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Decay of unstable states in presence of fluctuations[1])

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Abstract. We consider the non-linear diffusion of a single stochastic variable with white gaussian noise. For the equivalent Fokker–Planck description, we use its mathematical correspondence to the Schroedinger equation to provide a general recipe for constructing classes of exactly solvable models. We illustrate our method by considering particular examples: these include diffusion problems in double barrier potentials with discrete or continuous spectra. We then use these models to discuss the decay of an unstable or a metastable state in presence of fluctuations. The non-linear decay in such models is found to exhibit complex behaviour not deducible from a linearization procedure. All the results are analytic and exact.

1. Introduction

The evolution of systems far from their thermodynamical equilibrium is particularly sensitive to fluctuations near the unstable equilibria of their equations of motions. Indeed, it is the fluctuations which either drive the system to its final stationary state or which are able to produce phenomena of nucleation type where the final global state is reached by diffusion over an activation barrier. It is therefore essential to study in detail, various behaviours which may be expected when an initially unstable (or metastable) state decays in presence of fluctuations. Such situations find a natural mathematical framework in the study of stochastic differential equations (SDE) and, when the Markov property constitute a good idealization, their associated Fokker–Planck equations which describe the evolution of the transition probability densities (TPD).

One of the basic difficulties occurring in the study of the decay of unstable (or metastable) state in presence of fluctuations, lies in the fact that the

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linearization procedures fail to produce reliable approximate results. Indeed, the TPD's of Gaussian types, always associated with linear SDE with white Gaussian noise (WGN) fluctuations, never present a multi-modal character which will be recognised in this paper to be more the rule than the exception even when relatively short transients are considered. While, when the diffusion in double-well potential is considered, it is clear that a transition will occur from an initially delta peaked TPD to a bimodal TPD (according to the scenario discussed by Van Kampen [1]), it is far less intuitively clear what the behaviour of the TPD will be in a way, purely repulsive monotonic potential (single barrier). This is one of the questions we address ourselves in the present paper and which we shall discuss by means of various exactly soluble models.

Our work is organised as follows: In Section 2, we formulate diffusion problems in one dimension for a fluctuating variable x which obeys a stochastic differential equation. We use the well-known Feynmann–Kac functional integral representation of the TPD and its equivalent form consisting of a development over the complete set of eigenfunctions of an associated Schrödinger problem [1a] to establish elementary results. We use this correspondence to solve exactly *new* diffusion problems. The spectrum of these new models is obtained from the original TPD by a global translation (*shifted spectrum dynamics*), in which the parameter δ (see Section 2) provides a tuning for the potential of the diffusion problem. We then provide a general criterion for the occurrence of a transition from uni-modal TPD at short times to an even-modal TPD at time of the order of the large time scale introduced by Kramers [2].

In Section 3, we illustrate the results of Section 2 by considering specific models leading to simple calculations but which by no means exhaust the possibilities. We discuss diffusion in asymmetric double barrier potential with discrete spectrum, diffusion in symmetric double barrier with purely continuous spectrum and diffusion in a single symmetric barrier which has a continuous and discrete spectrum. The asymptotic behaviour of these potentials is either quadratic or linear. Some applications of these models are discussed at the end of Section 3.

In Section 4, we discuss the decay of initially unstable states in presence of noise using the models solved in Section 3. We observe that the decay of the TPD is complex and may, according to the values of external parameters controlling the shape of the potential, proceed via transitions from an odd (uni-) modal initial state to an even-modal shape, even when the drift is monotically repulsive (single barrier). The variety of possible behaviour are summarised in Table I at the end of Section 4.

In Section 5, we discuss the behaviour of the mean path in the diffusion problems for asymmetric double (or single) barriers with discrete spectrum (one of the models introduced in Section 3). For this class of processes, the first moment takes an extremely simple form which permits us to investigate its behaviour for any transient time. The first moment corresponds to a macroscopic order parameter for the fluctuating variable x. Phenomena like boomerang behaviour or short time stabilization of the mean are observed (see Fig. 4).

In Section 6, the reader will find a brief summary and the conclusions.

2. Diffusion problems in one-dimension

Let us consider the stochastic differential equation (SDE) of the form:

$$dx = \left(-\frac{d}{dx}\Omega(x)\right) dt + \sqrt{2}\, dW_t$$
$$= (-2\alpha f(x))\, dt + \sqrt{2}\, dW_t; \qquad \alpha \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{2.1}$$

where $\Omega(x)$ stands for a generalised thermodynamic potential and dW_t is the white Gaussian noise (WGN) with the statistical properties [2]:

$$\langle dW_t \rangle = 0 \tag{2.2a}$$
$$\langle dW_t\, dW_\tau \rangle = \delta(|t-\tau|) \tag{2.2b}$$

Due to the choice equations (2.2a, b), the stochastic process $x(t)$, the solution of equation (2.1), is itself Markovian and its transition probability density (TPD) obeys a Fokker–Planck equation (FPE) of the form [2]:

$$\frac{\partial}{\partial t} P(x, t \mid x_0, 0) = \mathscr{F} P(x, t \mid x_0, 0)$$
$$= 2\alpha \frac{\partial}{\partial x}(f(x) P(x, t \mid x_0, 0)) + \frac{\partial^2}{\partial x^2} P(x, t \mid x_0, 0) \tag{2.3}$$

with the usual properties:

$$P(x, t \mid x_0, 0) > 0 \qquad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}^+ \tag{2.3a}$$
$$P(x, 0 \mid x_0, 0) = \delta(x - x_0) \tag{2.3b}$$
$$\int_{\mathbb{R}} P(x, t \mid x_0, 0)\, dx \equiv 1, \quad \forall t \in \mathbb{R}^+, \quad \forall x_0 \in \mathbb{R}. \tag{2.3c}$$

Now we introduce the notation:

$$f(x) = \frac{d}{dx}[\ln(\phi(x))] \Leftrightarrow P_s(x) = C\phi^{-2\alpha}(x), \tag{2.4}$$

where $P_s(x)$ is the time-independent solution of equation (2.3) and C a constant. From equation (2.4), it is then obvious that:

$$\phi(x) > 0, \qquad \forall x \in \mathbb{R}. \tag{2.4a}$$

The TPD, solution of the FPE equation (2.3), can be written in terms of a functional integral in the form [3]:

$$P(x, t \mid x_0, 0) = \hat{C}\phi^{-\alpha}(x) \int_{x_0, 0}^{x, t} d\mu \exp\left\{-\int_0^t \left(\frac{\dot{x}^2}{4} + V(x)\right) dt\right\},^{2)} \tag{2.5}$$

where the dot stands for the derivative with respect to the time, $\hat{C}$ is a constant

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2) The proper definition of the measure $d\mu$ is considered in [3]. This integral is the well-known Feynmann–Kac formula [3].

which depends on x_0 and the effective potential $V(x)$ has the form:

$$V(x) = \alpha^2 f^2(x) - \alpha \frac{d}{dx} f(x)$$
$$= -\alpha\phi^{-1}(x)\left[\frac{d^2}{dx^2}\phi(x)\right] + \alpha(\alpha+1)\phi^{-2}(x)\left[\frac{d}{dx}\phi(x)\right]^2. \tag{2.6}$$

In a completely equivalent manner, the TPD, equation (2.5), can be developed on the basis of a complete set of eigenfunctions of a Schrödinger problem, namely [1a]:

$$P(x, t \mid x_0, 0) = \phi^{-\alpha}(x)\phi^{\alpha}(x_0) \sum\!\!\!\!\!\!\int_{\text{sp}} \mathcal{S}_s(x)\mathcal{S}_s(x_0) \times \exp\{-[E(s) - E(0)]t\}, \tag{2.7}$$

where $\mathcal{S}_s(x)$ are the $\mathcal{L}^2(R, dx)$ solution of the Schrödinger equation:

$$\frac{d^2}{dx^2}\mathcal{S}_s(x) + [E(s) - V(x)]\mathcal{S}_s(x) = 0 \tag{2.8}$$

In equation (2.7), the notation $\sum\!\!\!\!\!\!\int_{\text{sp}}$ stands for the summation over the entire spectrum which may contain, when $x \in R$, both discrete and continuous parts. Furthermore when the stationary state $P_s(x) = C\phi^{-2\alpha}(x)$ is normalizable, the spectrum $E(s)$ contains the eigenvalue $E(0)$ and the ground state of the Schrödinger problem reads:

$$\mathcal{S}_0 = \sqrt{C}\,\phi^{-\alpha}(x). \tag{2.9}$$

From equations (2.6) to (2.9), we find the following elementary results:

Lemma 1 (Shifted spectrum dynamics). *Assume that the* TPD $P(x, t \mid x_0, 0)$ *is known for a particular effective potential* $V(x)$ *corresponding to the* SDE:

$$dx = \left[-2\alpha\frac{d}{dx}\{\ln \phi(x)\}\right]dt + \sqrt{2}\,dW_t, \qquad x \in \mathbb{R}. \tag{2.10}$$

Then ∀ SDE *of the form:*

$$dx = \left[-2\alpha_\delta\frac{d}{dx}\{\ln \phi_\delta(x)\}\right]dt + \sqrt{2}\,dW_t, \qquad x \in \mathbb{R}, \tag{2.11}$$

where $\phi_\delta(x)$ *is a positive definite solution of the equation:*

$$-\alpha_\delta\phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right] + \alpha_\delta(\alpha_\delta+1)\phi_\delta^{-2}(x)\left[\frac{d}{dx}\phi_\delta(x)\right]^2 = V(x) + \delta, \tag{2.12}$$

the TPD *associated with the* SDE. *equation* (2.11), *is given by:*

$$P_\delta(x, t \mid x_0, 0) = C_\delta\phi^{\alpha}(x)\phi_\delta^{-\alpha_\delta}(x)e^{-\delta t}P(x, t \mid x_0, 0) \tag{2.13}$$

where the normalization factor C_δ *is:*

$$C_\delta = \phi^{-\alpha}(x_0)\phi_\delta^{\alpha_s}(x_0). \tag{2.14}$$

Proof of Lemma 1. Introduce equation (2.12) into equation (2.5) and note that the exact differentials with respect to the time factorise out of the functional integral; (i.e. each path tube is ponderated by the same factor). Then equation (2.13) follows immediately.

Finally, equation (2.14) follows from (i) the normalization of $P(x, t \mid x_0, 0)$, (ii) the fact that:

$$P_\delta(x, 0 \mid x_0, 0) = C_\delta \phi^{\alpha}(x)\phi_\delta^{-\alpha_\delta}(x)\, \delta(x - x_0) \tag{2.15}$$

and (iii) the conservation of probability.

Lemma 2 (Multimodal character of the TPD). *Assume that* $\phi(x) = \phi(-x)$, $\forall x \in R$ *and that* $E(0)$ *in the expansion equation* (2.7) *is an isolated eigenvalue in the spectrum. Let* $\Delta E(s) = \{E(0) - \inf_{s \neq 0}(E(s))\} < 0$ *stand for the smallest non-vanishing term in the exponential of equation* (2.7). *Then, for* $t \gtrsim |\Delta E(s)|^{-1}$, *the TPD associated with the SDE equation* (2.1), *behaves as*:

$$P(x, t \mid 0, 0) \quad is \quad \begin{Bmatrix} even\text{-}modal \\ odd\text{-}modal \end{Bmatrix} \quad if$$

$$sign\left\{\Delta E(s) - \left[2\alpha\phi^{-1}(x)\frac{d^2}{dx^2}\phi(x)\bigg|_{x=0}\right]\right\} \quad is \quad \begin{Bmatrix} >0 \\ <0 \end{Bmatrix}. \tag{2.16}$$

Proof of Lemma 2.

$$\phi(x) = \phi(-x) \Rightarrow P(x, t \mid 0, 0) = P(-x, t \mid 0, 0).$$

Moreover we have: $\frac{d}{dx}\phi(x)\Big|_{x=0} = 0$. Hence the curvature R of $P(x, t \mid 0, 0)$ at the origin is (using equation (2.3)):

$$R = \frac{d^2}{dx^2}P(x, t \mid 0, 0)\bigg|_{x=0} = P(x, t \mid 0, 0)\left\{\frac{\partial}{\partial t}\ln P(x, t \mid 0, 0) - \left[2\alpha\phi^{-1}(x)\frac{d^2}{dx^2}\phi(x)\bigg|_{x=0}\right]\right\}. \tag{2.17}$$

Hence from the expansion equation (2.7) the assertion follows.

Corollary 1. *Assume that* $P(x, t \mid x_0, t_0)$ *is known and is associated with the effective potential* $V(x)$ *as in equation* (2.6). *Assume further that we have*:

$$\lim_{t \to \infty} P(x, t \mid x_0, 0) = P_s(x) \tag{2.18}$$

with

$$\int_{\mathbb{R}} P_s(x)\, dx < \infty \tag{2.19}$$

Hence $\forall\delta>0$ and $\phi_\delta(x)>0$ *satisfying equation* (2.12), *the* TPD *has the form given by equation* (2.13) *so that:*

$$\lim_{t\to\infty} P_\delta(x,t\mid x_0,0)=0, \tag{2.20}$$

i.e. the stochastic processes generated by $\phi_\delta(x)$ *does not admit a finite stationary state.*

Moreover, we have in this case, for $t>\delta^{-1}$,

$$P_\delta(x,t\mid 0,0)\quad is\quad \begin{Bmatrix} even\text{-}modal \\ odd\text{-}modal \end{Bmatrix}\quad if:$$

$$\operatorname{sign}\left\{-\delta-2\alpha_\delta\phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right]\Big|_{x=0}\right\}\quad is\quad \begin{Bmatrix} >0 \\ <0 \end{Bmatrix}. \tag{2.21}$$

Proof of Corollary 1. Equation (2.20) follows obviously from equation (2.13). Equation (2.21) follows from equation (2.13) and (2.7) by noting that the smallest term in the exponential of the development equation (2.7) is, in this case, given by δ.

Property 1 (Extended Darboux–Zheng transform [4]). Consider the Schrödinger equation (2.8) with $V(x)$ defined by equation (2.6). Introduce the integral transform:

$$\mathcal{S}_s(x)=\phi^{-\alpha}(x)\int^{x_\alpha}\phi(z)\psi_s(z)\,dz. \tag{2.22}$$

The eigenfunction $\psi_s(x)$ obeys the equation:

$$\frac{d^2}{dx^2}\psi_s(x)+[\tilde{E}(s)-\tilde{V}(x)]\psi_s(x)=0 \tag{2.23}$$

where in equation (2.23), the transformed effective potential $\tilde{V}(x)$ is obtained from $V(x)$, equation (2.6) by the substitution $\alpha\mapsto-\alpha$.

In particular, for $\alpha=1$, the case is fully discussed in [4].

Let us close this section by noting that we are pursuing the extension of the present analysis to vectorial stochastic processes for which the stationary state obeys the detailed balance condition.

3. Exactly soluble diffusion processes

This section is devoted to illustrations of Lemma 1 introduced in Section 2. These illustrations are by no means exhaustive but are chosen to cover representative possibilities.

Models A

Let us choose:

$$\phi(x)=\exp\{\tfrac{1}{2}x^2\};\qquad x\in\mathbb{R},\quad \alpha\in\mathbb{R}^+ \tag{3.1}$$

corresponding to the class of SDE (see equations (2.1) and (2.4)):

$$dx = (-2\alpha x)\,dt + \sqrt{2}\,dW_t \tag{3.2}$$

which is the famous Ornstein–Uhlenbeck (OU) process for which the TPD reads [1a]:

$$P_{\mathrm{OU}}(x, t \mid x_0, 0) = \frac{\sqrt{\alpha}\exp\{-\alpha(x - x_0\exp(-2\alpha t))^2(1-\exp(-4\alpha t))^{-1}\}}{(\pi[1-\exp(-4\alpha t)])^{1/2}} \tag{3.3}$$

In view of equation (2.6), the effective potential $V(x)$ reads for the choice equation (3.1):

$$V(x) = \alpha^2 x^2 - \alpha. \tag{3.4}$$

From Lemma 1, we conclude that a class of shifted spectrum dynamics can be obtained by solving:

$$-\alpha_\delta \phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right] + \alpha_\delta(\alpha_\delta+1)\phi_\delta^{-2}(x)\left[\frac{d}{dx}\phi_\delta(x)\right]^2 = \alpha^2x^2 + \delta - \alpha \tag{3.5}$$

with:

$$\phi_\delta(x) > 0, \qquad \forall x \in \mathbb{R}. \tag{3.5a}$$

In particular for $\alpha_\delta = -1$, the equations (3.5), (3.5a) admit positive definite solution [5]:

$$\phi_\delta(x) = e^{-z^2/4}\left[{}_1F_1\left(\frac{\delta}{4\alpha}, \frac{1}{2}, \frac{z^2}{2}\right) + \beta z\,{}_1F_1\left(\frac{\delta}{4\alpha}+\frac{1}{2}, \frac{3}{2}, \frac{z^2}{2}\right)\right], \tag{3.6}$$

where:

$$z = \sqrt{2\alpha}\,x \tag{3.6a}$$

$$\delta > 0 \tag{3.6b}$$

and [5, 6]:

$$|\beta| < \beta_c = \sqrt{2}\,\Gamma(\delta + \tfrac{1}{2})(\Gamma(\delta))^{-1}. \tag{3.6c}$$

The restrictions equations (3.6b, c) are imposed to guarantee the property equation (3.5a). The function ${}_1F_1(a, b, z)$ stands for the Kummer function [7]. Note that for $\beta = 0$, $\phi_\delta(x)$ is symmetric whereas a nonzero β introduces asymmetry in the function $\phi_\delta(x)$.

