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Euclidean quantum mechanics and stochastic processes

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Abstract. The intimate relation between quantum mechanics in Euclidean time and stochastic processes is reviewed. This relation is shown to become even stronger in supersymmetric quantum mechanics, where the corresponding stochastic process is known as a 'Nicolai-map'. Supersymmetric quantum mechanics on a manifold is considered as an example.

1. Introduction

Stueckelberg has profoundly influenced and contributed to physics in quantum field theory, general relativity, and the theory of irreversible processes, and he was, of course, well aware of the connections between these seemingly different fields. In tribute to his memory I have therefore chosen a topic, which, to a certain extent, combines aspects of these different fields.

The idea to replace the physical pseudo-Euclidean or pseudo-Riemannian space-time by a formal Euclidean or Riemannian space-time and to recover physics by analytic continuation has occurred many times in physics on seemingly different and unrelated occasions. The first time this idea came up was after the discovery by Minkowski of the underlying unity of space-time of special relativity with its pseudo-Euclidean metric. In Minkowski's famous article on 'Raum und Zeit' [1] one finds the 'mystical' equation "$3 \times 10^5\,\mathrm{km} = \sqrt{-1}\,\mathrm{sec}$". Yet in classical relativity the formal rotation of the time-axis by $\pi/2$ in the complex plane, even though aesthetically pleasing, is not really important. It replaces the non-compact Lorentz group by the compact 4-dimensional rotation group and it makes the covariant equations look more symmetrical as it eliminates the difference between space and time coordinates. However, even though this higher symmetry is nowhere really necessary, it suggests that the fundamental difference between space and time is just this rotation in the complex plane.

A second and much more important occasion for the introduction of Euclidean time is quantum theory and quantum field theory. The formulation of these theories by functional integrals in physical time is often not well-defined as the integrals are not well-defined. However, as noted by Schwinger [2], Kac [3], Symanzik [4] and others (cf. [5], [6]) the integrals can be defined by first

formulating these theories in Euclidean time and passing over to physical time by analytic continuation. This operation is remarkable in several important ways:

(i) The very peculiar and counterintuitive properties of quantum mechanical probability in physical time (cf. e.g. [7]) become replaced, in Euclidean time, by notions of classical probability. Indeed, quantum mechanics is replaced by a classical, irreversible stochastic process in Euclidean time. The irreversibility of this process can be made manifest by the definition of an H-functional in the sense of Boltzmann, i.e. of an entropy which increases as the process evolves in Euclidean time.

(ii) The irreversibility of the stochastic process in Euclidean time leaves a trace in the analytical continuation to physical time by introducing a causal structure in physical time. The arrow of time which is visible in the causal relation between events can, in fact, be considered as a remnant of the analytic continuation from Euclidean times. In theories which do not use this analytical continuation causality has to be introduced 'by hand', while in theories constructed via analytic continuation from Euclidean time causality is automatically built-in.

(iii) Recently, the connection between Euclidean quantum mechanics and stochastic processes has become very interesting, again, because it was discovered by Parisi and Sourlas [8] and Nicolai [9] that this connection is surprisingly close for supersymmetric quantum theories. In a number of papers [10–15] this connection has been investigated in some detail.

It seems altogether remarkable that the formal connection of our world with a Euclidean world comes up again and again and seems to become closer with the deepening of our understanding of nature.

The paper is organized as follows. In the next section the stochastic process associated with quantum theory is constructed (see also [16, 17]). The properties of the stochastic process and their physical meaning are then considered. Turning to supersymmetric theories next we first formulate Nicolai's theorem. Then supersymmetric quantum theory on a manifold is considered. Finally, the associated stochastic process is constructed.

The last two sections of this lecture are based on and could not have been written without the joint work with Dirk Roekaerts [15], whom I would also like to thank for many discussions on the other subjects of this paper.

2. The stochastic process associated with quantum theory

Let us consider a non-relativistic spin-less particle of mass m and charge e in a static potential $V(\vec{x})$ and a static magnetic field described by the vector potential $\vec{A}(\vec{x})$. Its Schrödinger equation

$$i\hbar\psi = \left\{\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + V\right\}\psi \tag{2.1}$$

is assumed to possess a unique normalizable ground-state with no nodes

$$\psi_0(\vec{x}, t) = e^{-(i/\hbar)E_0 t}\varphi_0(\vec{x}) \tag{2.2}$$

$$\int d^3x\, |\varphi_0(\vec{x})|^2 = 1 \tag{2.3}$$

We define the function $\phi(\vec{x})$ by

$$|\varphi_0(\vec{x})| = \frac{1}{\sqrt{N_0}} \exp\left(-\frac{\phi(\vec{x})}{2\hbar}\right) \tag{2.4}$$

Changing the gauge of $\vec{A}$ and V by

$$\begin{aligned} V &\to V - \frac{1}{c}\frac{\partial \chi(\vec{x}, t)}{\partial t} \\ \vec{A} &\to \vec{A} + \frac{1}{e}\vec{\nabla}\chi(\vec{x}, t) \end{aligned} \tag{2.5}$$

the phase of the wave function is changed by

$$\psi \to \psi \exp\left(\frac{i\chi(\vec{x}, t)}{\hbar c}\right) \tag{2.6}$$

We make use of this freedom to arbitrarily change the phase of ψ by choosing the phase of $\psi_0(\vec{x}, t)$ real such that

$$\psi_0(\vec{x}, t) \equiv \varphi_0(\vec{x}) \equiv |\varphi_0(\vec{x})| \tag{2.7}$$

Obviously, the gauge freedom is then completely and uniquely fixed. In this gauge we have

$$E_0 = 0 \tag{2.8}$$

$$\hbar\vec{\nabla}\cdot\vec{A} - \vec{A}\cdot\vec{\nabla}\phi = 0 \tag{2.9}$$

$$V + \frac{e^2}{2mc^2}\vec{A}^2 = \frac{1}{4m}\left(\tfrac{1}{2}(\vec{\nabla}\phi)^2 - \hbar\nabla^2\phi\right) \tag{2,10}$$

Next, we perform a similarity transformation and introduce

$$\begin{aligned} \hat{\psi}(\vec{x}, t) &= \varphi_0(\vec{x})\psi(\vec{x}, t) \\ \hat{\psi}^+(\vec{x}, t) &= \psi^*(\vec{x}, t)/\varphi_0(\vec{x}) \end{aligned} \tag{2.11}$$

for the wave-function while for observables Ω we define

$$\hat{\Omega} = \varphi_0\Omega\frac{1}{\varphi_0} \tag{2.12}$$

The Schrödinger equation then takes the form

$$-i\dot{\hat{\psi}} = \frac{\hbar}{2m}\nabla^2\hat{\psi} - \vec{\nabla}\cdot(\vec{K}(\vec{x})\hat{\psi}) \tag{2.13}$$

with

$$\hat{K}(\vec{x}) = -\frac{1}{2m}\vec{\nabla}\phi(\vec{x}) + \frac{ie}{mc}\vec{A}(\vec{x}) \tag{2.14}$$

At this stage we perform the rotation to Euclidean time and introduce

$$t_E = it \tag{2.15}$$

The presence of an external magnetic field makes it necessary to rotate also the vector potential by introducing

$$\vec{A}_E = -i\vec{A} \tag{2.16}$$

Equation (2.16) follows e.g. from the equivalence of space and time after Euclidean rotation and the relations

$$\vec{B}_E = \vec{\nabla} \times \vec{A}_E = -i\vec{B}$$

$$\vec{E} = \frac{1}{c}\frac{\partial \vec{A}_E}{\partial t_E} - \vec{\nabla}\varphi$$

where $\vec{B}$, $\vec{E}$ are the electromagnetic field strengths and φ is the scalar potential.

