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Autor: Gallavotti, Giovanni

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Stability near resonances in classical mechanics[1])

By Giovanni Gallavotti

Dipartimento di Matematica, II Università di Roma, Via Raimondo, 00173 Roma, Italia

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1. Results on quasi-integrable systems

We consider perturbations of hamiltonian integrable systems, i.e. hamiltonian differential equations with hamiltonian function of the form

$$H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) = h(\mathbf{A}) + \varepsilon f(\mathbf{A}, \boldsymbol{\varphi}) \tag{1.1}$$

where h, f are analytic functions in

$$\begin{aligned}&\mathbf{A} = (A_1, \ldots, A_l) \in V_R \equiv \{\mathbf{A} \mid \mathbf{A} \in \mathbb{R}^l, |A_i| \leq R\}\\ &\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_l) \in T^l \equiv l\text{-dimensional torus} \equiv [0, 2\pi]^l\end{aligned} \tag{1.2}$$

The $\mathbf{A}$'s will be called 'action variables' and the $\boldsymbol{\varphi}$'s, which are their respective canonically conjugate variables, will be called the 'angle variables', [1].

The hamiltonian equations are therefore

$$\begin{aligned}\dot{\mathbf{A}} &= -\varepsilon \frac{\partial f}{\partial \boldsymbol{\varphi}}(\mathbf{A}, \boldsymbol{\varphi})\\ \dot{\boldsymbol{\varphi}} &= \boldsymbol{\omega}(\mathbf{A}) + \varepsilon \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}, \boldsymbol{\varphi})\end{aligned} \tag{1.3}$$

with $\boldsymbol{\omega}(\mathbf{A}) \equiv \partial h / \partial \mathbf{A}(\mathbf{A})$ being the gradient of h.

When $\varepsilon = 0$ the solution to (1.3) is obviously

$$\mathbf{A}(t) = \mathbf{A}(0), \qquad \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}(\mathbf{A}(0))t \tag{1.4}$$

showing that the natural foliation of the phase space $V \times T^l$ into tori of the form $\{\mathbf{A}\} \times T^l$ is an 'invariant foliation', i.e. motions starting on $\{\mathbf{A}\} \times T^l$ stay on it and, furthermore, such motions are quasi periodic with 'angular velocities' or 'spectrum' $\boldsymbol{\omega}(\mathbf{A})$.

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[1]) Talk presented at the Stueckelberg Memorial Lectures, Lausanne, 27–28 June 1985, Switzerland, partially supported by grant N.S.F. DMS 85-03333.

In the applications systems like (1.1) with $\varepsilon = 0$ occur often but usually they arise naturally in systems of canonical coordinates which are not the above action angle coordinates. Nevertheless the existence of a foliation of phase space into invariant tori is a coordinate independent property and it will manifest itself quite easily, once the motions are known (i.e. once the system is integrated).

For instance, if $l = 2$ and the system is integrable, we can restrict our attention to a given surface of constant energy $\bar{h}(\mathbf{p}, \mathbf{q}) = E$, if $\bar{h}$ is the function h in a generic system of canonical coordinates: this is a 3-dimensional surface in a 4-dimensional phase space (in general a $(2l-1)$-dimensional surface in a $2l$-dimensional phase space). The system being integrable, there is a second constant of motion which can be used to parametrize the various two-dimensional tori.

If one draws a plane π transversal to the tori they will be cut into circles parametrized, at fixed E, by the second constant of motion and therefore one will see the following picture (Fig. 1):

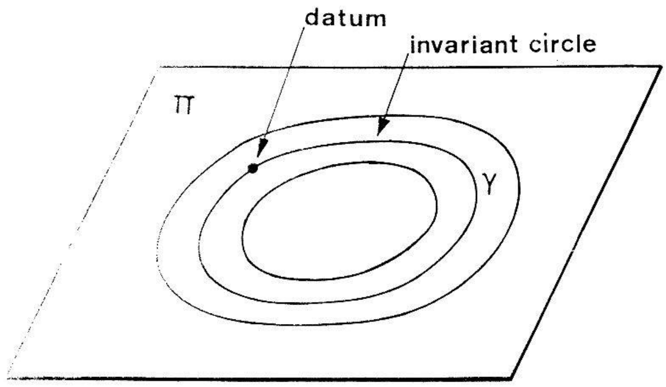

Figure 1.

if an initial datum is lying on a circle γ it will move leaving the circle, and eventually its trajectory will cross again π on another point of the same circle γ etc.: the successive images of the point on γ will generally fill γ densely.

Any initial datum will not only be on a given torus but, if we think of a 2-dimensional torus in a three-dimensional manifold as a 'closed tube', it will be 'inside' many other tori and 'outside' many others.

The above example is easily generalized to more than 2 degrees of freedom ($l > 2$): however the main difference will be that if $l > 2$ the invariant tori will be l-dimensional inside the $(2l-1)$-dimensional surface of constant energy and, therefore, it will no longer make sense to say that a given datum is inside or outside an invariant torus.

This fact has far-reaching consequences: imagine that the perturbed system still admits invariant tori close in shape to the unperturbed ones but a 'little less dense', i.e. not passing through every point of phase space. Then if $l = 2$, a given initial datum might be outside the set of invariant tori but still it will be enclosed in the tubular region between two invariant tori containing the point respectively inside and outside: hence, by uniqueness of motion, the trajectory of the point will be forever 'trapped' in the region between the two tori. This means that if

$l = 2$, there is a possibility of obtaining a priori estimates from the existence of enough invariant tori.

Such a possibility will be absent if $l \geqslant 3$, because l-dimensional tori in $(2l-1)$ dimensions do not have 'an inside and an outside', as soon as the set of the invariant tori is not closely packed.

