A general theory of Bose-Einstein condensation

Autor(en): **Berg, M. van den / Lewis, J.T. / Pulè, J.V.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **59 (1986)**

Heft 8

PDF erstellt am: **13.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-115776

Helvetica Physica Acta, Vol. 59 (1986) 1271–1288

0018–0238/86/081271–18$1.50 + 0.20/0

A general theory of Bose–Einstein condensation

By M. van den Berg and J. T. Lewis

Dublin Institute for Advanced Studies, 10, Burlington Road, Dublin 4, Ireland

and

J. V. Pulè

University College Dublin, Belfield, Dublin 4, Ireland

(1. VI. 1985)

Abstract. We give a unified treatment of Bose–Einstein condensation in non-interacting systems of bosons with a general single-particle Hamiltonian. We show that there are two critical densities: ρ_c, at which singularities in the thermodynamic functions occur, and ρ_m below which there can be no macroscopic occupation of the ground state. We identify the two asymptotic spectral distributions which determine the thermodynamic functions and the macroscopic occupation of the ground state respectively.

1. Introduction

The aim of this paper is two-fold: to summarize compactly the results of a large number of rigorous results on Bose–Einstein condensation in the free boson gas; to relate the mathematical structure of the phase-transition to its physical properties. We believe that this is a necessary preliminary to a successful assault on the problem of the existence of condensation in the interacting boson gas. There is also renewed interest among experimentalists in the free boson gas itself because of the prospect of attaining the Bose–Einstein critical density in magnetically trapped atomic hydrogen. The plan is this: in the introduction we establish some notation, give a brief account of the results and discuss their relation with earlier work; in Section 2 we recall some facts about the grand canonical ensemble and prove some preliminary lemmas; in Section 3 we establish the equation of state; in Section 4 we discuss generalized condensation, in Section 5 macroscopic occupation of the groundstate and in Section 6 the anomalous fluctuations.

The traditional description of Bose–Einstein condensation is this: in a system of non-interacting bosons in thermal equilibrium the excited states saturate at a critical value ρ_c of the density; when the density ρ is increased beyond this value the excess $\rho - \rho_c$ goes into the zero-energy state. The phenomenon is sometimes described as ‘condensation in momentum space’. The condensate has zero entropy as well as zero energy, and so makes no contribution to the pressure. Consequently, the pressure-density isotherm has a flat part; the pressure increases with increasing density for densities below ρ_c and thereafter remains constant. The occupation of the zero-energy state on a macroscopic scale has other consequences: there are changes in the coherence properties of the system and anomolous fluctuations in the particle number density. There is a basic difficulty which we have to face if we attempt a rigorous proof of these statements: a phase-transition manifests itself sharply in the mathematical behaviour of thermodynamic functions only in the thermodynamic limit in which the number of particles and the volume of the system both become infinite, but in this limit there is no unique precise formulation of the zero-energy state. For non-interacting particles in a box of finite volume the single-particle energy levels are well-defined and there is a unique ground state; as the volume increases, every energy-level tends to zero; for the infinite system the single-particle energy spectrum is a continuum filling the half-line but there are no eigen-states. There are two good candidates for the concept of macroscopic occupation of the zero-energy state: *macroscopic occupation of the ground state* is said to occur when the number of particles in the ground state becomes proportional to the volume; *generalized condensation* is said to occur when the number of particles whose energy levels lie in an arbitrary small band above zero becomes proportional to the volume. Obviously, the first implies the second. However, the second can occur without the first; we call this *non-extensive condensation.* In this paper we prove that there are two critical densities: there is ρ_c which is the density at which singularities in the thermodynamic functions occur; there is ρ_m which is the minimum density for macroscopic occupation of the ground state. In the common examples ρ_c and ρ_m are equal but this is not logically necessary. There are examples in which ρ_c and ρ_m are both finite and unequal and other examples where ρ_c is finite but ρ_m is infinite. We prove that generalised condensation occurs whenever ρ is greater than ρ_c and that anomolous fluctuations in the density occur if and only if there is macroscopic occupation of the ground state.

For non-interacting boson systems the thermodynamic limit is most conveniently studied in the grand canonical ensemble with the mean value of the particle number density held fixed. All we require for the description of the thermodynamic limit of such a system is a sequence $\{(h_l, V_l): l = 1, 2, \ldots\}$ of pairs, each pair consisting of a self-adjoint operator h_l, the single-particle hamiltonian of the system, and a real number V_l, the volume of the system. Our aim is to relate the asymptotic properties of this sequence to the phenomenon of Bose–Einstein condensation; it turns out that there is one function obtained from the sequence which describes completely the thermodynamic functions, and another which describes completely the anomolous fluctuations.

First we consider the sequence $\{p_l(\mu): l = 1, 2, \ldots\}$ of grand canonical pressures and define a sequence $\{\mu_l(\rho): l = 1, 2, \ldots\}$ of functions by taking $\mu_l(\rho)$ to be the unique root in $(-\infty, 0)$ of the equation

$$\frac{d}{d\mu} p_l(\mu) = \rho. \tag{1.1}$$

The pressure $\pi_l(\rho)$ at mean density ρ is then given by

$$\pi_l(\rho) = (p_l \circ \mu_l)(\rho). \tag{1.2}$$

We note that $\pi_l(\rho)$ does not depend directly on the eigen-values $\varepsilon_l(1) \le \varepsilon_l(2) \le \cdots$ of h_l but only on the differences $\lambda_l(k) = \varepsilon_l(k) - \varepsilon_l(1)$, $k = 1, 2, \ldots$. We define the single-particle partition function $\phi_l(\beta)$ by

$$\phi_l(\beta) = V_l^{-1} \sum_{k=1}^{\infty} \exp(-\beta\lambda_l(k)), \tag{1.3}$$

and prove that whenever $\phi(\beta) = \lim_{l\uparrow\infty} \phi_l(\beta)$ exists for all β in $(0, \infty)$ and is non-zero for some β in $(0, \infty)$ then:

1. The limit $p(\mu) = \lim_{l\uparrow\infty} p_l(\mu)$ exists for all μ in $(-\infty, 0)$ and is given by

$$\beta p(\mu) = \int_{[0,\infty)} \log(1 - e^{\beta(\mu-\lambda)})^{-1}\, dF(\lambda) \tag{1.4}$$

where the density of states $dF(\lambda)$ is defined uniquely by $\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda}\, dF(\lambda)$. We define $p(0) = \lim_{\mu\uparrow 0} p(\mu)$.
2. The limit $\mu(\rho) = \lim_{l\uparrow\infty} \mu_l(\rho)$ exists for all ρ in $(0, \infty)$ and is given by: $\mu(\rho)$ is the unique root in $(-\infty, 0)$ of the equation $dp/d\mu(\mu) = \rho$ for $\rho < \rho_c = \int_{[0,\infty)} (e^{\beta\lambda} - 1)^{-1}\, dF(\lambda)$ and $\mu(\rho) = 0$ for $\rho \ge \rho_c$.
3. The limit $\pi(\rho) = \lim_{l\uparrow\infty} \pi_l(\rho)$ exists and is given by $\pi(\rho) = (p \circ \mu)(\rho)$. This is the equation of state;

it follows from these results that the pressure-density isotherm is flat when ρ is greater than ρ_c. The grand canonical pressure $\rho(\mu)$ is the generating function for the thermodynamic functions; it follows that these are completely determined by the integrated density of states $F(\lambda)$ or, equivalently, by the limiting single-particle partition function $\phi(\beta)$.

