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Anomalies[1])

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Abstract. The lecture reviews the origin and the significance of chiral and gravitational anomalies. The phenomenon is analyzed in terms of the short distance singularities generated by free fermion fields, using two-dimensional space-time as a guide.

One of the basic features of the standard model is the fact that it contains gauge fields which only interact with left-handed fermions. The boson mediating the charged weak interaction, e.g., couples to the fermions through a current of the type $V - A$. It is of crucial importance for the consistency of the model that the gauge fields couple to conserved currents. The short distance singularities of the fermion fields in general however spoil the conservation of the axial currents: the conservation laws are afflicted with anomalies [1], which ruin gauge invariance. The standard perturbative analysis of gauge field theories with left-handed couplings is consistent only if these anomalies happen to cancel [2].

The first part of this review deals with free fermions. The Ward identities for the free currents contain anomalous contributions which originate in the short distance singularities of the Fermi fields. The implications of the free field analysis for gauge field theory are addressed in the second part of the talk. The third part concerns gravitational anomalies. Global anomalies [3, 4] and topological aspects [4–7] are not covered.

1. Green's functions of the free vector current

As announced above, I first consider a free Dirac field, for simplicity taken to be massless

$$-i\gamma^{\mu}\,\partial_{\mu}\psi(x) = 0 \qquad (1)$$

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The corresponding vector and axial currents

$$V^\mu = :\bar{\psi}\gamma^\mu\psi:; \qquad A^\mu = :\bar{\psi}\gamma^\mu\gamma_5\psi: \tag{2}$$

and the left-handed current

$$L^\mu = :\bar{\psi}\gamma^\mu\tfrac{1}{2}(1-\gamma_5)\psi: = \tfrac{1}{2}(V^\mu - A^\mu) \tag{3}$$

are conserved

$$\partial_\mu V^\mu = \partial_\mu A^\mu = \partial_\mu L^\mu = 0. \tag{4}$$

Let us first look at the Green's functions of the vector current. The two-point-function, e.g., is given by

$$\begin{aligned} G^{\mu\nu}(x-y) &= \langle 0|\, TV^\mu(x)V^\nu(y)\,|0\rangle \\ &= \operatorname{tr}\{\gamma^\mu S_0(x-y)\gamma^\nu S_0(y-x)\} \end{aligned} \tag{5}$$

where $S_0(x-y)$ is the free propagator

$$S_0(z) = -\frac{z_\mu\gamma^\mu}{2\pi^2(z^2-i\epsilon)^2} \tag{6}$$

The distribution $G^{\mu\nu}(z)$ is well defined only on test functions which vanish at $z=0$, together with their first and second derivatives. On these, it obeys the Ward identity

$$\partial_\mu G^{\mu\nu}(z) = 0 \tag{7}$$

The extension of $G^{\mu\nu}(z)$ to arbitrary test functions is not unique. There are extensions which obey (7) on all test functions and are Lorentz invariant. In fact, these two requirements determine $G^{\mu\nu}(z)$ up to one free parameter: two Lorentz invariant extensions which obey (7) differ by

$$\bar{G}^{\mu\nu}(z) = G^{\mu\nu}(z) + ic(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\delta(z) \tag{8}$$

The three-point-function $\langle 0|\, TV^\lambda V^\mu V^\nu\,|0\rangle$ vanishes on account of charge conjugation invariance. The four-point-function $\langle 0|\, TV^{\mu_1}(x_1)\cdots V^{\mu_4}(x_4)\,|0\rangle_{\text{conn}}$ is unique up to a local term proportional to $\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)$ (logarithmic divergence in the corresponding box graph). In fact, current conservation fixes the extension completely. Finally, Green's functions with more than 4 currents are unambiguous and are conserved. To summarize:

(i) The Green's functions of the free vector current can be chosen such that

$$\partial_{\mu_1}\langle 0|\, TV^{\mu_1}V^{\mu_2}\cdots V^{\mu_n}\,|0\rangle = 0 \tag{9}$$

(ii) There is a renormalization ambiguity only in the two-point-function.

2. Generating functional

It is convenient to collect all Green's functions of the vector current in the generating functional $Z_V(f)$, defined by

$$e^{iZ_V(f)} = \langle 0| \, Te^{i\int dx f_\mu(x) V^\mu(x)} \, |0\rangle \tag{10}$$

The individual connected Green's functions are obtained from $Z_V(f)$ by expanding in powers of the external field $f_\mu(x)$:

$$\begin{aligned} Z_V(f) = &\frac{i}{2!} \int dx_1 \, dx_2 f_{\mu_1}(x_1) f_{\mu_2}(x_2) \langle 0| \, TV^{\mu_1}(x_1) V^{\mu_2}(x_2) \, |0\rangle \\ &+ \frac{i^3}{4!} \int dx_1 \cdots dx_4 f_{\mu_1}(x_1) \cdots f_{\mu_4}(x_4) \langle 0| \, TV^{\mu_1}(x_1) \cdots V^{\mu_4}(x_4) \, |0\rangle_{\text{conn}} + \cdots \end{aligned}$$

The Ward identities (9) amount to the statement that the generating functional is invariant under the gauge transformation

$$\begin{aligned} f'_\mu(x) &= f_\mu(x) + \partial_\mu \alpha(x) \\ Z_V(f + \partial\alpha) &= Z_V(f) \end{aligned} \tag{11}$$

The fact that only the two-point-function contains a renormalization ambiguity, given by (8), is equivalent to the statement that the generating functional is unique up to a local polynomial in the external field:

$$\bar{Z}_V(f) = Z_V(f) + \frac{c}{4} \int dx (\partial_\mu f_\nu - \partial_\nu f_\mu)^2 \tag{12}$$