We now consider the case $\varepsilon \neq 0$ and we analyze two extreme cases:

I) the 'harmonic non-resonant oscillators', i.e.

$$h(\mathbf{A}) = \boldsymbol{\omega} \cdot \mathbf{A} \Rightarrow \boldsymbol{\omega}(\mathbf{A}) = \boldsymbol{\omega} \tag{1.5}$$

with $\boldsymbol{\omega}$ satisfying, for all integer components vectors $\mathbf{r} \in Z^l$, $\mathbf{r} = (r_1, \ldots, r_l)$, and for suitable C, $\alpha > 0$, a 'non resonance diophantine' condition:

$$|\boldsymbol{\omega} \cdot \mathbf{v}| \equiv |\omega_1 v_1 + \cdots + \omega_l v_l| \geqslant \frac{1}{C\left(\sum_{i=1}^{l} |v_i|\right)^{\alpha}} \tag{1.6}$$

II) the 'anisochronous rotators': if $I > 0$

$$h(\mathbf{A}) = \frac{1}{2}\frac{\mathbf{A}^2}{I} \Rightarrow \boldsymbol{\omega}(\mathbf{A}) = \frac{\mathbf{A}}{I} \tag{1.7}$$

or, more generally, h such that

$$\frac{\partial^2 h}{\partial \mathbf{A}\, \partial \mathbf{A}}(\mathbf{A}) \geqslant a > 0 \tag{1.8}$$

i.e. a 'strictly convex integrable hamiltonian'.

The main result on the above systems is the following theorem (which is one of the many versions of a set of results of the 'KAM theory'): this theorem holds both in cases I) and II), if to I) one adds the assumption that the average $\bar{f}$ of f over the angles φ is such that $\det(\partial^2 \bar{f}/\partial \mathbf{A}\, \partial \mathbf{A}) \neq 0$. In fact, in case II) the theorem below would hold under the weaker assumption $\det(\partial^2 h/\partial \mathbf{A}\, \partial \mathbf{A})(\mathbf{A}) \neq 0$. The theorem is presented and proved in the form quoted below in the review paper [2].

Theorem 1. i) *There exist two canonical maps of class C^∞ in $(\mathbf{A}, \boldsymbol{\varphi})$ with domain containing $V_r \times T^l$ for ε small enough of the form*

$$C_\varepsilon : \begin{cases} \mathbf{A} = \mathbf{A}' + \boldsymbol{\Xi}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') \\ \boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\Delta}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') \end{cases} \qquad C_\varepsilon^{-1} : \begin{cases} \mathbf{A}' = \mathbf{A} + \boldsymbol{\Xi}'_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) \\ \boldsymbol{\varphi}' = \boldsymbol{\varphi} + \boldsymbol{\Delta}'_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) \end{cases} \tag{1.9}$$

and $\boldsymbol{\Xi}_\varepsilon$, $\boldsymbol{\Xi}'_\varepsilon$, $\boldsymbol{\Delta}_\varepsilon$, $\boldsymbol{\Delta}'_\varepsilon \to_{\varepsilon\to 0} 0$, $\mathscr{C}_\varepsilon \mathscr{C}_\varepsilon^{-1} =$ identity on $V_R \times T^l$.

ii) *There exists a C^∞-function $\boldsymbol{\Omega}_\varepsilon(\mathbf{A}')$ on $\mathbf{A}' \in V_R$ for ε small enough such that $\boldsymbol{\Omega}_\varepsilon(\mathbf{A}) \to_{\varepsilon\to 0} \boldsymbol{\omega}(\mathbf{A})$ and if*

$$\mathbf{A}'(0) \in V_R^\varepsilon = \left\{ \mathbf{A}' \mid \mathbf{A}' \in V_R,\ |\boldsymbol{\Omega}_\varepsilon(\mathbf{A}') \cdot \mathbf{v}| > \frac{1}{C_\varepsilon |\mathbf{v}|^l},\ \forall \mathbf{v} \in \mathbb{Z}^l,\ \mathbf{v} \neq 0 \right\} \tag{1.10}$$

with C_ε suitably chosen and diverging as $\varepsilon \to 0$, then

$$\mathbf{A}'(t) = \mathbf{A}'(0), \qquad \boldsymbol{\varphi}'(t) = \boldsymbol{\varphi}'(0) + \boldsymbol{\Omega}_\varepsilon(\mathbf{A}')t \tag{1.11}$$

is a solution to the equations of motion in the $(\mathbf{A}', \boldsymbol{\varphi}')$-coordinates, for all $\boldsymbol{\varphi}' \in T^l$.

iii) (*Consequence of ii)*):

$$\text{Vol } V_R^\varepsilon \xrightarrow[\varepsilon \to 0]{} \text{Vol } V_R \tag{1.12}$$

The above theorem says that after perturbation, $\varepsilon \neq 0$, most of the invariant tori do still exist except that they are slightly deformed: the functions $\boldsymbol{\Xi}_\varepsilon$, $\boldsymbol{\Delta}_\varepsilon$ are in fact a measure of the deformation; all the sets of the form

$$\begin{aligned} \mathbf{A} &= \mathbf{A}' + \boldsymbol{\Xi}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') \\ \boldsymbol{\varphi} &= \boldsymbol{\varphi}' + \boldsymbol{\Delta}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}'), \end{aligned} \qquad \boldsymbol{\varphi}' \in T^l, \tag{1.13}$$

are, by the above theorem, tori which are invariant whenever $\mathbf{A}' \in V_R^\varepsilon$. Furthermore they are traversed quasi-periodically with angular velocities $\boldsymbol{\Omega}_\varepsilon(\mathbf{A}')$.

From the proof of the theorem it emerges that the distance between two invariant tori is of order $0(\varepsilon^\theta)$, $\theta < \frac{1}{2}$, this together with the fact that $\boldsymbol{\Xi}_\varepsilon$, $\boldsymbol{\Delta}_\varepsilon \to_{\varepsilon \to 0} 0$ as $0(\varepsilon)$ easily implies that if $l = 2$

$$|\mathbf{A}(t) - \mathbf{A}(0)| \leq 0(\varepsilon^\theta), \quad \forall t \tag{1.14}$$

i.e. the action variables admit an a priori bound and the points of phase space are forced to stay forever close to the unperturbed torus on which they originally lie. The fact that $\theta < \frac{1}{2}$ is probably an artifact of the proof, and one expects that θ could be chosen equal to $\frac{1}{2}$.

