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On the general theory of Bose–Einstein condensation and the state of the free boson gas

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1. Introduction

Following the paper by van den Berg, Lewis, Pule [7] which identified two critical densities ρ_c and ρ_m governing generalized condensation in the free boson gas, we aim to give a comprehensive description of the state of the gas when the thermodynamic limit is taken in a variety of ways. It was shown in [7] that both non-extensive condensation and macroscopic occupation of the ground state can occur if $\rho_c < \rho_m$. In this paper we use a similar method to that in [7], to treat the grand canonical ensemble, when the thermodynamic limit is taken keeping the mean value of the particle number density fixed.

In Section 2 we show precisely how the behaviour of the lower eigenvalues of the hamiltonian for large volume, through the re-scaled single particle partition function, determines the asymptotic behaviour of the state, under the additional assumption that the asymptotic behaviour of the Green's function of the hamiltonian parallels that of the partition function.

In Section 3 we consider two examples in which $\rho_c = \rho_m$. The first example is a greatly simplified derivation of the result of Lewis and Pule for an isotropically dilated region. The second example treats in detail one of Casimir's examples, a cuboid with edges V^{α_1}, V^{α_2}, V^{α_3}, $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 > 0$, $\sum \alpha_i = 1$. The single particle hamiltonian is taken to be $-\frac{1}{2}\Delta$ (the Laplacian with Dirichlet boundary conditions). In the case $\alpha_1 \geqslant \frac{1}{2}$, new results are found for both the diagonal and off-diagonal contribution of the condensate to the state.

In Section 4 we give new results for cuboids where the sides become large at very different rates, treating in particular the case $\rho_m > \rho_c$.

2. General results

Consider a sequence $(h_l, V_l): l = 1, 2, \ldots$ of pairs, h_l being a self-adjoint operator, the single particle hamiltonian of the system, and V_l being the volume of the system. In [7] the grand canonical pressures $p_l(\mu): l = 1, 2, \ldots$ are used to define the chemical potential, scaled by an exponential of the lowest eigenvalue, as the unique root $\mu_l(\rho)$ in $(-\infty, 0)$ of

$$\frac{d}{d\mu}\{p_l(\mu)\} = \rho. \tag{2.1}$$

If h_l has eigenvalues $\varepsilon_l(1) \leqslant \varepsilon_l(2) \leqslant \cdots$ with corresponding normalized eigenfunctions $\psi_l(x; 1)$, $\psi_l(x; 2), \ldots,$ we define the single particle partition function to be

$$\phi_l(\beta') = \frac{1}{V_l} \sum_{k=1}^{\infty} e^{-\beta' \lambda_l(k)}, \tag{2.2}$$

where $\lambda_l(k) = \varepsilon_l(k) - \varepsilon_l(1)$. We write

$$F_l(\lambda) = \frac{1}{V_l} \quad \#\{k : \lambda_l(k) \leqslant \lambda\}. \tag{2.3}$$

Then

$$\phi_l(\beta') = \int_{[0,\infty)} e^{-\beta'\lambda}\, dF_l(\lambda). \tag{2.4}$$

Invoking a similar treatment for the state of the system we define the spectral function (diagonal part)

$$F_l(\lambda; x) = \sum_{\{k : \lambda_l(k) \leqslant \lambda\}} |\psi_l(x; k)|^2, \tag{2.5}$$

and the transform

$$\phi_l(\beta'; x) = \int_{[0,\infty)} e^{-\beta'\lambda}\, dF_l(\lambda; x) \tag{2.6}$$

the Green's function of h_l. Assume $\phi_l(\beta'; x) \to \phi(\beta'; x)$ as $l \uparrow \infty$, pointwise in x. Let $F(\lambda; x)$ be the measure corresponding to ϕ.

We note that van den Berg, Lewis and Pule define the critical density as

$$\rho_c = \lim_{\varepsilon \downarrow 0} \lim_{l \uparrow \infty} \int_{[\varepsilon,\infty)} (e^{\beta\lambda} - 1)^{-1}\, dF_l(\lambda)$$

$$= \int_{[0,\infty)} (e^{\beta\lambda} - 1)^{-1}\, dF(\lambda), \tag{2.7}$$

where $F(\lambda)$ is the unique (non-defective) measure corresponding to $\phi(\beta') = \lim_{l\uparrow\infty} \phi_l(\beta')$; see [8].

Definition. Define the contribution of the condensate to the state (at constant density) to be

$$\lim_{\varepsilon\downarrow 0}\lim_{l\uparrow\infty} \int_{[0,\varepsilon)} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x). \tag{2.8}$$

Then it is easy to see that the contribution of the uncondensed gas is

$$\int_{[0,\infty)} (e^{\beta(\lambda-\mu)}-1)^{-1}\, dF(\lambda; x), \tag{2.9}$$

using Lemma 1 of [7], where μ is the limiting value of μ_l. This follows since for the free boson gas the diagonal part of the state is given by

$$K(x, x) = \lim_{l\uparrow\infty} \int_{[0,\infty)} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x). \tag{2.10}$$

In order to consider the macroscopic occupation of the ground state (see [7]) introduce the re-scaled partition function

$$\gamma_l(\beta') = V_l\phi_l(\beta' V_l). \tag{2.11}$$

If $\gamma(\beta') = \lim_{l\uparrow\infty} \gamma_l(\beta')$ is finite then

$$\mu_l = -V_l^{-1}b + o(V_l^{-1})$$

for constant b as V_l tends to infinity. Hence γ determines the asymptotic behaviour of μ_l. We require an extension of the argument in Theorem 3 of [7] in order to consider cases with differing asymptotic behaviour. Accordingly we look for α, $0 < \alpha \leqslant 1$ such that $\gamma_l(\beta') = V_l^\alpha \phi_l(\beta' V_l^\alpha)$ has a finite, non-zero limit as $l\uparrow\infty$.

Lemma 1. *If*

A1 $$\rho_\alpha = \lim_{x\uparrow\infty}\lim_{l\uparrow\infty} \int_{[x\cdot V_l^{-\alpha},\infty)} (e^{\beta\lambda}-1)^{-1}\, dF_l(\lambda)$$

exists and is finite,

A2 $$\gamma_\alpha(\beta') = \lim_{l\uparrow\infty} V_l^\alpha \phi_l(V_l^\alpha \beta')$$

exists and is non-zero for some $\beta' > 0$, *then*

$$b_l(\rho) = -V_l^\alpha \mu_l$$

has a finite limit $b(\rho)$ *which is the unique root of the equation*

$$\int_{[0,\infty)} e^{-\beta' b}\gamma_\alpha(\beta')\, d\beta' = \rho - \rho_\alpha \quad (\textit{for } \rho > \rho_\alpha),$$

A3 provided that the integral on the left exists for some $b>0$.

Remark. Since $\rho_c \leq \rho_\alpha \leq \rho_m$ we have in our case $\rho_c = \rho_\alpha$.

Proof. The proof follows much the same line as that given by [7]. We need to define $G_l(\lambda)$, the measure whose Laplace transform is given by $\gamma_l(\beta) = V_l^\alpha \phi_l(\beta V_l^\alpha)$, and the limiting measure $G(\lambda)$ whose transform is the limit $\gamma_\alpha(\beta)$. Define

$$\rho_\alpha^\pm(x) = \limsup_{l\uparrow\infty}\!\!\!\!\!\!\!\!\inf \int_{[x\cdot V_l^{-\alpha},\infty)} (e^{\beta\lambda}-1)^{-1}\, dF_l(\lambda). \tag{2.12}$$

Then

$$\rho_\alpha = \lim_{x\uparrow\infty} \rho_\alpha^\pm(x).$$

For $b>0$ and $\lambda \geq x\cdot V_l^{-\alpha}$

$$\left(\frac{e^{\beta x V_l^{-\alpha}}-1}{e^{\beta(x+b)V_l^{-\alpha}}-1}\right)\cdot (e^{\beta\lambda}-1)^{-1} \leq (e^{\beta(\lambda+bV_l^{-\alpha})}-1)^{-1} \leq (e^{\beta\lambda}-1)^{-1}. \tag{2.13}$$

