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Abstract. Harmonic features in the microwave spectrum of solar flares seem to be produced by non-linear wave coupling. Density, temperature and magnetic field in the vicinity of eruptive coronal regions have values for which the fluid concept of plasma is applicable. In this paper the current density for the interaction of three electromagnetic waves in a cold magnetized collisionless plasma is derived and applied to formulate the coupling rate of coherent waves in a first order approximation. This result is then extended to the rate of transfer of wave energy between incoherent spectra of extraordinary waves.

1. Introduction

In a recent investigation of the spectral properties of microwave radiation from solar flares, the occurrence of harmonic features with a 2:1 ratio of the emitted frequency bands has been discovered [1]. The generation of this harmonic emission is difficult to explain by those processes widely accepted to be responsible for various kinds of microwave flare radiation. In the processes of plasma radiation or of gyroradiation the fundamental frequencies would be strongly re-absorbed and the respective models require either steep density gradients or very high magnetic fields in order to become operative at the observed radio frequencies between about 3 and 10 GHz. The electron cyclotron maser process can be excluded because after the total reabsorption of the fundamental radiation not only the second and fourth but also the third harmonic would have to show up; this is not observed. Therefore, a concept of generation of the second harmonic by the intense fundamental through a parametric interaction within the magnetized plasma surrounding the radiation source may be considered a serious candidate for explaining the observed spectrum. The flux densities of the solar flare radiation with observed harmonic features were of the order or larger than 100 solar flux units (10^{-20} W/Hz m^2) at about 3 to 10 GHz. This can be computed back to the vicinity ($\approx$100 km) of the source region. A 1% bandwidth of this incoherent radiation will there yield already an intensity of several Watt per square meter.

This idea provoked the author to resume a conception which was developed long ago [2, 3], but was never applied because of lack of observational evidence in those years. It was thought that the non-linear wave coupling of wide band incoherent radiation can cause spectral redistribution. However, from an observations's point of view it is difficult to prove whether spectral equalizing is caused by non-linear wave coupling. Proof of this process is only possible if a distinction between fundamental and harmonic spectral bands can be recognized in the observation.

The concept of non-linear wave–wave interaction has been presented in many texts, e.g. [4, 5, 6, 7, 8]. The analysis has been applied to the parametric coupling of three waves, usually one of them being a longitudinal mode. The reason for this is that the dispersion relations of electromagnetic waves only allow for very few wave modes and/or very special geometry to satisfy the conservation of energy and momentum conditions simultaneously for three waves; this is particularly true for unmagnetized plasma.

Moreover, in the majority of the texts the interaction between exclusively coherent waves is treated because one had the application to laboratory plasmas in mind. The papers most relevant in the present context are by Stenflo [9] on three-wave interactions in cold magnetized plasma and by Tsytovich and Stenflo [10] on three-wave interaction in turbulent plasma. In [9] the coupling coefficients are derived for resonant interactions between three coherent waves with well defined phases propagating in different directions in order to satisfy the resonance conditions. In a turbulent plasma non-linear terms are present due to spatial gradients if the turbulence exceeds a critical limit [10]. High frequency electromagnetic waves can be absorbed via the non-linear decay into two longitudinal plasma waves and this process can be used for plasma heating.

Among the more recent publications on non-linear wave interaction a majority is also motivated by the possibilities of plasma heating. For this purpose the coupling between electrostatic modes and the extra-ordinary [11] or the ordinary wave close to the cutoff at the electron plasma frequency [12] are investigated and special problems as e.g. in the radio frequency heating of bumpy torus plasma [13], the non-linear mode conversion and anomalous absorption processes have been studied. With regard to the high intensity radiation in non-linear interaction it is important that attention has been drawn to the effects of high frequency induced magnetization causing an extra, ponderomotive, force in the plasma [14]. A necessary condition for the operation of wave coupling is that the dispersion relation allows sufficiently long interaction between the waves. A kinetic treatment, necessary for high temperature plasmas, shows the change of the dispersions relation for waves propagating in a magnetized plasma [15]. One of the rare papers on non-linear wave coupling in astrophysical conditions in particular with regard to solar radio bursts has been published by Wu et al. [16]. They propose a model based on the synchrotronmaser instability excited by a hollow beam of moderately relativistic electrons in a plasma in which the plasma frequency is much higher than the gyrofrequency. As a result of this instability, unpolarized electromagnetic waves with frequency near twice the electron plasma frequency can be amplified.