The 'trapping' between surviving invariant tori does not necessarily take place if $l \geq 3$: in this case there is no a priori bound following from the rather packed set (see (1.11)) of invariant tori surviving perturbation, and one can only state the obvious bound (with R = size of phase space, see (1.1)):

$$|\mathbf{A}(t) - \mathbf{A}(0)| < O(R) \tag{1.15}$$

no matter how small ε is, for all times for which the motion stays in $V_R \times T^l$.

The bound (1.15) is believed to be, generically, saturated for suitably chosen values of t: when this happens one says that 'Arnold's diffusion' takes place: the name is given because Arnold explained the basic mechanism ('wisked tori mechanism') [3] underlying the above 'diffusion' in phase space (through the invariant tori almost filling it) by providing a simple concrete example which we recall, for completeness.

Arnold's example deals with a $l = 3$ system consisting in an unperturbed system built with a pendulum, a rotator and a clock (Fig. 2),

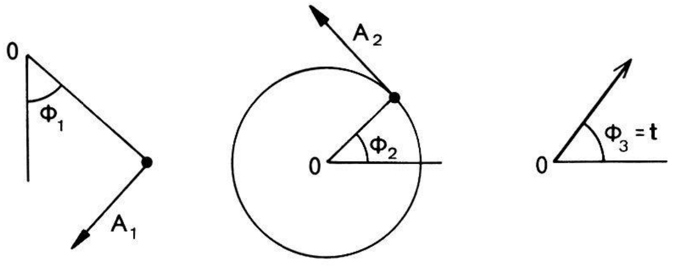

Figure 2.

i.e. a hamiltonian:

$$h(A_1, A_2, A_3, \varphi_1, \varphi_2, t) = \tfrac{1}{2}A_1^2 - (1 + \cos \varphi_1) + \tfrac{1}{2}A_2^2 + A_3 \tag{1.16}$$

note that $\dot{t} \equiv 1$, whence the name of 'clock'.

The perturbed hamiltonian is

$$H_\varepsilon = \tfrac{1}{2}A_1^2 - (1 + \cos \varphi_1)(1 + \varepsilon(\cos \varphi_2 + \sin t)) + \tfrac{1}{2}A_2^2 + A_3 \tag{1.17}$$

note that $\dot{t} = 1$.

Then Arnold shows [4], [3], that for fixed $\alpha_1 < \alpha_2$ there is ε_0 such that for all $|\varepsilon| < \varepsilon_0$ one can find initial data such that

$$A_2(0) < \alpha_1 \qquad A_2(\bar{t}) > \alpha_2 \tag{1.18}$$

for a suitable $\bar{t}$.

As one can see from (1.17) this is a very simple example which is special only because (1.17) has a structure which implies that

$$A_1 \equiv 0, \qquad \varphi_1 \equiv \pi, \qquad A_2 \equiv A \tag{1.19}$$

is a family of solutions to the equations of motions for *all* A's. The proof of the above basic result of Arnold is very close to the proof of the existence of a homoclinic point in a forced pendulum.

2. Nekhorossev theorem

So Arnold diffusion can really take place and therefore the question arises on how long does one have to wait 'to see it'.

The basic result on this problem is the following theorem of Nekhorossev [5]:

Theorem 2. *Consider a system like* (1.1) *with h given by I) or II) in Section* 1 ((1.5)–(1.8)), *i.e. being either a non-resonant harmonic oscillator or a rotator-like system.*

Then there exist constants a, b, T, $\varepsilon_0 > 0$ such that

$$|\mathbf{A}(t) - \mathbf{A}(0)| \leq R\left(\frac{\varepsilon}{\varepsilon_0}\right)^a \qquad \forall\, |t| < Te^{(\varepsilon/\varepsilon_0)^b} \equiv T_\infty(\varepsilon). \tag{2.1}$$

for $\varepsilon < \varepsilon_0$; here R is the size of phase space, see (1.1).

The interest of the above theorem lies in the fact that it holds basically 'under no assumptions' (i.e. no condition like $\mathbf{A}' \in V_R^\varepsilon$, (see (1.10)), typical of the KAM stability).

The theorem should be interpreted as saying that no Arnold diffusion can take place before a time scale $T_\infty(\varepsilon)$ which is 'perturbatively infinitely long'.

A new proof of Theorem 2 has been given recently by Benettin, Galgani, Giorgilli [6] who also determine explicitly the constants R, T, a, b, ε_0 following the scheme of proof of Nekhorossev, which is a recursive scheme along the lines of the proof of Arnold of Kolmogorov's theorem on the existence of quasi-periodic motions in quasi-integrable systems (essentially Theorem 1 of Section 1).

The above-mentioned proofs suggest that perturbation theory can be used for quantitative predictions on the details of the evolution up to exponentially long time scales (in terms of $1/\varepsilon$).

In a recent work by Benettin and Gallavotti [7] we have tried to make precise the latter statement, and our results are summarized below. At the same time we have produced a 'new' proof of Nekhorossev's theorem which is straightforwardly based on classical perturbation theory, i.e. without use of a recursive scheme.

In our approach the non-resonant harmonic oscillator case is treated first and with no extra assumptions besides the diophantine non-resonance condition (e.g. no assumption on $\partial^2 f/\partial\mathbf{A}\,\partial\mathbf{A}$ is required), [8], then we use the ideas of the above proof to treat the anisochronous cases (more interesting and technically less easy): however we feel that our paper does not contain new basic ideas beyond those already in Nekhorossev's work and represents, perhaps, an improvement from a technical point of view (different approach, better results in the harmonic case, better numerical bounds although the latter are not easy to compare because not all constants are worked out explicitly in the original paper).