Integrating and letting $l\uparrow\infty$, we have

$$\begin{aligned}\left(1+\frac{b}{x}\right)^{-1}\rho_\alpha^-(x) &\leq \liminf_{l\uparrow\infty}\int_{[x\cdot V_l^{-\alpha},\infty)} (e^{\beta(\lambda+bV_l^{-\alpha})}-1)^{-1}\, dF_l(\lambda)\\ &\leq \limsup_{l\uparrow\infty}\int_{[x\cdot V_l^{-\alpha},\infty)} (e^{\beta(\lambda+bV_l^{-\alpha})}-1)^{-1}\, dF_l(\lambda) \leq \rho_\alpha^+(x).\end{aligned} \tag{2.14}$$

So for all $b>0$,

$$\lim_{x\uparrow\infty}\lim_{l\uparrow\infty}\int_{[xV^{-\alpha},\infty)} (e^{\beta(\lambda+bV_l^{-\alpha})}-1)^{-1}\, dF_l(\lambda) = \rho_\alpha. \tag{2.15}$$

Using the inequality $0 < x^{-1} - (e^x-1)^{-1} < 1$, we derive

$$\lim_{l\uparrow\infty}\int_{[0,\infty)} (e^{\beta(\lambda+bV_l^{-\alpha})}-1)^{-1}\, dF_l(\lambda) = \rho_\alpha + \int_{[0,\infty)} \frac{dG(\lambda)}{\beta(\lambda+b)} \tag{2.16}$$

for all $b>0$. Choose $b(\rho)$ such that

$$\int_{[0,\infty)} (\lambda+b(\rho))^{-1}\, dG(\lambda) = \beta(\rho-\rho_\alpha). \tag{2.17}$$

Then we can conclude that $\lim_{l\uparrow\infty} b_l(\rho) = b(\rho)$.

We are now in a position to prove a general theorem concerning the contribution of the condensate. We need to make parallel assumptions about the behaviour of the Green's function.

B1 $\lim_{X\uparrow\infty}\lim_{l\uparrow\infty}\int_{[xV_l^{-\alpha},\infty)} (e^{\beta\lambda}-1)^{-1}\, dF_l(\lambda;x)$ exists and is finite.

B2 $\gamma_\alpha(\beta';x) = \lim_{l\uparrow\infty}\gamma_l(\beta';x)$, pointwise in x, where $\gamma_l(\beta';x) = V_l^\alpha\phi_l(\beta' V_l^\alpha; x)$.

B3 $\int_{[0,\infty)} e^{-\beta' b}\gamma_\alpha(\beta';x)\, d\beta'$ converges whenever $\int_{[0,\infty)} e^{-\beta' b}\gamma_\alpha(\beta')\, d\beta'$ converges.

Theorem 2. *For $\rho > \rho_\alpha$ under assumptions A and B, the contribution of the condensate to the state is*

$$\int_{[0,\infty)} \frac{dG(\lambda; x)}{\beta(\lambda + b(\rho))}, \tag{2.18}$$

where $G(\lambda; x)$ is the unique measure with transform $\gamma_\alpha(\beta'; x)$. Further

$$\lim_{X\uparrow\infty}\lim_{\varepsilon\downarrow 0}\lim_{l\uparrow\infty}\int_{[XV_l^{-\alpha},\varepsilon)} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x) = 0. \tag{2.19}$$

Proof. First we establish that

$$\lim_{l\uparrow\infty}\int_{[X\cdot V_l^{-\alpha},\varepsilon)} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x) \tag{2.20}$$

can be made arbitrarily small. Using $0 < x^{-1} - (e^x - 1)^{-1} < 1$ we have

$$\left|\int_{[XV_l^{-\alpha},\varepsilon)} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x) - \int_{[X,\varepsilon V_l^{\alpha})} \frac{dG_l(\lambda; x)}{\beta(\lambda + b_l)}\right| \leq F_l(\varepsilon; x). \tag{2.21}$$

Proceeding to the limit as $l\uparrow\infty$, $F_l(\varepsilon; x)\to 0$ as $\varepsilon\to 0$, given assumption B1. Now

$$0 < \int_{[X,\varepsilon V_l^{\alpha})} \frac{dG_l(\lambda; x)}{\beta(\lambda + b_l)} < \int_{[X,\infty)} \frac{dG_l(\lambda; x)}{\beta(\lambda + b_l)} \tag{2.22}$$

the last expression being finite by assumption B3. However we know $\{b_l\}$ are bounded and convergent. Further G_l converges to G and

$$\int_{[X,\infty)} \frac{dG(\lambda; x)}{\beta(\lambda + c)} \tag{2.23}$$

is uniformly convergent on compact sets, hence tending to 0 as $X\to\infty$. This proves the second statment in the theorem. The first statement follows in much the same way:

$$\left|\int_{[0,X/V_l^{\alpha})} (e^{\beta(\lambda-\mu_l)}-1)^{-1}\, dF_l(\lambda; x) - \int_{[0,X)} \frac{dG_l(\lambda; x)}{\beta(\lambda + b_l)}\right| < F_l\left(\frac{X}{V_l^{\alpha}}; x\right), \tag{2.24}$$

B1 gives $\lim_{l\uparrow\infty} F_l(X/V_l^{\alpha}; x) = 0$. Taking firstly the limit as $l\uparrow\infty$, secondly the limit as $X\uparrow\infty$, we easily see that the contribution of the condensate is:

$$\int_{[0,\infty)} \frac{dG(\lambda; x)}{\beta(\lambda + b)}. \tag{2.25}$$

This completes the proof of the theorem. ■

We conclude by noting that:

1. The contribution to the state for $\rho \leq \rho_c$ and for the uncondensed gas for

$\rho > \rho_c$ is

$$\int_{[0,\infty)} (e^{\beta(\lambda-\mu(\rho))}-1)^{-1}\, dF(\lambda; x), \tag{2.26}$$

where $\mu(\rho)$ is the unique solution of $p'(\mu)=\rho$ for $\rho \leqslant \rho_c$ and $\mu(\rho)=0$ for $\rho > \rho_c$.

2. The off-diagonal part of the Green's function can be treated in an exactly parallel manner if we define the function

$$F_l(\lambda; x, y) = \sum_{\{k:\lambda_l(k)\leqslant\lambda\}} |\psi_l(x;k)+\theta\psi_l(y;k)|^2, \tag{2.27}$$

where θ is an arbitrary positive constant. The off-diagonal expression for the state can then be recovered as the coefficient of θ.

3. Example A: Isotropic dilation of the boundary

Suppose that Λ_1 is a bounded, convex, open set $\subset \mathbb{R}^3$ with unit volume, and with closed boundary $\partial\Lambda_1$. Dilate istropically from an interior point 0, to give a sequence of regions $\{\Lambda_l\}$, $l=1,2,\ldots$ with $V_l=l^3$. Let $h_1=-\frac{1}{2}\Delta$, with Dirichlet boundary conditions on $\partial\Lambda_1$. Then it is well-known that [2]

$$\left|\sum_{k=1}^{\infty} e^{-t\varepsilon_l(k)}\psi_l(x;k)\psi_l(y;k) - \frac{e^{|x-y|^2/2t}}{(2\pi t)^{3/2}}\right| \leqslant \frac{6}{(2\pi t)^{3/2}} e^{(2\sqrt{2}-3)(2\delta_x^2/3t)}, \tag{3.1}$$

where $\delta_x = \inf_{y\in\partial\Lambda_l}|x-y|$, and

$$\left|e^{\varepsilon_l(1)t}\phi_l(t;x) - \frac{1}{(2\pi t)^{3/2}}\right| \leqslant \frac{6}{(2\pi t)^{3/2}} e^{-(2\delta_x^2/3t)}. \tag{3.2}$$

Note: $\varepsilon_l(k)=\varepsilon_1(k)V_l^{-2/3}$.

Hence we can conclude that the limiting transform $\phi(\beta';x)$ is $(2\pi\beta')^{-3/2}$, for convex bounded regions Λ_1.

Similarly the transform corresponding to the limiting measure, obtained from $F_l(\lambda;x,y)$ is $(2\pi\beta')^{-3/2}(1+2\theta\exp\{-(|x-y|^2/2\beta)\}+\theta^2)$. In this case ρ_c is finite [4]. Since $\lambda_l(k)=V_l^{-2/3}\lambda_1(k)$ we have:

$$\gamma_l(\beta') = V_l\phi_l(V_l\beta') = 1 + \int_\delta^\infty e^{-\beta' V_l^{1/3}\lambda}\, dN_1(\lambda) \tag{3.3}$$

where $N_1(\lambda) = \#\{k:\lambda_1(k)\leqslant\lambda\}$ and $0<\delta<\lambda_1(2)$.