To discuss condensation it is necessary to introduce the occupation number $n_l(k)$ of the k^{th} energy level of h_l. It turns out that we can regard then $n_l(k)$, $k = 1, 2, \ldots$, as independent random variables each having a geometric thermal distribution. It is convenient to introduce the family $\{N_l(\lambda) : \lambda \in (0, \infty)\}$ of random variables defined by

$$N_l(\lambda) = \sum_{\{k : \lambda_l(k) < \lambda\}} n_l(k). \tag{1.5}$$

We interpret $N_l(\lambda)$ as the number of particles with energy less than λ in volume V_l. Let $\mathbb{E}_\rho[X]$ denote the expectation value in the grand canonical ensemble at mean density ρ of the random variable X. We say that macroscopic occupation of the ground state occurs if the limit

$$v(1; \rho) = \lim_{l\uparrow\infty} \mathbb{E}_\rho[V_l^{-1} n_l(1)] \tag{1.6}$$

exists and is strictly positive. We say that generalized condensation occurs if the limit

$$v_0(\rho) = \lim_{\lambda\downarrow 0} \lim_{l\uparrow\infty} \mathbb{E}_\rho[V_l^{-1} N_l(\lambda)] \tag{1.7}$$

exists and is strictly positive. We prove that whenever $\phi(\beta)$ exists and is non-zero for some β then $\nu_0(\rho)$ exists and is given by

$$\nu_0(\rho) = (\rho - \rho_c)^+, \tag{1.8}$$

where $(x)^+$ stands for the positive part of x. Thus generalized condensation occurs whenever ρ is greater than ρ_c. Macroscopic occupation of the ground state depends on finer properties of the sequence $\{(h_l, V_l) : l = 1, 2, \ldots\}$. We show that for the limit

$$\nu(k; \rho) = \lim_{l \uparrow \infty} \mathbb{E}_\rho[V_l^{-1} n_l(k)] \tag{1.9}$$

to exist and be strictly positive it is necessary that $\lambda_l(k)$ go to zero like $V_l^{-1}x$ for some x. If the limit

$$\rho_m(x) = \lim_{l \uparrow \infty} \int_{[V^{-1}x, \infty)} (e^{\beta\lambda} - 1)^{-1} \, dF_l(\lambda) \tag{1.10}$$

exists, where $F_l(\lambda)$ is defined by $\phi_l(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} \, dF_l(\lambda)$, then we define ρ_m by $\rho_m = \lim_{x \uparrow \infty} \rho_m(x)$; thus ρ_m is the maximum denisty in the levels which go to zero more slowly than $V_l^{-1}x$ for every x. To describe macroscopic occupation of the ground state we introduce the re-scaled single-particle partition function $\gamma_l(\beta)$ by

$$\gamma_l(\beta) = \sum_{k=1}^{\infty} e^{-\beta V_l \lambda_l(k)} = \int_{[0,\infty)} e^{-\beta\lambda} \, dG_l(\lambda). \tag{1.11}$$

We prove that whenever ρ_m exists and the limit $\gamma(\beta) = \lim_{l \uparrow \infty} \gamma_l(\beta)$ exists and is finite for some β in $(0, \infty)$ then:

1. $\nu(1; \rho)$ exists and is given by $\nu(1; \rho) = 1/b(\rho)$ where $b(\rho) = \infty$ for $\rho \leq \rho_m$ and $b(\rho)$ is the unique root in $(0, \infty)$ of the equation

$$\int_0^\infty e^{-b\beta} \gamma(\beta) \, d\beta = \rho - \rho_m \quad \text{for} \quad \rho > \rho_m. \tag{1.12}$$

2. The limit

$$e^{-\psi(s;\rho)} = \lim_{l \uparrow \infty} \mathbb{E}_\rho[e^{-sV^{-1}N_l}] \tag{1.13}$$

exists and is given by $\psi(s; \rho) = s\rho$ for $\rho \leq \rho_m$ and by

$$\psi(s; \rho) = s\rho_m + \int_0^\infty \frac{1 - e^{-su}}{u} e^{-bu} \gamma(u) \, du \tag{1.14}$$

for $\rho > \rho_m$. Thus macroscopic occupation of the ground state and the fluctuations in the number density are described completely by the function $\gamma(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} \, dG(\lambda)$.

The phenomenon of Bose–Einstein condensation was described first by Einstein [1] in 1925. He based his prediction on a combination of Bose–Einstein statistics and the classical expression for the density of states in phase-space. In

1927 Uhlenbeck [2] objected that Einstein's results holds only 'when the quantization of translational motion is neglected'. The matter was not resolved until Kramers in 1937 pointed out the importance of the thermodynamic limit for the sharp manifestation of phase-transitions, whereupon Uhlenbeck [3] withdrew his objection and pointed out that Einstein's expression for the particle number-density is correct in the thermodynamic limit. In the same year London [4] introduced the concept of macroscopic occupation of the ground state and related it to coherence properties of the condensate. The concept of generalized condensation was introduced by Girardeau [5] in 1960. He claimed that in a model of impenetrable bosons in one-dimension there is generalized condensation but no macroscopic occupation of the ground state. Schultz [6] showed that this is not the case. The theoretical importance of the concept may nevertheless have been known well but not widely. Casimir [8] in 1967 pointed out the importance of distinguishing between the thermodynamic and the coherence properties of Bose–Einstein condensation. He sketched some examples to show that the coherence properties can be changed without changing the thermodynamic properties of the model; he includes an example in which there is condensation into 'a narrow band' of levels. In [9] van den Berg and Lewis showed that in two-dimensions in the presence of an external field there is macroscopic occupation of an infinite number of levels. In [10] they discussed in detail one of Casimir's examples: consider a prism of volume V with sides V^{α_1}, V^{α_2}, V^{α_3}, $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$, and take the single-particle hamiltonian to be the Laplacian with Dirichlet boundary conditions; then we have the following results:

$$\alpha_1 < \tfrac{1}{2}; \qquad \rho_c = \rho_m, \qquad \gamma(\beta) < \infty; \qquad \nu(1; \rho) = (\rho - \rho_c)^+.$$

$$\alpha_1 = \tfrac{1}{2}; \qquad \rho_c = \rho_m, \qquad \gamma(\beta) < \infty; \qquad \nu(k; \rho) > 0 \quad \text{for} \quad \rho > \rho_c$$

for an infinite number of levels and $\sum_k \nu(k; \rho) = (\rho - \rho_c)^+$.

$$\alpha_1 > \tfrac{1}{2}: \qquad \rho_c = \rho_m, \qquad \gamma(\beta) \equiv \infty; \qquad \nu(1; \rho) = 0 \quad \text{for all } \rho.$$

A more refined example was given by van den Berg [11] to show that it is possible to have $\rho_c < \rho_m < \infty$.