3. Green's functions of the free left-handed current

Consider now the generating functional associated with the Green's functions of the left-handed current

$$e^{iZ_L(f)} = \langle 0| \, Te^{i\int dx f_\mu(x) L^\mu(x)} \, |0\rangle \tag{13}$$

Again, the short distance singularities of the free propagator generate ambiguities in the Green's functions containing up to four currents. It is possible to renormalize the two- and the four-point-functions in such a manner that current conservation holds. In the case of the three-point-function this turns out to be impossible: independently of how one extends the distribution $\langle 0| \, TL^\lambda(x) L^\mu(y) L^\nu(z) \, |0\rangle$ to the space of all test functions, at least one of the three currents fails to be conserved at $x = y = z$. If the three-point-function is renormalized in such a manner that it is symmetric with respect to the interchange of any two of the three currents, one finds

$$\partial_\lambda \langle 0| \, TL^\lambda(x) L^\mu(y) L^\nu(z) \, |0\rangle = -\frac{1}{12\pi^2} \epsilon^{\mu\nu\alpha\beta} \, \partial^y_\alpha \, \partial^z_\beta \{\delta(x-y)\delta(x-z)\}$$

Accordingly, the generating functional is not gauge invariant:

$$Z_L(f + \partial\alpha) = Z_L(f) + \frac{1}{24\pi^2}\int dx\,\alpha(x)\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu f_\nu(x)\,\partial_\rho f_\sigma(x) \tag{14}$$

To see how the phenomenon arises, let us consider the left-handed current in two-dimensional space-time. With $\gamma_5 = \gamma_0\gamma_1$ the vector and axial currents are related by $A^0 = -V^1$, $A^1 = -V^0$. The component $L^- = L^0 - L^1$ of the left-handed current therefore vanishes identically, $L^-(x) = 0$. Furthermore, current conservation implies that $L^+ = L^0 + L^1$ is a free field, depending only in $x^0 - x^1$:

$$\partial_+ L^+(x) = 0 \tag{15}$$

The free propagator is

$$S_0(z) = \frac{z^\mu\gamma_\mu}{2\pi(z^2 - i\epsilon)} \tag{16}$$

and the two-point-function therefore becomes

$$\begin{aligned}\langle 0|\,TL^+(x)L^+(y)\,|0\rangle &= -\mathrm{tr}\,(\gamma^+ \not{z}\gamma^+\not{z})/4\pi^2(z^2 - i\epsilon)^2\\ &= \frac{\mathrm{i}}{\pi}\,\partial_-\,\partial_-\Delta(z)\end{aligned} \tag{17}$$

where $z = x - y$ and where $\Delta(z)$ is the scalar propagator

$$\begin{aligned}\Delta(z) &= \frac{1}{4\pi i}\ln(-z^2 + i\epsilon)\\ \Box\Delta(z) &= \delta(z)\end{aligned} \tag{18}$$

In two dimensions, the Ward identity for the two-point-function therefore contains an anomaly:

$$\partial_+\langle 0|\,TL^+(x)L^+(y)\,|0\rangle = \frac{i}{\pi}\,\partial_-\delta(z) \tag{19}$$

In $d = 2$ all connected Green's functions containing more than two currents vanish ($L^+(x)$ is a free field). The generating functional can therefore be given in closed form [8]:

$$Z_L(f) = \frac{1}{8\pi}\int dx\,dy\,\partial_- f_+(x)\Delta(x - y)\,\partial_- f_+(y) \tag{20}$$

Obviously, this expression is not gauge invariant.

If the dimension d of space-time is odd, there is no analogue of γ_5 (the product $\gamma_0\gamma_1\cdots\gamma_{d-1}$ is a multiple of the unit matrix); there are no left-handed or axial currents and there are no anomalies.

If d is even, there is an anomaly in the Ward identities for Green's functions with $d/2 + 1$ left-handed currents (vacuum polarization diagram in $d = 2$, triangle graph in $d = 4$, hexagon diagram in $d = 10$ etc.)

4. Fermions in an external field

The generating functional admits an alternative interpretation: it represents the vacuum-to-vacuum transition amplitude in the presence of an external field. To see this, consider the Lagrangian

$$\mathscr{L} = i\chi^+\tilde{\sigma}^\mu(\partial_\mu - if_\mu)\chi \tag{21}$$

where $\chi(x)$ is a two-component spinor describing left-handed fermions and where $f_\mu(x)$ is an external field. The 2×2 Weyl matrices σ^μ, $\tilde{\sigma}^\mu$ represent the right- and left-handed components of the Dirac matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \tilde{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}; \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

If the external field vanishes for $x^0 \to \pm\infty$, the field $\chi(x)$ develops from a free incoming field $\chi_{\text{in}}(x)$ to a free outgoing field $\chi_{\text{out}}(x)$. Denote the ground states of these free fields by $|0\ \text{in}\rangle$ and $|0\ \text{out}\rangle$ respectively. The probability for the external field not to create fermion pairs is given by $|\langle 0\ \text{out} \mid 0\ \text{in}\rangle|^2$, where

$$\begin{aligned}\langle 0\ \text{out} \mid 0\ \text{in}\rangle &= \langle 0\ \text{in}|\ Te^{i\int dx \mathscr{L}_{\text{int}}}\ |0\ \text{in}\rangle \\ &= \langle 0\ \text{in}|\ Te^{i\int dx f_\mu \chi^+_{\text{in}} \tilde{\sigma}^\mu \chi_{\text{in}}}\ |0\ \text{in}\rangle \end{aligned} \tag{23}$$