Therefore, introducing equation (3.6) into equation (2.11), taking into account that $\alpha_\delta = -1$, we obtain the class of dynamics:

$$dx = \left[2\frac{d}{dx}\ln(\phi_\delta(x))\right]dt + \sqrt{2}\,dW_t, \qquad x \in \mathbb{R} \tag{3.7}$$

for which the TPD reads: (using equation (2.13) with equation (3.6)),

$$P_A(x, t)x_0, 0) = P_{\mathrm{OU}}(x, t \mid x_0, 0)e^{-\delta t} \times \frac{{}_1F_1(\delta/4\alpha, \frac{1}{2}, \alpha x^2) + \beta\sqrt{2\alpha}\,x\,{}_1F_1(\delta/4\alpha + \frac{1}{2}, \frac{3}{2}, \alpha x^2)}{{}_1F_1(\delta/4\alpha, \frac{1}{2}, \alpha x_0^2) + \beta\sqrt{2\alpha}\,x_0\,{}_1F_1(\delta/4\alpha+\frac{1}{2}, \frac{3}{2}, \alpha x_0^2)} \tag{3.8}$$

and equation (3.6b) together with corollary 1 of Section 2 implies that

$$\lim_{t\to\infty} P_A(x, t \mid x_0, 0) = 0. \tag{3.8a}$$

The stochastic process equations (3.7) and (3.8) has been fully discussed for $\beta = 0$ in [8]. This situation corresponds to a symmetric $\phi_\delta(x)$. For $0 < |\beta| < \beta_c$ given by equation (3.6c) $\phi_\delta(x)$ is asymmetric. Let us briefly recall the properties of the thermodynamic potential $\Omega_\delta(x)$ associated with the SDE equation (3.7):

$$\Omega_\delta(x) = -2\ln[\phi_\delta(x)] \quad \text{is} \quad \begin{bmatrix} \text{(a symmetric double barrier)} & \text{when} & \beta = 0,\ \delta < \alpha \\ \text{(a symmetric single barrier)} & \text{when} & \beta = 0,\ \delta \geqslant \alpha \\ \text{(an asymmetric double barrier)} & \text{when} & 0 < \beta \ll \beta c,\ \delta < \alpha \\ \text{(an asymmetric single barrier)} & \text{when} & 0 < \beta \ll \beta c,\ \delta > 0 \end{bmatrix} \tag{3.9}$$

Finally $\forall\, |\beta| < \beta_c$ and $\forall \delta \in R^+$ we have:

$$\lim_{|x|\to\infty} \Omega_\delta(x) \simeq -\alpha x^2. \tag{3.10}$$

According to equations (3.9, a), the thermodynamic potential $\Omega_j(x)$ is always asympotically repulsive which immediately explains the result (equation (3.8a)).

Models B

Let us now consider the case:

$$\phi(x) = \cosh(x), \qquad \alpha \in \mathbb{R}^+, \quad x \in \mathbb{R} \tag{3.11}$$

from which we have, (see equations (2.1) and (2.4)):

$$dx = (-2\alpha \operatorname{tgh}(x))\, dt + \sqrt{2}\, dW_t, \qquad x \in \mathbb{R} \tag{3.12}$$

The TPD associated with equation (3.12) has been calculated in [9] and reads

$$P_W^\alpha(x, t \mid x_0, 0) = [\cosh(x)]^{-2\alpha}\left\{\Sigma + \int\right\}, \tag{3.13}$$

$$\Sigma = \Pi^{-1} \sum_{n=0}^{M} \frac{(\alpha - n)\exp(-n(2\alpha - n)t)}{n!\,\Gamma(2\alpha + 1 - n)}\, \theta_n(x)\theta_n(x_0) \tag{3.13a}$$

$$\theta_n(x) = (-1)^n 2^{\alpha - n}\Gamma(-n + \alpha + \tfrac{1}{2})[\cosh(x)]^{2\alpha+1} \times \frac{d^n}{d(\sinh(x))^n}[\cosh(x)]^{2n-2\alpha-1} \tag{3.13b}$$

and

$$\int = (2\pi)^{-1}\int_{\mathbb{R}^+} d\mu e^{-(\alpha^2+\mu^2)t}[\psi(\mu, x)\psi(-\mu, x_0) + \psi(-\mu, x)\psi(\mu, x_0)] \tag{3.13c}$$

with:

$$\psi(\mu, x) = e^{i\mu x}\cosh(x)\,{}_2F_1\left[\begin{matrix}-\alpha,\ \alpha+1, \\ 1+i\mu\end{matrix};\frac{1+\mathrm{tgh}(x)}{2}\right] \tag{3.13d}$$

where ${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}; z\right]$ stands for the Gauss hypergeometric function [7].

Furthermore in equation (3.13a) the summation is over

$$n = 0, 1, 2, \ldots, M, \qquad \alpha - 1 \le M < \alpha. \tag{3.14}$$

Hence, from equation (3.14) we see that if $\alpha < 1$, the eigenvalue spectrum is purely continuous.

According to equations (3.11) and (2.6), the effective potential $V(x)$ is, in this case:

$$V(x) = -\alpha + \alpha(\alpha+1)(\mathrm{tgh}(x))^2 = \alpha^2 - \alpha(\alpha+1)(\mathrm{sech}(x))^2. \tag{3.15}$$

Now, we use the Lemma 1 of Section 2 to generate a new class of dynamics by considering equation (2.12) which with equation (3.15) reads:

$$-\alpha_\delta \phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right] + \alpha_\delta(\alpha_\delta+1)\phi_\delta^{-2}(x)\left[\frac{d}{dx}\phi_\delta(x)\right]^2 = \alpha^2 - \alpha(\alpha+1)(\mathrm{sech}(x))^2 + \delta. \tag{3.16}$$

In particular, for $\alpha_\delta = -1$, we calculate in Appendix A, the solution of equation (3.16), which reads:[3])

$$\phi_\delta(x) = [\cosh(x)]^{-\sqrt{\delta-\alpha^2}}\ {}_2F_1\left[\begin{matrix}\dfrac{\gamma+\sqrt{\delta-\alpha^2}}{2}, & 1-\gamma-\sqrt{\delta-\alpha^2} \\ \tfrac{1}{2} & \end{matrix};(\mathrm{tgh}(x))^2\right] \tag{3.17}$$

with the definitions and restrictions:

$$\gamma = -\tfrac{1}{2} \mp \sqrt{\tfrac{1}{4} - \alpha(\alpha+1)}, \tag{3.17a}$$

$$\alpha \in \left[0, \frac{1}{\sqrt{2}} - \frac{1}{2}\right] \cong [0, 0.2071\cdots], \tag{3.17b}$$

$$\delta > \alpha^2 > 0, \tag{3.17c}$$

$$\gamma + \sqrt{\delta - \alpha^2} > 0. \tag{3.17d}$$

The asymptotic behaviour of $\phi_\delta(x)$, Eq. (3.17), reads (see Appendix A):

$$\lim_{|x|\to\infty} \phi_\delta(x) \sim [\cosh(x)]^{\sqrt{\delta-\alpha^2}} \sim \exp\{\sqrt{\delta-\alpha^2}\,|x|\}, \tag{3.18}$$

and from equation (3.16) itself with $\alpha_\delta = -1$, we have:

$$\phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right]\Bigg|_{x=0} = \delta - \alpha \tag{3.19}$$

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[3]) For the sake of simplicity, we confine ourselves to the symmetric solution. Asymmetric cases can also be considered.