As far as we know the first rigorous proof of the macroscopic occupation of the ground state of the Laplacian when the thermodynamic limit is taken by dilating an arbitrary star-shaped region was given in lectures by Kac in 1971; he obtained also the limiting distribution of the density in this case by computing $\lim_{l \uparrow \infty} \mathbb{E}_\rho[e^{-sV^{-1}N_l}]$. The mathematical details were supplied in 1972 in the thesis of Pule [12] (see also Lewis [13], Cannon [14], Lewis and Pule [15]). Kac's lectures remained unpublished until 1977 when they were incorporated in the review by Ziff, Uhlenbeck and Kac [16]. In many of these papers a substantial part is devoted to obtaining results about the asymptotic spectral properties of the sequence $\{(h_l, V_l) : l = 1, 2, \ldots\}$. We hope that we have succeeded in showing in this paper that if this rather technical part is hived-off the mathematical structure of the phase-transition becomes clear. Moves in this direction were made by Davies [17] and by Landau and Wilde [18].

2. The grand canonical ensemble

First we recall some facts about the grand canonical ensemble. For $l=1,2,\ldots$ and $n=1,2,\ldots$ and $\beta>0$, let $Z_l^\beta(n)$ be the canonical partition function for n particles in volume V_l at inverse temperature β; put $Z_l^\beta(0)=1$. The grand canonical pressure $p_l(\mu)$ is defined by

$$e^{\beta V_l p_l(\mu)} = \sum_{n=0}^{\infty} e^{n\beta\mu} Z_l^\beta(n) \tag{2.1}$$

for all values of μ for which the infinite series converges. Denote by N_l the total number of particles in volume V_l; we regard N_l as a random variable taking values in the non-negative integers. The probability $\mathbb{P}^\mu[N_l=n]$ that N_l takes the value n is given by

$$\mathbb{P}^\mu[N_l=n] = e^{\beta(n\mu - V_l p_l(\mu))} Z_l^\beta(n). \tag{2.2}$$

The probability distribution function K_l^μ of the particle number density is defined by

$$K_l^\mu(x) = \mathbb{P}^\mu\left[\frac{N_l}{V_l} \leq x\right] \tag{2.3}$$

and it is determined uniquely by its Laplace transform

$$\int_{[0,\infty)} e^{-sx}\, dK_l^\mu(x) = \mathbb{E}^\mu[e^{-sN_l/V_l}]. \tag{2.4}$$

Thus we have

$$\int_{[0,\infty)} e^{-sx}\, dK_l^\mu(x) = \exp \beta V_l\left\{p_l\left(\mu - \frac{s}{\beta V_l}\right) - p_l(\mu)\right\} \tag{2.5}$$

so that the mean $\mathbb{E}^\mu[N_l/V_l]$ and the variance $\mathbb{D}^\mu[N_l/V_l]$ are given by

$$\mathbb{E}^\mu\left[\frac{N_l}{V_l}\right] = p_l^{(1)}(\mu), \qquad \mathbb{D}^\mu\left[\frac{N_l}{V_l}\right] = \frac{\beta}{V_l} p_l^{(2)}(\mu). \tag{2.6}$$

(Throughout this paper $p^{(k)}$ will denote the kth partial derivative of p with respect to μ.) The mean value of the particle density changes as μ is varied. In finite volume the variance is always strictly positive so that $\mu \to p_l^{(1)}(\mu) = \mathbb{E}^\mu[N_l/V_l]$ is strictly increasing; hence the equation

$$\mathbb{E}^\mu\left[\frac{N_l}{V_l}\right] = \rho \tag{2.7}$$

has a unique solution $\mu_l(\rho)$ on $(0,\infty)$. The grand canonical pressure $\pi_l(\rho)$ at mean density ρ is given by

$$\pi_l(\rho) = (p_l \circ \mu_l)(\rho). \tag{2.8}$$

Thus the function $\mu \to p_l(\mu)$ determines the equation of state. We shall be interested in the equation of state in the thermodynamic limit in which l tends to infinity while ρ is held fixed; this will require an investigation of the existence of the limits

$$p(\mu)=\lim_{l\uparrow\infty} p_l(\mu), \tag{2.9}$$

$$\mu(\rho)=\lim_{l\uparrow\infty} \mu_l(\rho). \tag{2.10}$$

The function $\mu\to p(\mu)$ determines also the distribution function K_l^μ, as can be seen from (1.5) and the uniqueness theorem for the Laplace transform (see [19], ch. 13). We shall be interested in the weak convergence of the sequence of random variables N_l/V_l, $l=1,2,\ldots$, again at fixed mean density ρ; that is, in the convergence of the sequence of distribution functions

$$K_l(x;\rho)=K_l^{\mu_l(\rho)}(x), \quad l=1,2,\ldots. \tag{2.11}$$

It is important to notice that if we change the definition of $p_l(\mu)$ by substituting $e^{-n\beta\alpha_l}Z_l^\beta(n)$ for $Z_l^\beta(n)$ in (1.1) the function $\rho\to\pi_l(\rho)$ is unchanged. If we make the same substitution in (1.2) the function $\rho\to K_l(x;\rho)$ is unchanged.

Now we turn to the special case of a system of non-interacting bosons each having as its single-particle hamiltonian h_l acting on a hilbert space h_l and such that

$$\text{trace}\,[e^{-\beta h_l}]<\infty \tag{2.12}$$

for all β in $(0,\infty)$. Let $\varepsilon_l(1)<\varepsilon_l(2)\le\cdots$ be the eigen-values of h_l in ascending order. The canonical partition function for n bosons each having as its single-particle hamiltonian the self-adjoint operator h_l is

$$Z_l^\beta(n)=\sum_{\{n(k)\ge 0:\sum n(k)=n\}} e^{-\beta\sum_{k=1}^\infty n(k)\varepsilon_l(k)}. \tag{2.13}$$

It follows from a combinatorial identity (see [7]) that the grand canonical pressure $p_l(\mu)$ is given by

$$e^{\beta V_l p_l(\mu)}=\prod_{k=1}^\infty (1-e^{\beta(\mu-\varepsilon_l(k))})^{-1}. \tag{2.14}$$

It follows from the remark following equation (2.11) that the functions $\rho\to\pi_l(\rho)$ and $\rho\to K_l(x;\rho)$ depend only on the energy differences

$$\lambda_l(k)=\varepsilon_l(k)-\varepsilon_l(1). \tag{2.15}$$

It is convenient therefore to redefine $p_l(\mu)$ by subsituting $e^{\beta n\varepsilon_l(1)}Z_l^\beta(n)$ for $Z_l^\beta(n)$ so that $p_l(\mu)$ is now given by

$$e^{\beta V_l p_l(\mu)}=\prod_{k=1}^\infty (1-e^{\beta(\mu-\lambda_l(k))})^{-1}. \tag{2.16}$$

The domain of definition of each function $\mu\to p_l(\mu)$ is now the interval $(-\infty,0)$.

