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Renormalization group and theory of large fluctuations: some recent results in the study of coexistence of phases

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Abstract. In this paper we review certain developments of the Renormalization Group (RG) which are connected with the so called theory of Large deviations. They constitute a 'second generation' of RG ideas and provide a considerable enrichment of the subject. To illustrate the effectiveness of these ideas we give a simplified discussion of some recent results of Bleher on separation of phases in hierarchical models.

1. Renormalization group and large deviations

There is a feature of the modern versions of the Renormalization Group which deserves to be emphasized: the algorithmic character. This is not a purely mathematical aspect but reflects an important physical idea. It is possible to understand the global behaviour of a complicated system by studying it on different scales and establishing simple relationships among the different levels of description.

The fact that this program has been successful in many instances indicates that some kind of hierarchical structure may approximately characterize many relevant phenomena in complex physical systems. This became evident already in the early days of the development of the R.G. approach to critical phenomena. Certain hierarchical models in statistical mechanics introduced previously by Dyson were explicitly connected with Wilson approximate recursion formula [1]. It was via these models that the mathematical nature of the new R.G. was clarified and shown to have, once properly formulated, a deep probabilistic interpretation. This led to the recognition of R.G. theory as a new, albeit difficult, chapter in limit theorems for dependent variables. In this way the notion of self-similar (stable, automodel, . . .) random field was formulated as the analog of the fixed point Hamiltonian of the R.G. practitioner. The physicist's notion of universality class found its probabilistic counter part in that of domain of attraction [2].

The probabilistic interpretation led to a useful development. Guided by the theory of large deviations for the case of independent variables and its connection with limit theorems, it was found [3] that a quantity widely used by physicists, the effective potential, has a simple and direct probabilistic interpretation. It determines in fact the probability of large fluctuations of macroscopic quantities like, for instance, the magnetization in a large volume. Furthermore it provides the correction terms (preasymptotic behaviour) to limit theorems for interacting variables. This interpretation of the effective potential coupled with the well established probabilistic description of the R.G. leads quite naturally to the idea that recursive procedures typical of the new R.G. should be used for its calculation. In view of our previous remarks it is clear that hierarchiral models constitute an ideal testing ground for such an approach. A simple calculation to illustrate the point was given in [3].

In recent years Bleher has produced a series of very interesting papers [4] which, in my opinion, represent a rather complete implementation of the program sketched above, for the case of hierarchical models. He proves by recursion the existence of the effective potential (he does not use however such a terminology) for the Dyson models and he is able to characterize by the same approach the structure of the typical configurations in the phase coexistence region. In particular an interesting structure of the magnetization on different scales emerges from the calculation.

These articles have become available only recently and are rather technical. We shall give a simplified presentation of the result on typical configurations after a brief review of large fluctuation theory.

It looks to me that the approach we are going to describe is a good example of a second (or third?) generation of R.G. ideas. In fact it is possible to reconstruct a continuous thread (with some bifurcations) between them and the original work of Stueckelberg and Petermann [5]. In dedicating this paper to the memory of Stueckelberg I wonder whether he would recognize his descendants!

2. Review of large fluctuation theory

Suppose we consider a ferromagnetic lattice system of unbounded spins ϕ_i in a cubic region Λ described by a Gibbs measure μ_Λ^β. We assume $E(\phi_i) = 0$. A typical prediction of the theory of large fluctuations tells us that the probability that the mean spin $\phi_\Lambda = \sum_{i\in\Lambda} \phi_i/|\Lambda|$ be larger than a given value $\varphi > 0$ is of the order

$$P(\phi_\Lambda \geqslant \varphi) \approx e^{-|\Lambda|\Gamma(|\Lambda|,\varphi)} \tag{2.1}$$

for $|\Lambda|$, the volume of Λ, sufficiently large, Γ is given by

$$\Gamma(|\Lambda|, \varphi) = \sup_{\theta>0}\left\{\theta\varphi - \frac{1}{|\Lambda|}\ln E(e^{\theta\sum_{i\in\Lambda}\phi_i})\right\} \tag{2.2}$$

and is therefore the Legendre transform of $\dfrac{1}{|\Lambda|}\ln E(e^{\theta\sum_{i\in\Lambda}\phi_i})$. The dependence of

