Helvetica Physica Acta, Vol. 59 (1986) 229–233
0018–0238/86/020229–05$1.50 + 0.20/0

Renormalization properties and chiral anomalies in supersymmetric gauge theories[1])

By J. Wess

University of Karlsruhe, D-7500 Karlsruhe 1

(7. X. 1985)

The most striking features of supersymmetric field theories are their renormalization properties [1]. There are parameters in the theory (mass-parameters and coupling constants) that do not get any radiative correction. This property is almost sufficient to define a supersymmetric theory. If you start from a theory with spin 1/2 fermions and scalar bosons with an arbitrary renormalizable coupling and, if you ask at the one-loop level that mass parameters and coupling constants do not get any radiative corrections, you will end up essentially with a supersymmetric theory. Thus, in a way, supersymmetric theories are a logical extension of gauge theories if you go from the tree level to the one-loop level.

The non-renormalization properties are a result of cancellation between bosonic and fermionic contributions.

It has been known for a long time that in a free-field theory the normal ordering does not change the Lagrangian if there are as many bosonic as fermionic degrees of freedom belonging to the same mass. The c-number contributions from the normal ordering all cancel. This is already one of the non-renormalization properties and, as matter of fact, they all rest on the property of supersymmetric theories that there are as many bosonic as fermionic degrees of freedom degenerate in mass. This is an immediate consequence of the supersymmetry algebra:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta} P_m$$

and it holds for all representations of this algebra.

An important question to ask is: among which Feynman diagrams do the cancellations occur? The answer is readily given in terms of superfields. A superfield is a certain collection of bosonic and fermionic fields. Whenever in a Feynman diagram the internal lines represent a full superfield – all ordinary diagrams with the individual bosons and fermions of a superfield are supposed to be added up – then the cancellation takes place.

---

[1]) Talk given at the Stueckelberg Memorial Lectures, June 1985 at Geneva and Lausanne, Switzerland.

To explain the notion of a superfield, let me start from a free sypersymmetric Lagrangian:

$$\mathscr{L} = A^*\Box A + i\,\partial_m\bar\psi\sigma^m\psi + F^*F + m(AF + A^*F^* - \tfrac{1}{2}\psi\psi - \tfrac{1}{2}\bar\psi\bar\psi)$$

In this Lagrangian there appears the auxiliary field F in addition to the complex scalar field A and the Weyl spinor ψ. Because off-mass shell ψ represents two complex fields, another complex scalar field (F) has to be added to the multiplet to match the bosonic and fermionic degrees of freedom. On-mass shell, A and ψ represent exactly two degrees of freedom each. The following propagators can be derived from this Lagrangian:

$$\langle 0|\, T\{A(x)A^*(x')\}\,|0\rangle = \Delta(x-x')$$
$$\langle 0|\, T\{A(x)F(x')\}\,|0\rangle = -m\Delta(x-x')$$
$$\langle 0|\, T\{F(x),\, F^*(x')\}\,|0\rangle = \Box\Delta(x-x')$$
$$\langle 0|\, T\{\bar\psi_{\dot\alpha(x)}\bar\psi^{\dot\beta}(x')\}\,|0\rangle = -\delta^{\dot\beta}_{\dot\alpha} m\Delta(x-x')$$
$$\langle 0|\, T\{\psi_\alpha(x)\psi^\beta(x')\}\,|0\rangle = \delta^\beta_\alpha m\Delta(x-x')$$
$$\langle 0|\, T\{\psi_\alpha(x)\bar\psi_{\dot\beta}(x')\}\,|0\rangle = -i\sigma^n_{\alpha\dot\beta}\partial_n\Delta(x-x')$$

A very efficient way to summarize these propagators is achieved by introducing anticommuting parameters θ^α and $\bar\theta_{\dot\alpha}$. They are meant to be bookkeepers in combining the bosonic and fermionic fields in one single object, the superfield ϕ:

$$\phi(y,\theta) = A(y) + \sqrt{2}\,\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y)$$
$$y^n = x^n + i\theta\sigma^n\bar\theta$$

The variable y has been introduced in order to have a simple expression for the hermitean conjugate:

$$\phi^+(y^+,\bar\theta) = A^*(y^+) + \sqrt{2}\,\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(y^+) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}F^*(y^+)$$

A function in θ is always meant to be a power series in θ. The two superfields satisfy the constraint equations:

$$\bar D_{\dot\alpha}\phi = 0, \qquad D_\alpha\phi^+ = 0$$

with

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^n_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^n}$$
$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^n_{\alpha\dot\alpha}\frac{\partial}{\partial x^n}$$

which express irreducibility under supersymmetry transformations. A superfield, satisfying these constraints is called chiral. The above free Lagrangian has a simple form in terms of these superfields:

$$\mathscr{L} = \int d^2\theta\, d^2\bar\theta\,\phi^+\phi + \tfrac{1}{2}m\int d^2\theta\,\phi\phi + \tfrac{1}{2}m\int d^2\bar\theta\,\phi^+\phi^+$$