We begin by discussing the non-resonant harmonic oscillator, case I), Section 1.

Theorem 3. *There exist two analytic canonical maps with domain containing $V_R \times T^l$ for ε small enough, i.e. $\varepsilon < \varepsilon_0$ = suitable constant, of the form*

$$\begin{cases} \mathbf{A} = \mathbf{A}' + \Xi_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') \\ \boldsymbol{\varphi} = \boldsymbol{\varphi}' + \Delta_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') \end{cases} \qquad \begin{cases} \mathbf{A}' = \mathbf{A} + \Xi'_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) \\ \boldsymbol{\varphi}' = \boldsymbol{\varphi} + \Delta'_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) \end{cases} \tag{2.2}$$

with $R^{-1}|\Xi_\varepsilon|$, $|\Delta_\varepsilon|$, $R^{-1}|\Xi'_\varepsilon$, $|\Delta'_\varepsilon| \leq B(\varepsilon/\varepsilon_0)^a$ for some B, $a > 0$; $\mathscr{C}_\varepsilon \mathscr{C}_\varepsilon^{-1}$ = identity on $V_r \times T^l$ for $\varepsilon < \varepsilon_0$, and

$$H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) = h_\varepsilon(\mathbf{A}') + e^{-(\varepsilon/\varepsilon_0)^{-c}} f_\infty(\mathbf{A}', \boldsymbol{\varphi}', \varepsilon) \tag{2.3}$$

with f_∞ and h_ε analytic in $(\mathbf{A}', \boldsymbol{\varphi}') \in V_R \times T^l$, continuously[2]) *in ε for $|\varepsilon| < \varepsilon_0$, if c is a suitable positive constant.*

By writing the equations of motion in the $(\mathbf{A}', \boldsymbol{\varphi}')$-coordinates one sees that 'nothing happens' up to a time scale $T_\infty(\varepsilon) \simeq 0(e^{+(\varepsilon/\varepsilon_0)^b c})$.

The above also implies that 'exponentially close to H_ε', i.e. 'exponentially close' to any perturbation of a non-resonant harmonic oscillator, there is an integrable system [8].

The constant b depends on the number of degrees of freedom: our estimate is that it can be taken as

$$b = \frac{1}{4(l+1)} \tag{2.4}$$

The bounds on Ξ_ε and (2.3) obviously imply

$$|\mathbf{A}(t) - \mathbf{A}(0)| \leqslant R\left(\frac{\varepsilon}{\varepsilon_0}\right)^\alpha \tag{2.5}$$

All the constants can be determined explicitly and their values (better in [7] than in [8]) are reported from [7] in Appendix A.

An important application of the above theorem is to the Fermi–Ulam–Parta chain of N oscillators tied at points 0, L

$$\left[\sum_{i=1}^{N} \tfrac{1}{2} p_i^2 + \tfrac{1}{2} \sum_{i=0}^{N} (q_i - q_{i+1})^2\right] + \varepsilon \sum_{i=0}^{N} (q_i - q_{i+1})^4 q_0 = 0,\ q_{N+1} = L \tag{2.6}$$

The 'free part' h, in square brackets, does verify the diophantine condition (1.6), for most N, because

$$\omega_k = \sqrt{2\left(1 - \cos\frac{2\pi}{N+1} k\right)}, \qquad k = 1, \ldots, N. \tag{2.7}$$

as one sees by studying the free part in its natural action angle coordinates.

We now discuss the more interesting case of the anisochronous systems (case II), Section 1).

For the purpose of illustration we discuss here only the rotator case

$$h(\mathbf{A}) = \tfrac{1}{2}\mathbf{A}^2 \tag{2.8}$$

in the sense that we shall occasionally take advantage, to simplify the discussion, of the identity

$$\boldsymbol{\omega}(\mathbf{A}) = \mathbf{A} \tag{2.9}$$

but all we say can be easily extended to the general convex case (1.8).

The basic notion necessary to formulate and understand the result is the notion of 'resonance'.

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[2]) i.e. derivatives of any order in $\mathbf{A}'$, $\boldsymbol{\varphi}'$ are continuous in ε.

Let Z^l be the set of the integer component vectors $\mathbf{v} = (v_1, \ldots, v_l)$ and let $\mathcal{M}$ be a linear r-dimensional subspace in Z^l, $r = 0, 1, \ldots, l$. We determine $\mathcal{M}$ by giving a basis $\mathbf{v}_1, \ldots, \mathbf{v}_l$, generating it by linear combinations with rational coefficients (only the combinations leading to a result in Z^l are, of course, allowed). Since there are many bases for a given $\mathcal{M}$ we shall only consider 'minimal bases', i.e. bases for which the number

$$w(\mathcal{M}) = \sup_{i=1,\ldots,r} |\mathbf{v}_i| \tag{2.10}$$

is minimal ('wave number of $\mathcal{M}$').

A resonance surface associated with $\mathcal{M}$ is the set

$$\Sigma_{\mathcal{M}} = \{\mathbf{A} \mid \mathbf{A} \in V_R, \boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{v} = 0, \forall \mathbf{v} \in \mathcal{M}\} \tag{2.11}$$

(for $r = 0$, $\Sigma_{(\mathbf{0})} \equiv V_R$; for $r = l$, $\mathcal{M} = Z^l$, $\Sigma_{Z^l} = \{\mathbf{A} \mid \boldsymbol{\omega}(\mathbf{A}) = \mathbf{0}\}$).

In our case (2.8) the surfaces $\Sigma_{\mathcal{M}}$ are planes orthogonal to $\mathcal{M}$ and have dimension $l - r$ (Fig. 3).