Hence, the Lemma 1 of Section 2 implies that the SDE

$$dx = \left[2\frac{d}{dx}\{\ln \phi_\delta(x)\}\right] dt + \sqrt{2}\, dW_t; \qquad x \in \mathbb{R}, \tag{3.20}$$

with $\phi_\delta(x)$ given by equation (3.17) admits the TPD given by equation (2.13), namely in this case:

$$P_B(x, t \mid x_0, 0) = P_W^\alpha(x, t \mid x_0, 0)e^{-\delta t}$$

$$\times \frac{[\cosh(x)]^{\alpha-\sqrt{\delta-\alpha^2}}\, {}_2F_1\left[\begin{matrix}\dfrac{\gamma+\sqrt{\delta-\alpha^2}}{2}, & \dfrac{1-\gamma+\sqrt{\delta-\alpha^2}}{2} \\ \frac{1}{2} & \end{matrix}; (\mathrm{tgh}\,(x))^2\right]}{[\cosh(x_0)]^{\alpha-\sqrt{\delta-\alpha^2}}\, {}_2F_1\left[\begin{matrix}\dfrac{\gamma+\sqrt{\delta-\alpha^2}}{2}, & \dfrac{1-\gamma+\sqrt{\delta-\alpha^2}}{2}, \\ \frac{1}{2} & \end{matrix}; (\mathrm{tgh}\,(x_0))^2\right]} \tag{3.21}$$

Moreover from the corollary 1 of Section 2, we immediately have:

$$\lim_{t\to\infty} P_B(x, t \mid x_0, 0) = 0 \tag{3.22}$$

which is consistent with the fact that the potential $\Omega_\delta(x)$ is of repulsive nature. Indeed from equations (2.1), (2.11) and (3.18), we have:

$$\lim_{|x|\to\infty} \Omega_\delta(x) = \lim_{|x|\to\infty} -2\ln[\phi_\delta(x)] \sim -2\sqrt{\delta-\alpha^2}\,|x|. \tag{3.23}$$

Near the origin, the thermodynamic potential $\Omega_\delta(x)$ behaves like (see equation (3.19) and remember that $\phi_\delta(x) = \phi_\delta(-x) \Rightarrow \Omega_\delta(x) = \Omega_\delta(-x)$):

$\Omega_\delta(x)$ is [a symmetric single barrier for $\delta > \alpha$ (3.24a)
a symmetric double barrier for $0 < \delta < \alpha$ (3.24b)

Let us remark that the condition equation (3.24b) is compatible with the restrictions equations (3.17b, c, d) (take for instance: $\alpha = 0.1 \Rightarrow \gamma = -0.1258$, $\delta = 0.08$).

We close the discussion of the models B by pointing out the following additional features:

The fundamental difference between the diffusion problems in double (or single) barriers discussed in models A and B is as follows:

– The asymptotic behaviours of $\Omega_\delta(x)$ are

models A: *quadratic* (see equation (3.10)), (3.25a)

models B: *linear* (see equation (3.23)). (3.25b)

– The spectrum governing the dynamics are

models A: *purely discrete* (shifted spectrum of the OU process) (3.26a)

models B: *purely continuous* (shifted spectrum of the Wong process with $\alpha < 1$) (3.26b)

– Using the property 1 of Section 2, it is possible to solve exactly the double well problem (instead of the double barriers-model B). We therefore have the possibility of solving exactly a diffusion problem with a double-well thermodynamic potential whose dynamics is governed by a purely continuous spectrum. The same remark applies for the models A and has been fully discussed in [5].

Models C

Let us now consider the choice:

$$\phi(x) = \text{sech}\,(x), \qquad x \in \mathbb{R}, \tag{3.27}$$

giving rise to the SDE

$$dx = (2\alpha\,\text{tgh}\,(x))\,dt + \sqrt{2}\,dW_t. \tag{3.28}$$

This is a repulsive Wong process (i.e. $\alpha \to -\alpha$ in equation (3.12)).

The effective potential $V(x)$, equation (2.6), reads with equation (3.27):

$$V(x) = \alpha^2 - \alpha(\alpha-1)(\text{sech})\,(x))^2 \tag{3.29}$$

and hence, according to Lemma 1 of Section 2, a shifted spectrum dynamics results by solving the equation (see equation 2.12)

$$-\alpha_j\phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right] + \alpha_\delta(\alpha_\delta+1)\phi_\delta^{-2}(x)\left[\frac{d}{dx}\phi_\delta(x)\right]^2$$

$$= \alpha^2 - \alpha(\alpha-1)(\text{sech}\,(x))^2 + \delta, \tag{3.30}$$

$$\phi_\delta(x) > 0, \qquad \forall x \in \mathbb{R}. \tag{3.30a}$$

Choosing $\alpha_\delta = (\alpha - 1)$, a positive solution of the *non-linear* equation (3.30) is immediately found in form:

$$\phi_\delta(x) = \cosh\,(x), \tag{3.31}$$

with

$$\delta = 1 - 2\alpha. \tag{3.32}$$

Using equation (3.31), and the choice $\alpha_\delta = \alpha - 1$, equation (2.11) reads:

$$dx = [2(1-\alpha)\,\text{tgh}\,(x)]\,dt + \sqrt{2}\,dW_t, \qquad x \in \mathbb{R}. \tag{3.33}$$

Now, for $\alpha \geq 1$, equation (3.33) is identical in form with equation (3.12) (up to a unity shift of α in equation (3.12)). An application of Lemma 1 of Section 2, immediately gives the TPD associated with the repulsive Wong models, equation (3.28). Indeed we have:

$$P_W^{\alpha-1}(x, t \mid x_0, 0) = P_c(x, t \mid x_0, 0)e^{(2\alpha-1)t}\frac{[\cosh\,(x)]^{1-2\alpha}}{[\cosh\,(x_0)]^{1-2\alpha}} \tag{3.34a}$$

and hence

$$P_c^{\alpha}(x, t \mid x_0, 0) = P_W^{\alpha-1}(x, t \mid x_0, 0)e^{(1-2\alpha)t}\frac{[\cosh\,(x)]^{2\alpha-1}}{[\cosh\,(x_0)]^{2\alpha-1}}. \tag{3.34b}$$

In particular, for $\alpha = 1$, we have ($\alpha = 0$ in equation (3.12) gives rise to the Wiener process):

$$P_W^0(x, t \mid x_0, 0) = (4\pi t)^{-1/2} \exp\left\{-\frac{(x-x_0)^2}{4t}\right\} \tag{3.35}$$

and hence:

$$P_c^1(x, t \mid x_0, 0) = (4\pi t)^{-1/2} \frac{\cosh(x)}{\cosh(x_0)} e^{-t} \exp\left\{-\frac{(x-x_0)^2}{4t}\right\}. \tag{3.36}$$

The result equation (3.36) has previously been obtained in [10a]. For $\alpha = 1$, this model C has also been applied recently to the relation between stochastic processes and quantum measurements [10b].

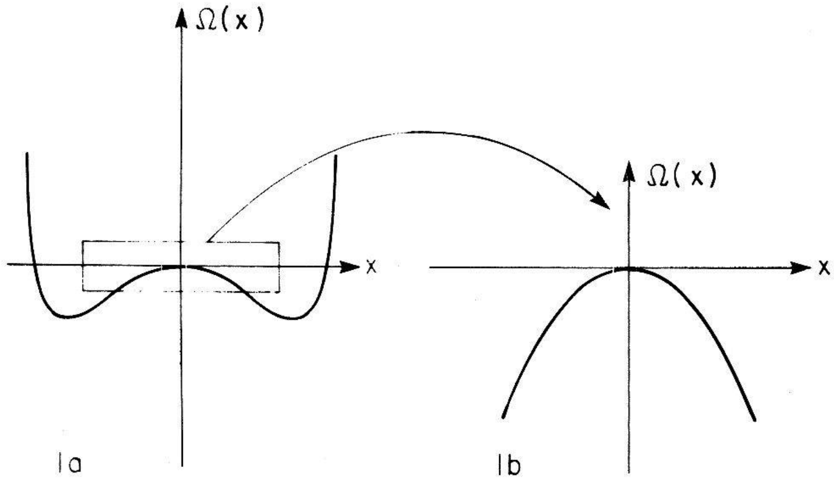
This model is also important in discussing the role of solutions from the Master equation and its approximate Fokker–Planck equation. Recently a form of Master equation with a one dimensional nonlinear, one step transition probability has been solved exactly [11]. We can show that this Master equation reduces in the Fokker–Planck approximation to the diffusion process of equation (3.33). The solution for such a Master equation [11], specifically its spectrum, is found to the qualitatively different than that for the corresponding Fokker–Planck equation.