The operator $\chi^+_{\text{in}}\tilde{\sigma}^\mu\chi_{\text{in}}$ is the left-handed current L^μ associated with the incoming free field. The vacuum-to-vacuum transition amplitude is therefore given by

$$\langle 0\ \text{out} \mid 0\ \text{in}\rangle = e^{iZ_L(f)} \tag{24}$$

where $Z_L(f)$ is the generating functional of the free left-handed current discussed above. The presence of an anomaly in the free left-handed current therefore implies that the vacuum-to-vacuum transition amplitude is not gauge invariant:

$$\langle 0\ \text{out} \mid 0\ \text{in}\rangle_{f+\partial\alpha} = \langle 0\ \text{out} \mid 0\ \text{in}\rangle_f e^{i/24\pi^2 \int dx \alpha \epsilon^{\mu\nu\rho\sigma} \partial_\mu f_\nu \partial_\rho f_\sigma} \tag{25}$$

The anomaly only affects the phase of the transition amplitude; the transition probability is gauge invariant.

5. Determinant

The vacuum-to-vacuum transition amplitude can be represented as a fermionic path integral

$$\langle 0\ \text{out} \mid 0\ \text{in}\rangle = \int [d\chi] e^{-i\int dx \chi^+ D_L \chi}$$

where D_L denotes the Weyl operator

$$D_L = -i\tilde{\sigma}^\mu(\partial_\mu - if_\mu) \tag{26}$$

Since the exponential is quadratic in the variables of integration, the integral can be done:

$$\langle 0 \text{ out} \mid 0 \text{ in} \rangle = \det D_L \tag{27}$$

To check that the determinant of the Weyl operator indeed coincides with $\exp iZ_L(f)$, we expand it in powers of the external field:

$$D_L = D_L^0 - f; \qquad f = \tilde{\sigma}^\mu f_\mu$$
$$\ln \det D_L = \ln \det D_L^0 - \operatorname{Tr}(fS_0) - \tfrac{1}{2}\operatorname{Tr}(fS_0fS_0) - \cdots$$

The symbol $S_0 = (D_L^0)^{-1}$ denotes the free propagator. If we normalize the determinant of the free Weyl operator to $\det D_L^0 = 1$, the series expansion for $\ln \det D_L$ is identical with the series expansion for $iZ_L(f)$. In particular, the renormalization ambiguities in $\ln \det D_L$ are the same as in the generating functional: the logarithm of the Weyl determinant is well defined up to a local polynomial in the external field.

To give unambiguous meaning to the determinant of the Weyl operator, we use Schwinger's method [9]: instead of specifying the determinant directly, we specify the change in $\det D_L$ produced by a change in the external field. Formally, the change is given by

$$\begin{aligned}\delta \ln \det D_L &= \operatorname{Tr}(\delta D_L D_L^{-1})\\ &= -\int dx\, \delta f_\mu(x) \operatorname{tr}\{\tilde{\sigma}^\mu S_f(x, x)\}\end{aligned} \tag{28}$$

where $S_f(x, y)$ is the propagator in an external field

$$-i\tilde{\sigma}^\mu(\partial_\mu - if_\mu(x))S_f(x, y) = \delta(x - y). \tag{29}$$

Since the propagator is singular at $x = y$, the quantity $S_f(x, x)$ does not make sense as it stands. In $d = 2$, e.g., the short distance behaviour of $S_f(x, y)$ is

$$S_f(x, y) = \frac{z^\mu \sigma_\mu}{2\pi(z^2 - i\epsilon)}\{1 + iz^\nu f_\nu(x)\} + \hat{S}_f(x, y) \tag{30}$$

where the remainder, $\hat{S}_f(x, y)$, approaches a well defined distribution $\hat{S}_f(x, x)$ as $x \to y$. The essential point here is that the terms which explode as $x \to y$ are local polynomials in the external field

$$S_f(x, y) = P_f(x, y) + \hat{S}_f(x, y) \tag{31}$$

In $d = 2$, the polynomial $P_f(x, y)$ is linear in $f_\mu(x)$, in $d > 2$ the singular part involves derivatives of f_μ as well as higher powers of f_μ. To give meaning to (28), we remove the singular terms and specify the change of the renormalized determinant as

$$\delta \ln \det D_L = -\int dx \delta f_\mu(x) \operatorname{tr}(\tilde{\sigma}^\mu \hat{S}_f(x, x)\} \tag{32}$$

If one considers arbitrary deformations of the external field, there is an integrability problem with this formula. We avoid this problem by considering only a special class of deformations which interpolate the external field $f_\mu(x)$ of interest with $f_\mu = 0$:

$$f^t_\mu(x) = tf_\mu(x); \qquad \delta f_\mu(x) = \delta t f_\mu(x) \tag{33}$$

Integrating the changes produced in ln det D_L as the parameter t is increased from 0 to 1, we get

$$\ln \det D_L = -\int_0^1 dt \int dx f_\mu(x) \operatorname{tr} \{\tilde{\sigma}^\mu \hat{S}_{f^t}(x, x)\} \tag{34}$$

This formula provides us with an explicit expression for the generating functional in terms of the external field propagator $S_f(x, y)$. To analyze the properties of the generating functional it therefore suffices to analyze the properties of the differential equation which determines the propagator.

