d^3p\,dp_0\,dq_0 f(p)(-g_{\mu\nu})\Delta_F(p_0,p,q_0,q)e^{-i(p_0x_0+y_0y_0)}\Lambda$$

where Δ_F *is defined according to 2.30.*

4. The free Dirac field

4.1. *Standard Fockspace*

The Fockspace $\mathbb{A}$ of the standard Dirac field is the iterated (partially antisymmetrical) tensorproduct $\mathbb{A}=(\mathbb{H}\otimes_a\mathbb{H})\otimes(\mathbb{H}\otimes_a\mathbb{H})$ where $\mathbb{H}=\overline{\bigoplus_{n=0}^{\infty}\mathbb{H}_1^{\otimes n}}$ is the completed direct sum of the n-fold antisymmetric tensorproducts of $\mathbb{H}_1=L^2_{\mathbb{C}}(\mathbb{R}^3, d^3p/2\Omega_p)$, $\Omega_p=\sqrt{M^2+p^2}$, where M is the fermion restmass.

4.2. *Standard free Dirac field*

The standard free Dirac field is defined for any $t\in\mathbb{R}$ as operatorvalued distribution in x

$$\psi_\nu(t,x):=\frac{\sqrt{2M}}{(2\pi)^{3/2}}\int\frac{d^3p}{2\Omega_p}\sum_{s=-1}^{s=+1}(e^{-i(\Omega_pt-px)}(u_s(p))_\nu{}^sa(p)+e^{i(\Omega_pt-px)}(v_s(p))_\nu{}^sc^+(p))$$

for $\nu=1,2,3,4$ where ${}^sa(p)$, ${}^sc^+(p)$ are the usual electron annihilation and positron creation operators ($s=$ spin) resp. and $u_s(p)$, $v_s(p)$ are the canonical spinor functions (cf. [20] 2-37).

4.3. *Standard free Dirac Hamiltonian*

The standard free Dirac Hamiltonian H_0^{el} is

$$H_0^{\text{el}} = \int \frac{d^3p}{2\Omega_p} \Omega_p \sum_{s=-1}^{+1} ({}^s a^+(p)^s a(p) + {}^s c^+(p)^s c(p))$$

4.4. *The cutoffs*

First we apply a particle number cutoff $\Omega \in \mathbb{N}_{(0)}$ (cf. 2.6) to each tensor factor $\mathbb{H}$ of $\mathbb{A}$ and to its annihilation and creation 'operators'. This gives rise to the modified Fockspace $\mathbb{A}^{4\Omega}$ and to ${}^{\pm 1}a_\Omega(p)$, ${}^{\pm 1}a^+_\Omega(q)$, ${}^{\pm 1}c_\Omega(p)$, ${}^{\pm 1}c_\Omega(q)$. Then we apply a space cutoff k and a UV cutoff h according to 2.8 and 2.11 in each tensor factor $\mathbb{H}^\Omega$, yielding

$${}^s b^{(+)}(p) := {}^s a^{(+)}_{hk\Omega}(p)$$

$${}^s d^{(+)}(p) := {}^s c^{(+)}_{hk\Omega}(p) \qquad (s = \pm 1) \text{ in } \mathbb{A}^{4\Omega}.$$

4.5. *Remark*

a) $\mathbb{R}^3 \ni p \mapsto {}^s b^{(+)}(p) : \mathbb{A}^{4\Omega} \to \mathbb{A}^{4\Omega}$ and $q \mapsto {}^s d^{(+)}(q) : \mathbb{A}^{4\Omega} \to \mathbb{A}^{4\Omega}$ are 1-internal operator valued functions (i.e. belonging to $\mathbb{M}_{(1)}$). They are uniformly bounded.
b) Furthermore ${}^s b(p)$ is adjoint to ${}^s b^+(p)$ and ${}^s d(q)$ is adjoint to ${}^s d^+(q)$.

4.6. *Cutoff anticommutators*

For $s, t = \pm 1$ the cutoff annihilation and creation operators fulfill the following relations

$$[{}^s b(p), {}^s b^+(q)]_+ \restriction \mathbb{A}^{\Omega-1} \qquad = [{}^t c(p), {}^t c^+(q)]_+ \restriction \mathbb{A}^{\Omega-1} = h(p)h(q)2\Omega_p 2\Omega_q K_M(p, q)$$

where

$$K_M(p, q) = \int_{\mathbb{R}^3} d^3u \frac{1}{2\Omega_u} k(p-u)k(q-u) \qquad \text{(cf. 2.10b)}$$

is a 1-internal normapproximation of $\delta^3(p-q)/2\Omega_q$. All 'other' anti-commutators vanish (cf. 2.13).

4.7. *Initial cutoff Dirac field*

The initial cutoff Dirac field $\theta_\nu(0, x)$ arises from $\psi_\nu(0, x)$ by inserting the respective cutoff annihilation and creation operators i.e.:

$$\theta_\nu(0, x) = \frac{\sqrt{2M}}{(2\pi)^{3/2}} \int \frac{d^3p}{2\Omega_p} \sum_{s=-1}^{+1} (e^{ipx}(u_s(p))^s_\nu b(p) + e^{-ipx}(v_s(p))^s_\nu d^+(p))$$

$\gamma = 1, 2, 3, 4$, converting it into a 1-internal uniformly bounded operator valued function.

4.8. *Modified Dirac Hamiltonian*

The modified Dirac Hamiltonian then becomes

$$H_1^{\mathrm{el}} = \int \frac{d^3p}{2\Omega_p} \Omega_p \sum_{s=-1}^{+1} ({}^sb^+(p)^sb(p) + {}^sd^+(p)^sd(p)) \qquad \text{(cf. 4.11).}$$

This is a 1-internal bounded selfadjoint operator over the cutoff Fockspace $\mathbb{A}^{4\Omega}$ (cf. 4.5).

4.9. Definition. Let $\mathbb{L} \subset \mathbb{A}^{\Omega-1}$ be the (external) subset consisting of all 0-quasistandard fermion states Λ whose components $\Lambda_{n_1,n_2,n_3,n_4}$ have support in $\mathbb{K}_P^{n_1} \times \mathbb{K}_P^{n_2} \times \mathbb{K}_P^{n_3} \times \mathbb{K}_P^{n_4}$ $(n_1+n_2+n_3+n_4 \leq \Omega - 1)$.