From Clark [9] we have: there exists A such that $N_1(\lambda)\leqslant A\lambda^{3/2}$ and hence the integral tends to zero as $l\uparrow\infty$. This gives $\gamma_1(\beta')=1$ for all $\beta'>0$ and the measure $G(\lambda)$ is an atom at $\lambda=0$. Theorem 3 of [7] gives

$$\frac{1}{\beta}\int_0^\infty e^{-b\beta'}\gamma(\beta')\, d\beta' = \rho-\rho_c; \tag{3.4}$$

that is, $b = (\beta(\rho - \rho_c))^{-1}$ and hence $\mu_l \sim -(\beta V_l(\rho - \rho_c))^{-1}$ asymptotically. We now check assumptions B1, B2, B3 for the Green's function. By Clark [9] it is clear that

$$\gamma_l(\beta'; x) = \sum_{k=1}^{\infty} e^{-\beta' V_l^{1/3}\lambda_l(k)} \left| \psi_1\left(\frac{x}{V_l^{1/3}}; k\right)\right|^2 \tag{3.5}$$

behaves is the same was as $\gamma_1(\beta')$. The measure $G(\lambda; x)$ is an atom at the origin, weighted by the term $|\psi_1(0; 1)|^2$, i.e. the ground state eigenfunction in the region of unit volume evaluated at 0 (the point of dilation).

Using Theorem 2 we conclude that the contribution of the condensate is

$(\rho - \rho_c)\,|\psi_1(0; 1)|^2$.

Denoting the state by $K(x, y) = \langle \psi^*(x)\psi(y)\rangle$,

$$K(x, y) = \begin{cases} \dfrac{1}{(2\pi\beta)^{3/2}} \displaystyle\sum_{n=1}^{\infty} \frac{e^{n\beta\mu(\rho)}}{n^{3/2}} e^{-(|x-y|^2/2n\beta)}, & \rho < \rho_c \\ (\rho - \rho_c)\,|\psi_1(0; 1)|^2 + \displaystyle\sum_{n=1}^{\infty} \frac{e^{-(|x-y|^2/2n\beta)}}{(2\pi\beta n)^{3/2}}, & \rho > \rho_c \end{cases} \tag{3.6}$$

which is the result obtained in [4].

Example B. Rectangular box with differing growth rates for the edges

For the second example we consider the case of the rectangular box with sides of length $V_l^{\alpha_1}$, $V_l^{\alpha_2}$, $V_l^{\alpha_3}$ where $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ and $\sum_{i=1}^{3} \alpha_i = 1$. Take $h_l = -\frac{1}{2}\Delta$ with Dirichlet boundary conditions. The eigenvalues are

$$\varepsilon_l(k) = \frac{\pi^2}{2} \sum_{i=1}^{3} \frac{k_i^2}{V_l^{2\alpha_i}}, \qquad k_i \in Z^+$$

with corresponding eigenfunctions

$$\psi_l(x; k) = \frac{1}{V_l^{1/2}} \prod_{i=1}^{3} \sqrt{2} \cdot \sin\left(\frac{\pi k_i}{V_l^{\alpha_i}}(x_i + \tfrac{1}{2}V_l^{\alpha_i})\right).$$

Now the single particle partition function is

$$\phi_l(\beta') = \prod_i \left(\frac{1}{V_l^{\alpha_i}} \sum_{k_i \in Z^+} \exp\left\{-\frac{\beta'\pi^2(k_i^2 - 1)}{2V_l^{2\alpha_i}}\right\}\right). \tag{3.7}$$

$\phi_l(\beta')$ is thus the product of three one-dimensional functions. Hence the limiting measure corresponding to $\phi(\beta')$ is the convolution of three one-dimensional measures, each with transform $(2\pi\beta')^{-1/2}$.

It is easy to see that the contribution to the state from the uncondensed gas is precisely the same as in Example A. To calculate the contribution of the condensate we need to consider the re-scaled partition function $\gamma_1(\beta')$.

(1) $\alpha_1 < \frac{1}{2}$. Choose

$$\gamma_l(\beta') = V_l\phi_l(V_l\beta') = \prod_1^3 \left(\sum_{k_i} \exp\left(-\frac{\pi^2}{2}\beta' V_l^{(1-2\alpha_i)}(k_i^2-1)\right)\right). \tag{3.8}$$

In the limit the one-dimensional re-scaled Green's function behaves like a Gaussian, variance $(\pi^2\beta' V_l^{(1-2\alpha_i)})^{-1}$. We see $\gamma(\beta')=1$ for all $\beta'>0$. The contribution to the state is identical to that in Example A.
(2) $\alpha_1 = \frac{1}{2}$. Then

$$\gamma_l(\beta') = V_l\phi_l(V_l\beta') = \prod_2^3 \left(\sum_{k_i} \exp\left(-\frac{\pi^2}{2}\beta' V_l^{(1-2\alpha_i)}(k_i^2-1)\right)\right). \left(\sum_{k_1} \exp\left(-\frac{\pi^2}{2}\beta'(k_1^2-1)\right)\right). \tag{3.9}$$

In the limit

$$\gamma(\beta') = \sum_{k_1=1}^{\infty} \exp\left(-\frac{\pi^2}{2}\beta'(k_1^2-1)\right). \tag{3.10}$$

Hence

$$F(\lambda) = \#\left\{k_1 : \frac{\pi^2}{2}(k_1^2-1) \leq \lambda\right\}, \tag{3.11}$$

i.e. $F(\lambda)$ is the number of eigenvalues $\leq\lambda$ for the one-dimensional hamiltonian in an interval of unit length. Similarly

$$\gamma(\beta'; x) = \sum_{k_1} 2\exp\left(-\frac{\pi^2}{2}\beta'(k_1^2-1)\right)\sin^2\frac{\pi k_1}{2}\left(2\sin^2\frac{\pi}{2}\right)^2, \tag{3.12}$$

reducing to the one-dimensional ground state in the x_2, x_3 directions. Thus $\mu_l \sim -1/AV_l$, [5], where

$$\frac{1}{\beta}\sum_{k=1}^{\infty}\left\{\frac{\pi^2}{2}(k^2-1)+\frac{1}{A}\right\}^{-1} = \rho - \rho_c,$$

and the condensate is given by

$$\left(2\sin^2\frac{\pi}{2}\right)^2 \sum_1^{\infty}\left(\frac{\pi^2}{2}(k_1^2-1)+\frac{1}{A}\right)^{-1}\cdot 2\sin^2\frac{\pi k_1}{2}. \tag{3.13}$$

In the limit as $l\uparrow\infty$ this expression is the same for both diagonal and off-diagonal terms. Similarly we may consider the barometric formula, obtained by re-scaling x_i to $V_l^{\alpha_i}u_i$ and looking at the diagonal part. For the uncondensed gas the limit will be ρ or ρ_c, since $x=y$ in (3.6). The expression for the barometric formula is:

$$\nu(u) = \rho_c + \sum_1^{\infty}\left(\frac{\pi^2}{2}(k_1^2-1)+\frac{1}{A}\right)^{-1} 2\sin^2\pi k_1(u_1+\tfrac{1}{2}) \times \prod_2^3 2\sin^2\pi(u_i+\tfrac{1}{2}) \quad \text{for} \quad \rho>\rho_c. \tag{3.14}$$

(3) $\alpha_1 > \frac{1}{2}$. This is potentially the most interesting case. If we consider the re-scaling $\gamma_l(\beta') = V_l \phi_l(\beta' V_l)$ then, since $V_l^{1-2\alpha_1} \to 0$, $\gamma_l(\beta') \to \infty$ as $l \uparrow \infty$. We must choose a different re-scaling and consider

$$\gamma_l(\beta') = V_l^{2(1-\alpha_1)} \phi_l(\beta' V_l^{2(1-\alpha_1)}), \quad \text{i.e.} \quad \alpha = 2(1-\alpha_1).$$

From [1] $\rho_\alpha = \rho_m = \rho_c$. Thus

$$\gamma_1(\beta') = \prod_2^3 \left(\sum_{k_i} \exp\left(-\frac{\pi^2}{2} \beta' V_l^{2(1-\alpha_1-\alpha_i)}(k_i^2 - 1) \right) \right).$$

$$V_l^{-(2\alpha_1-1)} \sum_{k_1} \exp\left(-\frac{\pi^2}{2} \beta' \frac{(k_1^2-1)}{V_l^{4\alpha_1-2}} \right). \tag{3.15}$$