As emphasized above, ln det D_L is well defined only up to a local polynomial. In the representation (34) of the determinant, this ambiguity shows up in the fact that the finite part $\hat{S}_f(x, x)$ of the propagator is not unique, because only the singular part of the polynomial $P_f(x,y)$ is unambiguous. In $d = 2$, e.g., the polynomial given in (30) can be replaced by

$$\bar{P}_f(x, y) = P_f(x, y) + c\sigma^\mu f_\mu(x) \tag{35}$$

This modification changes ln det D_L by a local polynomial:

$$\ln \overline{\det D_L} = \ln \det D_L + c \int dx f_\mu(x) f^\mu(x) \tag{36}$$

In fact, the explicit expression (20) for the generating functional in two dimensions differs from the determinant specified in (30), (34) by such a local polynomial:

$$iZ_L(f) = \ln \det D_L - \frac{i}{8\pi} \int dx f_\mu f^\mu \tag{37}$$

In two dimensions, the Weyl operator D_L only contains the component $f_+(x)$ of the external field – left handed fermions do not interact with $f_-(x)$. This property is borne out in the generating functional $Z_L(f)$, but it is not borne out in the above renormalization of the Weyl determinant which does depend on $f_-(x)$ through the local polynomial $f_\mu f^\mu = f_+ f_-$. It is clear where this comes from: the choice (30) of the local polynomial $P_f(x, y)$ involves a finite piece proportional to $f_-(x)$. The renormalization prescription (34) therefore contains a finite 'counter term' which depends on f_-. This is artificial – the singular part of $S_f(x, y)$ only contains f_+, because the Weyl operator only contains f_+.

The origin of the anomaly can easily be seen in the above representation for the generating functional. Under a gauge transformation, the Weyl operator transforms according to

$$\begin{aligned} f'_\mu &= f_\mu + \partial_\mu \alpha \\ D'_L &= e^{i\alpha} D_L e^{-i\alpha} \end{aligned} \tag{38}$$

The propagator $S_f(x, y)$ transforms in the same fashion; the polynomial $P_f(x, y)$ can be chosen such that the finite part $\hat{S}_f(x, x)$ is gauge invariant. We therefore obtain

$$\ln \det D'_L = \ln \det D_L - \int_0^1 dt \int dx\, \partial_\mu \alpha \operatorname{tr} \{\tilde{\sigma}^\mu \hat{S}_{f'}(x, x)\} \tag{39}$$

To evaluate the right-hand side we integrate by parts. The differential equation satisfied by the butchered propagator,

$$D_L \hat{S}_f(x, y) = -D_L P_f(x, y); \qquad x \neq y \tag{40}$$

implies

$$i\partial_\mu \operatorname{tr} \{\tilde{\sigma}^\mu \hat{S}_f(x, x)\} = \operatorname{tr} \{D^x_L P_f(x, y) + P_f(x, y) \overleftarrow{D}^y_L\}_{x=y} \tag{41}$$

In the two-dimensional case, the polynomial $P_f(x, y)$ was explicitly given above. Evaluating the derivatives and taking the trace over the Weyl matrices one finds ($\epsilon_{01} = 1$):

$$\partial_\mu \operatorname{tr} \{\tilde{\sigma}^\mu \hat{S}_f(x, x)\} = -\frac{i}{2\pi} \epsilon^{\mu\nu} \partial_\mu f_\nu(x) \tag{42}$$

Inserting this result in (39), we obtain

$$\ln \det D'_L = \ln \det D - \frac{i}{4\pi} \int dx\, \alpha \epsilon^{\mu\nu} \partial_\mu f_\nu \tag{43}$$

The above calculation shows why the renormalized determinant is not gauge invariant. To specify the determinant, we had to remove a certain piece from the external field propagator. As a result of this operation, the remainder fails to satisfy the equation of motion – this in turn implies that the renormalized current $\operatorname{tr} \{\tilde{\sigma}^\mu \hat{S}_f(x, x)\}$ is not conserved.

The anomaly of $Z_L(f)$ contains additional terms, because the local polynomial $\int dx f_\mu f^\mu$ appearing in (37) is not gauge invariant:

$$Z_L(f + \partial\alpha) = Z_L(f) - \frac{1}{4\pi} \int dx\, \alpha \{\epsilon^{\mu\nu} \partial_\mu f_\nu - \partial_\mu f^\mu - \tfrac{1}{2} \Box \alpha\} \tag{44}$$

It is a straightforward matter to extend this analysis to an arbitrary number of space-time dimensions. The calculation of the anomaly boils down to an analysis of the short distance singularities of the fermion propagator in an external field.

6. Right-handed current

The generating functional of the right-handed current is given by the determinant of the right-handed Weyl operator

$$e^{iZ_R(f)} = \det D_R$$
$$D_R = -i\sigma^\mu(\partial_\mu - if_\mu) \tag{45}$$

The operator D_R involves the matrices σ^μ instead of the matrices $\tilde{\sigma}^\mu$ which occur in D_L [if d is even, the two sets of matrices constitute inequivalent representations of the Weyl algebra.]

The generating functional of the vector current is given by the determinant of the Dirac operator:

$$e^{iZ_V(f)} = \det \not{D}$$
$$\not{D} = -i\gamma^\mu(\partial_\mu - if_\mu) = \begin{pmatrix} 0 & D_L \\ D_R & 0 \end{pmatrix} \tag{46}$$

It is possible to renormalize the determinants in such a manner that the product rule

$$\det \not{D} = \det D_R \cdot \det D_L \tag{47}$$

holds, i.e.

$$Z_V(f) = Z_R(f) + Z_L(f) \tag{48}$$

The difference $Z_R(f) - Z_L(f)$ is the generating functional of the axial current.

As discussed above, the generating functional of the vector current can be chosen to be gauge invariant. Under a gauge transformation, the determinant of D_R must therefore pick up a phase opposite to the phase of $\det D_L$; the anomalies in Z_R and in Z_L are of opposite sign.