4.10. Lemma. *If* $\Lambda \in \mathbb{L}$ *belongs to* $\mathbb{A}_n$ *it fulfills* $H_1^{\mathrm{el}}\Lambda \underset{0}{\approx} \sum_{i=1}^n \Omega_{p_i}(p_1, \ldots, p_n)$.

Proof. In the spirit of 2.19.

4.11. *Modified free Dirac field*

The modified free Dirac field $\theta(t, x)$ is defined according to $\theta_\nu(t, x) := e^{itH_1^{\mathrm{el}}}\theta_\nu(0, x)e^{-itH_1^{\mathrm{el}}}$, $\nu = 1, \ldots, 4$ where $\theta_\nu(0, x)$ and H_1^{el} are defined according to 4.7 and 4.8 respectively. For any $t \in \mathbb{R}_{(1)}$ the field $\theta(t, x)$ is a 1-internal uniformly bounded operator valued function in x over the cutoff Fockspace $\mathbb{A}^{4\Omega}$.

4.12. Lemma. *For any* $\Lambda \in \mathbb{L}_n$, $n < \Omega - 1$, *and any* $g \in \mathbb{G}_1$ (*cf.* 4.9, 2.18) *the following terms belong to* $\mathbb{L}$ *and fulfill*

$$\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} b^{(+)}(p)\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} a^{(+)}(p)\Lambda$$

$$\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} d^{(+)}(p)\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} c^{(+)}(p)\Lambda \qquad \text{(cf. 2.21).}$$

4.13. Theorem. *For any 0-finite time t and any* $\Lambda \in \mathbb{L}_n$, $n \leq \Omega - 1$ (*cf.* 4.9) *we have* $e^{itH_0^{\mathrm{el}}}\Lambda \underset{0}{\approx} e^{itH_1^{\mathrm{el}}}\Lambda$, *both terms belonging to* $\mathbb{L}_n$. *For the proof cf.* 2.22 *and its proof.*

4.1. Corollary. *For any 0-finite time t, any* $g: \mathbb{R}^3 \to \mathbb{C}$ *which is 0-quasistandard, and for any* $\Lambda \in \mathbb{L}$ *we have*

a) $e^{itH_1^{\mathrm{el}}} \displaystyle\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} b(p) e^{-itH_1^{\mathrm{el}}}\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} e^{-i\Omega_p t} g(p)^{\pm 1} b(p)\Lambda$

b) $e^{itH_1^{\mathrm{el}}} \displaystyle\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} d(p) e^{-itH_1^{\mathrm{el}}}\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} e^{-i\Omega_p t} g(p)^{\pm 1} d(p)\Lambda$

c) $e^{itH_1^{\mathrm{el}}} \displaystyle\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} b^+(-p) e^{-itH_1^{\mathrm{el}}}\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} e^{i\Omega_p t} g(p)^{\pm 1} b^+(-p)\Lambda$

d) $e^{itH_1^{\mathrm{el}}} \displaystyle\int \frac{d^3p}{2\Omega_p} g(p)^{\pm 1} d^+(-p) e^{-itH_1^{\mathrm{el}}}\Lambda \underset{0}{\approx} \int \frac{d^3p}{2\Omega_p} e^{i\Omega_p t} g(p)^{\pm 1} d^+(-p)\Lambda$

All eight terms belong to $\mathbb{L}$.

4.15. *Contractions*

Contractions of time dependent (Fermion-)operators are defined in the usual way:

$$\overbrace{A_1(x_0)A_2}(y_0) := T(A_1(x_0)A_2(y_0)) - :A_1(x_0)A_2(y_0)$$

where

$$T(A_1(x_0)A_2(y_0)) = \begin{cases} A_1(x_0)A_2(y_0) & \text{if} \quad x_0 > y_0 \\ -A_2(y_0)A_1(x_0) & \text{if} \quad y_0 > x_0 \end{cases}$$

and $:A_1(x_0)A_2(y_0):$ arises from $A_1(x_0)A_2(y_0)$ by transposing annihilation parts and creation parts, each time introducing a factor -1, until creation parts do not appear any more as factors on the right side of annihilation parts. (cf. 2.26 and [19] p. 102).

4.16. *Denotations*

By abuse of notation we will write

$$\theta_\nu(t, x) = \frac{1}{(2\pi)^{3/2}} \int d^3p\theta_\nu(t, p)e^{ipx}.$$

e.g.

$$\theta_\alpha(0, p) = \frac{\sqrt{2M}}{2\Omega_p} \sum_{s=-1}^{+1} \{(u_s(p))_\alpha{}^s b(p) + (v_s(-p))_\alpha{}^s d^+(-p)\}$$

It follows readily

4.17. Lemma

$$\overbrace{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q) = \begin{cases} \left[e^{ix_0H_1^{el}} \frac{\sqrt{2M}}{2\Omega_p} \sum_{s=-1}^{+1} (u_s(p))_\alpha{}^s b(p)e^{-ix_0H_1^{el}}, \right. \\ \qquad \left. e^{iy_0H_1^{el}} \frac{\sqrt{2M}}{2\Omega_q} \sum_{s'=-1}^{+1} (\bar{u}_{s'}(-q))_\beta{}^{s'} b^+(-q)e^{-iy_0H_1^{el}}\right] \\ \qquad\qquad \text{for} \quad x_0 > y_0 \\ -\left[e^{iy_0H_1^{el}} \frac{\sqrt{2M}}{2\Omega_q} \sum_{s'=-1}^{+1} (\bar{v}_{s'}(q))_\beta{}^{s'} d(q)e^{-iy_0H_1^{el}}, \right. \\ \qquad \left. e^{ix_0H_1^{el}} \frac{\sqrt{2M}}{2\Omega_p} \sum_{s=-1}^{+1} (v_s(-p))_\alpha{}^s d^+(-p)e^{-ix_0H_1^{el}}\right] \\ \qquad\qquad \text{for} \quad x_0 < y_0 \end{cases}$$