In order to define the analytical continuation back to physical time we introduce a parameter λ by writing

$$\vec{A}_E = \lambda\vec{A} \tag{2.17}$$

Physical time is then reached by taking

$$t_E \to it, \qquad \lambda \to -i \tag{2.18}$$

In order to formulate the Schrödinger equation in Euclidean representation we define

$$W(\vec{x}, t_E) = \frac{1}{N_1}\hat{\psi}(\vec{x}, t_E, \lambda) \tag{2.19}$$

Then the Schrödinger equation takes the form

$$\frac{\partial W}{\partial t_E} = -\vec{\nabla} \cdot (\vec{K}(\vec{x}, \lambda)W) + \frac{\hbar}{2m}\nabla^2 W \tag{2.20}$$

with

$$\vec{K}(\vec{x}, \lambda) = -\frac{1}{2m}\vec{\nabla}\phi - \frac{e\lambda}{mc}\vec{A} \tag{2.21}$$

and

$$\vec{A} \cdot \vec{\nabla}\phi - \hbar\,\vec{\nabla} \cdot \vec{A} = 0 \tag{2.22}$$

Due to the structure of equation (2.20) of a continuity equation the normalization

integral

$$\int W(\vec{x}, t_E)\, d^3x = 1 \tag{2.23}$$

is preserved under the Euclidean time-evolution. The condition (2.23) serves to fix the normalization constant in equation (2.19). Equation (2.20) can be interpreted as Fokker–Planck equation of a Markoffian stochastic process of a random variable $\vec{x}$. In this interpretation $W(\vec{x}, t_E)$ becomes the probability density of $\vec{x}$ at the time t_E. This interpretation is only possible, however, if $W(\vec{x}, t_E)$ is non-negative, which is guaranteed by the form of equation (2.20) provided that we allow only positive (normalized) initial conditions for $W(\vec{x}, t_E)$ at some initial time, e.g. $t_E = 0$. The most important initial condition of this kind is

$$W(\vec{x}, 0) = \delta^{(3)}(\vec{x} - \vec{x}_0) \tag{2.24}$$

which defines the conditional probability density of the stochastic process and the propagator of the associated Schrödinger equation.

Let us briefly consider two simple examples. The first is a particle in the Coulomb potential

$$V(\vec{x}) = -\frac{e^2}{|\vec{x}|} \tag{2.25}$$

with the ground state

$$\varphi_0(\vec{x}) = \frac{1}{\sqrt{N_0}} \exp\left(-\frac{e^2 m}{\hbar^2}|\vec{x}|\right) \tag{2.26}$$

The Fokker–Planck drift $\vec{k}(\vec{x})$ in equation (2.20) is, in this case, simply

$$\vec{K}(\vec{x}) = -\frac{e^2}{\hbar}\frac{\vec{x}}{|\vec{x}|} \tag{2.27}$$

As a second example consider a 3-dimensional isotropic harmonic oscillator in a homogeneous magnetic field $B = B\vec{e}_z$. For simplicity we take $m = e = \hbar = c = 1$. With

$$\begin{aligned} V(\vec{x}) &= \tfrac{1}{2}\vec{x}^2 \\ \vec{A}(\vec{x}) &= \frac{B}{2}(x\vec{e}_y - y\vec{e}_x) \end{aligned} \tag{2.28}$$

we obtain

$$\varphi_0(\vec{x}) = \frac{1}{\sqrt{N_0}} \exp\left(-\frac{1}{2}\sqrt{1 + \frac{B^2}{4}}\,(x^2 + y^2) - \tfrac{1}{2}z^2\right) \tag{2.29}$$

and the Fokker–Planck drift

$$\vec{K}(\vec{x}, \lambda) = \begin{pmatrix} -\sqrt{1+\frac{B^2}{4}}\,x + \frac{\lambda B}{2}\,y \\ -\sqrt{1+\frac{B^2}{4}}\,y - \frac{\lambda B}{2}\,x \\ -z \end{pmatrix} \tag{2.30}$$

We shall return to the second example in the following section.

Before proceeding further let us remark on possible generalizations of the scheme presented above. The inclusion of spin, in principle, presents no difficulty and can be handled by the introduction of an additional discrete random variable which is governed by a random jump process [18]. Otherwise the same steps leading from equation (2.1) to equation (2.20) can be carried through. The generalization to several interacting particles is also possible. The Schrödinger equation in this case takes the general form

$$i\hbar\dot{\psi}(\vec{x}_1, \ldots, \vec{x}_n, t) = \left\{ \sum_{i=1}^{n} \frac{1}{2m_i} \left(\vec{p}_i - \frac{e_i}{c} \vec{A}_i(\vec{x}_1, \ldots, \vec{x}_n) \right)^2 + V(\vec{x}_1, \ldots, \vec{x}_n) \right\} \psi(\vec{x}_1, \ldots, \vec{x}_n, t) \tag{2.31}$$

Here $\vec{A}_i(\vec{x}_1, \ldots, \vec{x}_n)$ is the sum of the vectorpotentials generated by all particles except particle i and the vectorpotential of an external magnetic field acting on the particle i.