It is convenient to introduce the functions $\phi_l(\beta)$ defined for all β in $(0, \infty)$ by

$$\phi_l(\beta) = V_l^{-1} \sum_{k=1}^{\infty} e^{-\beta\lambda_l(k)} \tag{2.17}$$

and the distribution function $F_l(\lambda)$ defined by

$$V_l F_l(\lambda) = \max \{k : \lambda_l(k) \leq \lambda\}. \tag{2.18}$$

We shall exploit the fact that $\phi_l(\beta)$ is the Laplace transform of $F_l(\lambda)$:

$$\phi_l(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} \, dF_l(\lambda). \tag{2.19}$$

Then $p_l(\mu)$ is given in terms of $F_l(\lambda)$ by

$$p_l(\mu) = \int_{[0,\infty)} p(\mu \mid \lambda) \, dF_l(\mu), \tag{2.20}$$

where the conditional pressure $p(\mu \mid \lambda)$ is defined by

$$\beta p(\mu \mid \lambda) = \log (1 - e^{\beta(\mu-\lambda)})^{-1}. \tag{2.21}$$

It follows from (2.16) that we can regard the random variable N_l as a sum

$$N_l = \sum_{k=1}^{\infty} n_l(k) \tag{2.22}$$

of independent random varibles $\{n_l(k) : l = 1, 2, \ldots\}$. We interpret $n_l(k)$ as the occupation number of the k^{th} level. Each $n_l(k)$ has an exponential distribution

$$\mathbb{P}^{\mu}[n_l(k) \geq n] = e^{+n\beta(\mu-\lambda_l(k))}. \tag{2.23}$$

For $|z| < 1$ and σ real define $g_\sigma(z)$ by the convergent power series

$$g_\sigma(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\sigma}. \tag{2.24}$$

It follows directly from the definitions that for $k = 0, 1, 2, \ldots$ and for $\mu < s \leq \lambda$ we have

$$0 < p^{(k)}(\mu \mid \lambda) \leq \beta^{k-1} e^{\beta(s-\lambda)} g_{1-k}(e^{\beta(\mu-s)}), \tag{2.25}$$

$$0 < p_l^{(k)}(\mu) \leq \beta^{k-1} \phi_l(\beta) g_{1-k}(e^{\beta\mu}). \tag{2.27}$$

Lemma 1. *Suppose that*

$$\phi(\beta) = \lim_{l \uparrow \infty} \phi_l(\beta) \tag{2.28}$$

exists for all β *in* $(0, \infty)$ *and is non-zero for some* β *in* $(0, \infty)$. *Let* $F(\lambda)$ *be the unique distribution function such that*

$$\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} \, dF(\lambda). \tag{2.29}$$

Then for each $k = 0, 1, 2, \ldots$ and each $\mu < s$ the sequence

$$p_l^{(k)}(\mu; s) = \int_{[s,\infty)} p^{(k)}(\mu \mid \lambda)\, dF_l(\lambda) \tag{2.30}$$

converges to

$$p^{(k)}(\mu; s) = \int_{[s,\infty)} p^{(k)}(\mu \mid \lambda)\, dF(\lambda). \tag{2.31}$$

Moreover, the convergence is uniform in μ on compacts in $(-\infty, s)$.

Proof. First we note that it follows from the inversion formula for Laplace transforms [19; ch. 13] that if $\phi(\beta) \neq 0$ for some β in $(0, \infty)$ then $\phi(\beta) \neq 0$ for all β in $(0, \infty)$. Following Feller [19; ch. 13] we introduce the probability distribution functions F_l^β defined by

$$dF_l^\beta(\lambda) = \phi_l(\beta)^{-1} e^{-\beta\lambda}\, dF_l(\lambda). \tag{2.32}$$

The Laplace transform of F_l^β is $\phi_l(\beta + t)/\phi_l(\beta)$ which by hypothesis converges to $\phi(\beta + t)/\phi(\beta)$ satisfying $0 < \phi(\beta + t)/\phi(\beta) \leq 1$ for all t in $(0, \infty)$. It follows that $\{F_l^\beta : l = 1, 2, \ldots\}$ converges to a distribution function F^β which is a (non-defective) probability distribution since $\lim_{t \downarrow 0} \phi(\beta + t)/\phi(\beta) = 1$. But $p_l^{(k)}(\mu; s)$ is given by

$$p_l^{(k)}(\mu; s) = \int_{[s,\infty)} p^{(k)}(\mu \mid \lambda) e^{\beta\lambda}\, dF_l^\beta(\lambda) \phi_l(\beta). \tag{2.33}$$

The function $\lambda \to p^{(k)}(\mu \mid \lambda) e^{\beta\lambda}$ is continuous and by (2.26) it is bounded on $[s, \infty)$ for all $\mu < s$. It follows from [19; ch. 8] that $\{p_l^{(k)}(\mu; s) : l = 1, 2, \ldots\}$ converges to $p^{(k)}(\mu; s)$. But $p^{(k)}(\mu; s)$ is strictly positive by (2.26) so that $\mu \to p_l^{(k)}(\mu; s)$ is convex; hence the convergence is uniform in μ on compacts in $(-\infty, s)$.