4.18. Theorem. *For any 0-finite time values $x_0 \neq y_0$, any (1-internal) $f: \mathbb{R}^3 \to \mathbb{C}$ which is 0-quasistandard and for any $\Lambda \in \mathbb{L}$ the following holds*

a) $$\int d^3qf(q)\overbrace{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)\Lambda \underset{0}{\approxeq} h(p)f(-p)\frac{e^{-i\Omega_p|x_0-y_0|}}{2\Omega_p}(M, \Omega_p, p)_{\alpha\beta}\Lambda$$

for $x_0 > y_0$

b) $$\int d^3qf(q)\overbrace{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(x_0, q)\Lambda \underset{0}{\approxeq} h(p)f(-p)\frac{e^{-i\Omega_p|x_0-y_0|}}{2\Omega_p}(+-\Omega_p, p)_{\alpha\beta}\Lambda$$

for $x_0 < y_0$

$$\text{c)}\quad \int d^3p f(p)\overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)\Lambda \underset{0}{\approxeq} h(q)f(-q)\frac{e^{-i\Omega_q|x_0-y_0|}}{2\Omega_q}(-\,-\not{(\Omega_q, q)})_{\alpha\beta}\Lambda$$

for $x_0 > y_0$

$$\text{d)}\quad \int d^3p f(p)\overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)\Lambda \underset{0}{\approxeq} h(q)f(-q)\frac{e^{-i\Omega_q|x_0-y_0|}}{2\Omega_q}(M-\not{(\Omega_q, q)})_{\alpha\beta}\Lambda$$

for $x_0 < y_0$, *where* $\not{(r_0, r)} := \sum_{\nu=0}^{3} r_\nu \gamma_\nu$.

Proof. For $x_0 > y_0$ we have

$$\int d^3q f(q)\,\overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q) = 2Me^{ix_0H_1^{cl}}\Bigg[\sum_{-1}^{+1}\frac{(u_s(p))_\alpha}{2\Omega_p}{}^s b(p),$$

$$e^{i(y_0-x_0)H_1^{cl}}\int d^3q f(q)\sum_{-1}^{+1}\frac{(\bar{u}_{s'}(-q))_\beta}{2\Omega_q}{}^{s'}b^+(-q)e^{-i(y_0-x_0)H_1^{cl}}\Bigg]_+ e^{-ix_0H_1^{cl}}\Lambda$$

$$\underset{0}{\approxeq} 2M\cdot e^{ix_0H_1^{cl}}\int d^3q \sum_{s,s'=-1}^{+1}(u_s(p))_\alpha(\bar{u}_{s'}(-q))_\beta\frac{e^{-i|x_0-y_0|\Omega_q}}{2\Omega_p 2\Omega_q}$$

$$\times f(q)[{}^s b(p), {}^{s'}b^+(-q)]_+ e^{-ix_0H^{cl}}\Lambda \quad \text{since} \quad x_0 > y_0$$

$$\underset{0}{\approxeq} 2M\left(\frac{M+\not{(\Omega_p, p)}}{2M}\right)_{\alpha\beta}\frac{e^{-i|x_0-y_0|\Omega_p}}{2\Omega_p}f(-p)h(p)\Lambda \qquad \text{(cf. [20] 2-40)}$$

All other cases work similarly.

4.19. *Modified Feynman propagator*

Definition of the modified Feynman propagator S_F according to

$$S_{F_{\alpha\beta}}(p_0, p, q_0, q) := -i\delta^1(p_0+q_0)\delta^3(p+q)\frac{(M+\not{(p_0, p)})_{\alpha\beta}}{p^2-p_0^2+M^2-i\varepsilon}h(p)$$

where $\varepsilon > 0$ is a fixed 0-infinitesimal number from $\mathbb{R}_{(1)}$.

4.20. Lemma. *For any 0-finite time values* $x_0 \neq y_0$, *for any 0-quasistandard* $f:\mathbb{R}^3 \to \mathbb{C}$ *and any* $\Lambda \in \mathbb{L}$ *the formulas*

$$\text{(I)}\quad \int d^3q\,\overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)f(q)\Lambda$$

$$\underset{0}{\approxeq} \frac{1}{2\pi}\int d^3q\, dp_0\, dq_0 f(q)e^{-i(p_0x_0+q_0y_0)}S_{F_{\alpha\beta}}(p_0, p, q_0, q)\Lambda$$

$$\text{(II)}\quad \int d^3p\, f(p)\overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)\Lambda$$

$$\underset{0}{\approxeq} \frac{1}{2\pi}\int d^3p\, dp_0\, dq_0 f(p)e^{-i(p_0x_0+q_0y_0)}S_{F_{\alpha\beta}}(p_0, p, q_0, q)\Lambda$$

hold.

Proof. For $x_0 > y_0$ we have

$$\frac{1}{2\pi}\int d^3q\, dp_0\, dq_0 f(q) e^{-i(p_0x_0+q_0y_0)} S_{F_{\alpha\beta}}(p_0, p, q_0, q)\Lambda$$

$$= -\frac{i}{2\pi} f(-p)h(p)\left(M + i\gamma_0 \frac{\partial}{\partial x_0} + \sum_{\nu=1}^{3} \gamma_\nu p_\nu\right)_{\alpha\beta} \int \frac{e^{-ip_0(x_0-y_0)}}{p^2 - p_0^2 + M^2 - i\varepsilon}\, dp_0 \Lambda$$

$$\underset{0}{\approxeq} -\frac{i}{2\pi} f(-p)h(p)\left(M + i\gamma_0 \frac{\partial}{\partial x_0} + \sum_{\nu=1}^{3} \gamma_\nu p_\nu\right)_{\alpha\beta} 2\pi i \frac{e^{-i\Omega_p|x_0-y_0|}}{2\Omega_p}\Lambda \qquad \text{(cf. 1.12)}$$

$$= \int d^3q f(q) \overline{\theta_\alpha(x_0, p)\bar{\theta}_\beta}(y_0, q)\Lambda \quad \text{since} \quad x_0 > y_0.$$

All other instances are proved similarly.

5. Interactions

5.1. *Fockspace*

The Fockspace of the interacting fields is the tensorproduct $\mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega}$ of the particle number cutoff Fockspaces $\mathbb{A}^{4\Omega}, \mathbb{B}^{4\omega}$ for fermions (electrons-positrons) and photons respectively (cf. 4.4 and 3.2). A special role will be played by the external subspace $\mathbb{L} \otimes \mathbb{D}$, where $\mathbb{L} \subset \mathbb{A}^{4\Omega}$ and $\mathbb{D} \subset \mathbb{B}^{4\omega}$ are 0-quasi standard fermion and photon states respectively with restricted support (cf. 4.9, 3.6 and 5.13b).