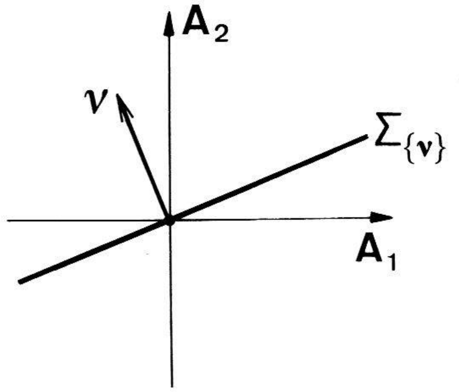

Figure 3.

The resonance surfaces of order ≥ 1 are the sets where perturbation theory is hard because its performance usually involves the operation of division by $\boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{v}$ ('small denominators problem').

Of course if perturbation theory runs into problems in dealing with data on some resonant surface, it will also run into problems in dealing with data too close to such surfaces.

This leads to the idea of defining 'resonance layers' around each resonance and then to classify the points of V_R according to the number of layers which contain them.

Since the resonant surfaces are dense in V_R it is necessary to realize that not all resonances are 'as bad': it will turn out from the discussion below, and this has been well known since Laplace, that the presence of some resonances affects the system only on a certain time scale, and for most resonances such a time scale is enormously large.

Hence one will simply disregard resonances whose corresponding time scale is beyond $T_\infty(\varepsilon)$. A heuristic argument suggests that a resonance of order $r = 1$, $\Sigma_{\{\mathbf{v}\}}$, has an associated time scale $\exp \xi |\mathbf{v}|$ for some $\xi > 0$; ξ depends on how regular is the analytic function $f(\mathbf{A}, \boldsymbol{\varphi})$ in the $\boldsymbol{\varphi}$-variables: usually ξ measures the size that the imaginary part of φ_j, $j = 1, \ldots, l$, can reach with $\boldsymbol{\varphi}$ still in the holomorphy domain of f.

The reason for the above estimate is very simple: the function εf can be written

$$\varepsilon f(\mathbf{A}, \boldsymbol{\varphi}) = \varepsilon \sum_{\mathbf{v} \in Z^l} f_{\mathbf{v}}(\mathbf{A}) e^{i\mathbf{v}\cdot\boldsymbol{\varphi}} \tag{2.12}$$

and hence the **r**th Fourier component of f (whose presence leads to the necessity of dividing by $\boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{v}$ in perturbation theory) has size $O(\varepsilon e^{-\xi|\mathbf{v}|})$. This means that one can neglect it for times shorter than $O(\varepsilon^{-1} e^{+\xi|\mathbf{v}|})$.

Since we are trying to reach a time scale of order $\exp[(\varepsilon_0/\varepsilon)^b c]$ it is natural to neglect all resonances $\mathcal{M}$ for which

$$w(\mathcal{M}) > N = \varepsilon^{-\tau} \tag{2.13}$$

where $\tau > 0$ is a parameter to be adjusted optimally, together with many others.

So we consider resonances $\mathcal{M}$ such that $w(\mathcal{M}) \leq N$.

Given $\sigma > 0$ we define for any $\mathcal{M}$ with $w(\mathcal{M}) \leq N$

$$\mathcal{V}^{\sigma,N}_{\mathcal{M}}(\varepsilon) = \{\mathbf{A} \mid \mathbf{A} \in V_R, \text{ distance of } \mathbf{A} \text{ from the real hyperplane generated by } \mathcal{M}^{\perp} \leq \varepsilon^{\sigma}\} \tag{2.14}$$

this will be called a 'resonance layer' for $\mathcal{M}$; then

$$\mathcal{B}^{\sigma,N}_{\mathcal{M}}(\varepsilon) = \mathcal{V}^{\sigma,N}_{\mathcal{M}}(\varepsilon) \Big/ \bigcup^{*}_{\substack{\mathcal{M}' \neq \mathcal{M} \\ w(\mathcal{M}') \leq N}} \mathcal{V}^{\sigma,N}_{\mathcal{M}'}(\tilde{\varepsilon}) \tag{2.15}$$

where * means that $\dim \mathcal{M}' = 1$, $\mathcal{M}' \not\subset \mathcal{M}$ and $\tilde{\varepsilon} = 3^{1/\sigma}\varepsilon$, which will be called the 'resonance set' for $\mathcal{M}$, and

$$\mathcal{B}^{r,\sigma,N}(\varepsilon) = \bigcup_{\substack{\dim \mathcal{M} = r \\ w(\mathcal{M}) \leq N}} \mathcal{B}^{\sigma,N}_{\mathcal{M}}(\varepsilon) \tag{2.16}$$

so that $\bigcup_{r=0}^{l} \mathcal{B}^{r,\sigma,N}(\varepsilon) = V_R$.

In our simple case ($\boldsymbol{\omega}(\mathbf{A}) = \mathbf{A}$) the sets $\mathcal{V}^{\sigma,N}_{\mu}(\varepsilon)$ are true layers with faces cut from hyperplanes: in general it is convenient to use a generalization of the above (2.14) ÷ (2.16) which it would be too long to describe here (and not too interesting).

By construction, the resonant sets are pairwise disjoint as $\mathcal{M}$ varies so that $w(\mathcal{M}) \leq N$: therefore to $\mathbf{A} \in V_R$ we can uniquely associate the corresponding 'leading resonance', $\mathcal{M}(\mathbf{A})$ such that $\mathcal{B}^{\sigma,N}_{\mathcal{M}(\mathbf{A})}(\varepsilon) \ni \mathbf{A}$.

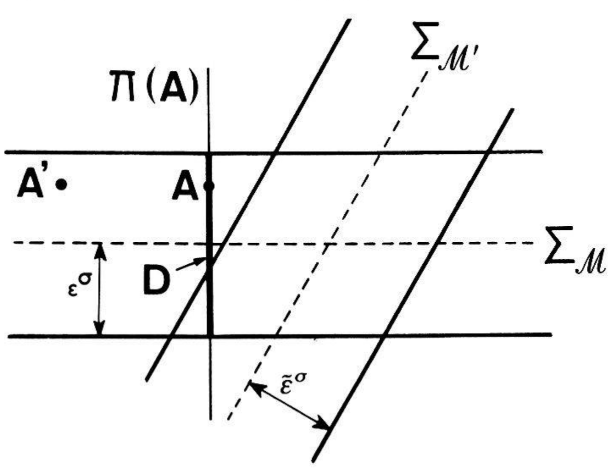

Figure 4.