In contrast to investigations presupposing either one of the interacting waves to be a longitudinal mode or different mutual propagation directions in order to satisfy the resonance conditions for parametric wave coupling, in the present paper the interacting waves are assumed to be electromagnetic waves and to propagate altogether in the same direction within a magnetized plasma. The conditions of the plasma in the surroundings of an active region of the solar corona are typically: temperature $10^5\,\mathrm{K} \le T \le 10^7\,\mathrm{K}$, electron density $10^{15}\,\mathrm{m}^{-3} \le n \le 10^{18}\,\mathrm{m}^{-3}$, magnetic field $0.01T \le B \le 0.1T$. This means that electron cyclotron and plasma frequencies can be within the same order of magnitude and are comparable with the frequency range of the observed electromagnetic waves. The temperature is such that the plasma can be treated with the fluid equations and at the given range of density the collision frequency is very low ($\ll 10^6$ Hz). One can also consider the plasma parameter $g = (n\lambda_D^3)^{-1}$, the inverse of the number of electrons in the Debye volume; this value is far below 10^{-4} showing that, together with a very low collision frequency, the fluid treatment, i.e. the concept of waves in a cold plasma is permitted. The plasma beta $\beta = nkT2\mu_0/B^2$, the ratio of kinetic energy of the particles and the energy contained in the magnetic field, is less or far less than 10^{-2}, i.e. the plasma behaviour is dominated by the magnetic field.

With regard to the high intensity of waves another criterion has to be satisfied. The electron oscillations induced by the wave must be sufficiently weak to avoid coupling to plasma waves in the polarization direction of the electromagnetic wave field, i.e. perpendicularly to its propagation direction. The mean amplitude of oscillation of an electron at the plasma frequency about its equilibrium position equals the Debye shielding distance λ_D. If the wave-induced oscillation is sufficiently less than λ_D, we may assume negligible coupling to a plasma wave. To estimate this we consider a free electron driven by a harmonic field $dv/dt = -(e/m)E_0 \sin \omega t$, from which the oscillation amplitude follows as $|x_0| = (e/m)E_0/\omega^2$. In the frequency range under consideration ($\omega \approx 10^{10}\,\mathrm{s}^{-1}$) it follows $x_0(m) \approx 10^{-9} E_0(V/m)$. As long as $n < 10^{18}$ ($\omega_p < 6\ 10^{10}\,\mathrm{s}^{-1}$) and $T > 10^5\,\mathrm{K}$ we get $\lambda_D > 2 \times 10^{-5}\,\mathrm{m}$, hence for $E_0 < 2 \times 10^4$ Volt per meter the direct coupling into a plasma oscillation will be negligible. Moreover an efficient coupling and propagation of plasma waves will only take place for signal frequencies within a few percent above ω_p.

It can also be shown that damping mechanisms, self absorption due to gyro resonance or bremsstrahlung as well as collision effects are sufficiently weak in the parameter range of interest, as to allow for efficient coupling of wave energy from the fundamental to the harmonic frequency even in the case of incoherent radiation.

It is the purpose of this paper to formulate the concept of non-linear coupling of electromagnetic waves in cold magnetized plasma. The application to the astrophysical problem which stimulated this study will be derived and the feasibility will be discussed in a separate paper at present in preparation.

2. Current density due to non-linear interaction of three electromagnetic waves

A sufficiently intense electromagnetic wave travelling through a plasma modulates density and velocity of the charged particles. A second wave entering the plasma meets with a medium whose propagation parameters change periodically in space and time. Hence this second wave is parametrically modulated and combination frequencies are produced. If not only the combination frequencies but also the combination wave vectors have well-defined relations to the respective quantities of the two original waves the combination waves may also be able to propagate.

Within the concept of cold plasma approximation, i.e. where all distribution functions are replaced by their respective moments, the momentum and the continuity equations yield products of velocity and density fluctuations caused by the waves. These products result in a second order term of the current density j and in this way enter the wave equation

$$\frac{\partial^2 \vec{E}_k}{\partial \vec{r}^2} - \frac{\partial}{\partial \vec{r}}\left(\frac{\partial}{\partial \vec{r}} \vec{E}_k\right) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_k}{\partial t^2} - \mu_0 \frac{\partial \vec{j}_k^{(1)}}{\partial t} = \mu_0 \frac{\partial \vec{j}_k^{(2)}}{\partial t} \tag{1}$$

where superscript (1) and (2) are set for first and second order contributions respectively and subscript k indicates the spectral component. Permittivity and permeability of free space are ε_0 and μ_0. This wave equation is applicable to any spectral contribution for which wave vector $\vec{k}$ and frequency ω_k satisfy a dispersion relation $\vec{k}(\omega_k)$ in the considered medium.