Next we define the critical density ρ_c: if $\lambda \to p^{(1)}(0 \mid \lambda)$ is integrable on $[0, \infty)$ with respect to F, put

$$\rho_c = \int_{[0,\infty)} p^{(1)}(0 \mid \lambda)\, dF(\lambda); \tag{2.34}$$

put $\rho_c = \infty$ otherwise. It follows from the dominated convergence principle [19; ch. 4] that if ρ_c is finite then

$$\rho_c = \lim_{\mu \uparrow 0} \int_{[0,\infty)} p^{(1)}(\mu \mid \lambda)\, dF(\lambda) = \lim_{\varepsilon \downarrow 0} \int_{[\varepsilon,\infty)} p^{(1)}(0 \mid \lambda)\, dF(\lambda). \tag{2.35}$$

It is clear that if $F(\lambda) \sim C\lambda^\alpha$ as $\lambda \downarrow 0$ with $\alpha > 1$ then ρ_c is finite, and if ρ_c is finite then $F(\lambda) \to 0$ as $\lambda \downarrow 0$. We shall need the following more precise estimate:

Lemma 2. *For $\varepsilon > 0$ we have*

$$F(\varepsilon) < \beta\varepsilon e^{\beta\varepsilon} \rho_c \tag{2.36}$$

Proof.

$$\rho_c = \int_{[0,\infty)} p^{(1)}(0 \mid \lambda)\, dF(\lambda) > \int_{[0,\varepsilon)} p^{(1)}(0 \mid \lambda)\, dF(\lambda)$$

$$> (e^{\beta\varepsilon} - 1)^{-1} \int_{[0,\infty)} dF(\lambda) = (e^{\beta\varepsilon} - 1)^{-1} F(\varepsilon). \tag{2.37}$$

Next we consider the equation of state for fixed l. Since $p_l^{(2)}(\mu)$ is strictly positive, the function $\mu \to p_l^{(1)}(\mu)$ is strictly increasing; since $\lambda_l(1)$ is zero, $p_l^{(1)}(\mu)$ tends to $+\infty$ as μ increases to zero and $p_l^{(1)}$ tends to zero as μ decreases to $-\infty$. It follows that the equation

$$p_l^{(1)}(\mu) = \rho \tag{2.38}$$

has a unique solution $\mu_l(\rho)$ in $(-\infty, 0)$ for each ρ in $(0, \infty)$. We can regard the grand canonical pressure as a function $\rho \to \pi_l(\rho)$ of the density given by

$$\pi_l(\rho) = (p_l \circ \mu_l)(\rho). \tag{2.39}$$

We are interested in the existence of the limit

$$\pi(\rho) = \lim_{l \uparrow \infty} \pi_l(\rho). \tag{2.40}$$

First we prove

Lemma 3. *For each ρ in $(-\infty, 0)$, let $\mu_l(\rho)$ denote the unique root in $(-\infty, 0)$ of the equation $p_l^{(1)}(\mu) = \rho$. Then $\lim_{l\uparrow\infty} \mu_l(\rho)$ exists and is equal to $\mu(\rho)$ where $\mu(\rho)$ is zero for $\rho \geq \rho_c$ and is the unique root in $(-\infty, 0)$ of the equation $p^{(1)}(\mu) = \rho$ for $\rho < \rho_c$.*

Proof. Putting $k = 1$ in (2.27) we have

$$\rho = (p_l^{(1)} \circ \mu_l)(\rho) \leq \phi_l(\beta) e^{\beta\mu_l(\rho)} (1 - e^{\beta\mu_l(\rho)})^{-1} \tag{2.41}$$

so that

$$0 \geq \limsup_{l\uparrow\infty} \mu_l(\rho) \geq \liminf_{l\uparrow\infty} \mu_l(\rho) \geq \beta^{-1} \log \frac{\rho}{\rho + \phi(\beta)}. \tag{2.42}$$

It follows that the sequence $\{\mu_l(\rho) : l = 1, 2, \ldots\}$ has at least one limit point, μ^* say. Let $\{\mu_{l(k)} : k = 1, 2, \ldots\}$ be a subsequence converging to μ^*. Suppose $\mu^* < 0$ and $\rho \geq \rho_c$ then the $\mu_{l(k)}$ lie eventually in a closed subinterval of $(-\infty, 0)$ on which, by Lemma 1, $\{p_l^{(1)} : l = 1, 2, \ldots\}$ converges uniformly. Thus we have

$$\lim_{k\uparrow\infty} p_{l(k)}^{(1)}(\mu_{l(k)}) = p^{(1)}(\mu^*). \tag{2.43}$$

Each μ_l satisfies $p_l^{(1)}(\mu_l) = \rho$; hence μ^* satisfies $p^{(1)}(\mu^*) = \rho$. But $p^{(1)}(\mu^*) < \rho_c$ for $\mu^* \in (-\infty, 0)$. By contradiction we conclude $\mu^* = 0$ for $\rho \geq \rho_c$.

Let $\rho < \rho_c$ and let $\mu(\rho)$ be the unique solution of $p^{(1)}(\mu) = \rho$. Suppose $\mu^* = 0$

for $\rho < \rho_c$. Then eventually $p_l^{(1)}(\mu(\rho)/2) < p_l^{(1)}(\mu_{l(k)}) = \rho$. But $\{p_l^{(1)}(\mu(\rho)/2) : l = 1, 2, \ldots\}$ converges by Lemma 1 uniformly to $p^{(1)}(\mu(\rho)/2)$ which is greater than ρ. By contradiction we conclude $\mu^* < 0$ for $\rho < \rho_c$. The limit point μ^* is unique since $p^{(1)}(\mu)$ is strictly increasing on $(-\infty, 0)$.

3. The equation of state

We are now ready to prove our first main result, the equation of state in the thermodynamic limit:

Theorem 1. *If $\phi(\beta) = \lim_{l\uparrow\infty} \phi_l(\beta)$ exists for all β in $(0, \infty)$ and is non-zero for some β in $(0, \infty)$ then $\pi(\rho) = \lim_{l\uparrow\infty} \pi_l(\rho)$ is given by*

$$\pi(\rho) = (p \circ \mu)(\rho), \tag{3.1}$$

where

$$p(\mu) = \int_{[0,\infty)} p(\mu \mid \lambda)\, dF(\lambda), \tag{3.2}$$

and $F(\lambda)$ is uniquely determined by

$$\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda}\, dF(\lambda). \tag{3.3}$$

Proof. For $\rho < \rho_c$ the result follows directly from Lemma 1 and 3. For $\rho \geq \rho_c$ and $\varepsilon > 0$ we have

$$\pi_l(\rho) = p_l(\mu_l) = \int_{[0,\varepsilon)} p(\mu_l \mid \lambda)\, dF_l(\lambda) + p_l(\mu_l; \varepsilon). \tag{3.4}$$

But for $0 < x < 1$ we have

$$0 < x^{1/2} \log x^{-1} < 2e^{-1} \tag{3.5}$$

so that

$$0 < \beta p(\mu \mid \lambda) < [p^{(1)}(\mu \mid \lambda)]^{1/2}; \tag{3.6}$$

using this and the Cauchy–Schwarz inequality, we have

$$0 < \int_{[0,\varepsilon)} \beta p(\mu_l \mid \lambda)\, dF_l(\lambda) < \int_{[0,\varepsilon)} [p^{(1)}(\mu_l \mid \lambda)]^{1/2}\, dF_l(\lambda)$$
$$\leq \left[\int_{[0,\varepsilon)} p^{(1)}(\mu_l \mid \lambda)\, dF_l(\lambda)\right]^{1/2} F_l(\varepsilon)^{1/2} \leq \rho^{1/2} F_l(\varepsilon)^{1/2}. \tag{3.7}$$

It now follows from Lemma 2 for $\rho_c < \infty$ that

$$\lim_{\varepsilon\downarrow 0} \limsup_{l\uparrow\infty} \int_{[0,\varepsilon)} p(\mu_l \mid \lambda)\, dF_l(\lambda) = \lim_{\varepsilon\downarrow 0} \liminf_{l\uparrow\infty} \int_{[0,\varepsilon]} p(\mu \mid \lambda)\, dF_l(\lambda) = 0, \tag{3.8}$$

so that for $\rho \geq \rho_c$, using Lemmas 1 and 3, we have

$$\lim_{l \uparrow \infty} \pi_l(\rho) = \lim_{\varepsilon \downarrow 0} p(0; \varepsilon). \tag{3.9}$$

But for $\rho_c < \infty$ the function $\lambda \to p(0 \mid \lambda)$ is integrable on $(0, \infty)$ with respect to F so that by the dominated converges principle $\lim_{\varepsilon \downarrow 0} p(0; \varepsilon) = p(0)$, and the result follows.