Γ on $|\Lambda|$ includes for example surface effects which can be parametrized by a power of $|\Lambda|$. $E(\ )$ denote expectation with respect to μ_Λ^β; $\approx$ means logarithmic equivalence i.e.

$$\lim_{|\Lambda|\to\infty} \Gamma(|\Lambda|, \varphi) = V(\varphi) = -\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln P(\phi_\Lambda \geqslant \varphi) \tag{2.3}$$

$V(\varphi)$ is usually called by physicists the effective potential. The validity of (2.1) or (2.3) can be shown by proving lower and upper bounds. Just to give an idea of why it is true we derive the upper bound which is the easier one. From the exponential Chebysheff inequality we have for $\theta > 0$, $\varphi > 0$

$$P(\phi_\Lambda \geqslant \varphi) \leqslant e^{-|\Lambda|\theta\varphi} E(e^{\theta \sum_{i\in\Lambda} \phi_i})$$

If we optimize with respect to θ we obtain

$$P(\phi_\Lambda \geqslant \varphi) \leqslant e^{-|\Lambda|\Gamma}$$

with Γ given by (2.2).

It is easy to see that the result remains the same if instead of $P(\phi_\Lambda \geqslant \varphi)$ we consider $P(\varphi \leqslant \phi_\Lambda < \varphi + \varepsilon)$. That is, if P admits a density $p(|\Lambda|, \varphi)$ besides (2.3) we have the relationship

$$V(\varphi) = -\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln p(|\Lambda|, \varphi) \tag{2.4}$$

It is clear that $V(\varphi)$ is a monotonic function of φ, for φ of constant sign. It is also clear that $\Gamma(|\Lambda|, \varphi)$ (and therefore $V(\varphi)$) is convex in φ, being the Legendre transform of a convex function.

In the phase coexistence region therefore $V(\varphi)$ is expected to have the shape indicated in Fig. 1 where M is the spontaneous magnetization at the given temperature. One can easily construct mean field examples of such behaviour.

It is convenient to introduce also the $|\Lambda|$ dependent effective potential $V(|\Lambda|, \varphi)$

$$V(|\Lambda|, \varphi) = -\frac{1}{|\Lambda|} \ln p(|\Lambda|, \varphi) \tag{2.5}$$

$V(|\Lambda|, \varphi)$ will not in general be convex for finite $|\Lambda|$. However the non convex part will be small for large $|\Lambda|$.

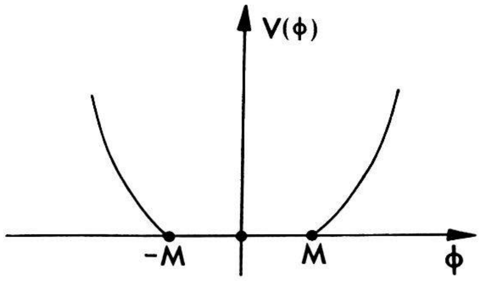

Figure 1

3. Hierarchical models

We recall the basic recursion relation of Dyson type hierarchical models. We have a countable set, $\mathbb{Z}$ for definiteness, and a decreasing sequence of partitions, $P_0 > P_1 \cdots$ satisfying 1) P_0 is the partition of $\mathbb{Z}$ into separate points 2) any element of the partition P_n consists of 2 elements of the partition P_{n-1}. To each point i of $\mathbb{Z}$ we associate a spin variable ϕ_i and we consider the mean spin of an element of the partition P_n, $\phi_n = \sum_1^{2^n} \phi_i/2^n$. If we define

$$p_n(\varphi)\,d\varphi = P(\varphi \leq \phi_n < \varphi + d\varphi) \tag{3.1}$$

the models are defined recursively by

$$p_n(\varphi) = L_n e^{\beta c^n \varphi^2} \int d\varphi' p_{n-1}(\varphi') p_{n-1}(2\varphi - \varphi') \tag{3.2}$$

where $1 < c < 2$ and L_n is fixed by the normalization condition. For each initial distribution $p_0(\varphi)$ of the spins ϕ_i we obtain a different model. Of special interest is the ϕ^4 hierarchical model defined by the distribution

$$p_0(\varphi) = L_0 \exp\left(-\tfrac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4\right) \tag{3.3}$$