The gauge transformation

$$\begin{aligned} V &\to V - \frac{1}{c} \frac{\partial \chi(\vec{x}_1, \ldots, \vec{x}_n, t)}{\partial t} \\ \vec{A}_i &\to \vec{A}_i + \frac{1}{e_i} \vec{\nabla}_i \chi(\vec{x}_1, \ldots, \vec{x}_n, t) \end{aligned} \tag{2.32}$$

leads to a change of the phase of the many-particle wavefunction

$$\psi \to \psi \exp\left(\frac{i\chi}{\hbar c}\right) \tag{2.33}$$

which we use again to make the wavefunction of the ground-state non-negative and time-independent

$$\psi_0(\vec{x}_1, \ldots, \vec{x}_n, t) \equiv |\psi_0(\vec{x}_1, \ldots, \vec{x}_n)| = \frac{1}{\sqrt{N_0}} \exp\left(-\frac{\phi(\vec{x}_1, \ldots, \vec{x}_n)}{2\hbar} \right) \tag{2.34}$$

The Fokker–Planck equation (2.20) then takes the form

$$\frac{\partial W}{\partial t} = \sum_{i=1}^{n} \left\{ -\vec{\nabla}_i \cdot (\vec{K}_i(\vec{x}_1, \ldots, \vec{x}_n, \lambda) W) + \frac{\hbar}{2m_i} \nabla_i^2 W \right\} \tag{2.35}$$

with the probability density

$$W = W(\vec{x}_1, \ldots, \vec{x}_n, t) \tag{2.36}$$

and the drift vectors

$$\begin{aligned} \vec{K}_i(\vec{x}_1, \ldots, \vec{x}_n, \lambda) = & -\frac{1}{2m_i}\vec{\nabla}_i\phi(\vec{x}_1, \ldots, \vec{x}_n) \\ & -\frac{e_i\lambda}{m_ic}\vec{A}_i(\vec{x}_1, \ldots, \vec{x}_n) \end{aligned} \tag{2.37}$$

One may also consider the generalization to explicitly time-dependent Hamiltonians. In this case the ground-state wave function $\psi_0(\vec{x}, t)$ in equation (2.2) must be replaced by another time-dependent and node-less solution $\psi_0(\vec{x}, t)$ of the Schrödinger equation. The gauge is fixed by requiring this solution to be positive which entails the conditions

$$\psi_0(\vec{x}, t) = \frac{1}{\sqrt{N_0}} \exp\left(-\phi(\vec{x}, t)/2\hbar\right) \tag{2.38}$$

$$\hbar\vec{\nabla}\cdot\vec{A} - \vec{A}\cdot\vec{\nabla}\phi + \frac{mc}{e}\dot{\phi} = 0 \tag{2.39}$$

$$V + \frac{e^2}{2mc^2}\vec{A}^2 = \frac{1}{4m}\left(\tfrac{1}{2}(\vec{\nabla}\phi)^2 - \hbar\nabla^2\phi\right) \tag{2.40}$$

The definitions (2.11), (2.21) then again lead to equation (2.20) also in the explicitly time-dependent case.

Another generalization is necessary, if the Hamiltonian does not admit a normalizable ground-state of the form (2.2), a case which occurs, e.g., if the ground-state is part of a continuous spectrum. Then $\varphi_0(\vec{x})$ in (2.4) and the following equations can be replaced by a non-normalizable solution of the form (2.2) which does not need to qualify as an eigenstate by itself. For the stochastic process (2.20) this means that it does not possess a normalizable steady state distribution. The simplest example of this case is a free particle, where $\varphi_0(\vec{x}) = 1$ in equation (2.2).

Finally, still in the framework of ordinary quantum mechanics it is also possible to consider relativistic generalizations. Some interesting results have been obtained recently [19], but the problem does not appear to be solved completely. For field theoretical generalizations see [5, 6].

3. Properties of the stochastic process and their physical meaning

The Green's function of the Fokker–Planck equation satisfies the initial condition

$$W(\vec{x}, 0) = \delta^{(3)}(\vec{x} - \vec{x}_0) \tag{3.1}$$

It is the conditional probability density

$$W(\vec{x}, t_E) = P(\vec{x} \mid \vec{x}_0, t_E; \lambda) \qquad (t_E \geqslant 0) \tag{3.2}$$

for $\vec{x}$ at time t_E if $\vec{x} = \vec{x}_0$ at $t_E = 0$. Its interpretation in physical time is the probability amplitude K for the transition from $\vec{x}_0$ to $\vec{x}$ in the time interval t

$$K(\vec{x} \mid \vec{x}_0, t) = \lim_{\delta \to (\pi/2)-0} \left\{ \frac{\varphi_0(\vec{x}_0)}{\varphi_0(\vec{x})} P(\vec{x} \mid \vec{x}_0, te^{i\delta}; e^{-i\delta}) \right\} \tag{3.3}$$

The linear relationship between K and P shows the correspondence between the superposition principle for probabilities (of mutually exclusive events) in Euclidean time and the superposition of probability amplitudes (of mutually exclusive events) in physical time.

Joint probability densities of the stochastic process in equilibrium correspond to time-ordered joint probability densities in the ground-state in physical time, e.g.

$$W_1(\vec{x}) = \frac{1}{N_0} \exp\left(-\frac{\phi(\vec{x})}{\hbar} \right) = |\varphi_0(\vec{x})|^2 \tag{3.4}$$

is the probability density in equilibrium and

$$W_2(\vec{x}_2 t_{E2}, \vec{x}_1 t_{E1}) = \varphi_0(\vec{x}_2) K(\vec{x}_2 \mid \vec{x}_1, \tau) \varphi_0(\vec{x}_1) \tag{3.5}$$

with

$$\tau = (t_{E2} - t_{E1}) e^{-i\pi/2} \qquad (t_{E2} > t_{E1}) \tag{3.36}$$

is the joint 2-point probability density. Correspondingly, multi-time correlation functions in equilibrium correspond to time-ordered ground-state expectation values

$$\langle x_{in}(t_{En}) \cdots x_{i_1}(t_{E1}) \rangle = \langle \varphi_0 \mid T(x_{i_n}(t_n) \cdots x_{i_1}(t_1)) \mid \varphi_0 \rangle \tag{3.7}$$

with $t_i = t_{Ei} e^{-i\pi/2}$, and T the time-ordering operator. Non-equilibrium averages of the stochastic process correspond to matrix elements with the ground state

$$\langle \hat{\Omega}(t_E) \rangle = \int d^3x \hat{\Omega}(\vec{x}) W(\vec{x}, t_E) = \frac{\langle \varphi_0 | \Omega | \psi(t) \rangle}{\langle \varphi_0 | \psi(t) \rangle} \tag{3.8}$$

with $t = e^{-i\pi/2} t_E$.

A very important new analytical description appearing as a result of the stochastic reinterpretation of the Euclidean Schrödinger equation are the Langevin equations corresponding to the Fokker–Planck equation

$$\dot{\vec{x}} = \vec{K}(\vec{x}, \lambda) + \sqrt{\frac{\hbar}{m}} \vec{\xi}(t_E) \tag{3.9}$$

with the Gaussian white noise sources

$$\langle \xi_i(t_E) \rangle = 0 \tag{3.10}$$

$$\langle \xi_i(t_{E2}) \xi_j(t_{E1}) \rangle = \delta_{ij} \delta(t_{E2} - t_{E1}) \tag{3.11}$$

They are the Euclidean version of the equations of motion in the Heisenberg picture. It is well known that stochastic differential equations of the form (3.9)–(3.11) require the specification of the stochastic calculus to which they refer. The two most important stochastic calculi are those of Ito [20] and Stratonovich [21], respectively. The distinction between these calculi is important either if the noise is $\vec{x}$-dependent (which it is not in our examples, so far, but see section 6) or if a non-linear change of variables to non-Cartesian coordinates is desired. In physical time the different operator ordering conventions correspond to the different stochastic calculi cf. [22]. Again, operator ordering conventions have to be specified for quantization in a non-trivial metric or in curvi-linear coordinates.