Other applications using for instance the quantum mechanical Morse potential as $V(x)$ can be also considered to generate non-linear soluble diffusion problems.

4. Decay of unstable states – study of the transition probability density

The problem of the precise role of the fluctuations near the unstable equilibria occurs in numerous situations encountered in systems undergoing phase transitions and has therefore stimulated much activity [1, 3c, 12–20]. The basic difficulty of this question, as clearly mentioned in [1a], lies in the fact that the linearization procedure fails to give reliable results. This is obvious if one considers the decay of an initially delta peaked TPD at $x = 0$ in Figs. 1a and 2a. It is indeed clear that any linearization procedure around $x = 0$ and for t in the medium or large transient will fail to relate the transition from uni- to multi-modal probability densities which are expected for these situations.

Somehow more surprising is the case of the decay of a purely repulsive situation Fig. 1b which is always recovered when small to medium time transient phenomena are discused for the case Fig. 1a. One can indeed raise the question: Does the linearization procedures give at least qualitatively reliable results for this case? This question is interesting since if for instance bimodal TPD arise in sitiutions like Fig. 1b, then the transition is not explained by the scenario introduced by Van Kampen [1]. Indeed according to [1], the bimodal shape of the TPD expected in situations as Fig. 1a results from the "compactification" of the wings of the TPD in the region where $\Omega(x)$ is attractive. While this is obviously sufficient to give rise to a multimodal character of the TPD, we will show that it is



Figures 1a, b
Qualitative shape of the thermodynamic potential $\Omega(x)$ for various situations considered in the text.

not necessary. To discuss this question, the shifted spectrum dynamics is particularly well-suited since it gives an insight of the possible behaviours.

Let us immediately start with the example (Model C of Section 3):

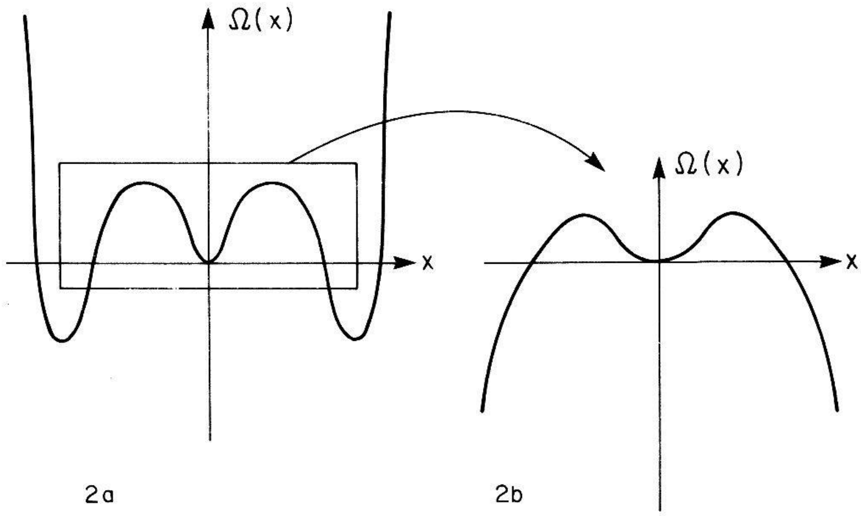
$$dx = (2\alpha \operatorname{tgh}(x))\, dt + \sqrt{2}\, dW_t. \tag{4.1}$$

For $\alpha = 1$, the TPD is given in equation (3.36) and it is advantageously rewritten in the form (when $x_0 = 0$):

$$P_c^1(x, t \mid 0, 0) = \frac{1}{4\sqrt{\pi t}}\left[\exp\left\{-\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right)^2\right\} + \exp\left\{-\left(\frac{-x}{2\sqrt{t}} + \sqrt{t}\right)^2\right\}\right], \tag{4.2}$$

which is a superposition of two Gaussians. Let us however remark that each of the Gaussians in equation (4.2) are *not* the solution of a diffusion process of the Wiener type which reads as equation (3.35).

Furthermore, from the sign of the curvature R (of the TPD equation (4.2)) at



Figures 2a, b
Qualitative shape of the thermodynamic potential $\Omega(x)$ for various situations considered in the text.

$x = 0$, we immediately deduce:

$$P(x, t \mid 0, 0) \text{ is } \begin{cases} \text{unimodal for } t < \frac{1}{2} \\ \text{bi-modal for } t > \frac{1}{2}. \end{cases} \tag{4.3}$$

Hence, the result equation (4.3) explicitly shows that for $t > \frac{1}{2}$, a simple linearization procedure (around $x = 0$) for which a uni-modal Gaussian will be obtained $\forall t \in R^+$, fails to describe the qualitative behaviour of $P(x, t \mid 0, 0)$.

As the simple solution, equation (4.2), is only valid for $\alpha = 1$ in equation (4.1), one can now infer whether the behaviour given by equation (4.3) is generic, $\forall \alpha \in R^+$. It is indeed important to realise that changing α in equation (4.1) corresponds in fact to changing the strength of the fluctuations in the model:

$$dx = (2 \operatorname{tgh}(x))\, dt + \sqrt{2}\, g\, dW_\tau, \tag{4.4}$$

where:

$$\tau = \alpha t \quad \text{and} \quad g = \alpha^{-1/2}. \tag{4.4a}$$

Let us now discuss the shape of the TPD associated with the repulsive Wong process equation (4.1). We shall consider two regimes ($\alpha \geq 1$ and $\alpha < 1$) separately.

(i) $\underline{\alpha \geq 1}$

This situation corresponds precisely to the SDE models C studied in Section 3. Let us now discuss the curvature R at the origin of the TPD equation (3.34b). We have:

$$\begin{aligned} R &= \frac{\partial^2}{\partial x^2} P_c^\alpha(x, t \mid 0, 0) = e^{(1-2\alpha)t} \frac{\partial^2}{\partial x^2} [P_W^{\alpha-1}(x, t \mid 0, 0)(\cosh(x))^{2\alpha-1}] \Big|_{x=0} \\ &= e^{(1-2\alpha)t} P_W^{\alpha-1}(x, t \mid 0, 0) \left\{ \left[\frac{\partial}{\partial t} \ln (P_W^{\alpha-1}(x, t \mid 0, 0) - 2(\alpha - 1)) \right] + 2\alpha - 1 \right\}, \end{aligned} \tag{4.5}$$

where the square bracket term in equation (4.5) has been obtained by using equation (2.17) with $\phi(x) = \cosh(x)$ and $\alpha \mapsto \alpha - 1$ corresponding to the Wong process $P_W^{\alpha-1}(x, t \mid x_0, 0)$. Now from [9] (see also equations (3.13a, c)) we have for $P_W^{\alpha-1}(x, t \mid 0, 0)$:

$$\Delta E(s) = E(0) - \inf_{\{s \neq 0\}} E(s) = \begin{cases} -(\alpha - 1)^2, & \text{for } 1 \leq \alpha \leq 2, \\ -2(\alpha - 1) + 1, & \text{for } \alpha > 2. \end{cases} \tag{4.6}$$

Hence from equation (4.5) and (4.6) and the Lemma 2 of Section 2, we end with:

$$e^{(2\alpha-1)t} R = \begin{cases} P_W^{\alpha-1}(x, t \mid 0, 0)[-(\alpha - 1)^2 + 1] & \text{for } 1 \leq \alpha \leq 2 \text{ and } t \lesssim |\alpha - 1|^{-2} \\ P_W^{\alpha-1}(x, t \mid 0, 0)[-2(\alpha - 2)] & \text{for } \alpha > 2 \text{ and } t \lesssim |3 - 2\alpha|^{-1} \end{cases} \tag{4.7}$$

From equation (4.7) we conclude that:

$$P_c(x, t \mid 0, 0) \quad \text{is} \quad \begin{cases} \text{even-modal} & \text{for} \quad 1 \leq \alpha \leq 2 \\ & \text{and} \quad t \lesssim |\alpha - 1|^{-2} \\ \text{odd-modal} & \text{for} \quad \alpha > 2 \\ & \text{and} \quad t \lesssim |3 - 2\alpha|^{-1} \end{cases} \qquad \begin{matrix} (4.8a) \\ \\ (4.8b) \end{matrix}$$

It is interesting to emphasize that the quantitative shape of $P_c(x, t \mid 0, 0)$ is then strongly related to the existence of a discrete spectrum. Indeed, we know from equation (3.14) that

$$P_W^{\alpha-1}(x, t \mid 0, 0) \quad \text{has} \quad \begin{cases} \text{(a purely continuous spectrum)} & \text{for} \quad 1 \leq \alpha \leq 2 \\ \text{(a discrete and continuous spectrum)} & \text{for} \quad \alpha > 2 \end{cases}$$

which then leads to the qualitatively different decays of the TPD as given in equations (4.8a, b).