The first two terms give unity in the limit and the third term is the standard one-dimensional Green's function which converges to $(2\pi\beta')^{-1/2}$. By Lemma 1, we have

$$\frac{1}{\beta} \int_0 e^{-b\beta'} (2\pi\beta')^{-1/2} \, d\beta = \rho - \rho_c. \tag{3.16}$$

Thus

$$b = \frac{1}{2(\rho-\rho_c)^2} \quad \text{and} \quad \mu_l \sim \frac{-1}{2(\rho-\rho_c)^2 V_l^{2(1-\alpha_1)}}. \tag{3.17}$$

We are now in a position to calculate the state. Consider

$$\gamma_l(\beta'; x) = V_l^{2(1-\alpha_1)} \phi_l(\beta' V_l^{2(1-\alpha_1)}; x)$$
$$= \prod_2^3 \sum_{k_i=1}^\infty \exp\left(-\left[\frac{\pi^2}{2} \beta' V_l^{2(1-\alpha_1-\alpha_i)}(k_i^2-1) \right] \right) \cdot 2\left(\sin \frac{\pi k_i}{V_l^{\alpha_i}} (x_i + \tfrac{1}{2} V_l^{\alpha_i}) \right)^2.$$

$$\frac{1}{V_l^{2\alpha_1-1}} \sum_{k_1=1}^\infty \exp\left[-\beta' \frac{\pi^2}{2} V_l^{-(4\alpha_1-2)}(k_1^2-1) \right] \cdot 2\left(\sin \frac{\pi k_1}{V_l^{\alpha_1}} (x_1 + \tfrac{1}{2} V_l^{\alpha_1}) \right)^2. \tag{3.18}$$

In the limit the first terms reduce to the value at $k_i = 1$, as in $\gamma_l(\beta')$. The last term is the one-dimensional Green's function. Thus in the infinite volume limit:

$$\gamma(\beta'; x) = \left(\prod_2^3 2 \sin^2 \frac{\pi}{2} \right) (2\pi\beta')^{-1/2}. \tag{3.19}$$

In order to calculate the barometric formula we rescale x so that $x_i = V_l^{\alpha_i} u_i$. Then

$$\gamma(\beta'; u) = \left(\prod_2^3 2 \sin^2 \pi (u_i + \tfrac{1}{2}) \right) (2\pi\beta')^{-1/2}.$$

The contribution of the condensate to the state is

$$\int_0^\infty \frac{dG}{\beta(\lambda+b)}, \quad \text{where} \quad b = \frac{1}{2(\rho-\rho_c)^2 \beta},$$

giving

$$(\rho-\rho_c)\left(2\sin^2\frac{\pi}{2}\right)^2. \tag{3.20}$$

This is clearly the contribution in the off-diagonal state also. The barometric formula $v(u)$ is:

$$v(u)=(\rho-\rho_c)\prod_2^3(2\sin^2\pi(u_i+\tfrac{1}{2}))+\rho_c, \quad \text{for} \quad \rho>\rho_c. \tag{3.21}$$

Thus although the distribution of the condensate in the two more slowly expanding directions depends on the ground state wave function in one dimension, the condensate is spread evenly in the direction of the longest side.

On considering the behaviour of $F_l(\lambda; x, y)$ and its corresponding transform

$$\phi_l(\beta'; x, y)=\int_0^\infty e^{-\beta'\lambda}\,dF_l(\lambda; x, y) \tag{3.22}$$

we can as before find the re-scaled transform

$$\gamma_l(\beta'; x, y)=V_l^{2(1-\alpha_1)}\phi_l(\beta' V_l^{2(1-\alpha_1)}; x, y). \tag{3.23}$$

In order to consider the persistence of off-diagonal long range order we also re-scale x and y, i.e. $x_i=V_l^{\alpha_i}u_i$ and $y_i=V_l^{\alpha_i}v_i$. The term in θ contains a one-dimensional Green's function of the form

$$\frac{1}{L}\sum_{k_1=1}^\infty \exp\left(-\beta'\frac{\pi^2}{2}L^{-2}(k_1^2-1)\right)\psi(u_1; k_1)\psi(v_1; k_1) \tag{3.24}$$

where $L=V_l^{2\alpha_1-1}$ and $\psi(z; k)=\sqrt{2}\sin\pi k(z+\tfrac{1}{2})$. Consequently in the limit as $L\uparrow\infty$, this term $\to 0$. Hence the off-diagonal contribution of the condensate disappears unless $u_1=v_1$.

4. Free boson gas in a cuboid

In this section we will consider a free boson gas with hamiltonian $-(\Delta/2)$ with Dirichlet conditions on the boundary of a cuboid L with sides $L_1\geqslant L_2\geqslant\cdots\geqslant L_d$ and volume V_L. The eigenvalues and eigenfunctions of this (single particle) hamiltonian are given by

$$E_k^L=\sum_{i=1}^d\frac{\pi^2k_i^2}{2L_i^2}, \qquad k_i=1, 2, \ldots, \qquad i=1, \ldots, d, \tag{4.1}$$

$$\phi_{k,L}(x)=\prod_{i=1}^d\left(\frac{2}{L_i}\right)^{1/2}\sin\frac{\pi k_i}{L_i}\left(\frac{L_i}{2}+x_i\right), \qquad -\frac{L_i}{2}<x_i<\frac{L_i}{2}, \qquad i=1, \ldots, d \tag{4.2}$$

The grand canonical equilibrium state for a free boson gas (at mean density ρ and inverse temperature 1) is determined by its "two-point" function

$$K_L(x, y) = \frac{1}{L_1 \cdots L_d} \sum_{(k)} \frac{\xi(L)}{e^{\eta_k^L} - \xi(L)} \phi_{k,L}(x)\phi_{k,L}(y), \tag{4.3}$$

where

$$\eta_k^L = E_k^L - \sum_{i=1}^{d} \frac{\pi^2}{2L_i^2}, \tag{4.4}$$

and $\xi(L)$ is the unique solution in $(0, 1)$ of

$$\rho = \frac{1}{L_1 \cdots L_d} \sum_{\{k\}} \frac{\xi(L)}{e^{\eta_k^L} - \xi(L)}. \tag{4.5}$$

In a previous paper [1] we have shown that the behaviour of the mean occupation density

$$\rho_{k,L} = \frac{1}{L_1 \cdots L_d} \frac{\xi(L)}{e^{\beta\eta_k^L} - \xi(L)} \tag{4.6}$$

in the limit $\{L_1 \uparrow \infty, \ldots, L_d \uparrow \infty\}$ depends on how that limit is taken. Since the results of [1] are of crucial importance for the computation of $\lim_{L\to\infty} K_L(x, y)$ we recall them here.

Theorem. *Let the infinite volume limit $L \to \infty$ be such that*

$$\lim_{L\to\infty} \frac{L_2 \cdots L_d}{L_1} = A \tag{4.7}$$

$$\lim_{L\to\infty} \frac{\log L_2}{L_3 \cdots L_d} = B \tag{4.8}$$

then for $\rho \leq \rho_m \equiv \rho_c + B/\pi$ none of the single particle states are macroscopically occupied. ρ_c (the density at which the thermodynamic functions behave singularly) is given by

$$\rho_c = \sum_{n=1}^{\infty} (2\pi n)^{-d/2}, \qquad d = 3, 4, \ldots. \tag{4.9}$$

For $\rho > \rho_m$ we have

$$\rho_k = \lim_{L\to\infty} \rho_{k,L} = \begin{cases} \left[\frac{\pi^2 A}{2}(k_1^2 - 1) + C\right]^{-1} & \textit{if} \quad k = (k_1, 1, \ldots, 1), 0 < A < \infty \\ 0 & \textit{if} \quad k \neq (k_1, 1, \ldots, 1), 0 < A < \infty \end{cases} \tag{4.10}$$

$$\rho_k = \lim_{L\to\infty} \rho_{k,L} = \begin{cases} \rho - \rho_m & \textit{if} \quad k = (1, \ldots,), A = \infty \\ 0 & \textit{if} \quad k \neq (1, \ldots, 1), A = \infty \end{cases} \tag{4.11}$$

$$\rho_k = \lim_{L\to\infty} \rho_{k,L} = 0 \quad \textit{for all } k \textit{ if } A = 0; \tag{4.12}$$

C is the unique positive solution of

$$\sum_{k=1}^{\infty}\left[(k^2-1)\frac{\pi^2 A}{2}+C\right]^{-1}=\rho-\rho_m. \tag{4.13}$$