5.2. *Interaction term*

Electromagnetic interactions are introduced by an additional summand H_I in the total Hamiltonian $H = H_1^{\text{el}} + H_1^{\text{ph}} + H_I$. We assume H_I to be of the form

$$H_I = -e \int :\bar{\theta}(0, x)\gamma_\mu \theta(0, x)(B_\mu(0, x) + E_\mu(0, x)):\, d^3x$$

where the 'external field operator' $E(0, x)$ is a 1-internal uniformly bounded operator on $\mathbb{B}^{4\omega}$ ($-e$ = charge of the electron).

5.3. *Examples*

a) The case where there is no exterior field corresponds to $E_\mu(0, x) = 0$.
b) The case of the Coulomb field corresponds to $E_\mu(0, x) = C_\mu(0, x)$ of 6.3.

5.4. *The interacting fields*

$\theta_\nu^{\text{HP}}(t, x)$ and $B_\rho^{\text{HP}}(t, x)$ in the Heisenberg picture are defined according to

$\theta_\nu^{\text{HP}}(t, x) := e^{it(H_1^{\text{el}}+H_1^{\text{ph}}+H_I)}\theta_\nu(0, x)e^{-it(H_1^{\text{el}}+H_1^{\text{ph}}+H_I)}$ etc.

5.5. *Interaction picture*

The interaction picture is introduced by the composed transformation

$\mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega}$ Heisenberg picture

$\downarrow e^{-it(H_1^{el}+H_1^{ph}+H_I)}$

$\mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega}$ Schrödinger picture

$\downarrow e^{it(H_1^{el}+H_1^{ph})}$

$\mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega}$ Interaction picture

as usual, yielding the equation of motion $(d/dt)\phi^{IP}(t) = -iH_I^{IP}(t)\phi^{IP}(t)$ for the state vector $\Phi^{IP}(t)$.

5.6. *Remark*

a) The existence of $e^{\pm it(H_1^{el}+H_1^{ph})}$ and $e^{\pm it(H_1^{el}+H_1^{ph}+H_I)}$ resp. is guaranteed since the exponents are (internal, nonstandard) bounded operators (cf. also the Dyson expansion 5.7).

b) The standard difficulties with the existence of the interaction picture of a translation invariant theory in connection with Haags theorem (cf. [21], chapter 6) are circumvented because of the nonstandard 'space cutoff' hidden in the Hamiltonian through the modified annihilation and creation operators (cf. 2.8, 3.5, 4.4).

5.7. *The Dyson expansion*

The transformation

$$U(t, s) := \mathrm{Id} + \sum_{n=1}^{\infty} (-i)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} H_I^{IP}(t_1) \cdots H_I^{IP}(t_n)\, dt_n \cdots dt_2\, dt_1$$

converges in the uniform operator topology and fulfills the equation of motion

$$\frac{d}{dt} U(t, s) = -iH_I^{IP}(t) U(t, s)$$

for any strongly continuous (internal) map $t \mapsto H_I^{IP}(t)$ of $\mathbb{R}_{(1)}$ into the (bounded) operators $\mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega} \to \mathbb{A}^{4\Omega} \otimes \mathbb{B}^{4\omega}$.

Proof. cf. [16] Theorem X.69 pg. 282.

5.8. Theorem. *The Dyson expansion can be rewritten in the form*

$$U(t, s) = \mathrm{Id} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_s^t \cdots \int_s^t T(H_I^{IP}(t_1) \cdots H_I^{IP}(t_n))\, dt_n \cdots dt_1.$$

The proof can be found in most of the textbooks (e.g. cf. [19] pg. 155).

5.9. *Wick's theorem*

The Theorem of Wick on T-products, normal products and contractions in standard quantum electrodynamics (cf. [19] pg. 161) carries over to our modified version unchanged.

5.10. *Feynman's rules*

Feynman's rules carry over in the sense that one has to use our modified free fields and contractions in configuration space instead of the usual free fields and contractions, respectively.

5.11. *First order perturbation theory*

First order perturbation theory deals with the summand

$$U_1(t,s) := -i\int_s^t H_I(x_0)\,dx_0 \quad \text{of} \quad U(t,s) \qquad \text{(cf. 5.8)}$$

In order to compare our modified $U_1(t, s)$ with the usual

$$S_1(t,s) := ie\int_s^t dx_0 \int_{\mathbb{R}^3} :\bar{\psi}(x_0,x)\gamma_\mu\psi(x_0,x)A_\mu(x_0,x):\,d^3x$$

assuming $E_\mu \equiv 0$, we need

5.12. Theorem. *Let t, s be 0-finite, then $(\phi, U_1(t,s)\Lambda) \underset{0}{\sim} (\phi, S_1(t,s)\Lambda)$ for any $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$ having support in $\mathbb{K}_{P/2}$ (cf. 2.11).*

Proof. It suffices to prove

$$\left(\phi, \int_s^t dx_0 \int \bar{\theta}^{\pm}(x_0,x)\gamma_\mu\theta^{\pm}(x_0,x)B_\mu^{\pm}(x_0,x)\,d^3x\Lambda\right)$$
$$\underset{0}{\sim}\left(\phi, \int dx_0 \int \bar{\psi}^{\pm}(x_0,x)\gamma_\mu\psi^{\pm}(x_0,x)A_\mu^{\pm}(x_0,x)\,d^3x\Lambda\right)$$

for any normal order of the fermion fields. For example

$$\left(\phi, \int_s^t dx \int \bar{\theta}^{-}(x_0,x)\gamma_\mu\theta^{-}(x_0,x)B_\mu^{+}(x_0,x)\,d^3x\Lambda\right)$$
$$\underset{0}{\sim}\frac{1}{(2\pi)^{3/2}}\int_s^t dx_0 \int d^3p\,d^3q\,e^{i(\omega_{p+q}-\Omega_p-\Omega_q)x_0}$$
$$\times(B_\mu^{-}(0,p+q)\phi, \theta^{-}(0,p)\gamma_\mu\theta^{-}(0,q)\Lambda)$$

For $p, q \in \mathbb{K}_{P/2}$ we have the approximation (cf. also 2.12b)

$$\underset{0}{\sim}\frac{1}{(2\pi)^{3/2}}\int_s^t dx_0 \int d^3p\,d^3q\,e^{i(\omega_{p+q}-\Omega_p-\Omega_q)x_0}$$
$$\times(A_\mu^{-}(0,p+q)\phi, \psi^{-}(0,p)\gamma_\mu\psi^{-}(0,q)\Lambda)$$
$$=\left(\phi, \int_s^t dx_0 \int \psi^{-}(x_0,x)\gamma_\mu\psi^{-}(x_0,x)A_\mu^{+}(x_0,x)\,d^3x\Lambda\right)$$

The other instances are proved similarly. qed.