4. Generalized condensation

In the course of the proof of Theorem 1 we showed that the low-lying levels make no contribution to the pressure in the thermodynamic limit at fixed mean number density:

$$\lim_{\varepsilon \downarrow 0} \limsup_{l \uparrow \infty} \int_{[0,\varepsilon)} p(\mu_l \mid \lambda)\, dF_l(\lambda) = \lim_{\varepsilon \downarrow 0} \liminf_{l \uparrow \infty} \int_{[0,\varepsilon)} p(\mu_l \mid \lambda)\, dF_l(\lambda) = 0. \tag{4.1}$$

The same is not true of the particle density: for $\rho \leq \rho_c$ they make no contribution, but for $\rho > \rho_c$ they always account for the excess $\rho - \rho_c$. This is made precise in our second main result, the existence of generalized condensation (we write $\mathbb{E}_\rho$ for $\mathbb{E}^{\mu_l(\rho)}$):

Theorem 2. *If $\phi(\beta) = \lim_{l \uparrow \infty} \phi_l(\beta)$ exists for all β in $(0, \infty)$ and is non-zero for some β in $(0, \infty)$ and ρ_c is finite then the limit*

$$\nu_0(\rho) = \lim_{\varepsilon \downarrow 0} \lim_{l \uparrow \infty} \mathbb{E}_\rho[V_l^{-1} N_l(\varepsilon)] \tag{4.2}$$

exists and is given by

$$\nu_0(\rho) = (\rho - \rho_c)^+. \tag{4.3}$$

Proof. From the definition (1.5) of $N_l(\lambda)$ we have

$$\begin{aligned} \mathbb{E}_\rho[V_l^{-1} N_l(\varepsilon)] &= \int_{[0,)} p^{(1)}(\mu_l \mid \lambda)\, dF_l(\lambda) \\ &= \rho - p_l^{(1)}(\mu_l; \varepsilon). \end{aligned} \tag{4.4}$$

By Lemma 1 and 3,

$$\lim_{l \uparrow \infty} p_l^{(1)}(\mu_l(\rho; \varepsilon)) = p^{(1)}(\mu(\rho); \varepsilon). \tag{4.5}$$

But, by the dominated convergence principle,

$$\lim_{\varepsilon \downarrow 0} (\rho - p^{(1)}(\mu(\rho); \varepsilon)) = (\rho - \rho_c)^+. \tag{4.6}$$

5. Macroscopic occupation of the ground state

As we have seen, the limit $\lim_{l\uparrow\infty} \mathbb{E}_\rho[V_l^{-1}N_l(\lambda)]$ exists under very weak conditions. The existence of the limit $\nu(k;\rho)=\lim_{l\uparrow\infty}\nu_l(k;\rho)$, where

$$\nu_l(k;\rho)=\mathbb{E}_\rho[V_l^{-1}n_l(k)] \tag{5.1}$$

requires stronger hypotheses. We see from the expression

$$\nu_l(k;\rho)=V_l^{-1}p^{(1)}(\mu_l(\rho)\mid\lambda_l(k)) \tag{5.2}$$

that

1. the individual occupation densities decrease with increasing k:

$$\nu_l(1;\rho)\geqslant\nu_l(2;\rho)\geqslant\cdots \tag{5.3}$$

2. for $\rho<\rho_c$ we have $\lim_{l\uparrow\infty}\nu_l(k;\rho)=0$, $k=1,2,\ldots$
3. let $b_l(\rho)=\beta V_l\mu_l(\rho)$; if $\nu(1;\rho)$ exists and is strictly positive then $b(\rho)=\lim b_l(\rho)$ exists and is given by

$$b(\rho)=-\nu(1;\rho); \tag{5.4}$$

if in addition $\nu(k;\rho)$ exists and is strictly positive then

$$\lim_{l\uparrow\infty}\beta V_l\lambda_l(k)=[\nu(k;\rho)^{-1}-\nu(1;\rho)^{-1}]^{-1}. \tag{5.5}$$

4. if $\nu(1;\rho)=0$ then

$$\lim_{l\uparrow\infty}\mathbb{D}_\rho\left[\frac{N_l}{V_l}\right]=0, \tag{5.6}$$

since for each l we have

$$\beta\nu_l(1;\rho)^2\leqslant\mathbb{D}_\rho\left[\frac{N_l}{V_l}\right]\leqslant\beta\rho\nu_l(1;\rho). \tag{5.7}$$

This follows directly from the expression

$$\mathbb{D}_\rho\left[\frac{N_l}{V_l}\right]=(\beta V_l)^{-1}p_l^{(2)}(\mu_l)=\frac{\beta}{V_l}\sum_{k=1}^{\infty}\frac{e^{\beta(\mu_l-\lambda_l(k))}}{(1-e^{\beta(\mu_l-\lambda_l(k))})^2}. \tag{5.8}$$

We see from (3) that if $\nu(k;\rho)$ is strictly positive then $\lambda_l(k)$ goes to zero like $V_l^{-1}x$ for some $x>0$, so we have to collect up the contribution to the density from such levels. The next theorem gives a sufficient condition for the existence of $\nu(1;\rho)$ and an equation to determine its value, since $-\nu(1;\rho)=1/b(\rho)$. First we must make a more general definition of ρ_m than the one we given in Section 1, where $\gamma(\beta)$ was also defined.