For the thermodynamic limit of the state we have

Theorem 3. $\lim_{L\to\infty} K_L(x,y)$ *exists and is given by*

$$K(x,y)=\sum_{n=1}^{\infty}\frac{\xi^n}{(2\pi n)^{d/2}}e^{-(|x-y|^2/2n)},\qquad \rho<\rho_c, \tag{4.14}$$

$$K(x,y)=\sum_{k=0}^{\infty}2^d[(k^2+k)\cdot 2\pi^2 A+C]^{-1}+\sum_{n=1}^{\infty}\frac{e^{-(|x-y|^2/2n)}}{(2\pi n)^{d/2}},\qquad B\geqslant 0,\quad 0<A<\infty,\quad \rho>\rho_m, \tag{4.15}$$

$$K(x,y)=2^{d-1}(\rho-\rho_m)+\sum_{n=1}^{\infty}\frac{e^{-(|x-y|^2/2n)}}{(2\pi n)^{d/2}},\qquad A=0,\quad B\geqslant 0,\quad \rho>\rho_m, \tag{4.16}$$

$$K(x,y)=2^{d-2}(\rho-\rho_c)+\sum_{n=1}^{\infty}\frac{e^{-(|x-y|^2/2n)}}{(2\pi n)^{d/2}},\qquad B>0,\quad 0\leqslant A\leqslant\infty,\quad \rho_c<\rho<\rho_m, \tag{4.17}$$

where ξ *is the unique positive solution of*

$$\sum_{n=1}^{\infty}\frac{\xi^n}{(2\pi n)^{d/2}}=\rho,\qquad \rho<\rho_c. \tag{4.18}$$

We remark that $K(x,y)$ is a function of $|x-y|$. (The state is translationally invariant). Hence the local particle density $K(x,x)$ is independent of x. In order to obtain the global particle density we introduce a scaling:

$$x_i=L_i w_i,\qquad i=1,\ldots,d. \tag{4.19}$$

For convenience we put

$$\tfrac{1}{2}+w_i=u_i,\qquad i=1,\ldots,d. \tag{4.20}$$

Define

$$\nu_L(u)=\sum_{\{k\}}\frac{\xi(L)}{e^{\eta_k^L}-\xi(L)}\prod_{i=1}^{d}\frac{2}{L_i}(\sin\pi k_i u_i)^2. \tag{4.21}$$

We have the following

Theorem 4. $\lim_{L\to\infty}\nu_L(u)$ *exists and is given by*

$$\nu(u)=\rho,\qquad \rho<\rho_c, \tag{4.22}$$

$$v(u)=\sum_{k=1}^{\infty}\frac{2(\sin\pi k u_1)^2}{(k^2-1)\dfrac{\pi^2 A}{2}+C}\prod_{i=2}^{d}2(\sin\pi u_i)^2+\frac{B}{\pi}\prod_{i=3}^{d}2(\sin\pi u_i)^2+\rho_c,$$

$$B\geqslant 0,\qquad 0<A\leqslant\infty,\qquad \rho>\rho_m,\quad (4.23)$$

$$v(u)=(\rho-\rho_m)\prod_{i=2}^{d}2(\sin\pi u_i)^2+\frac{B}{\pi}\prod_{i=3}^{d}2(\sin\pi u_i)^2+\rho_c,$$

$$A=0,\qquad B\geqslant 0,\qquad \rho>\rho_m,\quad (4.24)$$

$$v(u)=(\rho-\rho_c)\prod_{i=3}^{d}2(\sin\pi u_i)^2+\rho_c,\qquad B>0,\qquad 0\leqslant A\leqslant\infty,\qquad \rho_c<\rho<\rho_m. \quad (4.25)$$

Note that

$$\int_{(0,1)^d}v(u)\,du=\rho, \quad (4.26)$$

and that

$$K(x,x)=v(u)\quad\text{for}\quad u=(\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}),\qquad x\in R^d. \quad (4.27)$$

Though it is possible to prove part of Theorems 3 and 4 with the results obtained in Section 2 it is much more convenient to use the techniques developed in [4, 5, 1]. We defer the proofs to the appendix.

Appendix

Proof of (4.20). Since $\xi(L)\geqslant\xi e^{-E_1^L}$ (see the proof of Theorem 1 in [1]) we get

$$\begin{aligned}v_L(u)&=\sum_{n=1}^{\infty}\xi^n\sum_{\{k\}}e^{-nE_k^L}\prod_{i=1}^{d}2(\sin\pi k_i u_i)^2\\&\geqslant\sum_{n=1}^{\infty}\xi^n\cdot\frac{1}{(2\pi n)^{d/2}}\left(1-2de^{-(2L_d^2\cdot\partial_u^2/dn)}\right).\end{aligned} \quad (A1)$$

We have used here (11) of [2]. Since $e^{-x}\leqslant x^{-1/4}$ for $x>0$ we get

$$v_L(u)\geqslant\rho-\frac{d^{5/4}}{(L_d\,\partial_u)^{1/2}} \quad (A2)$$

where

$$\partial_u=\inf_{v\neq(0,1)^d}|v-u|. \quad (A3)$$

Moreover by (15) of [1]

$$\lim_{L\uparrow\infty}\xi(L)e^{E_1^L}=\xi, \quad (A4)$$

so that

$$v_L(u) = \sum_{n=1}^{\infty} (\xi(L)e^{E_1^L})^n \sum_{k=1}^{\infty} e^{-nE_k^L} \cdot \prod_{i=1}^{d} \frac{2}{L_i} (\sin \pi k_i u_i)^2$$
$$\leqslant \sum_{n=1}^{\infty} (\xi(L)e^{E_1^L})^n \frac{1}{(2\pi n)^{d/2}}. \tag{A5}$$

Hence $\limsup_{L\to\infty} v_L(u) = \rho$ for $\rho < \rho_c$; this completes the proof of (4.2).
The next lemma is the equivalent of Lemma 3 in [1].

Lemma A1. *For $z \in [0, 1]$ and $0 < u_i < 1$, $i = 1, \ldots, d$ and $d \geqslant 3$*

$$\lim_{L\uparrow\infty} \sum_{\{k:(k_3,\ldots,k_d)\neq(1,\ldots,1)\}} z(e^{\eta_k^L} - z)^{-1} \prod_{i=1}^{d} \frac{2}{L_i} (\sin \pi k_i u_i)^2 = \sum_{n=1}^{\infty} \frac{z^n}{(2\pi n)^{d/2}} \tag{A6}$$

Proof.

$$\sum_{\{k:(k_3,\ldots,k_d)\neq(1,\ldots,1)\}} z(e^{\eta_k^L} - z)^{-1} \prod_{i=1}^{d} \frac{2}{L_i} (\sin \pi k_i u_i)^2$$
$$\geqslant \sum_{\{k:(k_3,\ldots,k_d)\neq(1,\ldots,1)\}} z(e^{E_k^L} - z)^{-1} \prod_{i=1}^{d} \frac{2}{L_i} (\sin \pi k_i u_i)^2$$
$$\geqslant \sum_{\{k:k_d\neq 1\}} z(e^{E_k^L} - z)^{-1} \prod_{i=1}^{d} \frac{2}{L_i} (\sin \pi k_i u_i)^2$$
$$\geqslant \sum_{n=1}^{\infty} \frac{z^n}{(2\pi n)^{d/2}} - \frac{d^{5/4}}{(L_d\,\partial_u)^{1/2}}$$
$$- \sum_{n=1}^{\infty} \prod_{i=1}^{d-1} \left\{ \sum_{k_i=1}^{\infty} e^{-nE_{k_i}^{L_i}} \cdot 2(\sin \pi k_i u_i)^2 \right\} \cdot \frac{2}{V_L} \cdot e^{-(n\pi^2/2L_d^2)}. \tag{A7}$$

The third term above is smaller (in absolute value) than ($d \geqslant 3$)

$$\sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{(d-1)/2}} \cdot \frac{2}{L_d} \cdot e^{-(n\pi^2/2L_d^2)}$$
$$\leqslant \sum_{n=1}^{\infty} \frac{1}{\pi n L_d} \cdot e^{-n\pi^2/2L_d^2}$$
$$= -\frac{1}{\pi L_d} \log(1 - e^{-(\pi^2/2L_d^2)}), \tag{A8}$$

which tends to zero as $L \to \infty$. We have used in (A7) that

$$\sum_{k=1}^{\infty} e^{-n\pi^2k^2/(2L^2)} \cdot 2(\sin \pi k u)^2 \leqslant \frac{L}{(2\pi n)^{1/2}}. \tag{A9}$$