As emphasized by Jackiw [5], the convention (48) is not a natural one in $d = 2$. The natural renormalization of Z_L only involves f_+ while Z_R only contains f_-. If one uses (48) to define Z_V, one obtains a generating functional for the vector current which fails to be gauge invariant [the contributions to the anomalies of Z_R, Z_L proportional to the ϵ-tensor cancel, but the contributions proportional to $g_{\mu\nu}$ do not]. The disease is easily cured by adding a local term proportional to $\int dx f_\mu f^\mu$; the relation (48) then however fails to hold. The essential point here is that the generating functionals are well defined only up to local polynomials. Some of their properties like the form (43) of the anomaly or the product rule (48) do not hold for every choice of these local polynomials.

7. Fermions with internal quantum numbers

It is well known that one can bring all fermion fields to left-handed form. Instead of a single four-component Dirac spinor ψ_e describing right- and left-handed electrons and positrons, e.g., one can use two-component Weyl spinors χ_{e^+}, χ_{e^-} in terms of which the Dirac field takes the form

$$\psi_e = \begin{pmatrix} \epsilon\chi^*_{e^+} \\ \chi_{e^-} \end{pmatrix} \tag{49}$$

We collect all fermions in one left-handed field $\chi_a(x)$ where the internal quantum number $a = 1, \ldots, N$ labels the various different flavours and colours. To collect all Green's functions of the corresponding free left-handed currents

$$L^\mu_{ab} = :\chi^+_a \tilde{\sigma}^\mu \chi_b: \tag{50}$$

we need an $N \times N$ matrix $f_\mu(x)_{ab}$ of external fields:

$$e^{iZ_L(f)} = \langle 0|\, Te^{i\int dx\, \mathrm{tr}\,(f_\mu L^\mu)}\, |0\rangle \tag{51}$$

The corresponding Weyl operator

$$D_L = -i\tilde{\sigma}^\mu(\partial_\mu - if_\mu) \tag{52}$$

is a matrix in spin space as well as in the space of internal quantum numbers. Again, the generating functional is given by the determinant of this operator

$$e^{iZ_L(f)} = \det D_L \tag{53}$$

To discuss the Ward identities associated with the conservation of L^μ_{ab}, we consider the gauge transformation

$$\begin{aligned} f'_\mu &= Uf_\mu U^+ + iU\,\partial_\mu U^+ \\ D'_L &= UD_L U^+ \end{aligned} \tag{54}$$

The transformation properties of the determinant under this transformation were first calculated by Bardeen [10]. For an infinitesimal transformation

$$\begin{aligned} U &= 1 + i\alpha + \cdots \\ \delta f_\mu &= \nabla_\mu \alpha = \partial_\mu \alpha - i[f_\mu, \alpha] \end{aligned} \tag{55}$$

the result is

$$\delta \ln \det D_L = \frac{i}{24\pi^2} \int dx \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left[\alpha\left\{\partial_\mu f_\nu\, \partial_\rho f_\sigma - \frac{i}{2}\,\partial_\mu(f_\nu f_\rho f_\sigma)\right\}\right] \tag{56}$$

where the trace extends over the $N \times N$ matrices α, f_μ. This result shows that, in four dimensions, there are anomalies in the Ward identities only in the case of the three- and of the four-point-functions. For $N = 1$, the second term in (56) vanishes and the expression reduces to (14) as it should. [There are no anomalies in the four-point-function if there is only a single fermion flavour.] Note that the expression in the curly bracket is not gauge covariant. This is necessarily so: as was shown by Wess and Zumino [11], the algebraic structure of the anomalies is determined by the integrability conditions which are not consistent with a covariant expression for the anomalies.

The structure of the chiral anomalies in higher dimensions is described in [6, 7, 12]. The short distance approach to the problem was pioneered by Fujikawa [13]. An analysis of the d-dimensional Weyl determinant along the lines sketched in Section 5 is given in [14].

8. Fermion mass terms

In the preceding discussion we have exclusively dealt with free massless fermions. In the presence of mass terms, the free currents in general fail to be conserved. For the left-handed current connecting u and d quarks, e.g., we have

$$\partial_\mu\{\bar{u}\gamma^\mu\tfrac{1}{2}(1-\gamma_5)d\} = \frac{i}{2}\bar{u}\{(m_u - m_d) + \gamma_5(m_u + m_d)\}d \tag{57}$$

Even if there were no anomalies, the Green's functions of this current therefore fail to be conserved. To keep trace of the mass terms in the generating functional, one introduces external scalar and pseudoscalar fields and investigates the transformation properties of the generating functional in the presence of these additional fields [15]. One finds that the leading short distance singularities of the fermion propagator, which are responsible for the occurrence of anomalies, are not affected by scalar or pseudoscalar fields. Although mass terms do generate a right-hand side in the Ward identities for the various currents, they do not modify the anomalies, which can be analyzed on the basis of the corresponding massless theory. In the following, I will therefore drop all mass terms and disregard the Higgs field which generates them.

9. From external fields to interacting gauge fields

In the standard model, the fermions interact with a set of gauge fields $G^i_\mu(x)$ belonging to the gauge group $SU(3) \times SU(2) \times U(1)$. Let us denote the generators of the gauge group on the left-handed fermion fields by T_i, such that the gauge fields interact with the fermions through the matrix

$$G_\mu(x) = \sum_i G^i_\mu(x) T_i \tag{58}$$

The Lagrangian of the standard model is of the form

$$\mathscr{L} = \mathscr{L}_G + i\chi^+\tilde{\sigma}^\mu(\partial_\mu - iG_\mu)\chi + \mathscr{L}_H \tag{59}$$

The quantity $\mathscr{L}_G$ stands for the gauge field Lagrangian $\sim \mathrm{tr}\, G_{\mu\nu}G^{\mu\nu}/g^2$. The term $\mathscr{L}_H$ represents the part of the Lagrangian which involves the Higgs field. For the reason given in the last section, I disregard this part and consider the path integral

$$\int [dG][d\chi] e^{i\int dx \mathscr{L}_G - i\int dx \chi^+ D_L \chi}$$

$$D_L = -i\tilde{\sigma}^\mu(\partial_\mu - iG_\mu)$$

Integrating the fermions out, this becomes

$$\int [dG] e^{i\int dx \mathscr{L}_G} \det D_L \tag{61}$$

For the consistency of perturbation theory (graphs involving gauge fields as internal lines) it is crucial that gauge invariance can be maintained order bei order. This is the case only if $\det D_L$ is invariant under the transformations generated by the gauge group G, i.e., if there are no anomalies in the Ward identities for the currents $\chi^+ \tilde{\sigma}^\mu T_i \chi$.