In the following we assume a two-component plasma and use superscripts i and e for ions and electrons respectively. The current density

$$\vec{j} = e(Zn^i \vec{v}^i - n^e \vec{v}^e) \tag{2}$$

contains the product of the fluctuations of the velocities $\vec{v}^i$ and $\vec{v}^e$ and of the densities $Zn^i = n^e \equiv n$ of the charged particles.

The effect of the electromagnetic wave fields $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ and of the – homogeneously assumed – static magnetic field $\vec{H}_0$ on velocity and density can be derived from the equations of momentum (formulated here only for electrons, charge $-e$ and mass m)

$$\frac{\partial \vec{v}^e}{\partial t} + \left(\vec{v}^e \cdot \frac{\partial}{\partial \vec{r}}\right)\vec{v}^e = -\frac{e}{m}\{\vec{E} + \mu_0[\vec{v}^e \times (\vec{H}_0 + \vec{H})]\} \tag{3}$$

and from the continuity equation

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \vec{r}} \cdot (n\vec{v}) \tag{4}$$

With the Fourier transform

$$\vec{E}(\vec{r}, t) = \int \vec{E}_k e^{i(\omega t - \vec{k}\cdot\vec{r})}\, d\vec{k}\, d\omega \tag{5}$$

and its inverse

$$\vec{E}(\vec{k}, \omega) = \frac{1}{(2\pi)^4} \int \vec{E}(\vec{r}, t) e^{-i(\omega t - \vec{k}\cdot\vec{r})}\, d\vec{r}\, dt \tag{6}$$

the spectral component $\vec{E}(\vec{k}, \omega)$, (henceforth we abbreviate $\vec{E}(\vec{k}, \omega)$ by $\vec{E}_k$) is related to the space and time dependent wave field. The Fourier concept can be extended to products of harmonic wave quantities – say $A(\vec{k}_1, \omega_1)$ and $B(\vec{k}_2, \omega_2)$ – which may yield a product wave with wave vector $\vec{k}_0$ and frequency ω_0 if, in a first order consideration, only three waves are permitted (see e.g. [5]). The spectral presentation of this product wave is

$$(AB)_{k_0} = \int A_{k_1} B_{k_2}\, \delta(\vec{k}_0 - \vec{k}_1 - \vec{k}_2)\, \delta(\omega_0 - \omega_1 - \omega_2)\, d\vec{k}_1\, d\vec{k}_2\, d\omega_1\, d\omega_2 \tag{7}$$

The frequencies ω_1, ω_2, ω_0 and wave vectors $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_0$ of the original waves and the product wave together have to obey the conservation laws of energy and momentum

$$\omega_0 = \omega_1 + \omega_2 \tag{8}$$

$$\vec{k}_0 = \vec{k}_1 + \vec{k}_2 \tag{9}$$

respectively. Besides (8) and (9), each pair $\vec{k}$, ω_k is related by its proper dispersion relation $\vec{k}(\omega_k)$.

The spectral presentation of a product of two waves (7) applied to the momentum equation (3) of the electrons, making use of $\partial/\partial t = i\omega$ and $\partial/\partial \vec{r} = -i\vec{k}$, yields

$$i\omega_0 \vec{v}^e_{k_0} - i \int (\vec{v}^e_{k_1} \cdot \vec{k}_2) \vec{v}^e_{k_2}\, dk_{1,2}\, \delta_{\omega,k}$$
$$= -\frac{e}{m} \left\{ \vec{E}_{k_0} + \mu_0 [\vec{v}^e_{k_0} \times \vec{H}_0] + \mu_0 \int [\vec{v}^e_{k_1} \times \vec{H}_{k_2}]\, dk_{1,2}\, \delta_{\omega,k} \right\} \tag{10}$$

with the abbrevations

$$dk_{1,2} \equiv d\vec{k}_1\, d\vec{k}_2\, d\omega_1\, d\omega_2$$
$$\delta_{\omega,k} \equiv \delta(\omega_0 - \omega_1 - \omega_2)\, \delta(\vec{k}_0 - \vec{k}_1 - \vec{k}_2)$$