The propagators from above give rise to the superfield propagators:

$$\langle 0|\, T\{\phi(x,\theta,\bar\theta)\phi(x',\theta',\bar\theta')\}\,|0\rangle = \frac{m}{4}\bar D^2\delta(\theta-\theta')\delta(\bar\theta-\bar\theta')\Delta(x-x')$$

$$\langle 0|\, T\{\phi^+(x,\theta,\bar\theta)\phi^+(x',\theta',\bar\theta)\}\,|0\rangle = -\frac{m}{4}D^2\delta(\theta-\theta')\delta(\bar\theta-\bar\theta')\Delta(x-x')$$

$$\langle 0|\, T\{\phi(x,\theta,\bar\theta)\phi^+(x',\theta',\bar\theta')\}\,|0\rangle = \tfrac{1}{16}\bar D^2 D^2\delta(\theta-\theta')\delta(\bar\theta-\bar\theta')\Delta(x-x')$$

The superfield propagators and the propagators of the component fields contain exactly the same information. The θ variables are only useful to combine this information in a very comprehensive form. In the θ-variables, the superfield propagators are proportional to the δ-function and its derivatives. This will enable us to integrate out all the internal θ-variables – a process that corresponds to summing up all the individual bosonic and fermionic contributions to the standard Feynman diagrams.

The renormalizable interactions take a simple form in terms of superfields as well:

$$\mathscr{L}_{\text{int}} = \frac{1}{2}\int d^2\theta g_{ijk}\phi_i\phi_j\phi_k + hc$$

The index i lables the individual superfields in the model.

The form of the interaction shows that we can use superfield contractions directly to derive the Feynman rules. They will, then, be expressed in terms of the superfield propagators. Integrating out the internal θ variables, we find that the result for each Feynman diagram in terms of superfields leads to an expression of the form:

$$\int d^2\theta\, d^2\bar\theta\, d^4x_1,\ldots, d^4x_n F_1(x_1,\theta,\bar\theta)\cdots F_n(x_n,\theta,\bar\theta)G(x_1,\ldots,x_n)$$

The function $G(x_1,\ldots,x_n)$ is translationally invariant and the F's are products of superfields and their derivatives. No factors $\Box^{-1}$ appears in the F's.

For the vacuum-vacuum diagrams, the F's are just constants and because $\int d^2\theta\, d^2\bar\theta = 0$, there are no vacuum-vacuum contributions from the Feynman diagrams.

If the F's are all chiral superfields, then we are left with $\int d^2\bar\theta$, which is zero as well. This leads to the surprising result that mass and coupling terms, which enter the Lagrangian in terms of a product of chiral superfields only, are not renormalized. Furthermore, no higher dimensional momentum-independent chiral operators are introduced in the effective super-potential to any order in perturbation theory. This has a consequence that the minimum of the potential remains unchanged in perturbation theory.

The renormalization properties just discussed extend to super-symmetric gauge theories as well. Then, we have to introduce a superfield that combines the vector potential with its fermions. A proper choice is a real superfield V, i.e., it is subject to the constraint $V^+ = V$.

The most general Lagrangian for supersymmetric, renormalizable gauge theories is of the form:

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\phi^+ e^V \phi + \left\{ \int d^2\theta (\mathrm{Tr}\, W^\alpha W_\alpha + \tfrac{1}{2} m_{ij}\phi_i\phi_j + \tfrac{1}{3} g_{ijk}\phi_i\phi_j\phi_k) + hc \right\}$$

The vector superfield V is Lie algebra valued:

$$V^l T_l = V, \quad T^l\text{: generators}$$

The superfield $W_\alpha = -\frac{1}{4}\bar{D}^2 \bar{e}^V D_\alpha e^V$ enters the Yang–Mills part of the Lagrangian.
From this Lagrangian follows the conservation law:

$$\bar{D}^2 J^l = 0, \qquad J^l = \phi^+ e^V T^l \phi$$

This is Noether's theorem written in terms of superfields.

We know, however, that anomalies will arise in this conservation law. They have to be present because the supersymmetric theory can also be seen as an ordinary gauge theory with additional, gauge-invariant interactions of scalar fields. But we know that no such interactions are capable to remove the anomalies, which derive from the fermion loops.

