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Scalar fields in Kaluza–Klein theories[1])

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Abstract. We consider Kaluza–Klein theories in $(4+d)$-dimensions with internal space given by the manifold of an arbitrary group G. The G-invariant metric allows a set of scalar fields in it. With an additional Yang–Mills Lagrangian the Einstein and Yang-Mills equations are satisfied in $(4+d)$-dimensions by the physical ground state with compactified internal space. We study the scalar fields around the ground state. We find that the G-invariant metric contains no massless scalar fields, all their masses being real and of the order of the Planck mass.

A major problem of realistic Kaluza–Klein theories [1] is to obtain massless particles of right quantum numbers. Most of the results obtained so far in this direction concern spin 1/2 particles [2–4]. Search for massless scalar particles in such framework is also important, because it may lead, among other things, to an understanding of symmetry breaking mechanisms.

For this purpose it is desirable to study the mass spectrum of the scalars in as general a situation as possible. Here we obtain the mass spectrum of the scalar particles contained in any G-invariant metric [5, 6], the internal space being the group G. The Lagrangian is given by adding a Yang–Mills term with G as the gauge group to the $(4+d)$-dimensional gravity Lagrangian. Our result eliminates all these scalar fields as possible candidates for zero mass particles: they all have real masses but of the order of the Planck mass.

Introduce a set of coordinates y^m, $m=1,\ldots,d$ $(=\dim G)$, on the group manifold B of a compact nonabelian Lie group G. Then we define [7] a set of left (right) invariant vector fields $K_a = K_a^m\,\partial_m$ $(\tilde{K}_a = \tilde{K}_a^m\,\partial_m)$ on B which form bases for the Lie algebra of group G,

$$[K_a(y), K_b(y)] = C_{ab}^c K_c(y), \tag{1}$$

$$[\tilde{K}_a(y), \tilde{K}_b(y)] = -C_{ab}^c \tilde{K}_c(y), \tag{2}$$

C_{ab}^c being the structure constants[2]) of the group G. Here, and in the following, quantities like $K_a^m(y)$ with suffices $m, n, \ldots$ and $a, b, \ldots$ of the Latin alphabet transform as vectors (tensors) at the point y and at the origin respectively.

With the group manifold as the internal space, the isometry group [8] of the vacuum metric (eq. (7) below) is the product group $G_L \times G_R$ of G_L and G_R generated by the left and right invariant vector fields K_a and ${}^6K_a$ $(a=1,2,\ldots,d)$

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[2]) We take the Cartan–Killing metric to be diagonal. The upper and lower position of the indices on the structure constants is merely for convenience.

respectively, i.e. they are the Killing vectors of the vacuum metric. The $(4+d)$-dimensional G-invariant metric is obtained by requiring that the transformations generated by only one of these (we take it to be G_L in the following) be its isometries. Then the full metric has K_a's as its Killing vectors. Its components may be written as [5, 6],

$$\begin{aligned} g_{\mu\nu}(x) &= G_{\mu\nu} + A^a_\mu(x)\phi_{ab}(x)A^b_\nu(x), \\ g_{\mu m}(x, y) &= A^a_\mu(x)\phi_{ab}(x)\tilde{K}_m^{-1b}(y), \\ g_{mn}(x, y) &= \phi_{ab}(x)\tilde{K}_m^{-1a}(y)\tilde{K}_n^{-1b}(y). \end{aligned} \tag{3}$$

The Greek indices indicate physical space-time dimensions. $G_{\mu\nu}(x)$ is the physical 4-dimensional metric. $\tilde{K}^m_a(y)$ is considered as a matrix, the elements of its inverse being denoted as $\tilde{K}_m^{-1a}(y)$. Clearly all the fields in this metric are singlets of G_L. The gauge fields $A^a_{\mu(x)}$ belong to the adjoint representation, as usual, of G_R, while the scalar fields $\phi_{ab}(x)$ transform as the product of adjoint representation of G_R with itself.

The scalar curvature [3]) resulting from the above metric is

$$\begin{aligned} R = R^{(4)} &+ \tfrac{1}{4}\phi_{ab}F^{\mu\nu a}F^a_{\mu\nu} \\ &+ \tfrac{1}{2}(\phi^{-1})^{ab}\{\tfrac{1}{2}\phi_{cd}(\phi^{-1})^{ef}C^c_{ea}C^d_{fb} + C^c_{da}C^d_{cb}\} \\ &+ \nabla_\mu\{(\phi^{-1})^{ab}D^\mu\phi_{ab}\} + \tfrac{1}{4}(\phi^{-1})^{ab}(\phi^{-1})^{cd}\{D_\mu\phi_{ab}D^\mu\phi_{cd} + D_\mu\phi_{ac}D^\mu\phi_{bd}\}, \end{aligned} \tag{4}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc}A^b_\mu A^c_\nu, \tag{5}$$

$$D_\mu\phi_{ab} = \partial_\mu\phi_{ab} - A^d_\mu(C_{dac}\phi_{cb} + c_{dbc}\phi_{ac}). \tag{6}$$

Again $(\phi^{-1})^{ab}$ denotes the matrix elements of inverse of the matrix ϕ_{ab}. $R^{(4)}$ is the 4-dimensional curvature. ∇_μ is the covariant derivative with the usual affine connection.