The most prominent property of the stochastic process (3.9) or (2.20) is its irreversible approach of equilibrium. It can be exhibited by defining the entropy (cf. [23], [24])

$$S(t_E) = -\int d^3x W(\vec{x}, t_E) \ln \frac{W(\vec{x}, t_E)}{W_1(\vec{x})} \tag{3.12}$$

as a functional of $W(x, t_E)$. It satisfies

$$S(t_E) \leqslant 0 \tag{3.13}$$

where equality holds if and only if $W(\vec{x}, t_E) = W_1(\vec{x})$. This follows from the definition, $W(\vec{x}, t_E) > 0$, and

$$\int W(\vec{x}, t_E)\, d^3x = \int W_1(\vec{x})\, d^3x.$$

A second property of $S(t_E)$ is

$$\dot{S}(t_E) \geqslant 0 \tag{3.14}$$

if all states $\vec{x}$ are reached from any initial state $\vec{x}_0$ in the course of time. The property (3.14) follows from the above mentioned properties and the Fokker Planck equation (cf. e.g. [24], Section 3.1b). Equality in (3.14), under the stated conditions, implies again $W(\vec{x}, t_E) = W_1(\vec{x})$. The irreversible relaxation to equilibrium in Euclidean time corresponds, in physical time, to the forward propagation of positive energy states ($E > 0$) with an amplitude $\sim \exp(-iEt/\hbar)$, and hence causality. The arrow of time implicit in the causal relations between events thus can be viewed as a consequence of the increase of the entropy (3.12) in Euclidean time.

The time-reversal symmetry of the underlying Schrödinger equation gives rise to a corresponding detailed balance symmetry of the stochastic process

$$P(\vec{x} \mid \vec{x}_0, t_E; \lambda) W_1(\vec{x}_0) = W_1(\vec{x}) P(\vec{x}_0 \mid \vec{x}, t_E; -\lambda) \tag{3.15}$$

which, in turn, is equivalent to a set of Onsager–Casimir symmetry relations [25], [26] of the noise and drift coefficients of the Fokker–Planck equation (cf. [24]). In order to formulate these symmetry relations we define the operator of time-

reversal R for the stochastic process by

$$RF(\vec{x}, t_E, \lambda) = F(\vec{x}, -t_E, -\lambda) \tag{3.16}$$

Then we may decompose the drift $\vec{K}(\vec{x}, \lambda)$ uniquely into

$$\vec{K}(\vec{x}, \lambda) = \vec{d}(\vec{x}, \lambda) + \vec{r}(\vec{x}, \lambda) \tag{3.17}$$

with $\vec{d}$ and $\vec{r}$ defined by

$$R\vec{d}(\vec{x}, \lambda) = \vec{d}(\vec{x}, \lambda) \tag{3.18}$$

$$R\vec{r}(\vec{x}, \lambda) = -\vec{r}(\vec{x}, \lambda) \tag{3.19}$$

Explicitly, we find

$$d_i(\vec{x}, \lambda) = -\sum_j L_{ij} \frac{\partial \phi}{\partial x_j} \tag{3.20}$$

with

$$L_{ij} = L_{ji} = \frac{1}{2m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

Since $\vec{d}$ transforms oppositely to $\dot{\vec{x}}$ under time-reversal, it is the irreversible, dissipative part of the drift $\vec{K}$. Indeed, the 'potential' ϕ decreases under the motion $\dot{\vec{x}} = \vec{d}$,

$$\left(\frac{d\phi}{dt}\right)_{\dot{\vec{x}}=\vec{d}} = -\frac{1}{2m}(\vec{\nabla}\phi)^2 \leqslant 0 \tag{3.22}$$

The symmetry relation $L_{ij} = L_{ji}$ is Onsager's symmetry relation of transport coefficients for variables even under time-reversal (like $\vec{x}$).

On the other hand, we find

$$\vec{r}(\vec{x}, \lambda) = -\frac{e\lambda}{mc}\vec{A}(\vec{x}) \tag{3.23}$$

for $\vec{r}$ which transforms like $\dot{\vec{x}}$ under time-reversal and therefore constitutes the reversible part of the drift $\vec{K}$. From the gauge condition (2.9) for $\vec{A}$ and equation (3.23) it follows that

$$\vec{\nabla} \cdot (\vec{r} \exp(-\phi/\hbar)) = 0 \tag{3.24}$$

and hence [27]

$$r_i e^{-\phi/\hbar} = \hbar \sum_j \frac{\partial}{\partial x_j}(A_{ij} e^{-\phi/\hbar}) \tag{3.25}$$

with an antisymmetric matrix

$$A_{ij}(\vec{x}, \lambda) = -A_{ji}(\vec{x}, \lambda) \tag{3.26}$$

and

$$A_{ij}(\vec{x}, \lambda) = -A_{ij}(\vec{x}, -\lambda) \tag{3.27}$$

due to (3.23). Therefore, $\vec{r}$ admits a representation in the form

$$r_i(\vec{x}, \lambda) = -\sum_j A_{ij}(\vec{x}, \lambda) \frac{\partial \phi(\vec{x})}{\partial x_j} + \hbar \sum_j \frac{\partial A_{ij}(\vec{x}, \lambda)}{\partial x_j} \tag{3.28}$$

Using (3.17), (3.20) and (3.28) we obtain the following representation of the drift vector $\vec{K}$

$$K_i(\vec{x}, \lambda) = -\sum_j D_{ij}(\vec{x}, \lambda) \frac{\partial \phi}{\partial x_j} + \hbar \sum_j \frac{\partial D_{ij}(\vec{x}, \lambda)}{\partial x_j} \tag{3.29}$$

with

$$D_{ij}(\vec{x}, \lambda) = L_{ij}(\vec{x}) + A_{ij}(\vec{x}, \lambda) \tag{3.30}$$

satisfying the Onsager–Casimir symmetry relations

$$D_{ij}(\vec{x}, \lambda) = D_{ji}(\vec{x}, -\lambda) \tag{3.31}$$

As a final property of the stochastic process we mention its functional integral representation [24]

$$P(\vec{x} \mid \vec{x}_0, t_E; \lambda) = \int D[q] \exp\left\{-\int_{\vec{x}_0, 0}^{\vec{x}, t_E} d\tau L(\dot{\vec{x}}(\tau), \vec{x}(\tau))\right\} \tag{3.32}$$

with the formal measure of integration

$$D[q] = \prod_{\tau=0}^{t_E} \frac{d^3x(\tau)}{(2\pi\hbar\, d\tau/m)^{3/2}} \tag{3.33}$$

and the 'Onsager Machlup function' [28, 24]

$$L(\dot{\vec{x}}, \vec{x}, \lambda) = \frac{m}{2}(\dot{\vec{x}} - \vec{K}(\vec{x}, \lambda))^2 + \tfrac{1}{2}\vec{\nabla} \cdot \vec{K}(\vec{x}, 2) \tag{3.34}$$

The interpretation of the stochastic integral in equation (3.32) is in the sense of Stratonovich. The functional integral representation (3.32) of the conditional probability density corresponds to Feynman's path integral representation of the propagator in physical time. L becomes then the Lagrangian related to the Hamiltonian of the Schrödinger equation by the usual Legendre transformation. The Hamiltonian version of the quantum mechanical functional integral also exists in the Euclidean description and serves there as a unifying representation of correlation functions and response functions [29]. In closing this section let us also comment on the difficulties of the present approach and their possible resolution.