A simple model for which there is an anomaly that ruins gauge invariance is Quantum Electrodynamics with only left-handed electrons (and right-handed positrons):

$$\mathscr{L} = -\frac{1}{4e^2} G_{\mu\nu} G^{\mu\nu} + i\chi^+ \tilde{\sigma}^\mu(\partial_\mu - iG_\mu)\chi \tag{62}$$

For ordinary QED there is no problem: the anomalies generated by the left-handed electron field cancel the anomalies produced by the left-handed positron field. Gauge fields coupled to fermions through vector currents are always anomaly free.

It is not difficult to find out under what conditions cancellation of anomalies takes place. A glance at (56) shows that infinitesimal gauge transformations of the form $\alpha = \sum_i \alpha^i T_i$ are anomaly free if and only if the fermion representation satisfies the condition

$$\operatorname{tr}(T_i\{T_k, T_l\}) = 0 \tag{63}$$

In QED there is a single generator, $T_i = Q$, where Q is the electric charge matrix and the condition (63) amounts to $\operatorname{tr} Q^3 = 0$. For left-handed QED, Q is a 1×1 matrix with $\operatorname{tr} Q^3 \neq 0$ and the condition is violated, whereas for the standard electromagnetic interaction of the electron, there are two left-handed fields with opposite charge such that $\operatorname{tr} Q^3 = 0$.

In the standard model based on the gauge group $SU(3) \times SU(2) \times U(1)$ the condition (63) would be violated if there were only quarks, no leptons (or vice versa). The condition is obeyed if the fermions occur in copies of the first generation (ν_e, e, u, d), provided the electric charge of the quarks is related to the electric charge of the electron in the familiar manner. In order for the standard model to be a consistent gauge theory, it is thus necessary that the hydrogen atom is electrically neutral, $Q_e + 2Q_u + Q_d = 0$.

10. Anomalies in external currents

Suppose that those currents to which the gauge fields are coupled are anomaly free. What about the remaining currents which do not occur in the Lagrangian of the theory? To analyze the properties of these 'external' currents we again introduce a set of external fields and consider the generating functional

$$e^{iZ_L(f)} = \int [dG] e^{i\int \mathcal{L}_G \, dx} \det D_L \qquad D_L = -i\tilde{\sigma}^\mu(\partial_\mu - iG_\mu - if_\mu) \tag{64}$$

In order for the gauge invariance associated with the gauge group G to remain intact, we have to restrict ourselves to gauge invariant external currents, i.e.

$$[T_i, f_\mu] = 0 \tag{65}$$

The Ward identities obeyed by the external currents are easily worked out from the general formula (56) with the replacement $f_\mu \to G_\mu + f_\mu$. They involve terms proportional to the winding number density $\epsilon^{\mu\beta\rho\sigma} G_{\mu\nu} G_{\rho\sigma}$ of the gauge fields.

Why are the Ward identities for external currents of more than academic interest? The point is that conserved currents correspond to symmetries of the Hamiltonian. If the conservation of an external current is found to be ruined by an anomaly, the corresponding symmetry is lost. For the standard model, anomalies in external axial currents are vital for our understanding of the decay $\pi^0 \to 2\gamma$ and of the mass spectrum of the low lying pseudoscalar mesons.

11. Gravitational anomalies

Finally, I turn to fermions moving in an external gravitational field. In a curved space, the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x) \tag{66}$$

leads to coordinate dependent γ-matrices which can be represented in the form

$$\gamma^\mu(x) = e^\mu_a(x)\gamma^a \tag{67}$$

where γ^a is a representation of the Dirac algebra in flat space and where the vielbein e^μ_a obeys

$$g^{\mu\nu}(x) = e^\mu_a(x) e^\nu_b(x) \mathring{g}^{ab} \tag{68}$$

The vielbein is determined by the geometry $g_{\mu\nu}(x)$ only up to a local Lorentz transformation ('frame rotation')

$$e^\mu_a(x)' = \Lambda^{-1}(x)^b_a e^\mu_b(x) \tag{69}$$

The corresponding change in the γ-matrices is given by

$$\gamma^\mu(x)' = S[\Lambda(x)]\gamma^\mu(x) S[\Lambda(x)]^{-1} \tag{70}$$

where $S[\Lambda]$ is the spinor representation associated with the Lorentz transformation Λ.

For massless fermions in an external gravitational field, the Dirac equation reads

$$-i\gamma^{\mu}(x)\{\partial_{\mu}+\omega_{\mu}(x)\}\psi(x)=0 \tag{71}$$

The spin connection ω_{μ} insures invariance under local frame rotations:

$$\omega'_{\mu}=S\omega_{\mu}S^{-1}+S\,\partial_{\mu}S^{-1} \tag{72}$$

The connection can be expressed in terms of the vielbein and its first derivative as

$$\omega_{\mu}=\tfrac{1}{4}\gamma^{\nu}\{\partial_{\mu}\gamma_{\nu}-\Gamma^{\lambda}_{\mu\nu}\gamma_{\lambda}\} \tag{73}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is the Christoffel symbol associated with the metric $g_{\mu\nu}$.