The plasma has been assumed homogeneous to first order, therefore, $d\vec{v}/d\vec{r}$ (the second term l.h.s. of (3)) is exclusively determined by the wave induced fluctuations. The corresponding equation for the ions contains Ze/M instead of $-e/m$ with ion mass M. The spectral presentation of the continuity equation (4) yields terms like

$$\omega_0 n^e_{k_0} = \vec{k}_0 \int n^e_{k_1} \vec{v}^e_{k_2}\, dk_{1,2}\, \delta_{\omega,k} \tag{11}$$

By cyclic interchange of the wave vectors $\vec{k}_0$, $\vec{k}_1$, $\vec{k}_2$ and frequencies ω_0, ω_1, ω_2 in equations (10) and (11), all contributions to the three waves by the non-linear interaction of the corresponding pair can be obtained.

By use of Maxwell's equation $(\vec{k} \times \vec{E}_k) = \omega\mu_0 \vec{H}_k$, the magnetic wave field can be substituted. With some vector algebra applied to (10), we obtain the spectral component $\vec{k}_0$, ω_0 of the electron velocity caused by the first and second order effects of the wave fields $\vec{k}_0$, ω_0 and $\vec{k}_1$, ω_1, $\vec{k}_2$, ω_2 respectively.

$$\vec{v}^e_{k_0} = \frac{ie}{\omega_0 m}[\vec{E}_{k_0} + \mu_0(\vec{v}^e_{k_0} \times \vec{H}_0)] + \frac{ie}{\omega_0 m}\int \frac{\vec{k}_2}{\omega_2}(\vec{v}^e_{k_1}\cdot \vec{E}_{k_2})\,dk_{1,2}\,\delta_{\omega,k}$$
$$+\frac{1}{\omega_0}\int (\vec{v}^e_{k_1}\cdot \vec{k}_2)\left[\vec{v}^e_{k_2} - i\frac{\vec{E}_{k_2}}{\omega_2}\frac{e}{m}\right]dk_{1,2}\,\delta_{\omega,k} \tag{12}$$

The first term at the r.h.s. of (12) is obviously the linear relation between velocity and wave field of the identical spectral range. The second and third terms represent the second order contributions to the velocity fluctuation $\vec{v}_{k_0}$ due to the products of wave fields and first order velocity fluctuations in the spectral ranges $\vec{k}_1$, ω_1 and $\vec{k}_2$, ω_2. Again, cyclic interchange will yield the complete picture. The implicitely appearing velocity v_{k_0} in the first term at the r.h.s. of (12) can be approximated by the solution of the linear relation alone as long as the effect of the non-linear terms on $\vec{v}_{k_0}$ is weak.

If higher order terms, i.e. combinations of multiples of $\vec{k}_i$ and ω_i respectively, are not considered the relation (12) for $\vec{k}_0$ and the corresponding ones for $\vec{k}_1$ and $\vec{k}_2$ the spectral components of the velocity $\vec{v}_k$ can be represented by the first two terms of a series expansion like

$$\vec{v}_k = \vec{v}_k^{(1)} + \vec{v}_k^{(2)} \tag{13}$$

where k stands for k_0 or k_1 or k_2.

Let us first regard only the linear term in (12). The presence of a static magnetic field causes the relationship between electron velocity $v_k^{e(1)}$ and the wave field E_k

$$(v_k^{e(1)})_p = (T_k^e)_{pq}\frac{ie}{m\omega_k}(E_k)_q \tag{14}$$

to be determined by a tensor

$$(T_k^e)_{pq} = \frac{1}{1-\left(\frac{\omega_c^e}{\omega_k}\right)^2}\begin{bmatrix} 1-\left(\frac{\omega^e_{cx}}{\omega_k}\right)^2 & i\frac{\omega^e_{cz}}{\omega_k} - \frac{\omega^e_{cx}\omega^e_{cy}}{\omega_k^2} & -i\frac{\omega^e_{cy}}{\omega_k} - \frac{\omega^e_{cx}\omega^e_{cz}}{\omega_k^2} \\ -i\frac{\omega^e_{cz}}{\omega_k} - \frac{\omega^e_{cx}\omega^e_{cy}}{\omega_k^2} & 1-\left(\frac{\omega^e_{cy}}{\omega_k}\right)^2 & i\frac{\omega^e_{cx}}{\omega_k} - \frac{\omega^e_{cy}\omega^e_{cz}}{\omega_k^2} \\ i\frac{\omega^e_{cy}}{\omega_k} - \frac{\omega^e_{cx}\omega^e_{cz}}{\omega_k^2} & -i\frac{\omega^e_{cx}}{\omega_k} - \frac{\omega^e_{cy}\omega^e_{cz}}{\omega_k^2} & 1-\left(\frac{\omega^e_{cz}}{\omega_k}\right)^2 \end{bmatrix} \tag{15}$$