We now define the effective potential at level n

$$V_n(\varphi) = -\frac{1}{2^n} \ln p_n(\varphi) \tag{3.4}$$

which by (3.2) satisfies the recursion equation

$$V_n(\varphi) = -\frac{1}{2^n}\ln L_n - \beta\left(\frac{c}{2}\right)^n \varphi^2 - \frac{1}{2^n}\ln \int d\varphi' \exp\left[-2^{n-1}(V_{n-1}(\varphi') + V_{n-1}(2\varphi - \varphi'))\right] \tag{3.5}$$

To understand the difference between (3.5) and the usual R.G. recursion for hierarchical models we consider first the trivial gaussian case obtained by setting $\lambda = 0$ in (3.3). This can be easily solved exactly and gives for β sufficiently small

$$V_n(\varphi) = \tfrac{1}{2}\left[1 - \beta c \sum_{k=0}^{n} \left(\frac{c}{2}\right)^k\right]\varphi^2 + v_n \tag{3.6}$$

Therefore going to the limit $n \to \infty$

$$V(\varphi) = \tfrac{1}{2}\left[1 - \frac{2\beta c}{2 - c}\right]\varphi^2 \tag{3.7}$$

from which we obtain the critical temperature

$$\beta_c = \frac{1}{c} - \frac{1}{2} \tag{3.8}$$

If we had considered the usual recursion for the renormalized density distribution

$$f_n(x) = p_n(c^{-n/2}x) \tag{3.9}$$

the critical point would have been defined as the only β for which f_n converges to the gaussian fixed point $f(x) = A \exp\left(-\frac{\beta c}{2-c}x^2\right)$ and an easy calculation shows that this corresponds to the β_c for which the difference between two divergent expressions

$$2\beta \sum_{k=0}^{n} \left(\frac{2}{c}\right)^k - \left(\frac{2}{c}\right)^{n+1} \tag{3.10}$$

is finite. It is immediately seen that this β_c coincides with (3.8).

The recursion (3.5) converges for all $\beta < \beta_c$ and for each n one can calculate an approximate β_{nc} as the β for which the coefficient of φ^2 vanishes.

4. Typical configurations in the phase coexistence region

The problem we want to solve is the following. At the level n we have two blocks containing each 2^{n-1} spins interacting through an Hamiltonian

$$H_n = c^n\left(\frac{\varphi_{n-1} + \varphi'_{n-1}}{2}\right)^2 = c^n\varphi_n^2 \tag{4.1}$$

where φ_{n-1}, φ'_{n-1}, φ_n are the mean magnetizations of the two blocks of size 2^{n-1} and of the resulting block of size 2^n, respectively. Suppose now that φ_n is assigned a value αM, $0 < \alpha < 1$ where M is the spontaneous magnetization corresponding to the given temperature β and the given coupling λ. We want to understand how φ_n is related to φ_{n-1} and φ'_{n-1}. More precisely we want to calculate the conditional distribution of φ_{n-1} or φ'_{n-1} given φ_n, for large n. The remarkable result is that one of the two quantities φ_{n-1} or φ'_{n-1} with probability very close to 1 is equal to the full magnetization M.

This means that if we follow the magnetization through some levels of the hierarchy, we find a structure (for definiteness suppose $\alpha < \frac{1}{2}$) like in Fig. 2. We now indicate the argument leading to this conclusion. To calculate the

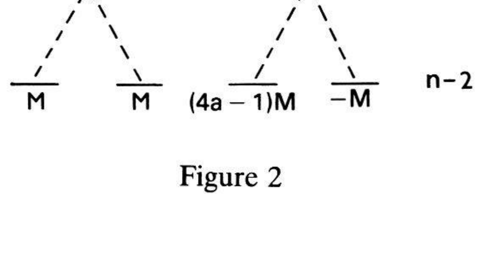

Figure 2

desired distribution we have to estimate $p_n(\varphi)$ or what is the same V_n for $n \to \infty$. From the discussion in Section 2 and 3 we expect (see equation (3.5)).

$$V_n(\varphi) = V(\varphi) + \left(\frac{c}{2}\right)^n Y(\varphi) + \cdots \tag{4.2}$$

where $V(\varphi)$ is the effective potential and $Y(\varphi)$ describes the weak interaction between the two sub blocks: $(c/2)^n$ may be considered as a 'surface' term.