Now the ground state of a Kaluza–Klein theory is assumed to be given by the product $M^{(4)} \times G$ of 4-dimensional Minkowski space $M^{(4)}$ and a compact internal space of group manifold G, corresponding to the metric

$$\mathring{g}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \mathring{g}_{mn} \end{pmatrix}, \qquad M, N = 1, 2, \ldots, (4+d) \tag{7}$$

with

$$\mathring{g}_{mn}(y) = \frac{1}{M^2}\tilde{K}_m^{-1a}(y)\tilde{K}_n^{-1b}(y)\,\delta_{ab}. \tag{8}$$

As the internal coordinates are taken dimensionless, the constant M has the dimension of mass, giving the scale of the compactified internal dimensions.

The physical spectrum of the theory is determined by studying the quadratic terms around the ground state. Accordingly, we set for the scalar field matrix

$$\phi = \frac{1}{M^2}(1 + \beta\varphi), \tag{9}$$

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[3]) Our sign conventions for R and the flat metric η are the same as of Weinberg [9], naturally extended to $(4+d)$-dimensions.

where the constant β fixes the scale of the scalar fields. Since φ is given the dimension of mass, β has the dimension of inverse mass.

The curvature scalar (4) up to quadratic terms is φ and excluding its interactions with gauge fields is then

$$R = R^{(4)} + \frac{1}{4M^2} F^a_{\mu\nu} F^{a\mu\nu} - \frac{M^2}{4} dC_2^{(A)} + \frac{M^2}{4} \beta C_2^{(A)} \varphi_{aa} + \frac{\beta^2}{4} (\partial_u \varphi_{aa}\, \partial^\mu \varphi_{bb} + \partial_\mu \varphi_{ab}\, \partial^\mu \varphi_{ab} - M^2 C_{abc} C_{ade} \varphi_{bd} \varphi_{ce}), \tag{10}$$

where $C_2^{(A)}$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation. For later use, we note for any representation (R),

$$T_a^{(R)} T_a^{(R)} = -C_2^{(R)} \cdot 1 \tag{11}$$

where $T_a^{(R)}$ are the generators of the group in representation (R).

The compactified vacuum metric (7) does not actually satisfy the Einstein equations obtained by varying the $(4+d)$-dimensional gravity Lagrangian. One must incorporate a compactifying mechanism in the Lagrangian, before one can read off properties of fields appearing in it. Various mechanisms have been proposed [10, 11]. As mentioned earlier we shall consider here adding a Yang–Mills Lagrangian in $(4+d)$-dimensions to that for gravity, proposed originally in ref. [12] and studied extensively by others [13, 14, 3].

The full $(4+d)$-dimensional Lagrangian density is then

$$\bar{\mathscr{L}} = -\sqrt{\bar{g}}\left(\frac{1}{16\pi\bar{G}} R + \frac{1}{4\bar{e}^2} G^a_{MN} G^{aMN} + V_0\right) \tag{12}$$

Here $\bar{g} = -\det g_{MN} \cdot \bar{G}$, $\bar{e}$ and V_0 are the $(4+d)$-dimensional gravitational and Yang–Mills coupling constants and the cosmological constant respectively. The gauge fields B^a_M in Yang–Mills Lagrangian belong to the same gauge group G as the gauge fields A^a_μ.

That the Lagrangian (12) leads to the desired compactification has been shown by Luciani [14]. Assume that the ground state is given by the metric (7) and the non-vanishing internal components of the elementary gauge fields,

$$B^a_\mu = 0, \qquad B^a_m(y) = \lambda K_m^{-1a}(y) \tag{13}$$

where λ is a constant. For this ground state, the Einstein and Yang–Mills equations are satisfied if

$$\lambda = \tfrac{1}{2}, \qquad M^2 = \frac{\bar{e}^2}{2\pi\bar{G}}, \qquad V_0 = \frac{\bar{e}^2}{(8\pi\bar{G})^2} \frac{dC_2^{(A)}}{4} \tag{14}$$

We do not consider here the scalar fields arising from the internal components of the elementary gauge fields. Retaining up to quadratic terms in the scalar fields contained in the metric, we get

$$G^a_{mn} G^{amn} = \frac{M^4}{16} (dC_2^{(A)} - 2C_2^{(A)} \beta \varphi_{aa} + 2C_2^{(A)} \beta^2 \varphi_{aa}^2 + \beta^2 C_{abc} C_{ade} \varphi_{bd} \varphi_{ce}). \tag{15}$$