5.13. *Remark*

a) Assuming that the cutoff value P be rather large (cf. 6.24) Theorem 5.12 guarantees that the first order perturbation theory of our modified Q.E.D. agrees with the standard one up into highly relativistic regions.
b) The role which is always played by the external sets $\mathbb{D}$ and $\mathbb{L}$ suggests that $\mathbb{L}\otimes\mathbb{D}$ be called the set of physical states.

6. Coulomb scattering and vacuum polarization

6.1. *The modified Coulomb field*

The Coulomb field is usually given by the vectorpotential

$$c_0(q) = \frac{Ze}{(2\pi)^{3/2}} \frac{1}{q^2}, \qquad c_1(q) = c_2(q) = c_3(q) = 0, \; q \in \mathbb{R}^3,$$

in the momentum representation. In connection with our nonvanishing, infinitesimal photon mass $m \in \mathbb{M}_{(0)}$ and our UV cutoff, we suggest the modified form

$$C_0(q) := \frac{Ze}{(2\pi)^{3/2}} \frac{h(q)}{\omega_q^2} \; (\omega_q = \sqrt{m^2 + q^2}), \; C_1(q) = C_2(q) = C_3(q) = 0$$

which is everywhere defined, as well as its Fourier transformed $C_\nu(x)$, $x \in \mathbb{R}^3$.

6.2. *The exterior field operator*

We view $C_0(q)$ as operating multiplicatively on the Fockspace $\mathbb{B}^{4\omega}$ of photon. We then define the 'time dependent' operator $C_0(t, q) := e^{itH_I^{ph}} C_0(q) e^{-itH_I^{ph}}$ obviously is independent of t, $C_0(t, q) = C_0(q)$.

6.3. *The interaction Hamiltonian H_I*

H_I is defined according to

$$H_I := -e \int :\bar{\theta}(0, x)\gamma_\mu \theta(0, x)(B_\mu(0, x) + C_\mu(0, x)): d^3x$$

where

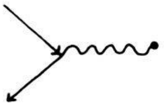
$$C_\mu(0, x) := \frac{1}{(2\pi)^{3/2}} \int e^{iqx} C_\mu(q) \, d^3q$$

6.4. *Coulomb scattering*

Coulomb scattering during the time interval $[-t, +t]$ is given by the summand

$$s(-t, t) := +ie \int_{-t}^{+t} dx_0 \int d^3x \, :\bar{\theta}(x_0, x)\gamma_\mu \theta(x_0, x) C_\mu(x):$$

of the S-matrix, corresponding to the Feynman graph



For the remainder of this chapter we assume t to be a 0-finite time value.

6.5. Theorem. *For any two one-electron states* $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$ *Coulomb scattering yields*

$$(\phi, s(-t, t)\Lambda) \underset{0}{\sim} +\frac{ie}{(2\pi)^{3/2}} \int_{-t}^{+t} dx_0 \times \int d^3p\, d^3q\, e^{ix(\Omega_p - \Omega_q)}(\phi, \bar{\theta}^+(0, -p)\gamma_0 \theta^-(0, q) C_0(p-q)\Lambda)$$

where

$$\theta^-(0, w) := \frac{\sqrt{2M}}{2\Omega_w} \sum_{-1}^{+1} u_s(w)^s b(w), \text{ etc. (cf. 4.17)}$$

Proof. Since ϕ, Λ are one-electron states it follows that

$$(\phi, s(-t, t)\Lambda) = \left(\phi, \frac{ie}{(2\pi)^{3/2}} \int_{-t}^{+t} dx_0 \int d^3p\, d^3q\, e^{ix_0 H_1^{el}} \bar{\theta}^+(0, -p) e^{-ix_0 H_1^{el}} \times \gamma_0 e^{ix_0 H_1^{el}} \theta^-(0, q) e^{-ix_0 H_1^{el}} C_0(p-q)\Lambda\right)$$

$$\underset{0}{\sim} \left(\phi, \frac{ie}{(2\pi)^{3/2}} \int_{-t}^{+t} dx_0 \int d^3p\, d^3q e^{ix_0\Omega_p} \bar{\theta}^+(0, -p) \times \gamma_0 e^{-ix_0\Omega_q} \theta^-(0, q) C_0(p-q)\Lambda\right.$$

(since $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$, cf. 4.14). qed.

We get immediately

6.6. Corollary. *For any two one-electron states* $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$ *Coulomb scattering gives rise to*

$$(\phi, s\Lambda) := \lim_{t \to \infty} 0\text{-st}\, (\phi, s(-t, t)\Lambda)$$

$$= \frac{ie}{\sqrt{2\pi}} 0\text{-st}\left(\int d^3q\, d^3q\, \delta^1(\Omega_p - \Omega_q)(\phi, \bar{\theta}^+(0, -p)\gamma_0 \theta^-(0, q) C_0(p-q)\Lambda)\right)$$

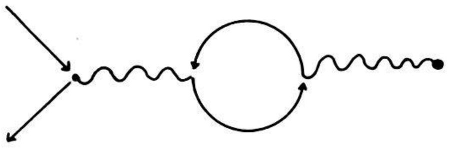
6.7. *Vacuum polarization*

Vacuum polarization results from the summand

$$\sigma(-t, t) := (ie)^3 \int_{-t}^{+t} dz_0 \int_{-t}^{+t} dy_0 \int_{-t}^{+t} dx_0 \int d^3z\, d^3y\, d^3x$$

$$:\bar{\theta}(z_0, z)\gamma_m \theta(z_0, z) \overbrace{B_m(z_0, z) B_n}(y_0, y) \times \bar{\theta}(y_0, y)\gamma_n \theta(y_0, y)\bar{\theta}(x_0, x)(\gamma_r)\theta(x_0, x) C_r(x):$$

of the S-matrix, corresponding to the Feynman graph



Exactly as in the standard case one gets

6.8. Lemma

$$\sigma(-t, t) = -i^3 e^3 \int_{-t}^{+t} dz_0\, dy_0\, dx_0 \int d^3z\, d^3y\, d^3x$$

$$\times : \bar{\theta}(z_0, z)\gamma_m \theta(x_0, z) \overbrace{B_m(z_0, z) B_n}(y_0, y)$$

$$\times \mathrm{Tr}(\overbrace{\theta(x_0, x)\bar{\theta}}(y_0, y)\gamma_n \overbrace{\theta(y_0, y)\bar{\theta}}(x_0, x)\gamma_r) C_r(x):$$