Since in the phase coexistence region $V(\varphi) = \text{const.}$, the whole φ dependence is given by $Y(\varphi)$. In order to isolate this contribution we perform a subtraction and consider $V_n(\varphi) - V_n(\varphi_0)$ with φ_0 also in the flat region of $V(\varphi)$. Then

$$V_n(\varphi) - V_n(\varphi_0) \approx \left(\frac{c}{2}\right)^n (Y(\varphi) - Y(\varphi_0)).$$

Therefore if we succeed in establishing a recursion formula for the above difference we can obtain $Y(\varphi)$ as the main asymptotic term in the phase coexistence region. From equation (3.5) it is clear that $\delta_n(\varphi) = (2/c)^n(V_n(\varphi) - V_n(\varphi_0))$ satisfies an equation of the form

$$\delta_n(\varphi) = -c^{-n} \ln A_n - \beta\varphi^2$$
$$-c^{-n} \ln \int d\varphi' \exp\left[-c^{n-1}(\delta_{n-1}(\varphi + \varphi') + \delta_{n-1}(\varphi - \varphi'))\right] \tag{4.3}$$

A_n is determined by the condition $\delta_n(\varphi_0) = 0$. Let us choose now $\varphi_0 = M$. By the symmetry of the problem $\delta_n(\pm M) = 0$. Then for $0 \leqslant \varphi \leqslant M$ and large n the main contribution to the integral on the right hand side of (4.3) comes from the region $\varphi \pm \varphi' = M$, while for $-M \leqslant \varphi \leqslant 0$ the largest contribution will be associated to the region $\varphi \pm \varphi' = -M$. We can write therefore the approximate recursion equations

$$\begin{aligned} \delta_n(\varphi) &\approx \beta(M^2 - \varphi^2) + c^{-1}\delta_{n-1}(2\varphi - M) \qquad 0 \leqslant \varphi \leqslant M \\ \delta_n(\varphi) &\approx \beta(M^2 - \varphi^2) + c^{-1}\delta_{n-1}(2\varphi + M) \qquad -M \leqslant \varphi \leqslant 0 \end{aligned} \tag{4.4}$$

This type of equations was rigorously studied by Bleher and the solutions for $n \to \infty$ explicitly determined. Already a few iterations of (4.4) permit to visualize its behaviour.

The probability of interest to us is given by

$$P\{|\varphi_{n-1} - M| < \varepsilon M| \; \varphi_n = \alpha M\}$$
$$= \frac{\int_{|\varphi' - M| < \varepsilon M} d\varphi' P_{n-1}(\varphi') P_{n-1}(2\alpha M - \varphi')}{\int_{-\infty}^{\infty} d\varphi' P_{n-1}(\varphi') P_{n-1}(2\alpha M - \varphi')} \tag{4.5}$$

and $P_n(\varphi) \simeq \text{const}_n \, e^{-c^n \delta(\varphi)}$.

An argument similar to the one used before now shows that the integral in the indicated region actually exhausts the normalization condition and the probability (4.5) is ~1 for large n.

REFERENCES

[1] G. A. BAKER JR., Phys. Rev. *5B*, 2622 (1972).
[2] For a review of these concepts see M. CASSANDRO, G. JONA-LASINIO, Adv. Phys. *27*, 913 (1978) and YA. G. SINAI, *Theory of Phase Transitions. Rigorous Results,* Chapter 4, Budapest, Acad. Kiado, 1982.
[3] G. JONA-LASINIO, *Large Fluctuations of Random Fields and Renormalization Group. Some Perspectives,* in "Scaling and Self-Similarity in Physics" J. Fröhlich ed. Birkhäuser, Boston 1983.
[4] P. M. BLEHER, *Theorem on Large Deviations in ϕ^4 Hierarchical Models,* preprint 1984 (Russian). P. M. Bleher, P. Major, Comm. Math. Phys. *95*, 487 (1984). P. M. Bleher, Teor. i Mat. Physica, *61*, 226 (1984) (Russian).
[5] E. C. G. STUECKELBERG and A. PETERMANN, Helv. Phys. Acta *26*, 499 (1953).