Inserting (10) and (15) in (12) we get the total 4-dimensional Lagrangian density up to quadratic terms in scalar fields,

$$\mathscr{L}(x)=\int d^d y \bar{\mathscr{L}}(x,y)$$
$$=-\frac{1}{16\pi\bar{G}}\int d^d y\sqrt{\mathring{g}}\left\{R^{(4)}+\frac{1}{4M^2}F^{a\mu\nu}F^a_{\mu\nu}-\frac{\beta^2}{4}\mathscr{L}\varphi\right\} \tag{16}$$

where $\mathring{g}=(M^d \det\tilde{K})^{-2}$, and

$$\mathscr{L}_\varphi=-[\partial_\mu\varphi_{aa}\,\partial^\mu\varphi_{bb}+\partial_u\varphi_{ab}\,\partial^u\varphi_{ab}+M^2(C_2^{(A)}\varphi_{aa}^2-\tfrac{1}{2}C_{abc}C_{ade}\varphi_{bd}\varphi_{ce})] \tag{17}$$

Now define the 4-dimensional gravitational constant G, gauge coupling e and the scale factor β by

$$\frac{1}{\bar{G}}\int d^d y\sqrt{\mathring{g}}=\frac{1}{G}, \tag{18}$$

$$\frac{1}{8\bar{e}^2}\int d^d y\sqrt{\mathring{g}}=\frac{1}{e^2}, \tag{19}$$

and

$$\frac{\beta^2}{64\pi\bar{G}}\int d^d y\sqrt{\mathring{g}}=1, \tag{20}$$

to get

$$M^2=\frac{e^2}{16\pi G},\qquad \beta=\frac{2e}{M}. \tag{21}$$

The Lagrangian (16) becomes

$$\mathscr{L}(x)=-\frac{1}{16\pi G}R^{(4)}-\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu}+\mathscr{L}_\varphi \tag{22}$$

It is now simple to diagonalise $\mathscr{L}_\varphi$. Consider first the last term in $\mathscr{L}_\varphi$ (17),

$$\begin{aligned}&C_{abc}C_{ade}\varphi_{bd}\varphi_{ce}\\&\quad=\varphi_{bd}(T^{(A)}\otimes T^{(A)})_{bd,ce}\varphi_{ce}\\&\quad=\tfrac{1}{2}\varphi_{bd}(T_a^{(P)}T_a^{(P)})_{bd,ce}\varphi_{ce}-\varphi_{bc}(T_a^{(A)}T_a^{(A)})_{cd}\varphi_{db},\end{aligned} \tag{23}$$

where $T_a^{(A)}$ and $T_a^{(P)}$ are the generators in the adjoint and in the product of two adjoint representations, $T_a^{(P)}=T_a^{(A)}\otimes 1+1\otimes T_a^{(A)}$. Decompose the scalar fields as

$$\varphi_{ab}=\sum_{(R),k}\varphi^{(R)k}C_{ab}^{(R)k}, \tag{24}$$

where the sum over (R) runs through all the symmetric irreducible representations contained in the product of the adjoint representation with itself and k labels the fields in each of these representations. $C_{ab}^{(R)k}$ are the Clebsch–Gordon coefficients. Then (23) reduces to

$$-\sum_{(R)k}(\tfrac{1}{2}C_2^{(R)}-C_2^{(A)})(\varphi^{(R)k})^2. \tag{25}$$

So $\mathscr{L}_\varphi$ may be written in the diagonalised basis as

$$\mathscr{L}_\varphi = -\left[\partial_u \varphi^{(s)}\,\partial^\mu \varphi^{(s)} + \sum_{(R),k}\left\{\partial_\mu \varphi^{(R)k}\,\partial^\mu \varphi^{(R)k} + \frac{M^2}{2}\left(\tfrac{1}{2}C_2^{(R)} + C_2^{(A)}\right)(\varphi^{(R)k})^2\right\}\right] \quad (26)$$

where the superscript (S) denotes singlet. The masses of all the scalar particles are seen to be real and of the order of $M(\sim eM_{\text{Planck}})$.

The above result holds for all G-invariant metrics on $M^{(4)}\times G$, given by the parametrization (3). More general metrics constructed without imposing G-invariance may contain additional scalar fields. Thus our result does not preclude the existence of massless scalars in such theories. However, by restricting our consideration to those scalars allowed by G-invariance of the metric, we could easily obtain their mass spectrum of *all* Kaluza–Klein theories with internal space taken as the group manifold.

A similar analysis can be done on any coset space G/H, H being a subgroup of G. However, there arises constraint on the scalar field matrix, which is difficult to solve in a general way. Also the gauge group reduces to $N(H)/H$, where $N(H)$ is the normaliser of H in G [5, 8]. A simple deviation of this metric as well as the mass spectrum of the scalars contained in such a metric will be reported elsewhere.

A recent work [15] obtaining the mass spectrum of scalars for theories with internal space as the coset space $SO(d+1)/SO(d)$ also finds no massless scalars except for $d=4$.

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