A straightforward evaluation leads to

6.9. Theorem. *For any two one-electron states $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$, vacuum polarization yields*

$$(\phi, \sigma(-t, t)\Lambda) \underset{0}{\sim} e^3 \Bigg(\phi, \int_{-t}^{+t} dx_0\, dy_0\, dz_0 \int d^3p\, d^3q\, d^3w\, du_0\, dw_0\, dk_0$$

$$\times \frac{1}{2\pi} g_{mn} e^{iz_0(\Omega_p - \Omega_q - u_0)} e^{iy_0(u_0 + w_0 - k_0)} e^{ix_0(-w_0 + k_0)}$$

$$\times \frac{1}{(2\pi)^{3/2}} \theta^+(0, -p)\, \gamma_m \frac{1}{(2\pi)^{3/2}} \theta^-(0, q) \cdot \frac{h(p-q)}{(p-q)^2 - u_0^2 + m^2 - i\varepsilon}$$

$$\times \frac{1}{(2\pi)^2} \mathrm{Tr}\left(\frac{M + \cancel{w_0, w}}{w^2 - w_0^2 + M^2 - i\varepsilon} h(w)\gamma_n \frac{M + \cancel{k_0, p-q+w}}{(p-q+w)^2 - k_0^2 + M^2 - i\varepsilon} \gamma_0 h(p-q+w)\right)$$

$$\times \frac{1}{(2\pi)^{3/2}} C_0(p-q)\Lambda\Bigg)$$

6.10. Corollary. *For any two one-electrons states $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$ vacuum polarization yields*

$$(\phi, \sigma\Lambda) := \lim_{t \to \infty} (\phi, \sigma(-t, t)\Lambda)$$

$$= e^3 \frac{1}{(2\pi)^{9/2}} \text{0-st} \left\{ \int d^3p\, d^3q\, \delta^1(\Omega_p - \Omega_q) \cdot \Big(\phi, \bar{\theta}^+(0, -p)\gamma_m \theta^-(0, q) \right.$$

$$\left. \times \frac{g_{mn} h(p-q)}{(p-q)^2 - (\Omega_p - \Omega_q)^2 + m^2 - i\varepsilon} \Pi_{no}(\Omega_p - \Omega_q, p-q) C_0(p-q)\Lambda\Big) \right\}$$

where

$$\Pi_{\mu\nu}(u_0, u) := \int dw_0\, d^3w \operatorname{Tr}\left(\frac{M + \not{w_0, w}}{w^2 - w_0^2 + M^2 - i\varepsilon}\right.$$
$$\left.\times\, h(w)\gamma_\mu \frac{M + \not{u_0 + w_0, u + w}}{(u+w)^2 - (u_0 + w_0)^2 + M^2 - i\varepsilon}\, h(u+w)\gamma_\nu\right).$$

Standard methods yield

6.11. Lemma

$$\Pi_{\mu\nu}(u_0, u) = \int_0^1 dv \int dw_0\, d^3w$$
$$\times \frac{\operatorname{Tr}\{(M + \not{w_0 - vu_0, w - vu})\gamma_\mu(M + \not{w_0 + (1-v)u_0, w + (1-v)u})\gamma_\nu\}}{(-(w_0^2 - w^2) + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon)^2}$$
$$\times\, h(w + (1-v)u)h(w - vu)$$

as well as

6.12. Lemma

$$\operatorname{Tr}\{(M + w_0 - vu_0, w - vu)\gamma_\mu(M + w_0 + (1-v)u_0, w + (1-v)u)\gamma_\nu)\}$$
$$= 4\{g_{\mu\gamma}(M^2 - (w_0, w)^2 + (2v-1)(w_0, w)(u_0, u) - (v^2 - v)(u_0, u^2)$$
$$+ 2w^\mu w^\nu + 2(v^2 - v)u^\mu u^\nu + (1-2v)$$
$$\times (w^\mu u^\nu + w^\nu u^\mu)\} \quad \textit{where} \quad u^\mu := g_{\mu\nu}u_\nu \textit{ etc.}$$

6.13. Corollary

$$\Pi_{\mu\nu}(u_0, u) = 4\int_0^1 dv \int dw_0\, d^3w h(w - vu)h(w + (1-v)u)$$
$$\frac{g_{\mu\nu}\{M^2 - (w_0, w)^2 + (2v-1)(w_0, w)(u_0, u) - (v^2 - v)(u_0, u)^2\} + 2w^\mu w^\nu + 2(v^2 - v)u^\mu u^\nu + (1-2v)(w^\mu u^\nu + w^\nu u^\mu)}{(-(w_0, w)^2 + (v^2 - v)(u_0, u)^2 + M^2 - i\varepsilon)^2}$$

6.14. *Remark*. The following integrals, occurring in $\Pi_{\mu\nu}$, fulfill

a) $$\int_{-\infty}^{+\infty} dw_0 \frac{1}{(w^2 - w_0^2 + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon)^2}$$
$$= \frac{\pi i}{2} \frac{1}{(\sqrt{w^2 + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon})^3}$$

b) $$\int_{-\infty}^{+\infty} dw_0 \frac{w_0}{(w^2 - w_0^2 + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon)^2} = 0$$

c) $$\int_{-\infty}^{+\infty} dw_0 \frac{w_0^2}{(w^2 - w_0^2 + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon)^2}$$
$$= -\frac{\pi i}{2} \frac{1}{\sqrt{w^2 + (v^2 - v)(u_0^2 - u^2) + M^2 - i\varepsilon}}$$

where $\sqrt{w^2+(v^2-v)(u_0^2-u^2)+M^2-i\varepsilon}$ is the root with negative imaginary part.

Proof. By applying the residues calculus.