As one can see from the picture, given $\mathbf{A} \in \mathscr{B}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ the r-dimensional plane $\pi(\mathbf{A})$ intersects $\mathscr{V}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ in a set D not entirely contained in $\mathscr{B}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$, in general (it is contained in the case of $\mathbf{A}'$ but not of A in Fig. 4).

However it is easy to see, by some geometric considerations, that if $\mathbf{A} \in D$, $\mathbf{A} \in \mathscr{B}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ then

$$|\mathbf{A} \cdot \mathbf{v}| > (\beta\varepsilon)^{\sigma} \qquad \forall \mathbf{v} \in Z^{l}, \qquad \mathbf{v} \notin \mathcal{M} \tag{2.17}$$

where β is a suitably chosen constant. Call $\mathcal{N}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ the set of $\mathbf{A} \in V_R$ for which (2.17) holds.

The above geometric definitions tell us that we can classify the points of V_R according to their leading resonances: and if a point $\mathbf{A}$ has leading resonance $\mathcal{M}$ then, $\forall \mathbf{v} \notin \mathcal{M}$, $\boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{v}$ is not 'too small', see (2.17).

Furthermore it is quite clear that if $\tau \ll \sigma$ then the leading resonance of most points in V_R will be $\mathcal{M} = \{0\}$; then most of the remaining points will be in the resonant sets with $r = 1$ 'first order resonances', etc.

This can easily be seen by observing that the volume of $\mathscr{V}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ is $O(R^{l-r}\varepsilon^{\sigma N})$ (because $\mathscr{V}_{\mathcal{M}}^{\sigma,M}(\varepsilon)$ is a 'tube' with r dimensions of order ε^{σ} and $l - r$ of order R); furthermore the number of resonances of order r is estimated by $\binom{N^l}{r} = O(N^{lr}) = \{$bound on the number of r-ples such that $|\mathbf{v}_i| \leq N\}$. Hence, recalling that $N = \varepsilon^{-\tau}$:

$$\operatorname{Vol} \mathscr{B}^{r,\sigma,N}(\varepsilon) \leq \binom{N^l}{r} \max_{\dim \mathcal{M} = r} \operatorname{Vol} \mathscr{B}_{\mathcal{M}}^{\sigma,N}(\varepsilon) \leq O(N^{lr}\varepsilon^{\sigma r}R^{l-r}) \leq Q\varepsilon^{(\sigma - l\tau)r} \operatorname{Vol} V_R \tag{2.18}$$

for some $R > 0$.

Therefore we choose $\sigma > l\tau$: to have interesting phenomena we shall fix $\sigma \ll \frac{1}{2}$ too (choices of σ less than $l\tau$ would lead to trivial results even if the theorem below were true with such choices: in fact it turns out that there are other technical conditions on σ, τ which force one to take $\tau = O(\sigma/l^2)$).

Our results [7] can be formulated as follows:

Theorem 4. *There is* $\varepsilon_0 > 0$, $0 < \sigma < \frac{1}{2}$, $\tau > 0$, $\tau < \sigma/l$, b_1, b_2, T_1, K *such that*
i) *if* $\mathbf{A}(0) \in \mathscr{B}_{\mathcal{M}}^{\sigma,N}(\varepsilon)$ *then* $\mathbf{A}(t) \in \mathcal{N}_{\mathcal{M}}^{\sigma,N}(\varepsilon/2)$ *and*

$$|\mathbf{A}(t) - \mathbf{A}(0)| \leq C_1\left(\frac{\varepsilon}{\varepsilon_0}\right)^{b_1} \tag{2.19}$$

for all $|t| \leq T_\infty(\varepsilon)$

$$T_\infty(\varepsilon) = T_1 e^{(\varepsilon_0/\varepsilon)^{b_2}} \tag{2.20}$$

ii) *on each set* $\mathcal{N}_{\mathcal{M}}^{\sigma,N}(\varepsilon/2)$ *one can define an 'adapted' system of canonical coordinates* $(\mathbf{S}, \mathbf{F}; \boldsymbol{\sigma}, \boldsymbol{\varphi})$, *'slow and fast coordinates', with* $\mathbf{S} = (S_1, \ldots, S_r)$, $\mathbf{F} = (F_1, \ldots, F_{l-r})$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)$, $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{l-r})$, $r = \dim \mathcal{M}$ *such that* H_ε *takes the form*

$$H_\varepsilon = \hat{h}_\varepsilon(\mathbf{S}, \mathbf{F}) + \varepsilon\hat{G}_\varepsilon(\mathbf{S}, \mathbf{F}; \boldsymbol{\sigma}) + e^{-(\varepsilon_0/\varepsilon)^{2b_2}}\hat{f}_\infty(\mathbf{S}, \mathbf{F}; \boldsymbol{\sigma}, \boldsymbol{\varphi}, \varepsilon) \tag{2.21}$$

with $\hat{h}_\varepsilon$, $\hat{G}_\varepsilon$, $\hat{f}_\infty$ *analytic in* $(\mathbf{S}, \mathbf{F}, \boldsymbol{\sigma}, \boldsymbol{\varphi})$ *continuously in* ε *for* $|\varepsilon| < \varepsilon_0$.

iii) *The set* $\mathcal{N}^{\sigma,N}_{\mathcal{M}}(\varepsilon/2)$ *is contained in the set where*

$$\left|\frac{\partial \hat{h}_\varepsilon(\mathbf{S},\mathbf{F})}{\partial S_j}\right| \leqslant \varepsilon^\sigma K, \qquad j=1,\ldots,r \tag{2.22}$$

if K is a suitably chosen constant.