Suppose that the space is asymptotically flat, $e^{\mu}_{a}(x)\to\delta^{\mu}_{a}$ as $x^{0}\to\pm\infty$. What is the probability for the gravitational field not to create any fermion pairs? The probability amplitude is given by

$$\begin{aligned}&\langle 0\text{ out}\mid 0\text{ in}\rangle=\det \not{D}\\&\not{D}=-i\gamma^{\mu}(\partial_{\mu}+\omega_{\mu})\end{aligned} \tag{74}$$

Assume that the geometry $g_{\mu\nu}(x)$ is given. To calculate the transition amplitude, the geometry alone is however not sufficient: the Dirac operator explicitly contains the vielbein field e^{μ}_{a}. We therefore need to know whether any vielbein which describes the same geometry also leads to the same transition amplitude. Vielbeins belonging to the same geometry differ at most by a frame rotation. Under a frame rotation, the Dirac operator transforms according to

$$\not{D}'=S\not{D}S^{-1} \tag{75}$$

The problem therefore boils down to the question of whether or not the determinant of $\not{D}$ is invariant under the transformation (75). For the Dirac operator, this is indeed the case: the transition amplitude only depends on the geometry. For left-handed fermions, however, there is a problem [4, 7, 16]: the determinant of the Weyl operator is not invariant under frame rotations, if the dimension d of space-time is of the form $d=4n+2=2, 6, 10, \ldots$.

To understand the origin of this 'Lorentz anomaly', it is instructive to again consider a space-time of two dimensions [17, 18]. In $d=2$, the σ-matrices are numbers. In flat space, we have $\tilde{\sigma}^{0}=\tilde{\sigma}^{1}=1$ and the free Weyl operator reduces to $D_{L}=-i(\partial_{0}+\partial_{1})$. In curved space it is of the form

$$D_{L}=-i\tilde{\sigma}^{\mu}(x)\{\partial_{\mu}+\omega_{\mu}(x)\} \tag{76}$$

The vector $\tilde{\sigma}^{\mu}(x)$ is a linear combination of the zweibein vectors e^{μ}_{0} and e^{μ}_{1}:

$$\tilde{\sigma}^{\mu}(x)=e^{\mu}_{0}(x)+e^{\mu}_{1}(x);\qquad \sigma^{\mu}(x)=e^{\mu}_{0}(x)-e^{\mu}_{1}(x) \tag{77}$$

The spin connection is given by a derivative of the zweibein vectors

$$\omega_{\mu}=\tfrac{1}{4}g_{\mu\nu}(\tilde{\sigma}^{\alpha}\,\partial_{\alpha}\sigma^{\nu}-\sigma^{\alpha}\,\partial_{\alpha}\tilde{\sigma}^{\nu}) \tag{78}$$

Under a frame rotation

$$\tilde{\sigma}^{\mu\prime} = e^{2\lambda}\tilde{\sigma}^{\mu}; \qquad \omega'_{\mu} = \omega_{\mu} + \partial_{\mu}\lambda \tag{79}$$

The spin connection thus plays a role analogous to an imaginary electromagnetic field; a frame rotation changes the spin connection by a gauge transformation.

In two dimensions, one can always bring the metric to conformally flat form, $g_{\mu\nu}(x) = F(x)\mathring{g}_{\mu\nu}$. This property allows one to explicitly calculate the determinant of the Weyl operator [19]. In the presence of both an external gravitational field and an external electromagnetic field $G_{\mu}(x)$,

$$D_L = -i\tilde{\sigma}^{\mu}\{\partial_{\mu} + \omega_{\mu} - iG_{\mu}\} \tag{80}$$

the vacuum-to-vacuum transition amplitude is given by [18]

$$\det D_L = \exp\left\{\frac{i}{3} Z_g(\omega) + iZ_g(G)\right\} \tag{81}$$

where the functional $Z_g(f)$ is quadratic in the vector field f_{μ}:

$$\begin{aligned} Z_g(f) &= \frac{1}{8\pi}\int dx\, dy \hat{f}(x)\Delta_g(x, y)\hat{f}(y) \\ \hat{f}(x) &= \partial_{\mu}\{(\epsilon^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu})f_{\nu}(x)\} \end{aligned} \tag{82}$$

The kernel $\Delta_g(x, y)$ denotes the scalar Feynman propagator in an external gravitational field

$$\partial_{\mu}\sqrt{-g}\, g^{\mu\nu}\, \partial_{\nu}\Delta_g(x, y) = \delta(x - y) \tag{83}$$

Note that $\ln \det D_L$ is unique only up to a local polynomial in the external fields. The expression given in [18] differs from (82) by a local term proportional to $\int dx\sqrt{-g}\, f_{\mu}f^{\mu}$ (compare Section 5).

In flat space, the spin connection vanishes and the term involving $Z_g(\omega)$ is therefore absent. Furthermore, we have

$$\partial_{\mu}\{(\epsilon^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu})f_{\nu}\} = -\partial_{-}f_{+} \tag{84}$$

such that $Z_g(G)$ reduces to the generating functional (20) associated with the left-handed current in flat space, as it should.

The determinant of the Weyl operator is neither invariant under gauge transformations of the electromagnetic field, nor under local Lorentz transformations, because the functional $Z_g(f)$ is not invariant under $f_{\mu} \rightarrow f_{\mu} + \partial_{\mu}\alpha$. The transition amplitude therefore not only fails to be independent of the gauge chosen for the electromagnetic field, it also depends on the zweibein chosen to represent the geometry. The failure of $\det D_L$ to be invariant under frame rotations is referred to as a Lorentz anomaly. The problem only shows up in the phase of the transition amplitude – the transition probability only depends on the geometry and is invariant under gauge transformations of the electromagnetic field.