Here the cyclotron frequency for electrons is $\vec{\omega}^e_c = -\mu_0\; e/m\; \vec{H}_0$, with its components in an x, y, z orthogonal coordinate system $(\omega^e_c)^2 = (\omega^e_{cx})^2 + (\omega^e_{cy})^2 + (\omega^e_{cz})^2$. Tensors and vectors used in tensor relations are set between brackets and their coordinates are indicated by subscripts (suffix notation). The anti-symmetry of the imaginary parts in this tensor is caused by the anisotropy of the magnetic field. In the corresponding tensor for the ions the signs of the imaginary parts are reversed and the value of ω_c^i is reduced by the electron to ion mass ratio as compared with ω_c^e.

In order to simplify the following computation without loosing the essential generality of the analysis we assume the static magnetic field to be aligned along the z-axis, i.e. $\omega_c \equiv \omega_{cz}$, $\omega_{cx} = \omega_{cy} = 0$.

The tensor (15) is then simplified to

$$(T_k^e)_{pq} = \frac{1}{1-\left(\frac{\omega_c^e}{\omega_k}\right)^2} \begin{bmatrix} 1 & i\frac{\omega_c^e}{\omega_k} & 0 \\ -i\frac{\omega_c^e}{\omega_k} & 1 & 0 \\ 0 & 0 & 1-\left(\frac{\omega_c^e}{\omega_k}\right)^2 \end{bmatrix} \tag{16}$$

A series expansion of the particle density with respect to linear and non-linear interaction, $n_k = n_k^{(1)} + n_k^{(2)}$, as for the velocity is not necessary if we limit the non-linear terms to second order, i.e. to products of constants and second order quantities and to products of two first order quantities. Because of the assumed stationarity of the plasma ($v_0 = 0$) there remain only 3 terms when density and velocity are multiplied

$$n\vec{v} = n_0\vec{v}^{(1)} + n^{(1)}\vec{v}^{(1)} + n_0\vec{v}^{(2)} \tag{17}$$

There is no constant term, one linear term and two terms which determine the non-linear contribution to the current density. The linear portion of n with wave number k_0 due to electrons follows in analogy to (11) if (14) is used in the product $n_0 v_{k_0}^{(1)}$

$$n_{k_0}^{e(1)} = i\frac{en_0}{m\omega_0^2}(k_0)_p(T_{k_0}^e)_{pq}(E_{k_0})_q \tag{18}$$

The current density with wave vector $\vec{k}_0$ can be formulated with the products of spectral components of density and velocity as

$$\vec{j}_{k_0} = e\int (Zn_{k_1}^i\vec{v}_{k_2}^i - n_{k_1}^e\vec{v}_{k_2}^e)\, dk_{1,2}\, \delta_{\omega,k} \tag{19}$$

The linear portion of the current density is

$$(j_{k_0}^{(1)})_p = -i\frac{\varepsilon_0}{\omega_{k_0}}[\omega_p^{i2}(T_{k_0}^i)_{pq} + \omega_p^{e2}(T_{k_0}^e)_{pq}](E_{k_0})_q \tag{20}$$

where the plasma frequencies squared

$$\omega_p^{i2} = \frac{n_0^iZ^2e^2}{M\varepsilon_0} \quad \text{and} \quad \omega_p^{e2} = \frac{n_0^ee^2}{m\varepsilon_0} \tag{21}$$

of ions and electrons respectively are introduced.

The second order portion of the current density with wave number k_0 is due to the non-linear action of the waves k_1 and k_2. As already indicated with (17) and (19), only the resulting expressions of products $n_{k_1}^{(1)} v_{k_2}^{(1)} + n_0 v_{k_0}^{(2)}$ are presented with $v_{k_0}^{(2)}$ taken from (12). The respective products with k_1 and k_2 interchanged yield formally equal expressions and are suppressed in the following result. With this prerequisite, the second order current density contains six terms, three stem from the action of the wave fields on the ions and three from the electrons. One of the three terms of each species originates from the product of first order expressions of density and velocity fluctuations (second term r.h.s. of (17)) and two terms are due to the second order expression of the velocity fluctuations (last term in (17) where the two last terms of (12) have to be applied).