(ii) case $\alpha < 1$

This particular regime requires some attention as to the associated Schrödinger problem reads (see equations (4.1) and (2.8)) [21]:

$$\frac{d^2}{dx^2} \mathcal{S}_s(x) + [E(s) + d^2 - \alpha(\alpha - 1)(\operatorname{sech}(x))^2] \mathcal{S}_s(x) = 0, \tag{4.9}$$

which for $\alpha < 1$, corresponds to a scattering problem in presence of the potential sketched in Fig. 3.

The spectrum is therefore purely continuous and reads:

$$E(s) - E(0) = \alpha^2 + s, \qquad s \in \mathbb{R}^+ \tag{4.10}$$

and hence the smallest term in the exponential of equation (2.7) is:

$$\Delta E(s) = -\alpha^2 \tag{4.10a}$$

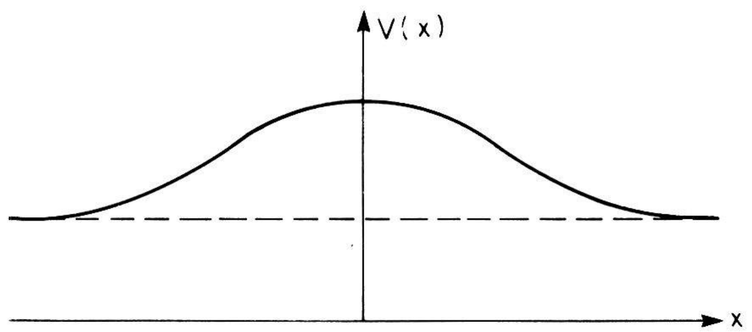


Figure 3
Sketch of the potential $V(x)$ for Model C, with $\alpha < 1$ studied in Section 4: see equation (4.9).

With equation (4.10a), the Lemma 2 of Section 2 implies:

$$\operatorname{sign}[-\alpha^2+2\alpha]>0 \quad \text{for} \quad 0<\alpha<1 \qquad t \gtrsim \alpha^{-2} \tag{4.11}$$

$\Rightarrow P_c(x, t \mid 0, 0)$ is even-modal.

Let us however remark that for $\alpha = 0$ (no drift term), we have the Wiener TPD which, being Gaussian, stays uni-modal $\forall t \in R^+$. This fact is consistent with equation (4.1) which gives no information for this particular limiting case.

From the above analysis, we conclude that for $\alpha > 2$ the TPD associated with the SDE equation (4.1) stays odd-modal for $t \gtrsim |3-2\alpha|^{-1}$ while for $0<\alpha\leq 2$ it suffers a transition to an even-modal shape for $t \gtrsim \alpha^{-2}$. This clearly shows the complexity of the decay of the TPD which according to equation (4.4a) stays odd-modal when small fluctuations (small $g \Rightarrow$ large α) are considered; while large fluctuations, even of the additive type, generate new dynamical features, i.e. even-modal TPD.

Let us now investigate some other situations. Let us consider the Models A of Section 3 in the symmetric case, namely $\beta = 0$ in equations (3.6) and (3.8). In this case the corollary 1 immediately holds and we have; using equations (2.21) and (3.5) with $\alpha_\delta = -1$:

$$\begin{aligned}
&\operatorname{sign}\left[-\delta + 2\phi_\delta^{-1}(x)\left[\frac{d^2}{dx^2}\phi_\delta(x)\right]\Big|_{x=0}\right] \\
&\quad = \operatorname{sign}[\delta - 2\alpha] \\
&\quad = \begin{cases} <0, & \delta < 2\alpha \\ >0, & \delta > 2\alpha. \end{cases}
\end{aligned} \tag{4.12}$$

Hence from the corollary 1, we deduce for $t \gtrsim |\delta|^{-1}$

$$P_A(x, t \mid 0, 0) \quad \text{is} \quad \begin{cases} \text{odd-modal for} & \delta < 2\alpha \\ \text{even-modal for} & \delta > 2\alpha \end{cases} \tag{4.13}$$

and here again transitions are observed. In fact in [8], we have studied these models in full detail and shown that the exact critical time t_c for which the transition odd to even modal (in this case uni- to bi-modal) occurs obeys the equation

$$t_c = \alpha^{-1} \ln\left(\frac{\delta}{\delta - 2\alpha}\right), \qquad \alpha \geq 0 \tag{4.14}$$

which is positively defined for

$$0 < 2\alpha < \delta. \tag{4.14a}$$

Finally, we investigate the behaviour of the Models B of Section 3. In this case we have (equations (2.21) with $\alpha_\delta = -1$ and 3.19):

$$\operatorname{sign}\{-\delta + 2(\delta - \alpha)\} = \delta - 2\alpha \tag{4.15}$$

for which we draw the same conclusion as from equation (4.13).

To close this section, we recapitulate in Table 1 below, the various features of the models studied.

Table 1

Models	$\Omega(x)$ (qualitative)	Spectrum	Shape of the TPD (qualitative)
A*	$0<\delta<\alpha$ double-barrier	discrete	unimodal $\forall t \in \mathbb{R}^+$
A*	$\alpha\leq\delta\leq 2\alpha$ single-barrier	discrete	unimodal $\forall t \in \mathbb{R}^+$
A*	$\delta>2\alpha$ single-barrier	discrete	bimodal for $t\geq\alpha^{-1}\ln(\delta/\delta-2\alpha)$
B†	$0<\delta<\alpha$ double-barrier	continuous	unimodal $\forall t \in \mathbb{R}^+$
B†	$\alpha\leq\delta\leq 2\alpha$ single-barrier	continuous	unimodal $\forall t \in \mathbb{R}^+$
B†	$\delta>2\alpha$ single-barrier	continuous	bimodal for $t\gtrsim|\delta|^{-1}$
C†	$0\leq\alpha\leq 2$ single-barrier	continuous	bimodal for $t\gtrsim\alpha^{-2}$
C†	$\alpha>2$ single-barrier	continuous & discrete	unimodal $\forall t \in \mathbb{R}^+$

*) We have proved in [8] } that (odd = uni / even = bi) for these situations.
†) We conjecture

5. Behaviour of the mean $\langle x(t)\rangle$

In [8], we have studied the behaviour of the mean $\langle x(t)\rangle$ defined by:

$$\langle x(t)\rangle=\int_{\mathbb{R}} xP(x,t\mid x_0,0)\,dx, \tag{5.1}$$

for the case $\beta=0$ of the Models A of Section 3. Here, we shall extend the discussion to the asymmetric thermodynamic potential $\Omega_\delta(x)$ obtained from equation (3.6) and (3.9) for $0\neq|\beta|<\beta_c$ defined by equation (3.6c). In Appendix B, we indicate the calculation of $\langle x(t)\rangle$, equation (5.1), when the TPD is as in equation (3.8). We find:

$$\langle x(t)\rangle=x_0\left[e^{-\alpha t/2}+\gamma(\alpha,x_0)\sinh\left(\frac{\alpha t}{2}\right)\right] \tag{5.2}$$

where

$$\gamma(\alpha,x_0)=1-\frac{1}{2\alpha x_0}\left(\frac{d}{dx}\Omega_\delta(x)\bigg|_{x=x_0}\right) \tag{5.2a}$$

From equation (5.2) we conclude that the velocity of the mean $\frac{d}{dt}\langle x(t)\rangle=\langle v(t)\rangle$ may vanish at the critical time t^* defined by:

$$\langle v(t^*)\rangle=0\Rightarrow$$
$$t^*=\alpha^{-1}\ln(2\gamma^{-1}-1), \tag{5.3}$$