We introduce a second critical density ρ_m as follows:

$$\rho_m^+(x)=\limsup_{l\uparrow\infty}\int_{[V^{-1}x,\infty)}p^{(1)}(0\mid\lambda)\,dF_l(\lambda), \tag{5.9}$$

and

$$\rho_m^-(x) = \liminf_{l\uparrow\infty} \int_{[V_l^{-1}x,\infty)} p^{(1)}(0 \mid \lambda)\, dF_l(\lambda). \tag{5.10}$$

If the limits as $x \uparrow \infty$ of $\rho_m^+(x)$ and $\rho_m^-(x)$ exist and are equal then we define ρ_m to be their common value:

$$\rho_m = \lim_{x\uparrow\infty} \rho_m^+(x) = \lim_{x\uparrow m} \rho_m^-(x). \tag{5.11}$$

Recall that ρ_c can be expressed as

$$\rho_c = \lim_{\varepsilon\downarrow 0} \lim_{l\uparrow\infty} \int_{[\varepsilon,\infty)} p^{(1)}(0 \mid \lambda)\, dF_l(\lambda). \tag{5.12}$$

It is clear that $\rho_m \geq \rho_c$ since x/V_l is eventually less than ε for each $\varepsilon > 0$. We interpret ρ_m as the contribution to the density of those levels which go to zero more slowly than x/V_l for every x.

Theorem 3. *Suppose that*
C1: *ρ_m exists,*
C2: *$\gamma(\beta) = \lim_{l\uparrow\infty} \gamma_l(\beta)$ exists,*
C3: *$\int_0^\infty e^{-b\beta}\gamma(\beta)\, d\beta < \infty$ for some b in $(0, \infty)$.*
Then $b(\rho) = \lim_{l\uparrow\infty} b_l(\rho)$ exists. For $\rho_c \leq \rho \leq \rho_m$ we have $b(\rho) = \infty$, while $b(\rho)$ is the unique root in $(0, \infty)$ of $\int_0^\infty e^{-b\beta}\gamma(\beta)\, d\beta = \rho - \rho_m$ for $\rho > \rho_m$.

Proof. For $b \geq 0$ and $\lambda \geq V_l^{-1}x$ we have

$$\left[1 - \frac{e^{\beta b V^{-1}} - 1}{1 - e^{\beta x V_l^{-1}}}\right] p^{(1)}(0 \mid \lambda) \leq p^{(1)}(bV_l^{-1} \mid \lambda) \leq p^{(1)}(0 \mid \lambda), \tag{5.13}$$

so that

$$\begin{aligned}(1 - b/x)\rho_m^-(x) &\leq \liminf_{l\uparrow\infty} \int_{[xV_l^{-1},\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) \\ &\leq \limsup_{l\uparrow\infty} \int_{[xV_l^{-1},\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) \leq \rho_m^+(x).\end{aligned} \tag{5.14}$$

From the inequality $0 < x^{-1} - (e^x - 1)^{-1} < 1$ we have for b in $(0, \infty)$

$$\left| \int_{[0,xV_l^{-1})} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) - \int_{[0,x)} \frac{dG_l(\lambda)}{\lambda + b} \right| \leq F_l\left(\frac{x}{V_l}\right). \tag{5.15}$$

But $\lim_{l\uparrow\infty} F_l(V_l^{-1}x) = 0$ since ρ_c is finite. Hence

$$\lim_{l\uparrow\infty} \int_{[0,xV_l^{-1})} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) = \int_{[0,x)} \frac{dG(\lambda)}{\lambda + b}. \tag{5.16}$$

It follows, putting (5.14) and (5.16) together, that

$$\int_{[0,x)} \frac{dG(\lambda)}{\lambda+b} + (1-b/x)\rho_m^-(x) \leq \liminf_{l\uparrow\infty} \int_{[0,\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda)$$
$$\leq \limsup_{l\uparrow\infty} \int_{[0,\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) \leq \int_{[0,x)} \frac{dG(\lambda)}{\lambda+b} + \rho_m^+(x). \quad (5.17)$$

Invoking C1 we have

$$\lim_{l\uparrow\infty} \int_{[0,\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda) = \int_0^\infty e^{-b\beta}\gamma(\beta)\, d\beta + \rho_m. \quad (5.18)$$

It follows from the convexity of $b \to p^{(1)}(V_l^{-1}b \mid \lambda)$ that convergence is uniform in b on compact subsets of $(0,\infty)$. Suppose that the sequence $\{b_l = V_l^{-1}\mu_l(\rho) : l = 1, 2, \dots\}$ is unbounded; given b in $(0,\infty)$ we have, for b sufficiently large,

$$\rho = \int_{[0,\infty)} p^{(1)}(b_l V_l^{-1} \mid \lambda)\, dF_l(\lambda) < \int_{[0,\infty)} p^{(1)}(bV_l^{-1} \mid \lambda)\, dF_l(\lambda). \quad (5.19)$$

Thus $\rho < \int_0^\infty e^{-b\beta}\gamma(\beta)\, d\beta + \rho_m$, and letting $b \uparrow \infty$ we have $\rho \leq \rho_m$, contradicting the hypothesis; hence $\{b_l\}$ is bounded above. On the other hand, we have

$$\liminf_{l\uparrow\infty} b_l > \lim_{l\uparrow\infty} V_l \log(1 + (\rho V_l)^{-1}) = \rho^{-1}.$$

Let b^* be a limit point of $\{b_l\}$; then there is a subsequence lying in a compact subset of $(0,\infty)$ and converging to b^*. Using this subsequence we have

$$\rho = \int_0^\infty e^{-b^*\beta}\gamma(\beta)\, d\beta + \rho_m. \quad (5.20)$$

Thus $b^* = b(\rho)$ the unique root in $(0,\infty)$ of $\int_0^\infty e^{-b\beta}\gamma(\beta)\, d\beta = \rho - \rho_n$; hence $\{b_l\}$ converges to $b(\rho)$.

6. The anomalous fluctuations

We saw in Section 5 that the fluctuations in the number density satisfy the inequality

$$\beta v_l(1;\rho)^2 \leq \mathbb{D}_\rho[V_l^{-1}N_l] \leq \beta\rho v_l(1;\rho), \quad (6.1)$$

so that if $v_l(1;\rho)$ does not converge to zero then neither does $\mathbb{D}_\rho[V_l^{-1}N_l)$. Thus the anomalous fluctuations occur whenever the ground state is macroscopically occupied. We shall investigate them, following Kac [16], by determining the distribution function

$$K(x;\rho) = \lim_{l\uparrow\infty} \mathbb{P}_\rho[V_l^{-1}N_l \leq x], \quad (6.2)$$

which is known as the Kac distribution function. We determine $K(x;\rho)$ by computing its Laplace transform

$$e^{-\psi(s;\rho)} = \lim_{l\uparrow\infty} \mathbb{E}_\rho[e^{-sV_l^{-1}N_l}]. \tag{6.3}$$

Here $\psi(s;\rho) = \lim_{l\uparrow\infty} \psi_l(s;\rho)$ where

$$\psi_l(s;\rho) = \beta V_l \int_{[0,\infty)} \left\{p(\mu_l \mid \lambda) - p\left(\mu_l - \frac{s}{\beta V_l} \mid \lambda\right)\right\} dF_l(\lambda). \tag{6.4}$$