The chiral anomalies have been studied by O. Piguet and K. Sibold [2]. They have found that there is no anomaly for the supersymmetry current itself. Therefore, it should be possible to write the Adler–Bardeen anomaly in a manifest supersymmetric form, i.e. in terms of superfields. It turned out to be much more difficult to construct this supersymmetric anomaly then expected. The reason is that the anomaly is not of a polynomial type in e^V, but is an infinite power series [3].

There are two ways of constructing the anomaly. One is to find a non-trivial solution to the supersymmetric consistency condition or secondly to compute it in a field theoretical model.

Several authors have tried to derive the supersymmetric anomaly from a field theoretical model [4]. A conceptional simple way is to use Pauli–Villars regularization in terms of regulator superfields and to treat the vector superfield as external. The calculation associates with every n-point one-loop diagram, an expression, where $X = e^V - 1$ enters to the $(n-1)$st power. The anomaly is the sum overall n, and we find that it does not terminate at a finite n. With a suitable definition of the current, we can derive the supersymmetric anomaly in the following form [5]:

$$(\Lambda \circ G - \Lambda^+ \circ G^+) = \mathrm{Tr} \int d^2\theta\, d^2\bar{\theta}\, d^4x \{\Lambda(x, \theta, \bar{\theta}) G(x, \theta, \bar{\theta}) \\ - \Lambda^+(x, \theta, \bar{\theta}) G^+(x, \theta, \bar{\theta})\}$$

The chiral superfield Λ is the supersymmetric extension of the gaugetransformation parameter and G is found to be:

$$G = \frac{1}{(4\pi)^2} \sum_{n=2}^{\infty} \frac{(-)^n}{n(n-1)} G_n$$

$$G_n = -\tfrac{1}{4} \sum_{l,k=1}^{n-1} \sigma^a_{\alpha\dot\alpha} l\, \partial^{(l)}_a \bar D^{\dot\alpha} (X^{k-1} D^\alpha X) X^{n-k-1} + \frac{1}{n+1} \sum_{l,k=1}^{n-1} [l(l-n) - \tfrac{1}{2}|l-k|]\, \partial^{(l)}_a \partial^{a(k)} X^{n-1}$$

Notation: $\partial^{(l)}_a$ means $\partial/\partial x^a$ acting on the term in the lth position of the following expression.

This is a fairly complicated result, and we have not been able to simplify it in an essential way by redefining the current. We have, however, been able to show that this expression satisfies the supersymmetric consistency condition to all order in n:

$$\delta_2 i(\Lambda_1 \circ G - \Lambda_1^+ \circ G^+) - \delta_1 i(\Lambda_2 \circ G' - \Lambda_2^+ \circ \sigma^+) = [\Lambda_1, \Lambda_2] \circ G - [\Lambda_1^+, \Lambda_2^+] \circ G^+$$

The above anomaly is not trivial because it contains the usual Adler–Bardeen anomaly. It is, however, not in a form that vanishes for such models where the usual Adler–Bardeen anomaly vanishes. A supersymmetric normal term, which can be obtained by variation from a local functional in V, has to be subtracted. However, the form of this functional, or, what is the same, a proper definition of the current in the field theoretical model, is not yet known. The fact that it should exist, follows from a result by G. Girardi, R. Grimm and R. Stora [6] who have succeeded in constructing the full supersymmetric anomaly as a solution of the consistency condition using methods of differential topology in superspace. Their anomaly is parametrized by the d-symbols of the gauge group. Thus, it vanishes for a vanishing Adler–Bardeen anomaly. We have also been able to show that our anomaly is equivalent to the anomaly of Girardi, Grimm and Stora up to the 4th power in X, i.e., we have been able to construct a normal term up to the 5th order in X, whose variation gives the difference between the two anomalies.

To conclude, I should like to remark that supersymmetric gauge theories have a much improved structure concerning their renormalization properties, but the structure of the chiral anomaly is exceedingly more complex.

REFERENCES

[1] For notation and details for the first part of the talk see: J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Series in Physics, Princeton University Press (1983).
[2] O. Piguet and K. Sibold, Nucl. Phys. *B247* (1984) 484.
[3] S. Ferrara, L. Girardello, O. Piguet and R. Stora, CERN-TH 4134/85.
[4] N. K. Nielsen, Nucl. Phys. *B244* (1984) 499. A. W. Fisher, Karlsruhe preprint KA-THEP 85-1 April 1985. E. Guadagnini, K. Konishi and M. Mintchev, Pisa preprint IFUP-TH-10/85 February 1985. K. Harada and K. Shizuya, Tohoku University preprint TU/85/280 1985.
[5] R. Garreis, M. Scholl and J. Wess, Karlsruhe preprint KA-THEP-85-4, May 1985.
[6] G. Girardi, R. Grimm and R. Stora, Annecy preprint LAPP-TH-130, March 1985.