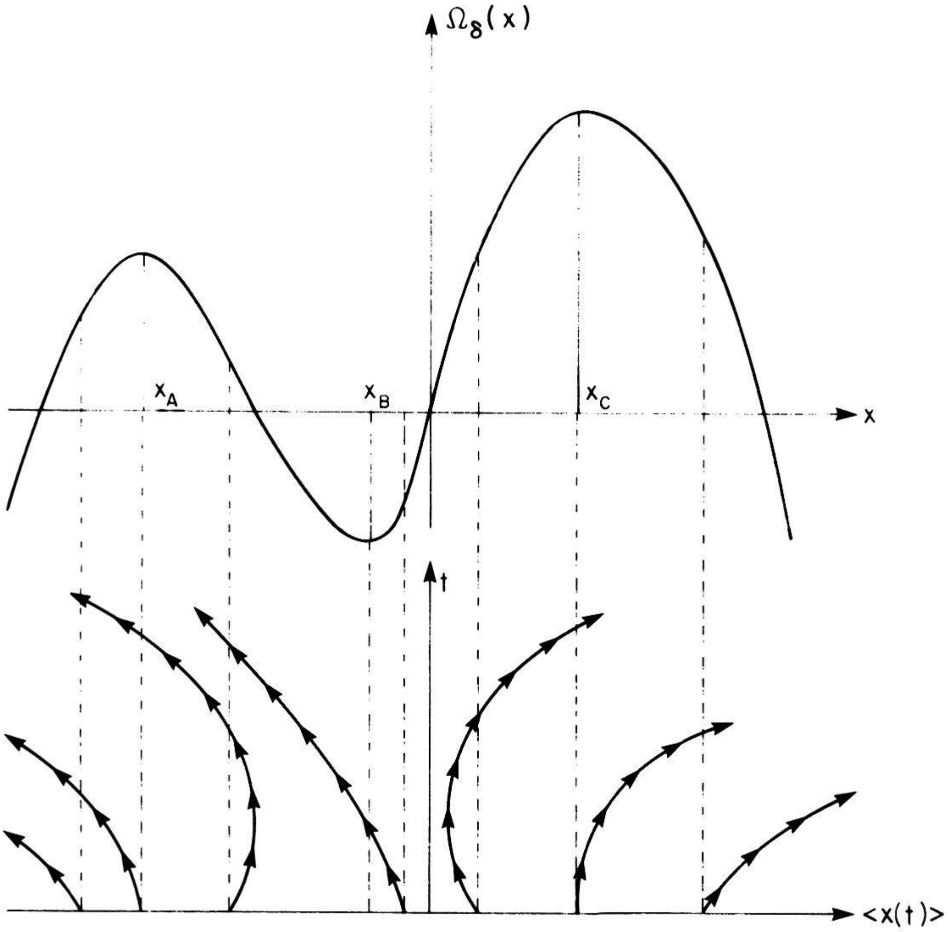


Figure 4a
Qualitative behaviour of the mean $\langle x(t) \rangle$ for initial conditions.

which is positively defined if:

$$\gamma \leq 1. \tag{5.4}$$

From equation (5.4), the following behaviours may occur (see Fig. 4a, b):

(i) For X_0 located at the extrema of $\Omega_\delta(x)$, t^* is identically zero. ($\gamma = 1 \Rightarrow t^* = 0$)

(ii) $0 < \delta/4\alpha < \frac{1}{4}$, $|\beta| \ll \beta_c$, (Fig. 4a)

For $0 < X_0 < X_c$ and $X_A < X_0 < X_B$, equation (5.4) is obviously satisfied and we observe boomerang behaviours of the mean path $\langle X(t) \rangle$. This phenomena is intrinsically due to the non-linearities as it is intuitively clear that it comes from the diffusion over the barrier.

(iii) $\delta/4\alpha > \frac{1}{4}$, (Fig. 4b)

For $0 < X_0 < X_c$, the condition, equation (5.4), is obviously satisfied and the mean $\langle x(t) \rangle$ presents the boomerang behaviour sketched in Fig. (4b). Therefore for short transient, the mean $\langle x(t) \rangle$ is *stabilized* as compared to the deterministic path. While this is intuitively clear from the diffusion over the barrier X_c, such behaviour is of course not obtainable from any linearization procedures.

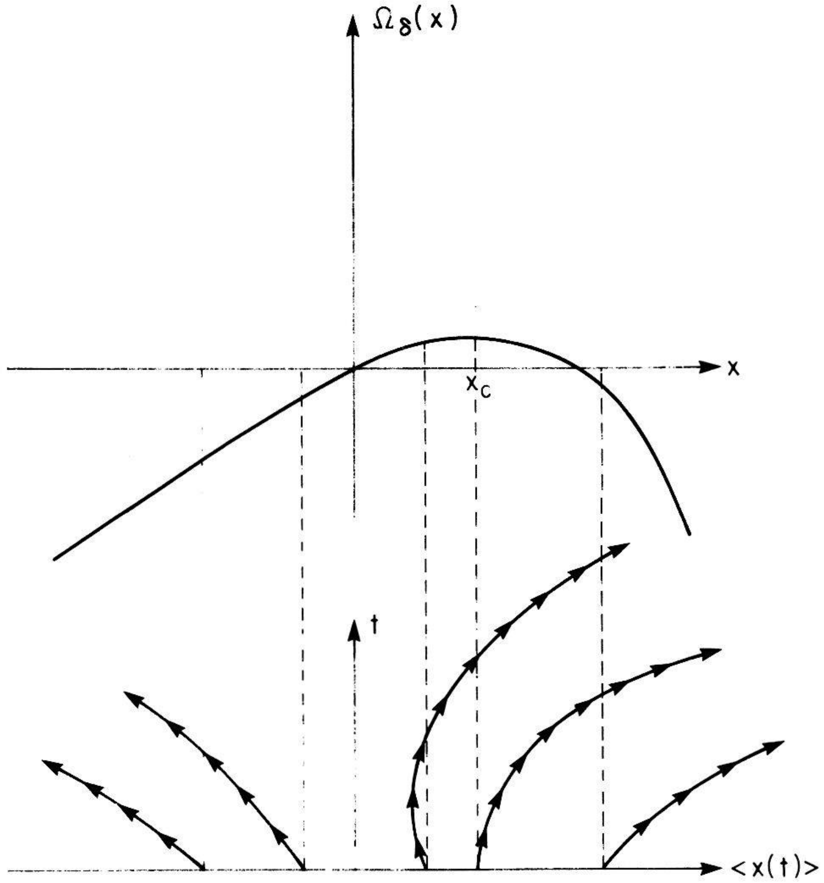


Figure 4b
Qualitative behaviour of the mean $\langle x(t) \rangle$ for initial conditions.

Let us close this section by mentioning that the above behaviours are typical for processes like nucleation and boomerang behaviours of $\langle x(t) \rangle$ are of course not restricted to the particular diffusion processes viz. the Models A of Section 3, as it is clear from an intuitive point of view.

6. Summary and conclusions

Our major interest in this paper has been to explore the behaviour of the transition probability density (TPD) characterizing the decay of an unstable (and metastable) state in presence of fluctuations. As clearly remarked by Van Kampen [1], the linearization procedures are not available to discuss this type of problem for medium and large transient time. Therefore we carried out our study using exactly soluble models. We started in Section 2 by introducing a general procedure (*shifted spectrum dynamics* in Lemma 1) to construct exactly soluble models of diffusion equations from already known ones. The new models so generated (see our illustrations in Section 3) are particularly well-suited to discuss the decay of the TPD of unstable states which we discussed in Section 4. The models treated include the diffusion in asymmetric double or single barrier potentials with purely discrete spectrum, symmetric double or single barriers with purely continuous spectrum, and a single symmetric barrier with mixed continuous and discrete spectrum. The shape of the TPD was analysed for large times (t^{-1} = first eigenvalue of the spectrum [2]) and presented complex behaviour in certain cases directly related to the existence or non-existence of discrete eigenvalues in the spectrum. We observed in particular the transition from uni- to even-modal shape of the TPD for monotonic repulsive potentials; such transitions cannot be explained in the context of Van Kampen's scenario for the decay. Indeed due to the fact that the potential is, in our case, purely repulsive, there is no possibility of "compactification" of the wings of the TPD in regions where the potential is attractive. In Section 5, we briefly discussed the time dependence of the mean path in asymmetric double- (or single-) barriers. Boomerang behaviours and short time stabilization due to fluctuations were observed, both behaviours being obviously not obtainable via linearization procedures.