6.15. Theorem. *For any two one-electron states $\phi, \Lambda \in \mathbb{L}\otimes\mathbb{D}$ vacuum polarization yields:*

$$
\begin{aligned}
(\phi, \sigma\Lambda) &:= \lim_{t\to\infty} \text{0-st}\,(\phi, \sigma(-t,t)\Lambda) \\
&= \frac{e^3}{(2\pi)^{9/2}}\,\text{0-st}\left\{\iint d^3p\,d^3q\,\delta^1(\Omega_p-\Omega_q)(\phi, \bar\theta^+(0,-p)\gamma_0\theta^-(0,q)\right. \\
&\quad \left.\times\frac{h(p-q)}{(\Omega_p-\Omega_q, p-q)^2+m^2-i\varepsilon}\cdot\Pi_{00}(\Omega_p-\Omega_q, p-q)C_0(p-q)\Lambda)\right\}
\end{aligned}
$$

Proof. In view of 6.10 it suffices to show that $\delta^1(\Omega_p-\Omega_q)\Pi_{n0}(\Omega_p-\Omega_q, p-q)=0$ for $n\neq 0$ which follows from Corollary 6.13 since each summand in the enumerator which does not have $(\Omega_p-\Omega_q)$ as a factor contributes 0 according to 6.14b). qed.

6.16. *Remark.* The expression $\Pi_{00}(\Omega_p-\Omega_q, p-q)$ in Theorem 6.15 can be replaced by $\Pi_{00}(0, p-q)$, because of the occurrence of $\delta^1(\Omega_p-\Omega_q)$.

6.17. Lemma

$$\Pi_{00}(0,u) \,\widetilde{\scriptstyle 0}\, 4\int_0^1 dv\int_{+\infty}^{-\infty} dw_0\int_{G(v,u)} d^3w\,\frac{2(v^2-v)u^2-(2v-1)wu}{(w^2-w_0^2+(v-v^2)u^2+M^2-i\varepsilon)^2}$$

where $G(v,u)\subseteq\mathbb{R}^3$ is the support of the function $w\mapsto h(w-vu)h(w+(1-v)u)$.

Proof

$$
\begin{aligned}
\Pi_{00}(0,u) = 4\int_0^1 dv\int dw_0\int_{G(v,u)} d^3w \\
\times\frac{M^2+(v^2-v)u^2+w_0^2+w_1^2+w_2^2+w_3^2-(2v-1)wu}{(w^2-w_0^2+(v-v^2)u^2+M^2-i\varepsilon)^2}
\end{aligned}
$$

(cf. Corollary 6.13 for $u_0=0$). It suffices to know that

$$\int dw_0\,\frac{w^2+w_0^2+(v-v^2)u^2+M^2}{(w^2-w_0^2+(v-v^2)u^2+M^2-i\varepsilon)^2}\,\widetilde{\scriptstyle 0}\,0$$

which follows from 6.14a,c. qed.

6.18. Theorem. *For $|u|<M<P/2$ we have the approximation*

$$
\begin{aligned}
4\int_0^1 dv\int_{G(v,u)} d^3w\int_{-\infty}^{+\infty} dw_0 \\
\times\frac{2(v^2-v)u^2}{((w^2-w_0^2)+(v-v^2)u^2+M^2-i\varepsilon)^2} \simeq -8\pi^2 iu^2\left\{\frac{1}{3}\left(\ln\frac{2P}{M}-1\right)-\frac{u^2}{30M^2}\right\}
\end{aligned}
$$

with an error smaller than

$$|u|^2\, 8\pi^2\left(\frac{u^4}{6M^4}+\frac{M}{P}+\frac{2}{3}\frac{M^2}{P^2}\right)$$

Proof. By a long chain of elementary approximations (for details cf. [23]).

6.19. Theorem. *For $|u|<M<P$ we have the approximation*

$$4\int_0^1 dv\int_{G(v,u)}^{d^3w}\int dw_0\,\frac{-(2v-1)wu}{(w^2-w_0^2+(v-v^2)u^2+M^2-i\varepsilon)^2}\simeq-\frac{4\pi^2 i}{9}|u|^2$$

with an error smaller than

$$|u|^2\frac{M}{P}\left(4\pi^2+\frac{\pi^2}{3}\frac{M}{P}+4(1+2\pi^2)\frac{M^2}{P^3}\right)$$

Proof. By a long chain of elementary approximations involving the explicit shape of $G(v, u)$. For details cf. [23].

6.20. Corollary. *For $|u|<M<P/2$ we have the approximation*

$$\Pi_{00}(0, u)\simeq-8\pi^2 iu^2\left\{\frac{1}{3}\left(\ln\frac{2P}{M}-\frac{5}{6}\right)-\frac{u^2}{30M^2}\right\}$$

with an error smaller than

$$\frac{4}{3}\pi^2u^2\left(\frac{u^4}{M^4}+\frac{M}{P}(9+Q(M, P))\right)$$

where $Q(M, P)$ is a linear combination of M/P, M/P^2, with coefficients <10.

Proof. Apply 6.19 and 6.20 to 6.17.

6.21. Theorem. *For any two one-electron states $\phi, \Lambda\in\mathbb{L}\otimes\mathbb{D}$ with support in $\mathbb{K}_{r/2}$, $r<M<P/2$, vacuum polarization yields*

$$(\phi, \sigma\Lambda)\simeq-\frac{ie}{\sqrt{2\pi}}\,\text{0-st}\left\{\int d^3p\,d^3q\,\delta^1(\Omega_p-\Omega_q)(\phi, \bar{\theta}^+(0, -p)\gamma_0\theta^-(0, q)\right.$$
$$\times\left[\frac{\alpha}{3\pi}\left(\ln\frac{P^2}{M^2}+\ln 4-\frac{5}{3}\right)-\frac{\alpha(p-q)^2}{15\pi M^2}\right]$$
$$\left.\times\ C_0(p-q)\Lambda)\right\}\qquad\left(\text{where } \alpha:=\frac{e^2}{4\pi}\right)$$

with an error smaller than

$$\frac{1}{12\pi^2}\left(\frac{r^4}{M^4}+\frac{M}{P}(9+Q(M, P))\right)|(\phi, s\Lambda)|$$