One difficulty is associated with the necessity to rotate not only time but also the vector potential to the imaginary axis. In order to analytically continue the final solution back to the real axis it is necessary to solve the stochastic process for arbitrary λ, which is, in general, very difficult. (Nevertheless, the inverse of this procedure has been found useful for solving some Fokker–Planck equations analytically in [30].) A possible way around this difficulty is to carry out the analytical continuation in λ directly in the Langevin equation [31]. This does not seem to cause any serious problems, if the $(V + (e^2/2mc^2)\vec{A}^2)$ term, which is not affected by the analytical continuation, dominates the terms of the Hamiltonian

which are linear in $\vec{A}$. Adopting such a procedure we have to solve

$$\dot{\vec{x}} = \vec{K}(\vec{x}, -i) + \sqrt{\frac{\hbar}{m}}\,\vec{\xi}(t_E) \tag{3.35}$$

for a complex (vector-) valued process $\vec{x}$ and real $\vec{\xi}$. Let us see how this works in the exactly solvable example of an isotropic harmonic oscillator in a homogeneous magnetic field. The Langevin equations (3.35) with (2.30) read

$$\begin{aligned}
\dot{x} &= -\sqrt{1+\frac{B^2}{4}}\,x - \frac{i}{2}By + \xi_x(t_E) \\
\dot{y} &= -\sqrt{1+\frac{B^2}{4}}\,y + \frac{i}{2}Bx + \xi_y(t_E) \\
\dot{z} &= -z + \xi_z(t_E)
\end{aligned} \tag{3.36}$$

Solving for $x(t_E)$ we find

$$\begin{aligned}
x(t_E) = \frac{1}{2}\int_{-\infty}^{t_E} d\tau \{ & (e^{-\omega_1(t_E-\tau)} + e^{-\omega_2(t_E-\tau)})\xi_x(\tau) \\
& - i(e^{-\omega_1(t_E-\tau)} - e^{-\omega_2(t_E-\tau)})\xi_y(\tau)\}
\end{aligned} \tag{3.37}$$

with

$$\omega_{1,2} = \sqrt{1+\frac{B^2}{4}} \mp \frac{B}{2} \tag{3.38}$$

The Euclidean correlation function in the steady state

$$\langle x(t_E)x(t'_E)\rangle = \frac{1}{2(\omega_1+\omega_2)}(e^{-\omega_1(t_E-t'_E)} + e^{-\omega_2(t_E-t'_E)}) \tag{3.39}$$

indeed, yields the correct time-ordered expectation value

$$\langle \varphi_0 | T(x(t)x(t')) | \varphi_0 \rangle = \frac{1}{2(\omega_1+\omega_2)}(e^{-i\omega_1|t-t'|} + e^{-i\omega_2|t-t'|}) \tag{3.40}$$

A second and indeed quite serious difficulty of the present approach is the fact that $\phi(\vec{x})$ has no simple counterpart in the quantum process. In order to determine ϕ the complete ground-state solution φ_0 of the Schrödinger equation must be known. The absolute value of φ_0 is required for ϕ, (2.4); the phase of φ_0 is required for pićking the correct gauge (2.5, (2.6) before going to Euclidean time. The requirement to determine φ_0 severely limits the practical usefulness of the stochastic process (2.20). Therefore, it is very remarkable that this difficulty disappears in some supersymmetric theories. There, $\phi(\vec{x})$ turns out to be the 'superpotential' and can be read off immediately from the super-space formulation of the Hamiltonian. Therefore, we consider supersymmetric theories in the remaining sections of this paper.

4. Supersymmetry – Nicoalai's theorem

There are several good reasons why supersymmetry as a beautiful mathematical structure is interesting for physics (cf. e.g. [32]). There is a widespread feeling among theoreticians that because of its intrinsic beauty nature should make use of it. There is the hope that it may help in the unsolved problem of quantizing gravity, which would make it a relevant symmetry at the Planck scale of energy. Another hope is that supersymmetry may help to solve the 'hierarchy problem' of particle physics, which would imply its relevance at ordinary particle energies. Further physical applications of supersymmetry are in the theory of nuclear spectra [23] and even atomic spectra [34]. But even apart from such direct physical applications supersymmetry has turned out to be useful as a technical tool in applications. The basic idea is to uncover 'hidden supersymmetries', which often means to enlarge the original system in such a way that it becomes supersymmetric and then to use that supersymmetry in order to simplify the original problem. A simple example is the quantum mechanics of a spin-less particle in a potential well. It is possible to extend this problem (artificially) to the supersymmetric quantum theory of a particle with spin-$\frac{1}{2}$ in a spin-dependent potential. Supersymmetry guarantees that the energy-levels occur in degenerate spin-doublets, one member of which describes the states of the original problem. The advantage of supersymmetry is due to the fact that it may be easier to determine the energy levels and states from the second member of the doublet, which is brought in by the supersymmetry. Exact solutions may become possible in this way, or improvements in approximate analytical calculations with Ritz's method [35], WKB-approximations [36] or $1/d$-expansions, where d is the dimensionality of the space considered [37]. Other examples where 'hidden supersymmetries' are beneficial occur in statistical mechanics in the random field problem of an Ising ferromagnet [8], critical dynamics, or the problem of an electron with impurity scattering in a strong magnetic field [38].