Appendix A

For the choice $\alpha\delta = -1$, equation (3.16) reads:

$$\frac{d^2}{dx^2}\phi_\delta(x) = (\delta + \alpha^2 - \alpha(\alpha+1)(\operatorname{sech}(x))^2)\phi_\delta(x). \tag{A1}$$

Introducing

$$\gamma(\gamma-1) = -\alpha(\alpha+1) \Rightarrow \gamma = -\tfrac{1}{2} \mp \sqrt{\tfrac{1}{4} - \alpha(\alpha+1)} \tag{A2}$$

$$\phi_\delta(x) = [\cosh(x)]^\gamma \psi(-\sinh^2(x)), \tag{A3}$$

equation (A1) takes the form:

$$\begin{aligned}[-\sinh^2(x)][1+\sinh^2(x)]\frac{d^2}{d(-\sinh^2(x))}\psi(-\sinh^2(x)) \\ +[\tfrac{1}{2}+(r\gamma)\sinh^2(x)]\frac{d}{d(-\sinh(x))}\psi(-\sinh^2(x)) \\ -\left(\frac{\gamma+\delta-\alpha}{4}\right)\psi(-\sinh^2(x)) = 0.\end{aligned} \tag{A4}$$

Equation (A4) is immediately recognized to be the Gauss hypergeometric equation [22] with the solution (we confine ourselves to the symmetric cases):

$$\psi(-\sinh^2(x)) = {}_2F_1\left[\begin{matrix}\dfrac{\gamma+\sqrt{\delta-\alpha^2}}{2}, & \dfrac{\gamma-\sqrt{\delta-\alpha^2}}{2}, & \\ \tfrac{1}{2}, & ; & -\sinh^2(x)\end{matrix}\right] \tag{A5}$$

Now we use the properties [22]:

$$ {}_2F_1\begin{pmatrix} a, b, & \\ c, & ; z \end{pmatrix} = (1-z)^{-a}\,{}_2F_1\begin{pmatrix} a, c-b, & \\ c, & \dfrac{z}{z-1} \end{pmatrix} $$

$$ = (1-z)^{-b}\,{}_2F_1\begin{pmatrix} c-a, b, & \\ c, & \dfrac{z}{z-1} \end{pmatrix} $$

and equation (A5) with (A3) takes the respective forms:

$$ \phi_\delta(x) = $$

$$ [\cosh(x)]^{-\sqrt{\delta-\alpha^2}}\,{}_2F_1\begin{bmatrix} \dfrac{\gamma+\sqrt{\delta-\alpha^2}}{2}, & \dfrac{1-\gamma+\sqrt{\delta-\alpha^2}}{2}, & \operatorname{tgh}^2(x) \\ \frac{1}{2} & ; & \end{bmatrix} $$

$$ = [\cosh(x)]^{\sqrt{\delta-\alpha^2}}\,{}_2F_1\begin{bmatrix} \dfrac{\gamma-\sqrt{\delta-\alpha^2}}{2}, & \dfrac{1-\gamma-\sqrt{\delta-\alpha^2}}{2}, & \operatorname{tgh}^2(x) \\ \frac{1}{2} & ; & \end{bmatrix}. \quad \text{(A7b)} $$

By imposing the coefficients in equation (A7a) to be positive, we get the conditions of equation (2.17) which in turn imply $\phi_\delta(x) > 0$, $\forall x$ from the expansion of ${}_2F_1\begin{pmatrix} ab \\ c \end{pmatrix}; z)$.

Equation (A7b) is used to get the asymptotic development. Indeed we have [22]:

$$ {}_2F_1\begin{pmatrix} a, b, & \\ c, & ; z \end{pmatrix} \quad \begin{array}{l}\text{is absolutely convergent on the} \\ \text{unit circle for } \operatorname{Re}(a+b-c) < 0.\end{array} \quad \text{(A8)} $$

and we have [22]:

$$ {}_2F_1\begin{pmatrix} a, b & \\ c, & ; 1 \end{pmatrix} = \frac{\Gamma_{(c)}\Gamma_{(c-a-b)}}{\Gamma_{(c-a)}\Gamma_{(c-b)}}. \quad \text{(A9)} $$

From equation (A7b) we have:

$$ \operatorname{Re}(a+b-c) = -2\sqrt{\delta-\alpha^2} < 0 \quad \text{(A10)} $$

and:

$$ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{\Gamma(\frac{1}{2})\Gamma(2\sqrt{\delta-\alpha^2})}{\Gamma(\gamma+\sqrt{\delta-\alpha^2})\Gamma(\sqrt{\delta-\alpha^2}-\gamma)}. \quad \text{(A11)} $$

Using equations (3.17a, d) we see that the constant equation (A11) is positively defined and hence the asymptotic behaviour of (A7b) reads:

$$ \phi_\delta(x) \xrightarrow[|x|\to\infty]{} [\cosh(x)]^{\sqrt{\delta-\alpha^2}}. \quad \text{(A12)} $$

Appendix B

Here we calculate the integrals of the type:

$$\langle x^m(t)\rangle = N(t)\int_{\mathbb{R}} x^m \exp\{-\alpha(t)[x-\gamma(t)]^2\} \times \left[{}_1F_1\left(\zeta, \frac{1}{2}, \frac{x^2}{2}\right) + \beta x\, {}_1F_1\left(\mu, \frac{3}{2}, \frac{x^2}{2}\right)\right] dx \qquad \text{(B1)}$$

where $\alpha(t)$, $\gamma(t)$, μ and ζ are defined in equations (3.8) and (3.3).

We introduce the integral representations [22]:

$${}_1F_1\left(\zeta, \frac{1}{2}, \frac{x^2}{2}\right) = \frac{2}{\Gamma(\zeta)}\int_{\mathbb{R}^+} e^{-\lambda^2}\lambda^{2\zeta-1}\cosh(\sqrt{2}\,\lambda x)\, d\lambda \qquad \text{(B2)}$$

and

$${}_1F_1\left(\mu, \frac{3}{2}, \frac{x^2}{2}\right) = \frac{\sqrt{2}}{\Gamma(\mu)z}\int_{\mathbb{R}^+} e^{-\lambda^2}\lambda^{2\mu-1}\sinh(\sqrt{2}\,\lambda x) d\lambda. \qquad \text{(B3)}$$

Introducing equations (B2) and (B3) into equation (B1), exchanging the orders of integration yields after simple but long calculations:

$$\langle x(t)\rangle = x_0\left\{e^{-\alpha t/2} \times 2\sinh(\alpha t/2) \times \frac{2\zeta x_0\, {}_1F_1(\zeta+1, 3/2, x_0^2/2) + \beta\, {}_1F_1(\mu, 1/2, x_0^2/2)}{{}_1F_1(\zeta, 1/2, x_0^2/2) + \beta x_0\, {}_1F_1(\mu, 3/2, x_0^2/2)}\right\}. \qquad \text{(B4)}$$

Defining:

$$y_1(a, \hat{x}) = \exp\left\{-\frac{\hat{x}^2}{4}\right\} {}_1F_1\left(\frac{a}{2}+\frac{1}{4}, \frac{1}{2}, \frac{\hat{x}^2}{2}\right) \qquad \text{(B5)}$$

$$y_2(a, \hat{x}) = \hat{x}\exp\left\{-\frac{\hat{x}^2}{4}\right\} {}_1F_1\left(\frac{a}{2}+\frac{3}{4}, \frac{3}{2}, \frac{\hat{x}^2}{2}\right) \qquad \text{(B6)}$$

and using [7]:

$$\frac{d}{d\hat{x}} y_1(a, \hat{x}) + \tfrac{1}{2}\hat{x} y_1(a, \hat{x}) = (a+\tfrac{1}{2}) y_2(a+1, \hat{x}) \qquad \text{(B7)}$$

$$\frac{d}{d\hat{x}} y_2(a, \hat{x}) + \tfrac{1}{2}\hat{x} y_2(a, \hat{x}) = y_1(a+1, \hat{x}), \qquad \text{(B8)}$$

the form equation (B4) can be reduced to equations (5.2) with (5.2a).

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