12. Energy-momentum tensor

The Lorentz anomaly also shows up in the properties of the energy-momentum tensor, which determines the response of the system to a change in the metric

$$\delta \ln \langle 0 \text{ out} \mid 0 \text{ in} \rangle = -\frac{i}{2} \int dx\, \delta g_{\mu\nu}(x) T^{\mu\nu}(x) \tag{85}$$

Since the transition amplitude is not fixed by the geometry alone, but explicitly depends on the vielbein, this formula does not specify $T^{\mu\nu}(x)$ unambiguously. What is well defined is the response of the system to a deformation of the vielbein

$$\delta \ln \langle 0 \text{ out} \mid 0 \text{ in} \rangle = i \int dx\, \delta e_a^\mu(x) t_\mu^a(x) \tag{86}$$

The deformation of the metric fixes δe_a^μ up to an infinitesimal frame rotation. Since an infinitesimal frame rotation at the point x produces a change in the transition amplitude which only depends on the properties of the vielbein in the vicinity of the point x, the quantity $T^{\mu\nu}$ is well defined up to a symmetric local polynomial $p^{\mu\nu} = p^{\nu\mu}$

$$T^{\mu\nu} = t^{\underline{\mu\nu}} + p^{\mu\nu} \tag{87}$$

where $t^{\underline{\mu\nu}}$ is the symmetric part of the tensor $t^{\mu\nu} = g^{\mu\alpha} e_a^\nu t_\alpha^a$.

If there is an external electromagnetic field, there is no reason for the energy-momentum tensor to be conserved, because energy may flow in and out of the external electromagnetic field. Let us therefore switch the electromagnetic field off and consider pure gravity. If the gravitational field $g_{\mu\nu}(x)$ is the only external field, coordinate invariance implies that $T^{\mu\nu}$ is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0 \tag{88}$$

It turns out, however, that independently of how one chooses the local polynomial $p^{\mu\nu}$, this relation fails to be satisfied. In two dimensions, one instead finds

$$\nabla_\mu T^{\mu\nu} = \frac{1}{96\pi} \epsilon^{\mu\nu} \partial_\nu R \tag{89}$$

where R is the scalar curvature.

At first sight, this relation appears to indicate a breakdown of coordinate invariance. Since the quantity $Z_g(\omega)$ is however explicitly coordinate invariant, this is not the proper conclusion to draw. Instead, the relation (89) expresses the fact that energy flows in and out of the degree of freedom which specifies the direction of the zweibein – the metric is not the only external field the system interacts with.

13. Lorentz anomalies versus coordinate anomalies

As was shown by Bardeen and Zumino [16], one can always modify the determinant of the Weyl operator in such a fashion that it does become frame independent – at the cost of coordinate invariance. In fact, in $d = 2$, it suffices to add the local functional [17]

$$F = \frac{i}{24\pi} \int dx \sqrt{-g}\, \{\tfrac{1}{4} R \ln [k_\mu \tilde{\sigma}^\mu(x)] + g^{\mu\nu} \omega_\mu \omega_\nu\} \tag{90}$$

to $\ln \det D_L$. The quantity

$$\ln \overline{\det D_L} = \ln \det D_L + F \tag{91}$$

is invariant under local Lorentz transformations, but breaks coordinate invariance through the constant vector k_μ. In other words, $\det D_L$ does not have a Lorentz anomaly; instead it has a coordinate anomaly.

The functional F is local, but it is not a polynomial. The short distance singularities of the Weyl operator only generate polynomial ambiguities in $\ln \det D_L$ (local polynomials of the vielbein matrix, of its inverse and of its derivatives.) The counter terms needed are polynomials – the modification given above is therefore outside the class of counter terms which occur in the renormalization procedure. The short distance singularities break frame independence, but they do not break coordinate independence. In this sense, Lorentz anomalies and coordinate anomalies are not equivalent. (These statements also hold in higher dimensions [20]).

Left-handed fermions generate a Lorentz anomaly in $d = 2, 6, 10, \ldots$. The phenomenon does not occur in four dimensions, but it does occur in Kaluza–Klein theories. The Lorentz anomaly is a purely gravitational effect; the manner in which the phase of the vacuum-to-vacuum transition amplitude changes under a frame rotation only depends on the curvature of the space and is not modified if gauge fields are present. On the other hand, the gravitational field does modify the structure of the chiral anomalies discussed in the first part of this lecture (there are no mixed Lorentz anomalies, but there are mixed chiral anomalies). Chiral anomalies occur whenever the dimension of space-time is even. In four dimensions, e.g., the Ward identity for the left-handed $U(1)$ current $\chi^+ \tilde{\sigma}^\mu \chi$ receives a purely gravitational contribution proportional to the square of the Riemann tensor [21]. An external gravitational field therefore ruins gauge invariance, unless the generators of the fermion representation are traceless, $\operatorname{tr} T_i = 0$. If this condition is not satisfied, the quantum theory of the gauge field does not survive the perturbation produced by an external gravitational field (in the standard model, the condition holds, generation by generation).

In the case of the gravitational field, a consistent perturbative quantization is not available, even if there are no fermions. It is therefore not possible to extend the preceding discussion from external gravitational fields to interacting gravitational fields. Superstrings may provide a consistent framework within which gravity as we know it emerges as an effective low energy approximation. At low energies, the internal degrees of freedom of the string are frozen and the theory reduces to a local field theory involving a rich spectrum of fields – a gravitational field, as well as gauge fields and fermions. For some of these theories, the structure of the string Lagrangian implies that the anomalies in the corresponding effective low energy theory cancel [22] – as if a consistent theory for quantum gravity was not enough of a miracle already.

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