An important characterization of supersymmetric quantum theories is given by Nicolai's theorem [9] which we now formulate. Let

$$S(g, \phi) = S_B(g, \varphi) + \tfrac{1}{2} \int \bar{\psi} M(g, \varphi) \psi \tag{4.1}$$

be the action of a supersymmetric theory, where g is a set of coupling constants, ϕ denotes a set of superfields (or supervariables) with the bosonic components φ and the fermionic components ψ. Then (with $\hbar = 1$)

$$d\mu(g, \phi) \sim \exp(-S(g, \phi)) D[\varphi] D[\bar{\psi}] D[\psi] \tag{4.2}$$

is the (formal) Euclidean measure of that theory, in which the φ are c-number variables and the ψ, $\bar{\psi}$ are Grassmann variables. Integrating over the Grassmann variables a reduced c-number measure is obtained, which is of the form

$$d\bar{\mu}(g, \varphi) \sim \exp(-S_B(g, \varphi)) D(g, \varphi) D[\varphi] \tag{4.3}$$

where $D(g, \varphi)$ is a determinant resulting from the integration over the Grassmann variables. Nicolai's theorem now states that for any supersymmetric S of the form (4.1) there exists a map $\varphi \to \varphi'$

$$\varphi' = F(g, \varphi) \tag{4.4}$$

such that the action S_B is reduced to that of a free field

$$S_B(g, \varphi) = S_B(0, \varphi') + \text{const} \tag{4.5}$$

and the functional determinant of (4.4) cancels $D(g, \varphi)$

$$\det\left(\frac{\delta\varphi'}{\delta\varphi}\right) = D(g, \varphi) \tag{4.6}$$

Hence, in the new variables φ' the Euclidean measure (4.3) becomes simply the Gaussian measure of a free field

$$d\bar{\mu}(g, \varphi) \sim \exp(-S_B(0, \varphi'))D[\varphi'] \tag{4.7}$$

The importance of Nicolai's theorem in our present context is the fact that it associates a stochastic problem (4.4) with any supersymmetric Euclidean quantum theory [10]. Equation (4.4) is a stochastic problem (if the functional $F(g, \varphi)$ on the right-hand side is known explicitly), because the random field (or variable) φ' on the left-hand side is given with all its stochastic properties by (4.7), and the task is to solve (4.4) for the unknown φ in terms of the known φ'.

Let us consider a simple example – supersymmetric quantum theory in $d = 1$ [32], [39], [40]. Let Q_i $(i = 1 \cdots N)$ be the symmetry generators satisfying

$$[Q_i, H] = 0, \qquad \{Q_i, Q_j\} = \delta_{ij}H \tag{4.8}$$

and consider the simplest case $N = 2$ where we may use the representation

$$Q_1 = \frac{1}{2}\left(-i\sigma_1\frac{\partial}{\partial x} + g\sigma_2 W(x)\right) \tag{4.9}$$

$$Q_2 = \frac{1}{2}\left(-i\sigma_2\frac{\partial}{\partial x} - g\sigma_1 W(x)\right)$$

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{g^2}{2}W^2(x) + \frac{g}{2}\sigma_3\frac{\partial W(x)}{\partial x} \tag{4.10}$$

with the Pauli matrices σ_1, σ_2, σ_3 and an arbitrary function $gW(x)$. The action (4.1) then takes the form

$$S = \int d\tau\left(\frac{\dot{x}^2}{2} + \frac{g^2}{2}W^2(x)\right) + \int d\tau\psi^*\left(\frac{d}{d\tau} - gW'(x)\right)\psi \tag{4.11}$$

where x has to be identified with φ.

Since the Hamiltonian commutes with σ_3 the theory splits into the $\sigma_3 = \pm 1$ sectors. In each sector a seperate Nicolai map applies [12–14]. The results of

Section 2 make it obvious that

$$\dot{x} + gW(x) = \xi(\tau) \qquad (\sigma_3 = 1) \tag{4.12}$$

$$\dot{x} - gW(x) = \xi(\tau) \qquad (\sigma_3 = -1) \tag{4.13}$$

are the desired maps, if $\int^t \xi(\tau)\, d\tau$ is identified with the free Gaussian variable φ'. Indeed, S_B of equation (4.1), which is given by the first term on the right-hand side of equation (4.11), when written in terms of ξ, takes the Gaussian form

$$S_B = \frac{1}{2}\int d\tau \xi^2(\tau) + \text{const} \tag{4.14}$$

where the constant depends on the fixed values of x at the initial and final point.

Furthermore, the functional determinant [24]

$$\det\left(\frac{\delta\xi}{\delta x}\right) \sim \exp\left(\mp\frac{g}{2}\int d\tau \frac{\partial W}{\partial x}\right)$$

$$\sim \int D[\psi^*]D[\psi] \exp\left\{-\int d\tau \psi^*\left(\frac{d}{d\tau} - gW'\right)\psi\right\} \tag{4.15}$$

cancels the determinant due to integration over the Grassmann variables in the $\sigma_3 = \pm 1$ sectors.

To summarize, the Nicolai map (4.4) is nothing but the Langevin equation of the stochastic process of Section 2 associated with the various sectors of the theory (distinguished by eigenvalues of σ_3 or 'fermion number'). Furthermore, the drift $\pm gW(x)$ of the Langevin (or Fokker–Planck) equation appears explicitly in the Q_i and in H. Unfortunately, in the general case, the Nicolai map is no longer local in time, nor is it even known explicitly. Rather, it must be constructed perturbatively and may then be only of limited practical value. However, there are (besides ordinary supersymmetric quantum mechanics) some quantum field theories (e.g. the Wess–Zumino model with $d = 2$, $N = 2$ [8] or a certain sector of the supersymmetric Yang–Mills theory with $d = 4$, $N = 1$ [41, 42]) where the Nicolai map is known exactly and is still local in time.

In the remaining 2 sections we will restrict our attention to ordinary quantum mechanics in d dimensions and ask: what is the most general form of the Langevin equation corresponding to $N = 2$ supersymmetric quantum mechanics? The answer for the zero- or d-fermion sector turns out to be a stochastic process on a manifold with purely irreversible drift (cf. Section 3). This result was obtained independently in Refs. [13], [15]. For the general n-fermion sector ($0 \leqslant n \leqslant d$) with $n \neq 0$ or d it is necessary to consider Nicolai maps in the form of discontinuous random jump processes and the answer is not yet known.

5. Supersymmetric quantum theory on a manifold

Let us construct the most general $N = 2$ supersymmetric Lagrangian on a d-dimensional Riemannian manifold. Local coordinates of the manifold are

denoted by q^ν ($\nu = 1, \ldots, d$), its metric tensor is $g^{\nu\mu}(q)$, a local orthogonal 'Vielbein' basis is $e_i^\nu(q)$ with

$$\begin{aligned} e_i^\nu(q)e_i^\mu(q) &= g^{\nu\mu}(q) \\ e_i^\nu(q)e_{j\nu}(q) &= \delta_{ij} \end{aligned} \tag{5.1}$$

where the summation convention is implied in both equations and in the following. The anticommuting partners of the q^ν are ψ^ν and its complex conjugate $\psi^{\nu *}$ with

$$\{\psi^\nu, \psi^\mu\} = \{\psi^{\nu *}, \psi^\mu\} = 0 \tag{5.2}$$

It is convenient to introduce also anticommuting partners η and η^* of t. Then the supersymmetry transformation for $N = 2$ can be written as

$$\begin{aligned} &t \to t + \varepsilon^*\eta + \eta^*\varepsilon \\ &\eta \to \eta + i\varepsilon, \qquad \eta^* \to \eta^* - i\varepsilon^* \end{aligned} \tag{5.3}$$

with ε, ε^* anticommuting.