Given **F** *let* $\mathbf{S}^*(\mathbf{F})$ *be defined by*

$$\frac{\partial \hat{h}_\varepsilon}{\partial \mathbf{S}}(\mathbf{S}^*(\mathbf{F}),\mathbf{F}) = \mathbf{0} \tag{2.23}$$

(which always admits a solution $(\mathbf{S}^*(\mathbf{F}),\mathbf{F}) \in \Sigma_{\mathcal{M}}$*).*

iv) *Let* $(\mathbf{S}_0, \mathbf{F}_0, \boldsymbol{\sigma}_0, \boldsymbol{\varphi}_0) \in (\mathcal{B}^{\sigma,N}_{\mathcal{M}}(\varepsilon)\times T^l)$[3]) *and write the motion with this initial datum as follows*

$$\begin{aligned}&\mathbf{S}(t) = \mathbf{S}^*(\mathbf{F}_0) + \sqrt{\varepsilon}\,\mathbf{s}(\sqrt{\varepsilon}\,t),\ \boldsymbol{\sigma}(t) = \boldsymbol{\gamma}(\sqrt{\varepsilon}\,t),\\ &\mathbf{F}(t) = \mathbf{F}_0 + \sqrt{\varepsilon}\,\mathbf{f}(\sqrt{\varepsilon}\,t),\ \boldsymbol{\varphi}(t) = \boldsymbol{\delta}(\sqrt{\varepsilon}\,t)\end{aligned} \tag{2.24}$$

Then the new variables $(\mathbf{s},\mathbf{f},\boldsymbol{\gamma},\boldsymbol{\delta})$ *are canonical coordinates whose motion is described by a hamiltonian of the form*

$$\frac{1}{\sqrt{\varepsilon}} h_{\mathbf{F}_0}(\mathbf{f}) + \{\tfrac{1}{2}L_{\mathbf{F}_0}(\mathbf{f})\mathbf{s}\cdot\mathbf{s} + V_{\mathbf{F}_0}(\boldsymbol{\gamma})\} + \sqrt{\varepsilon}\,V^{(1)}_{\mathbf{F}_0}(\mathbf{s},\mathbf{f};\boldsymbol{\gamma};\varepsilon) + e^{-(\varepsilon_0/\varepsilon)^{b_2}}\tilde{f}_\infty(\mathbf{f},\mathbf{s};\boldsymbol{\delta},\boldsymbol{\gamma};\varepsilon) \tag{2.25}$$

where $h_{\mathbf{F}_0}$, $V_{\mathbf{F}_0}$, $V^{(1)}_{\mathbf{F}_0}$ *are linear scalar functions of h, f and their first two derivatives and* $L_{\mathbf{F}_0}$ *is a* $r\times r$*-matrix valued positive definite function of* **f** *linearly depending on* $\partial^2 h/\partial\mathbf{A}\,\partial\mathbf{A}$. *They are analytic in* **f**, **s**, $\boldsymbol{\delta}$, $\boldsymbol{\gamma}$ *continuously in* ε *with domain of definition* $(\mathcal{M}^{\sigma,N}_{\mathcal{M}}(2\varepsilon)\times T^l)$ *together with* f_∞.

v) *Given h,* $\mathbf{F}_0$ *and varying f one can give to* $V_{\mathbf{F}_0}(\boldsymbol{\gamma})$ *any prescribed form* $V(\boldsymbol{\gamma})$ *analytic on* T^l.

Remarks. 1) At fixed $Y>0$ (e.g. $Y=1$) the initial data

$$\begin{cases}\mathbf{S}(0) = \mathbf{s}(0)\sqrt{\varepsilon}, & |\mathbf{s}(0)|<Y\\ \mathbf{F}(0)=\mathbf{F}_0 & \text{with } (\mathbf{S}(0),\mathbf{F}(0))\in\mathcal{B}^{\sigma,N}_{\mathcal{M}}(\varepsilon)\ ^{4})\end{cases} \tag{2.26}$$

are in the domain of applicability of the above theorem if ε is small enough.

2) All the constants can be determined explicitly.

3) See next section for a deeper discussion of the meaning of the above theorem.

3. Time scales near a resonance

The following interpretation of the above theorem is suggestive.

The hamiltonian (2.25) shows that in the perturbed motions one should

[3]) By this we mean that the datum in the old coordinates is in $\mathcal{B}^{\sigma,N}_{\mathcal{M}}(\varepsilon)\times T^l$.

[4]) See preceding footnote.

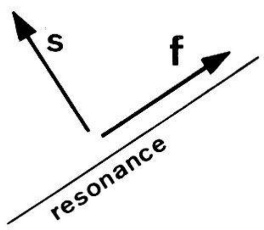

Figure 5.

distinguish three time scales:

1) $t \leqslant O(1/\sqrt{\varepsilon})$ 'unperturbed time scale'
2) $t \geqslant T_\infty(\varepsilon)$ 'strongly perturbed regime'
3) $1/\sqrt{\varepsilon} \leqslant t \leqslant T_\infty(\varepsilon)$ 'perturbative regime'.

During the perturbative regime there is a sharp distinction between slow and fast variables (Fig. 5). The fast variables have constant momenta ($\mathbf{f} = \text{const}$) and the corresponding angles rotate at angular speed of $O(1)$.

These pairs of variables evolve uninterestingly: on the other hand the slow variable evolution (after a rescaling of time of $\sqrt{\varepsilon}$) evolve as described by the hamiltonian with r degrees of freedom:

$$\tfrac{1}{2}L\mathbf{s}\cdot\mathbf{s} + V(\boldsymbol{\gamma}) + O(\sqrt{\varepsilon}) \tag{3.1}$$

and $V(\boldsymbol{\gamma})$ is an essentially arbitrary function on T^r (linearly depending on f): it is arbitrary as long as we are allowed to vary f.