Define

$$b(L, n; u) = \sum_{k=2}^{\infty} e^{-(n\pi^2/2L^2)(k^2-1)} \cdot 2(\sin \pi k u), \qquad n > 0. \tag{A10}$$

Similarly to (30) of [1] we get for the left hand side of (A6)

$$\frac{1}{V_L}\sum_{n=1}^{\infty} z^n\Big\{\sum_{i=3}^{d} b(L_i, n; u_i)\prod_{k\neq i} 2(\sin \pi u_k)^2$$
$$+\sum\sum_{1\leq i<j\leq d} b(L_i, n; u_i)b(L_j, n; u_j)\cdot \prod_{k\neq i, k\neq j} 2(\sin \pi u_k)^2$$
$$+\sum_{\cdots\cdots}\sum\sum \cdots+\cdots+\prod_{i=1}^{d} b(L_i, n; u_i)\Big\}. \quad \text{(A11)}$$

The estimates of the terms in (A11) with Σ, $\Sigma\,\Sigma$ up to $(d-1)-\Sigma$'s go similarly as (31), (32) etc. in [1] since $b(L, n; u)\leq 2a(L, n)$. Finally for $d\geq 3$

$$\frac{1}{V_L}\sum_{n=1}^{\infty} z^n\prod_{i=1}^{d} b(L_i, n; u_i)\leq \sum_{n=1}^{[L_d]} z^n\cdot \exp\frac{n\pi^2}{2}\left(\frac{1}{L_1^2}+\cdots+\frac{1}{L_d^2}\right)\cdot\frac{1}{(2\pi n)^{d/2}}$$
$$+\sum_{n=[L_d]+1}\left(\frac{2}{\pi n}\right)^{d/2}\leq e^{d\pi^2/L_d}\sum_{n=1}^{\infty}\frac{z^n}{(2\pi n)^{d/2}}+\left(\frac{2}{\pi}\right)^{d/2}\cdot\frac{2}{d-2}\cdot L_d^{1-d/2}, \quad \text{(A12)}$$

where $[x]$ denotes the integer part of x. the combination of (A7), (A8) and (A12) proves Lemma A1 ■

Note that we have used in (A12)

$$b(L, n; u)\leq e^{n\pi^2/2L^2}\cdot\frac{L}{(2\pi n)^{1/2}}. \quad \text{(A13)}$$

Lemma A2. *For* $n>0$ *and* $u\in(0, 1)$ *we have*

$$0\leq\sum_{k=1}^{\infty} e^{-n\pi^2k^2/2L^2}\cos 2\pi k u\leq\frac{1}{u}+\frac{1}{1-u}. \quad \text{(A14)}$$

Proof. Using the Poisson summation formula (see §10 of [6])

$$\sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)}\cos 2\pi k u=\sum_{l=-\infty}^{+\infty}\int_0^{\infty} dx\cdot(\cos 2\pi l x)\cdot(\cos 2\pi u x)\cdot e^{-(n\pi^2x^2/2L^2)}$$
$$=\int_0^{\infty} dx\cdot\cos 2\pi u x\cdot e^{-(n\pi^2x^2/2L^2)}+\sum_{l=1}^{\infty}\int_0^{\infty} dx\{\cos 2\pi x(l+u)$$
$$+\cos 2\pi x(l-u)\}\cdot e^{-(n\pi^2x^2/2L^2)}$$
$$=\frac{L}{(2\pi n)^{1/2}}\left\{e^{-(2L^2u^2/n)}+\sum_{l=1}^{\infty} e^{-(2L^2(l+u)^2/n)}+\sum_{l=1}^{\infty} e^{-(2L^2(l-u)^2/n)}\right\}; \quad \text{(A15)}$$

this proves the left-hand side of the lemma. It follows that

$$\sum_{k=1}^{\infty} e^{-(n\pi^2k^2)/2L^2)}\cos 2\pi k u\leq\frac{L}{(2\pi n)^{1/2}}\left\{e^{-(2L^2u^2/n)}+e^{-(2L^2(1-u)^2/n)}+2\sum_{l=1}^{\infty} e^{-(2L^2l^2/n)}\right\}$$
$$\leq\frac{L}{(2\pi n)^{1/2}}\left\{\left(\frac{n}{2L^2u^2}\right)^{1/2}+\left(\frac{n}{2L^2(1-u)^2}\right)^{1/2}+2\int_0^{\infty} e^{-(2L^2l^2/n)}\,dl\right\}$$
$$=\frac{1}{2u\cdot\pi^{1/2}}+\frac{1}{2(1-u)\cdot\pi^{1/2}}+\frac{1}{2}\leq\frac{1}{u}+\frac{1}{1-u}. \quad ■ \quad \text{(A16)}$$

Lemma A3. *For $n > 0$ and $u \in (0, 1)$ we have*

$$\left|\sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)} \sin 2\pi ku\right| \leqslant 13\left(\frac{1}{u^2} + \frac{1}{(1-u)^2}\right). \tag{A17}$$

Proof.

$$\sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)} \sin 2\pi ku = \sum_{l=-\infty}^{+\infty} \int_0^{\infty} dx \cdot (\cos 2\pi lx) \cdot (\sin 2\pi xu) \cdot e^{-(n\pi^2x^2/2L^2)}$$

$$= \sum_{l=1}^{\infty} \int_0^{\infty} dx \{\sin 2\pi x(u+l) + \sin 2\pi x(u-l)\} e^{-(n\pi^2x^2/2L^2)}$$

$$+ \int_0^{\infty} dx \cdot (\sin 2\pi xu) e^{-(n\pi^2x^2/2L^2)}$$

$$= \sum_{l=1}^{\infty} \left\{(u+l)\int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{L^2(u+l)^2(t-1)/n} + (u-l)\int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} \cdot e^{L^2(u-l)^2(t-1)/n}\right\}$$

$$\cdot \frac{2^{3/2} \cdot L^2}{\pi n} + \frac{L^2u \cdot 2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{L^2u(t-1)/n}. \tag{A18}$$

We have used the Poisson summation formula and two integral formulas of [3] (3.896.3 and 9.211.1). First we will compute an upperbound:

$$\leqslant \sum_{l=1}^{\infty} \left\{\frac{L^2(u+l)2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{l^2L^2(t-1)/n} + \frac{L^2(u-l)2^{3/2}}{\pi n}\right.$$

$$\left. \times \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{l^2L^2(t-1)/n}\right\} + \frac{L^2u2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{u^2L^2(t-1)/n}$$

$$\leqslant \sum_{l=1}^{\infty} \left\{\frac{L^2u2^{5/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{l^2L^2(t-1)/n}\right\} + \frac{L^2u2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{u^2L^2(t-1)/n}. \tag{A19}$$

But for $y > 0$

$$\int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{y(t-1)} \leqslant \frac{2^{1/2}}{y}, \tag{A20}$$

so that

$$\sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)} \sin 2\pi ku \leqslant \sum_{l=1}^{\infty} \left\{\frac{l^2u \cdot 2^{5/2}}{\pi n} \cdot \frac{2^{1/2}n}{l^2L^2}\right\} + \frac{L^2u \cdot 2^{3/2}}{\pi n} \cdot \frac{2^{1/2}n}{u^2L^2}$$

$$= \frac{4\pi u}{3} + \frac{4}{\pi u}. \tag{A21}$$

The next step is to find a lowerbound:

$$\sum_{l=1}^{\infty} \left\{\frac{L^2l2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{L^2(1+l)^2(t-1)/n} - \frac{L^2l2^{3/2}}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{L^2(u-l)^2 \cdot (t-1)/n}\right\}$$

$$= -\frac{2^{3/2}L^2}{\pi n} \int_{-1}^{1} \frac{dt}{(1+t)^{1/2}} e^{L^2(u-1)^2(t-1)/n}$$

$$+\sum_{l=2}^{\infty}\frac{L^2 l 2^{3/2}}{\pi n}\left\{\int_{-1}^{1}\frac{dt}{(1+t)^{1/2}}\left(e^{L^2(1+l)^2(t-1)/n}-e^{L^2(1-l)^2(t-1)/n}\right)\right\}$$

$$\geqslant -\frac{4}{\pi(1-u)^2}-\frac{2^{5/2}L^2}{\pi n}\sum_{l=2}^{\infty}\frac{2^{1/2}}{L^2(1-l)^2}-\frac{12}{\pi}$$

$$\geqslant -\frac{13}{(1-u)^2}. \qquad \blacksquare \tag{A22}$$

Lemma A4. *For $\rho>\rho_c$ and $\alpha>0$*

$$\lim_{L\uparrow\infty}\frac{L_1L_2}{L_1\cdots L_d}\sum_{n=1}^{\infty}\frac{(\xi(L))^n}{2\pi n}\cdot e^{-(\alpha n/L_2^2)}=\inf\{\rho,\rho_m\}-\rho_c. \tag{A23}$$