The supervariables ϕ^ν are introduced via

$$\phi^\nu = q^\nu + \eta\psi^\nu + \psi^{\nu *}\eta^* + \eta^*\eta F^\nu \tag{5.4}$$

where F^ν is an auxiliary commuting variable. An invariant derivative is defined by

$$D_\eta = \frac{\partial}{\partial\eta} + i\eta^*\frac{\partial}{\partial t} \tag{5.5}$$

A Lagrangian can now be constructed from the invariants $V(\phi)$, where the function $V(q)$ is called superpotential, and

$$\tfrac{1}{2}(D_\eta\phi^\mu)^* g_{\mu\nu}(\phi)(D_\eta\phi^\nu) \tag{5.6}$$

It is important to note that a non-vanishing invariant of the form $A_\nu(\phi)D_\eta\phi^\nu$ is not available, since $A_\nu(\phi)$ would have to be anti-commuting and cannot be non-zero without violating the supersymmetry. Hence

$$\begin{aligned} S &= \int dt\, d\eta\, d\bar{\mu}[\tfrac{1}{2}(D_\eta\phi^\mu)^* g_{\mu\nu}(\phi)(D_\eta\phi^\nu) - V(\phi)] \\ &= \int dt(L_B + L_F) \end{aligned} \tag{5.7}$$

with [32],

$$\begin{aligned} L_B &= -\tfrac{1}{2}g^{\mu\nu}(q)p_\nu p_\mu + p_\mu(\dot{q}^\mu - g^{\mu\nu}V_{;\nu}) + \dot{q}^\nu V_{;\nu} \\ L_F &= \frac{1}{2}\left(\psi_i^*\frac{D\psi_i}{Dt} - \frac{D\psi_i^*}{Dt}\psi_i\right) \\ &\quad - \psi^{\nu *}V_{i\nu\mu}\psi^\mu - \tfrac{1}{4}\psi^{\nu *}\psi^{\mu *}\psi^\kappa\psi^\lambda R_{\nu\mu\kappa\lambda} \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} p_\nu &= g_{\nu\mu}(F^\mu + \dot{q}^\mu + \Gamma^\mu_{\kappa\lambda}\psi^{\kappa *}\psi^\lambda) \\ \psi_i &= e_{i\nu}\psi^\nu \end{aligned} \tag{5.9}$$

$\Gamma^{\nu}_{\kappa\lambda}$, $R_{\nu\mu\kappa\lambda}$ are the affine connection and the Riemann curvature tensor associated with $g^{\nu\mu}(q)$, respectively, $V_{;\nu}$ is the covariant derivative of V with respect to q^{ν}, and

$$\frac{D\psi_i}{Dt} = \frac{\partial \psi_i}{\partial t} + e_{i\nu}\left(\frac{\partial e^{\nu}_j}{\partial q^{\lambda}} + \Gamma^{\nu}_{\lambda\kappa} e^{\kappa}_j\right)\dot{q}^{\lambda}\psi_j \tag{5.10}$$

is the covariant time-derivative of ψ_i. It is not difficult to check that p_{ν} transforms as a covariant vector in the tangent space and ψ_i, ψ^*_i transform as Euclidean vectors under local rotations of the 'Vielbein'. Hence, S, besides being a supersymmetry invariant, transforms as a scalar under local general coordinate transformations and local rotations of the 'Vielbein'. However, it is important to note that the general rules of calculus are used in proving these facts. This makes it necessary to use Stratonovich's calculus in our later consideration of the Nicolai map. We also remark that the action S is actually not of the form (4.1) but contains a term which is quartic in the Grassmann variables.

As a last remark we note that each term of L_F contains an equal number of ψ^* and ψ, i.e. the total fermion number

$$p = \psi^*_i \psi_i \tag{5.11}$$

is conserved. Due to the Pauli principle we have $0 \leqslant p \leqslant d$. The total Fock space of the problem can be decomposed into the sectors with $p = 0, 1, 2, \ldots, d$.

6. Construction of the associated stochastic process [15]

Let us now restrict our attention to the $p = 0$ sector of the theory. The problem then is to integrate the total Euclidean measure of the theory

$$d\mu \sim \exp\left(-\int_0^T dt(L_b + L_f)\right) D[q]D[p]D[\bar{\psi}]D[\psi] \tag{6.1}$$

over ψ, $\bar{\psi}$ and p. The method used in [15] for accomplishing this is to consider the path $q(\tau)$ between given end-points at $\tau = 0$ and $\tau = T$ as fixed and to use coordinates and a 'Vielbein' basis along this path in which

$$\frac{D\psi_i}{D\tau} = \frac{d\psi_i}{d\tau} \tag{6.2}$$

The integrals over ψ, $\bar{\psi}$ are then not difficult to evaluate with boundary conditions corresponding to the zero-fermion sector. The result obtained in these special coordinates must be a coordinate scalar, and it can be written as a manifest scalar, namely

$$\begin{aligned} &\int D[\bar{\psi}]D[\psi] \exp\left(-\int_0^T L_F \, d\tau\right) \\ &\quad = \exp\left[\frac{1}{2}\int_0^T d\tau (g^{\mu\nu} V_{i\mu\nu} - \tfrac{1}{4}R)\right] \\ &\quad = J[q] \end{aligned} \tag{6.3}$$

Hence, it is also valid in this form in general coordinates and for arbitrary orientation of the 'Vielbein'.

Next we carry out the integration of $d\mu$ over the auxiliary variables p_ν and obtain

$$\int D[p] \exp\left(-\int_0^T L_B \, d\tau\right)$$
$$= \prod_{\tau=0}^{T} ((2\pi \, d\tau)^d \det g^{\nu\mu}(q(\tau)))^{-1/2} \exp\left(-\int_0^T \bar{L}_B \, d\tau\right) \tag{6.4}$$

with

$$\bar{L}_B = \tfrac{1}{2} g_{\nu\mu}(q)\dot{q}^\nu \dot{q}^\mu + \tfrac{1}{2} g^{\nu\mu}(q) V_{i\nu} V_{i\mu} \tag{6.5}$$

where $g_{\nu\mu}$ is the inverse of $g^{\nu\mu}$.

Now we turn to the problem of finding the Nicolai map. Trying an 'ansatz' in the form of a stochastic differential equation in the sense of Stratonovich we put

$$\dot{q}^\nu = -g^{\nu\mu}(q) V_{i\mu} + e_i^\nu(q) \circ \xi_i = l^\nu(q) \tag{6.6}$$

with the Gaussian white noise

$$\langle \xi_i \rangle = 0$$
$$\langle \xi_i(\tau)\xi_j(0) \rangle = \delta_{ij} \, \delta(\tau) \tag{6.7}$$

The term $l^\nu(q)$ is necessary in Stratonovich's calculus in order to ensure the invariance of $\dot{q}^\nu$ under local rotations of the 'Vielbein', and it takes the form

$$l^\nu(q) = \tfrac{1}{2} g^{\nu\mu} e_i^\lambda \left(\frac{\partial e_{i\mu}}{\partial q^\lambda} - \partial e_{i\lambda} / \partial q^\mu \right) \tag{6.8}$$

As a result of this term the form of the map (6.6) depends explicitly on the 'Vielbein' basis. However, the different forms obtained by locally rotating the 'Vielbein' are all stochastically equivalent and all correspond, e.g., to the same Fokker–Planck equation.