The main point is that (3.1) is *not* a small perturbation of an integrable system. Hence if $r \geqslant 2$, i.e. if the resonance is of order higher than the first, (3.1) is susceptible to produce 'irreversible behaviour', i.e. strong dependence on initial conditions and chaotic phenomena. And using the arbitrariness of V one can produce explicit examples of homoclinic points near a given resonance surface $\Sigma_{\mathcal{M}}$ by suitably choosing $V(\boldsymbol{\gamma})$. The presence of the $O(\sqrt{\varepsilon})$ corrections will not perturb their presence since they are structurally stable objects.

In any event, the control of the motion up to a time scale $T_\infty(\varepsilon)$ is basically expressed by (3.1) and hence is 'computable' in perturbation theory, at least if one is able to investigate the desired properties of (3.1).

If the leading resonance of the initial datum is of order $r = 0$, (3.1) 'does not exist' (*no* **S** variables), the very long intermediate scale 'disappears' and the unperturbed scale extends up to $T_\infty(\varepsilon)$: since, as noted, most of phase space consists of points with trivial leading resonance ($\mathcal{M} = (\mathbf{0})$, i.e. $\mathbf{A} \in \mathscr{B}^{0,\sigma,N}(\varepsilon)$) we see here the mechanism underlying the connection between the Nekhorossev theorem and the KAM theorem.

If the leading resonance is of order $r = 1$ (3.1) is a one-dimensional 'pendulum': in this case the intermediate time scale exists but the motions are integrable (because one-dimensional pendula are integrable) and any measurement of quantities linked to chaotic behaviour (like Lyapunov exponents) is bound to give trivial results unless extended over a time scale exceeding $T_\infty(\varepsilon)$.

If $l = 2$ most of phase space, except perhaps the little box $|A_1| < \varepsilon^\sigma$ will be either in $\mathscr{B}^{0,\sigma,N}$ or in $\mathscr{B}^{1,\sigma,N}$: hence in such a case we cannot expect to see any chaotic phenomena, before a time scale $T_\infty(\varepsilon)$, by randomly sampling the initial data far away from the origin.

Appendix A. Values of the constants in Theorem 3 (harmonic case, from [7])

The function f is supposed holomorphic in

$$W = \{|\mathrm{Re}\, A_i| < R + \rho,\ |\mathrm{Im}\, A_i| < \rho,\ |\mathrm{Im}\, \varphi_i| < \xi\} \tag{A.1}$$

for some ρ, $\xi > 0$. For any g holomorphic on W we set, if g is C^p-valued:

$$\|g\| = \sup_{\substack{i=1,\dots,p \\ (\mathbf{A},\boldsymbol{\varphi})\in W}} |g^{(i)}(\mathbf{A},\boldsymbol{\varphi})| \tag{A.2}$$

Then let

$$E = \left\|\frac{\partial h}{\partial \mathbf{A}}\right\| \equiv \left\|\frac{\partial f}{\partial \mathbf{A}}\right\| + \frac{1}{\rho}\left\|\frac{\partial f}{\partial \boldsymbol{\varphi}}\right\| \tag{A.3}$$

(in this case, since $h = \boldsymbol{\omega}\cdot\mathbf{A}$, $E = \sup_i |\omega_i|$): the identity in (A.3) is an hypothesis which is not restrictive because we have ε free in (1.1) and we are interested in $\varepsilon \to 0$. Then

$$\varepsilon_0 = (2^{10l+g}(l+1)^{l+1}\xi^{-2l-1}CE)^{-2}$$

$$T_\infty = \frac{1}{E\sqrt{\varepsilon}}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/\varepsilon^b} \qquad b = \frac{1}{4(l+1)}$$

$$\|\mathbf{A}(t) - \mathbf{A}(0)\| \le R\left(\frac{1}{2}\frac{\rho}{R}\right)\sqrt{\frac{\varepsilon}{\varepsilon_0}}, \qquad |t| \le T_\infty \tag{A.4}$$

Appendix B. Values of the constants in Theorem 4 (from [7])

Using the notations (A.1), (A.2), (A.3):

$$\varepsilon_c = \left(B_e^{-1}\left(\frac{E}{\rho I}\right)^2 \xi^{-2l-2}\right)^8, \qquad \varepsilon_1 = \left(\frac{\rho I}{2E}\right)^8$$

$$\varepsilon_0 = \min\left(\tfrac{1}{2}\varepsilon_c, \varepsilon_1\right), \qquad B_l = 2^{22l+18}\rho^{4l+1}$$

$$b = \tau = \frac{1}{8l^2}, \qquad \sigma = \tfrac{1}{16} \tag{B.1}$$

$$\|\mathbf{A}(t) - \mathbf{A}(0)\| \le \frac{4E}{I}(l+1)\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-4\sigma} \qquad \forall\, |t| \le \frac{1}{I\rho}e^{(\xi/8)\varepsilon^{-\tau}}$$

also

$$\|\boldsymbol{\Xi}_\varepsilon\|, \|\boldsymbol{\Xi}'_\varepsilon\| \le 8l\frac{E}{I}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-4\sigma}$$

$$\|\boldsymbol{\Delta}_\varepsilon\|, \|\boldsymbol{\Delta}'_\varepsilon\| \le \xi\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-4\sigma} \tag{B.2}$$

where I is a constant introduced, for dimensional reasons, to write $h(\mathbf{A}) = \frac{1}{2}(\mathbf{A}^2/I)$.

Also, $\forall \varepsilon < \varepsilon_0$:

$$\frac{1}{\rho}\left|\frac{\partial f_\infty}{\partial \boldsymbol{\varphi}}(\mathbf{A}, \boldsymbol{\varphi})\right| + \left|\frac{\partial f_\infty}{\partial \mathbf{A}}(\mathbf{A}, \boldsymbol{\varphi})\right| \leqslant \xi \varepsilon_0^{\sigma+\tau}. \tag{B.3}$$

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