Proof. We write

$$\frac{L_1L_2}{L_1\cdots L_d}\sum_{n=1}^{\infty}\frac{(\xi(L))^n}{2\pi n}\cdot e^{-(\alpha n/L_2^2)}$$

$$=\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}(\xi(L))^n a(L_1,n)\cdot a(L_2,n)$$

$$+\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}(\xi(L))^n a(L_1,n)a(L_2,n)\cdot\left(e^{-(\alpha n/L_2^2)}-1\right)$$

$$+\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}(\xi(L))^n a(L_1,n)\left(\frac{L_2}{(2\pi n)^{1/2}}-a(L_2,n)\right)e^{-(\alpha n/L_2^2)}$$

$$+\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}(\xi(L))^n\cdot\frac{L_2}{(2\pi n)^{1/2}}\left(\frac{L_1}{(2\pi n)^{1/2}}-a(L_1,n)\right)e^{-(\alpha n/L_2^2)}$$

$$=I+II+III+IV. \tag{A24}$$

Term I converges to $\inf\{\rho,\rho_m\}-\rho_c$ by (35), (40), (44) of [1]. Furthermore since

$$e^{-(n\pi^2/L^2)}\frac{L}{(2\pi n)^{1/2}}\geqslant a(L,n)\geqslant\frac{L}{(2\pi n)^{1/2}}-\frac{3}{2}, \tag{A25}$$

we get

$$(II)\quad\leqslant\sum_{n=1}^{\infty}\frac{L_1}{(2\pi n)^{1/2}}\cdot\frac{L_2}{(2\pi n)^{1/2}}\cdot e^{-(\pi^2 n/L_2^2)}\cdot\frac{\alpha n}{L_2^2}\cdot\frac{1}{L_1\cdots L_d}\leqslant\frac{\alpha}{2\cdot\pi^3(L_3\cdots L_d)}, \tag{A26}$$

$$(III)\quad\leqslant\sum_{n=1}^{\infty}\frac{L_1}{(2\pi n)^{1/2}}\cdot\frac{3}{2}\cdot e^{-(\alpha n/L_2^2)}\cdot\frac{1}{L_1\cdots L_d}\leqslant\frac{3}{(8\alpha)^{1/2}}\cdot\frac{1}{L_3\cdots L_d}, \tag{A27}$$

$$(IV)\quad\leqslant\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}\frac{L_2}{(2\pi n)^{1/2}}\cdot\frac{3}{2}\cdot e^{-(\alpha n/L_2^2)}$$

$$\leqslant\frac{3}{(8\alpha)^{1/2}}\cdot\frac{1}{L_1\cdots L_d}. \qquad \blacksquare \tag{A28}$$

We are now in a position to prove the following

Lemma A5. *For $\rho > \rho_c$ and $u_i \in (0, 1)$, $i = 1, 2$*

$$\lim_{L\to\infty} \frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n b(L_1, n; u_1) b(L_2, n; u_2) = \inf\{\rho, \rho_m\} - \rho_c. \tag{A29}$$

Proof. From Lemma A2 and Lemma A3 it follows that

$$\begin{aligned} b(L, n; u) &\leq e^{-(n\pi^2/L^2)} \sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)} \cdot 2(\sin \pi(k+1)u)^2 \\ &\leq e^{n\pi^2/L^2}\left\{\frac{L}{(2\pi n)^{1/2}} - \sum_{k=1}^{\infty} e^{-(n\pi^2k^2/2L^2)}\{(\cos 2\pi k u) \right. \\ &\quad \left. \cdot (\cos 2\pi u) - (\sin 2\pi k u)\cdot(\sin 2\pi u)\}\right\} \\ &\leq e^{-(n\pi^2/L^2)}\left\{\frac{L}{(2\pi n)^{1/2}} + 14(u^{-2} + (1-u)^{-2})\right\}. \end{aligned} \tag{A30}$$

Combining (A30) and (A13) we get

$$\begin{aligned} b(L_1, n; u_1) b(L_2, n; u_2) &\leq \frac{L_1 L_2}{2\pi n} e^{-(n\pi^2/2L_{\frac{3}{2}})} \\ &\quad + \frac{14 L_1}{(2\pi n)^{1/2}} \cdot e^{-(n\pi^2/L_{\frac{3}{2}})} \cdot \left(\frac{1}{u_2^2} + \frac{1}{(1-u_2)^2}\right) \end{aligned} \tag{A31}$$

But

$$\begin{aligned} &\frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n \cdot \frac{L_1}{(2\pi n)^{1/2}} \cdot e^{-n\pi^2/L_{\frac{3}{2}}} \cdot 14\left(\frac{1}{u_2^2} + \frac{1}{(1-u_2)^2}\right) \\ &\quad \leq \frac{14}{L_2 \cdots L_d} \cdot \left(\frac{1}{u_2^2} + \frac{1}{(1-u_2)^2}\right); \end{aligned} \tag{A32}$$

Because of Lemma A4, (A32) and (A31) we conclude that for $\rho > \rho_c$

$$\limsup_{L\to\infty} \frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n b(L_1, n; u_1) b(L_2, n; u_2) = \inf\{\rho, \rho_m\} - \rho_c. \tag{A33}$$

Furthermore for $u_i \in (0, 1)$

$$2e^{-(n\pi^2/L^2)} \cdot \frac{L}{(2\pi n)^{1/2}} \geq b(L, n; u) \geq e^{-(n\pi^2/L^2)}\left\{\frac{L}{(2\pi n)^{1/2}} - 14\left\{\frac{1}{u^2} + \frac{1}{(1-u)^2}\right)\right\}, \tag{A34}$$

so that

$$\begin{aligned} &\frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n \prod_{i=1}^{2} b(L_i, n; u_i) \\ &\geq \frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n e^{-(n\pi^2/L_{\frac{3}{2}})} \cdot \frac{L_2}{(2\pi n)^{1/2}} \left\{\frac{L_1}{(2\pi n)^{1/2}} - 14\left(\frac{1}{u_1^2} + \frac{1}{(1-u_1)^2}\right)\right\} \\ &\quad - \frac{1}{V_L} \sum_{n=1}^{\infty} (\xi(L))^n e^{-(n\pi^2/L_{\frac{3}{2}})} \cdot 14\left(\frac{1}{u_2^2} + \frac{1}{(1-u_2)^2}\right) \cdot \frac{L_1}{(2\pi n)^{1/2}}. \end{aligned} \tag{A35}$$

But

$$\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n e^{-(n\pi^2/L_2^2)}\cdot\frac{L_1}{(2\pi n)^{1/2}}\leqslant\frac{1}{L_3\cdots L_d},\tag{A36}$$

and

$$\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n\cdot e^{-(n\pi^2/L_2^2)}\cdot\frac{L_2}{(2\pi n)^{1/2}}\leqslant\frac{L_2}{L_1}\cdot\frac{1}{L_3\cdots l_d},\tag{A37}$$

so that (A35), (A36), (A37) and Lemma A4 gives us

$$\liminf_{L\to\infty}\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n b(L_1, n; u_1)b(L_2, n; u_2)=\inf\{\rho,\rho_m\}-\rho_c. \quad\blacksquare\tag{A38}$$

Corollary A1. *For $\rho>\rho_c$ and $u_i\in(0,1)$, $i=1,\ldots,d$*

$$\lim_{L\to\infty}\frac{1}{V_L}\sum_{\{k:k_2\neq 1,(k_3,\ldots,k_d)=(1,\ldots,1)\}}\xi(L)\cdot(e^{\eta_k^L}-\xi(L))^{-1}\prod_{i=1}^{d}\frac{2}{L_i}(\sin\pi k_i u_i)^2$$

$$=(\inf\{\rho,\rho_m\}-\rho_c)\cdot\prod_{i=3}^{d}2(\sin\pi u_i)^2.\tag{A39}$$

Proof. Since

$$\lim_{L\to\infty}\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n b(L_2, n; u_2)\leqslant\lim_{L\to\infty}\frac{2}{V_L}\sum_{n=1}^{\infty}e^{-(n\pi^2/L_2^2)}\frac{L_2}{(2\pi n)^{1/2}}=0,\tag{A40}$$

and because of (A38) and (A33) we arrive at A(39). ■

The next lemma enables us to compute the contribution from the levels $k_1=1,2,\ldots,(k_2,\ldots,k_d)=(1,\ldots,1)$.