In order to verify that (6.6) is, indeed, a Nicolai map we have to evaluate its Jacobian. Again we can do this by using the fixed path $q(\tau)$ between $q(0)$ and $q(T)$ and the special coordinates and the 'Vielbein' basis employed in the derivation of equation (6.3). In these coordinates we have along the path $q(\tau)$

$$l^\nu = 0$$
$$l^\nu_{;\nu} = -\tfrac{1}{4} R \tag{6.9}$$

It is then easy to show that, indeed

$$\det\left(\frac{\delta \xi_i}{\delta q^\nu}\right) = J[q] \tag{6.10}$$

along $q(\tau)$. Furthermore, in the same coordinates we obtain from (6.5)

$$\int_0^T d\tau \bar{L}_B = \frac{1}{2} \int_0^T d\tau \xi_i(\tau)\xi_i(\tau) + V(q(T)) - V(q(0)) \tag{6.11}$$

i.e. the right-hand side is of the form (4.5). Thus for the particular path chosen and the particular coordinates and 'Vielbeine' that come with it via equations (6.9), the stochastic differential equation (6.6) is a Nicolai map. Strictly speaking, a different Nicolai map has to be used for each path $q(\tau)$, but, fortunately, all the different maps are stochastically equivalent and correspond to the same stochastic process, described by the Fokker–Planck equation

$$\frac{\partial W(q,t)}{\partial t} = \frac{\partial}{\partial q^\nu}(g^{\nu\mu}(q)V_{;\mu}W(q,t)) + \frac{1}{2}\frac{\partial}{\partial q^\nu}\frac{g^{\nu\mu}(q)}{\sqrt{g(q)}}\frac{\partial}{\partial q^\mu}\sqrt{g(q)}W(q,t) \tag{6.12}$$

with

$$g(q) = \det(g^{\nu\mu}(q)) \tag{6.13}$$

and the probability density $W(q, t)$.

The result, therefore, is a stochastic process whose drift vector, in covariant form [27], is a pure gradient. In the notions introduced in equations (3.17)–(3.23) the drift is purely dissipative and of the form (3.20), the reversible part of the drift (3.23) vanishes. The basic reason behind this result is the non-existence of a supersymmetric invariant of the form $A_\nu(\phi)D_\mu\phi^\nu$ in the Lagrangian. The superpotential $V(q)$ can be identified with $\phi(q)$ by

$$\phi(q) = 2V(q) \tag{6.14}$$

and, hence $\phi(q)$ can be read off the Lagrangian or the Hamiltonian expressed in terms of the supervariables. The ground-state wave-function is, therefore, also known and reads

$$\psi_0(q) = \frac{1}{\sqrt{N_0}}\exp(-V(q)) \tag{6.15}$$

if we adopt the normalization convention

$$\int \frac{dq}{\sqrt{g(q)}}|\psi(q)|^2 = 1 \tag{6.16}$$

on manifolds. On non-compact manifolds equation (6.15) may not be normalizable, in which case equation (6.12) does not admit a time-independent probability density (we exclude here the possibility discussed in [15] that V but not $V_{;\nu}$ is a multi-valued function on the manifold) and the zero-fermion sector does not contain a normalizable zero-energy ground state.

Very similar results are obtained for the $p = d$ fermion sector. The Nicolai map for this case is obtained by changing the sign of the first term on the right-hand side of equation (6.6). Hence, the zero-energy state in this sector takes the form

$$\psi_d(q) = \frac{1}{\sqrt{N_d}}\exp(V(q)) \tag{6.17}$$

provided that this expression is normalizable (as it always is on compact manifolds). Not much is known about the general p-fermion sector ($p \neq 0, d$). It contains $\binom{d}{p}$ fermion Fock states. i.e. the Nicolai maps in these sectors are given by random jump processes with $\binom{d}{p}$ states coupled to the continuous d-dimensional random process associated with $q^{\nu}(\tau)$. A very interesting result has been found by Witten [43] who showed that the number b_p of zero-energy ground states in the p-fermion sector, which equals the number of normalizable equilibrium distributions of the $\binom{d}{p}$-state jump process coupled to the d-dimensional continuous random process, is given by the p-th Betti number of the manifold. E.g. for a compact 2-surface with n handles $b_0 = b_2 = 1$, $b_1 = 2n$, i.e. in this case there are $2n$ additional zero-energy ground-states in the 1-fermion sector which contains 2 different kinds of states associated with two different kinds of fermions.

If, actually, no normalizable zero energy state exists in any of the p-fermion sectors ($0 \leq p \leq d$) the ground state of the system must have positive energy and must be degenerate, i.e. supersymmetry is spontaneously broken [32].

A simple example, discussed in [39], is the motion of a particle of unit mass on the real line with a superpotential $V(q)$ of the form

$$V(q) = -\alpha q + \tfrac{1}{3}\beta q^3 \qquad (\alpha, \beta > 0) \tag{6.18}$$

For the stochastic process (6.6) in this potential there is a finite escape time after which the process, initially localized at the minimum of $V(q)$, will escape to $q = -\infty$. In Kramer's approximation [44] this time is given as

$$T_K = \frac{1}{V''\left(\sqrt{\frac{\alpha}{\beta}}\right)} e^{2\Delta V} \tag{6.19}$$

where

$$2\Delta V = \frac{8}{3}\sqrt{\frac{\alpha^3}{\beta}} \tag{6.20}$$

is the height of the potential barrier.

For the quantum mechanical problem T_K^{-1} is a typical frequency of tunnelling which leads to a (non-perturbative) increase of the ground-state energy E due to 'instanton effects' [39] by

$$E = \frac{2\pi}{T_K} \tag{6.21}$$

7. Conclusion

In conclusion two points seem worth stressing. The first point is that the relation between physical space-time and Euclidean space-time seems to become

stronger with increasing level of understanding. Originally, in classical physics, it looked like an aesthetical trick which turns Lorentz invariance into rotational invariance. After quantization it acquired a completely new aspect by essentially turning probability amplitudes into probabilities. In supersymmetric theories, finally, the superpotential and the potential of the stochastic process become the same quantity.

Nevertheless, it must be said that the relation between the Euclidean and the physical world remains formal. There remains the tantalizing question whether it is possible to give a physical meaning to the Euclidean rotation of time and whether one could reach a deeper understanding of quantum fluctuations in this way. Also, one should point out that unsolved difficulties even with the definition of the Euclidean rotation of time still exist in gravitational fields without time-like Killing vectors [45].

As a second concluding remark let us point out the practical value which a representation of quantum theory in terms of stochastic differential equations may have. New analytical approaches become possible, and, with increasing importance, new ways for numerical solutions are opened up.

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