We state our results as

Theorem 4. *Suppose that hypotheses* C1, C2 *and* C3 *of Theorem* 3 *holds; then the Kac distribution* $K(x;\rho)$ *is infinitely divisible and is given by*

$$\int_{[0,\infty)} e^{-sx}\, dK(x;\rho) = e^{-\psi(s;\rho)}, \tag{6.5}$$

where $\psi(s;\rho)$ *is given by*

$$\psi(s;\rho) = \begin{cases} s\rho, & \rho \leqslant \rho_m, \\ s\rho_m + \displaystyle\int_0^\infty \frac{1-e^{-su}}{u}\, dP(u), & \rho > \rho_m, \end{cases} \tag{6.6}$$

with

$$dP(u) = e^{-b(\rho)u} V(u)\, du. \tag{6.7}$$

Proof. the infinite divisibility of K is evident from the form of $\psi(s;\rho)$; see [19; ch. 13]. To prove that $\psi(s;\rho)$ has the stated form, we break $\psi_l(s;\rho)$ up into two parts and prove:

(1)
$$\lim_{l\uparrow\infty} \beta V_l \int_{[0,xV^{-1})} \left\{p\left(\mu_l - \frac{s}{\beta V_l} \,\middle|\, \lambda\right) - p(\mu_l \mid \lambda)\right\} dF_l(\lambda) = \int_0^\infty \frac{1-e^{-su}}{u} e^{-bu} V(u;x)\, du, \tag{6.8}$$

where

$$V(u;x) = \int_{[0,x)} e^{-u\lambda}\, dG(\lambda). \tag{6.9}$$

(2)
$$(1-b/x)\frac{s}{1+s(x+b)^{-1}} \rho_m^-(x) \leqslant \liminf_{l\uparrow\infty} \beta V_l \int_{[xV_l^{-1},\infty)} \left\{p\left(\mu_l - \frac{s}{\beta V_l} \,\middle|\, \lambda\right) - p(\mu_l \mid \lambda)\right\}$$
$$\leqslant \limsup_{l\uparrow\infty} \int_{[xV_l^{-1},\infty)} \left\{p\left(\mu_l - \frac{s}{\beta V_l} \,\middle|\, \lambda\right) - p(\mu_l \mid \lambda)\right\} dF_l(\lambda) \leqslant s\rho_m^+(x). \tag{6.10}$$

Putting the two inequalities together and letting $x \to \infty$ we get the stated result. The proofs of (1) and (2) follow a by now familiar pattern and we sketch them here:

A straightforward manipulation yields

$$\beta V_l \int_{[0,xV^{-1})} \left\{ p\left(\mu_l - \frac{s}{\beta V_l} \,\middle|\, \lambda\right) - p(\mu_l \mid \lambda) \right\} dF_l(\lambda) = \int_0^\infty \frac{1-e^{-su}}{u} \, dP_l(u;x), \tag{6.11}$$

where

$$P_l(u;x) = \int_{[0,x)} \frac{1-e^{-u(\lambda+b_l)}}{\lambda+b_l} \, dG_l(\lambda). \tag{6.12}$$

But

$$\lim_{l\uparrow\infty} \int_{[0,\infty)} e^{-tu} \, dP_l(u;x) = \lim_{l\uparrow\infty} \int_0^\infty e^{-tu} e^{-b_l u} \gamma_l(u;x) \, du$$

$$= \int_0^\infty e^{-tu} e^{-bu} \gamma_l(u;x) \, du, \tag{6.13}$$

so that $P_l(u)$ converges to the distribution $P(u)$ with density $e^{-bu}\gamma(u)$, and (1) follows.

Next we use the inequality $y(1+y)^{-1} < \log(1+y) < y$ for y in $(0,\infty)$; proceeding along the lines of the proof of Theorem 3, (2) follows.

Finally we note that if C1 holds then for $\rho \le \rho_m$ we have $\psi(s,\rho) = s\rho$, independently of whether C2 and C3 hold. Similarly $\psi(s;\rho) = s\rho$ for all $\rho > 0$ if C2 holds with $\gamma(\beta) = \infty$.

REFERENCES

[1] EINSTEIN, A., *Quantentheorie des einatomigen idealen Gases*. Sitzungsberichte der Preussischen Akademie der Wissenschaften. I, 3–14 (1925).

[2] UHLENBECK, G. E., Thesis. Leyden: 1927.

[3] KAHN, B. and UHLENBECK, G. E., *On the theory of condensation*. Physica *5*, 399–415 (1938).

[4] LONDON, F.: On the Bose–Einstein condensation. Phys. Rev. *54*, 947–954 (1938).

[5] GIRARDEAU, M., *Relationship between systems of impenetrable bosons and fermions in one dimension*. Journ. Math. Phys. *1*, 516–523 (1960).

[6] SCHULTZ, T., *Note on the one-dimensional gas of impenetrable point particle bosons*. Journ. Math Phys. *4*, 666–671 (1963).

[7] RIORDAN, J., *An Introduction to combinatorial analysis*, Wiley (1958).

[8] CASIMIR, H. B. G., On Bose–Einstein condensation. *Fundamental Problems in Statistical Mechanics III*, 188–196 (1968) ed. E. G. D. Cohen, North-Holland Publishing Company–Amsterdam.

[9] VAN DEN BERG, M. and LEWIS, J. T., *On the free boson gas in a weak external potential*. Commun. Math. Phys. *81*, 475–494 (1981).

[10] VAN DEN BERG, M. and LEWIS, J. T., *On generalized condensation in the free boson gas*. Physica 110A, 550–564 (1982).

[11] VAN DEN BERG, M., *On condensation in the free boson gas and the spectrum of the laplacian*. Journ. Stat. Phys. *31*, 623–637 (1983).

[12] PULÈ, J. V., D. Phil, Thesis, Oxford: 1972.

[13] LEWIS, J. T., The Free Boson Gas; Proceedings of the LMS Instructional Conference, Bedford College 1971, *Mathematical Problems of Modern Physics*, Academic Press (1972).

[14] CANNON, J. T., *Infinite volume limits of the canonical free Bose gas states on the Weyl algebra*. Commun. Math. Phys. *29*, 89–104 (1973).

[15] LEWIS, J. T. and PULÈ, J. V., *The equilibrium states of the free boson gas*. Commun. Math. Phys. *36*, 1–18 (1974).

[16] ZIFF, R. M., UHLENBECK, G. E. and KAC, M., *The ideal Bose-gas, revisited*. Physics Reports 32C, 169–248 (1977).

[17] DAVIES, E. B., *The ideal boson gas in an external scalar potential.* Commun. Math. Phys. *30*, 229–247 (1973).
[18] LANDAU, L. J. and WILDE, I. F., *On Bose–Einstein condensation of an ideal gas.* Commun. Math. Phys. *70*, 43–51 (1979).
[19] FELLER, W., *An introduction to probability theory and its applications II*, Wiley & Sons, New York (1966).