Lemma A6. *For $z\in[0,1]$ and $L_1>1$*

$$\left|\sum_{k=2}^{\infty}\left\{\frac{z\cdot 2(\sin\pi k u_1)^2}{(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}+1-z}-\frac{z\cdot 2(\sin\pi k u_1)^2}{\exp(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-z}\right\}\right|\leqslant 2L_1 e^{\pi^2/2}.\tag{A41}$$

This is of course nothing else but (47) of [1]. Note the misprint in [1].

Proof.

$$\sum_{k=2}^{\infty}\frac{2z(\sin\pi k u_1)^2}{\exp(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-z}\leqslant\sum_{k=2}^{\infty}\frac{2z(\sin\pi k u_1)^2}{(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-z},\tag{A42}$$

and

$$\sum_{k=2}^{\infty}\left\{\frac{2z(\sin \pi k u_1)^2}{(k^2-1)\dfrac{\pi^2}{2L_1^2}+1-z}-\frac{2z(\sin \pi k u_1)^2}{\exp (k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-z}\right\}$$

$$\leq 2\sum_{k=2}^{\infty}\frac{\exp (k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-1-(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}}{(k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}\cdot\left(\exp (k^2-1)\cdot\dfrac{\pi^2}{2L_1^2}-1\right)}$$

$$\leq \frac{4L_1^2}{\pi^2}\sum_{k=D+1}^{\infty}(k^2-1)^{-1}+2D\sup_{(3\pi^2/2L_1^2)\leq y\leq(D^2-1)(\pi^2/2L_1^2)}\frac{e^y-1-y}{y(e^y-1)}$$

$$\leq \frac{4L_1^2}{\pi^2 D}+De^{(D^2-1)(\pi^2/2L_1^2)}. \tag{A43}$$

We choose $D=[L_1]$ so that for $L_1>1$ we have $1/D<2/L_1$ and

$$\frac{4L_1^2}{\pi^2 D}+De^{(D^2-1)(\pi^2/2L_1^2)}\leq\frac{8L_1}{\pi^2}+L_1e^{\pi^2/2}\leq 2L_1e^{\pi^2/2}. \quad ■ \tag{A44}$$

Corollary A2. *For $\rho>\rho_m$ and $0<A\leq\infty$ we have*

$$\lim_{L\uparrow\infty}\sum_{\{k:(k_2,\dots,k_d)=(1,\dots,1)\}}\frac{\xi(L)}{e^{\eta_k^L}-\xi(L)}\prod_{i=1}^{d}\frac{2}{L_i}(\sin \pi k_i u_i)^2$$

$$=\sum_{k=1}^{\infty}\frac{2(\sin \pi k u_1)^2}{(k^2-1)\dfrac{\pi^2 A}{2}+C}\cdot\prod_{i=2}^{d}2(\sin \pi u_i)^2. \tag{A45}$$

Proof. From Theorem (3) of [1] we have for $\rho>\rho_m$, $0<A\leq\infty$

$$\xi(L)\sim 1-\frac{C}{V_L}, \tag{A46}$$

so that (A45) follows from (A46) and Lemma A6. ■

Corollary A3. *For $\rho_c<\rho<\rho_m$ and $0\leq A\leq\infty$ we have no contribution from the levels $k_1=1,2,\dots,(k_2,\dots,k_d)=(1,\dots,1)$ to $\nu(u)$.*

Proof. Note that $b(L,n;u)\leq 2a(L,n)$ and use (40) of [1]. ■

Lemma A7. *For $\rho>\rho_m$, $A=0$ we have*

$$\sum_{\{k:(k_2,\dots,k_d)=(1,\dots,1)\}}\frac{\xi(L)}{e^{\eta_k^L}-\xi(L)}\prod_{i=1}^{d}\frac{2}{L_i}(\sin \pi k_i u_i)^2=(\rho-\rho_m)\prod_{i=2}^{d}2(\sin \pi u_i)^2. \tag{A47}$$

Proof. We write the left hand side of (A47) as follows

$$\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n\sum_{k=1}^{\infty}e^{-(n\pi^2/2L_1^2)(k^2-1)}-\frac{1}{V_L}\sum_{n=1}^{\infty}(\xi(L))^n\sum_{k=1}^{\infty}e^{-(n\pi^2/2L_1^2)(k^2-1)}\cos 2\pi k u_1. \tag{A48}$$

The first term tends to $(\rho-\rho_m)\prod_{i=2}^{d}2(\sin\pi u_i)^2$ by (44) of [1]. For the second term we have

$$\begin{aligned}
&\left|\frac{1}{L_1\cdots L_d}\sum_{n=1}^{\infty}(\xi(L))^n\sum_{k=1}^{\infty}e^{-(n\pi^2/2L_1^2)(k^2-1)}\cos 2\pi k u_1\right|\\
&\leqslant\frac{1}{L_1\cdots L_d}\sum_{n=1}^{[L_1^2]}(\xi(L))^n\left|\sum_{k=1}^{\infty}e^{-(n\pi^2/2L_1^2)(k^2-1)}\cos 2\pi k u_1\right|\\
&\quad+\frac{1}{L_1\cdots L_d}\sum_{n=[L_1^2]+1}^{\infty}(\xi(L))^n\sum_{k=1}^{\infty}e^{-(n\pi^2/2L_1^2)(k^2-1)}\\
&\leqslant\frac{1}{L_1\cdots L_d}\sum_{n=1}^{[L_1^2]}(\xi(L))^n\cdot e^{\pi^2/2}\cdot\left(\frac{1}{u}+\frac{1}{1-u}\right)\\
&\quad+\frac{1}{L_1\cdots L_d}\sum_{n=[L_1^2]+1}^{\infty}(\xi(L))^n\cdot\left(1+\frac{L_1}{(2\pi n)^{1/2}}\right)\\
&\leqslant e^{\pi^2/2}\cdot\frac{1}{L_1\cdots L_d}\cdot\frac{\xi(L)}{1-\xi(L)}\cdot\left(\frac{1}{u}+\frac{1}{1-u}\right)\\
&\quad+\frac{1}{L_2\cdots L_d}\int_{L_1^2}^{\infty}e^{-(n/(1+6\rho L_2\cdots L_d)^2)}\cdot\frac{dn}{(2\pi n)^{1/2}}.
\end{aligned}\tag{A49}$$

The first term tends to zero by (51) of [1]. The second term becomes less than

$$\left(\frac{2}{\pi}\right)^{1/2}\int_{\{L_1/(L_2\cdots L_d)\}^2}^{\infty}e^{-(L_2\cdots L_d)/1+6\rho L_2\cdots L_d)^2\cdot u^2}\,du. \tag{A50}$$

For $L_2\cdots L_d\cdot 6\rho>1$ we have

$$\leqslant\left(\frac{2}{\pi}\right)^{1/2}\int_{\{L_1/(L_2\cdots L_d)\}^2}^{\infty}e^{-(u^2/144\rho^2)}\,du, \tag{A51}$$

which tends to zero since $L_1/L_2\cdots L_d\to\infty$ by assumption. ■

Proof of Theorem 4. For $\rho>\rho_c\xi(L)\uparrow 1$. So Lemma A1 provides the contribution ρ_c. Corollary A1 proves the appearance of $\{\inf(\rho,\rho_m)-\rho_c\}\prod_{i=3}^{d}2(\sin\pi u_i)^2$ and the Corollaries A2, A3 and Lemma A7 give the "one-dimensional" contribution to $v(u)$. ■

Indication of the proof of Theorem 3. The contribution from the normal fluid to $K(x,y)$ follows from (6) of [2]. The condensate part (diagonal and off-diagonal) can be obtained by putting $u_1=\cdots=u_d=\frac{1}{2}$ in the condensate part of

Theorem 1. The reason is that

$$\left|\sin\frac{\pi k}{L_i}\left(x+\frac{L_i}{2}\right)-\sin\frac{\pi k}{2}\right|\leq\frac{\pi k}{L_i}, \tag{A52}$$

is small for small k. The contribution from large k values in (4.3) is small because η_k^L is large.

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