Proof. By a straightforward application of 6.20, 6.15 and 6.6. From 6.21 we get

6.22. Theorem. *For any two one-electron states $\phi, \Lambda \in \mathbb{L} \otimes \mathbb{D}$ with support in $\mathbb{K}_{r/2}$, $r < M < P/2$ Coulomb scattering together with vacuum polarization yield*

$$(\phi, (s+\sigma)\Lambda)$$
$$\simeq \frac{iZe^2}{(2\pi)^2} \text{0-st} \left\{ \iint d^3p\, d^3q\, \delta^1(\Omega_p - \Omega_q) \left(\phi, \bar{\theta}^+(0, -p)\gamma_0 \theta^-(0, q) \right.\right.$$
$$\left.\left. \times \left[1 - \frac{\alpha}{3\pi}\left(\ln \frac{P^2}{M^2} + \ln 4 - \frac{5}{3} \right) + \frac{\alpha(p-q)^2}{15\pi M^2} \right] C_0(p-q)\Lambda \right) \right\}$$

with an error smaller than

$$\frac{1}{12\pi^2}\left(\frac{r^4}{M^4} + \frac{M}{P}(9 + Q(M, P)) \right) |(\phi, s\Lambda)|$$

6.23. *Charge renormalization and Uehling effect*

The approximate result 6.22 for $(\phi, (s+\sigma)\Lambda)$ can also be obtained by computing $(\phi, s\Lambda)$ alone (cf. 6.6) using a 'corrected' form $D_0(q)$ instead of our Coulomb field $C_0(q)$, namely

$$D_0(q) := \left(1 - \frac{\alpha}{3\pi}\left(\ln \frac{P^2}{M^2} + \ln 4 - \frac{5}{3} \right) + \frac{\alpha \cdot q^2}{15\pi M^2} \right) \frac{h(q)}{q^2 + m^2} \frac{Ze}{(2\pi)^{3/2}}$$

As long as

$$C := \frac{\alpha}{3\pi}\left(\ln \frac{P^2}{M^2} + \ln 4 - \frac{5}{3} \right) > 0$$

is small (cf. 6.24) one can interpret $\sqrt{1-C}$ as a renormalization factor for the charge e. The term $\alpha \cdot q^2/15\pi M^2$ gives rise to the socalled Uehling effect, a displacement of s-levels in hydrogenlike atoms (cf. [20], pg. 327).

6.24. *Charge renormalization and UV cutoff*

The following choices for the *UV* cutoff *P* yield

a) $P = M \cdot 10^{30}$ $\quad C = \frac{\alpha}{3\pi}\left(\ln \frac{P^2}{M^2} + \ln 4 - \frac{5}{3} \right) \simeq 0.105 \quad \sqrt{1-C} \simeq 0.95$

b) $P = M \cdot 10^{20}$ $\quad C \simeq 0.070 \quad \sqrt{1-C} \simeq 0.96$

c) $P = M \cdot 10^{10}$ $\quad C \simeq 0.035 \quad \sqrt{1-C} \simeq 0.98$

REFERENCES

[1] N. N. Bogolubov, A. A. Logunov and I. T. Todorov, *Introduction to axiomatic quantum field theory*, Benjamin, 1975.
[2] A. S. Wightman, *Constructive field theory*. In: *Fundamental interactions in physics and astrophysics*, New York, 1973.
[3] A. Robinson, *Nonstandard analysis*, North-Holland Publ. Co., 1966.
[4] Ph. Blanchard and J. Tarski, *Renormalizable interactions in two dimensional sharp-time fields*, Acta Physica Austriaca Vol. *49*, 1978.

[5] M. O. FARRUKH, *Application of nonstandard analysis to quantum mechanics*, J. Math. Phys. Vol. *16*, No. 2, 1975.
[6] J. E. FENSTAD, *Nonstandard methods in stochastic analysis and mathematical physics*, Jber. d. Dt. Math. Verein. *82* (1980).
[7] C. E. FRANCIS, *Applications of nonstandard analysis to relativistic quantum mechanics I*, J. Phys. A: Math. Gen. *14* (1981).
[8] R. KAMBE, *On quantum field theory I*, Progr. of Theor. Phys., Vol. *52*, No. 2 (1974).
[9] P. J. KELEMAN, *Applications of nonstandard analysis to quantum mechanics and quantum field theory*, in: *Victoria symposium on nonstandard Analysis* 1972. Lecture Notes in Math. Vol. *369*, Springer Verlag 1974.
[10] E. NELSON, *Internal set theory*, Bull. Am. Math. Soc., Vol. *83*, No. 6, 1977.
[11] A. ROBINSON and P. J. KELEMAN, *The nonstandard* $\lambda : \phi_2^4(x)$: *model I, II*, J. Math. Phys. Vol. *13*, No. 12, 1972.
[12] J. TARSKI, *Short introduction to nonstandard analysis and its physical applications*. In: *Many degrees in freedom in field theory*, Plenum Press, 1978.
[13] J. K. THURBER and J. KATZ, *Applications of fractional powers of delta functions*. In: *Victoria symposium on nonstandard analysis* 1972. Lecture Notes in Math. Vol. *369*, Springer Verlag 1974.
[14] K. D. STROYAN and W. A. J. LUXEMBURG, *Introduction to the theory of infinitesimals*, Academic Press 1976.
[15] J. L. KRIVINE, *Introduction to axiomatic set theory*, D. Reidel Publ. Co., 1971.
[16] M. REED and B. SIMON, *Methods of modern mathematical physics II*, (Fourier analysis, selfadjointness), Academic Press, 1975.
[17] K. YOSIDA, *Functional Analysis*, Grundlehrndlehren der Mathematischen Wissenschaften Vol. *123*, Springer Verlag 1971 (3rd edition).
[18] Ch. N. BURGER, *Nonstandard complex analysis*. In: *Alternative mathematics*, Malz. Press, Hopfen, 1974.
[19] A. I. AKHIEZER and V. B. BERESTETSKII, *Elements of quantum electrodynamics*, Oldbourne Press, London, 1962.
[20] C. ITZYKSON and J. B. ZUBER, *Quantum field theory*, McGraw-Hill, 1980.
[21] A. S. WIGHTMAN, *Introduction to some aspects of the relativistic dynamics of quantized fields*. In: *Cargèse Summer School of Theoretical Physics*, 1964.
[22] S. ALBEVERIO, J. E. FENSTAD and R. HØEGH-KROHN, *Singular Perturbations and Nonstandard Analysis*, Trans. A.M.S., Vol. *252* (1979).
[23] R. FITTLER, *Some Nonstandard Quantum Electrodynamics*, Preprint Nr. 142, Fachbereich Mathematik, Freie Universität Berlin, 1982.