$$
\begin{aligned}
(j_{k_0}^{(2)})_p = &-\frac{Ze}{M}(T_{k_0}^i)_{pq}\frac{\varepsilon_0\omega_p^{i2}}{\omega_0}\left\{\int\int\frac{(k_2)_q}{\omega_1\omega_2}(T_{k_1}^i)_{rs}(E_{k_1})_s(E_{k_2})_r\,dk_{1,2}\,\delta_{\omega,k}\right.\\
&\left.+\int\frac{(k_2)_r}{\omega_1\omega_2}(T_{k_1}^i)_{rs}(E_{k_1})_s[(T_{k_2}^i)_{qt}(E_{k_2})_t-(E_{k_2})_q]\,dk_{1,2}\,\delta_{\omega,k}\right\}\\
&-\frac{Ze}{M}\varepsilon_o\omega_p^{i2}\int\frac{(k_1)_r}{\omega_1^2\omega_2}(T_{k_1}^i)_{rs}(E_{k_1})_s(T_{k_2}^i)_{pq}(E_{k_2})_q\,dk_{1,2}\,\delta_{\omega,k}\\
&+\frac{e}{m}(T_{k_0}^e)_{pq}\frac{\varepsilon_0\omega_p^{e2}}{\omega_0}\left\{\int\int\frac{(k_2)_q}{\omega_1\omega_2}(T_{k_1}^e)_{rs}(E_{k_1})_s(E_{k_2})_r\,dk_{1,2}\,\delta_{\omega,k}\right.\\
&\left.+\int\frac{(k_2)_r}{\omega_1\omega_2}(T_{k_1}^e)_{rs}(E_{k_1})_s[(T_{k_2}^e)_{qt}(E_{k_2})_t-(E_{k_2})_q]\,dk_{1,2}\,\delta_{\omega,k}\right\}\\
&+\frac{e}{m}\varepsilon_0\omega_p^{e2}\int\frac{(k_1)_r}{\omega_1^2\omega_2}(T_{k_1}^e)_{rs}(E_{k_1})_s(T_{k_2}^e)_{pq}(E_{k_2})_q\,dk_{1,2}\,\delta_{\omega,k}
\end{aligned}
\tag{22}
$$

The indices p, q, r, s, t are used as subscripts to indicate the tensor and vector coordinates, the subscript p is also used for plasma frequency.

Equation (22) only shows the effect of waves with wave vectors $\vec{k}_1$ and $\vec{k}_2$ on the creation of a current density fluctuation with wave vector $\vec{k}_0 = \vec{k}_1 + \vec{k}_2$ and frequency $\omega_0 = \omega_1 + \omega_2$, i.e. the generation of higher energy photons by pairs of lower energy photons. The inverse process, the decay of the higher energy photon into two of lower energy according to the same conservation laws (8), (9), not formulated here, can formally be derived according to the same procedure.

3. Rate of coupling of coherent waves

With the current densities computed for the linear, (20), and the non-linear contribution, (22), we can enter the wave equation in order to find a relation for the rate of wave coupling. Before doing so, a few simplifying assumptions will be made, certainly permitted in the astrophysical applications for which the present concept has been developed, but also feasible in many laboratory plasma situations. We suppose that the dimensions of the plasma are much larger than the characteristic length along which substantial non-linear wave coupling takes place. Second, the properties of the plasma relevant to the wave interaction change slowly and adiabatically over the characteristic interaction length. Third, the following considerations are only valid for a weak non-linear process such that the characteristic interaction length is much larger than the wave length and the characteristic interaction time much longer than the wave oscillation period. If, additionally, we take a localized source of intense radiation and study the wave coupling along the propagation path, then the assumed quasi-stationarity permits to approximate d/dt by $i\omega$ in the wave equation while time derivatives of $|E|$ and $|j|$ are negligible compared with $i\omega$. The same assumptions allow to limit the spatial derivatives to the first order. We set $\vec{E}_k \equiv \vec{F}_k \exp i(\omega t - \vec{k}\cdot\vec{r})$ in order to distinguish between amplitude (including polarization) and oscillatory property of the wave field; with this we get

$$\frac{d(E_k)_p}{d(r)_q} = \left[-i(k)_q(F_k)_p + \frac{\partial(F_k)_p}{\partial(r)_q}\right] \exp i(\omega t - \vec{k}\cdot\vec{r})$$

where again index notation of the vectors is used.

In the second order derivatives, one has to distinguish between

$$\frac{d}{d(r)_q}\frac{d(E_k)_p}{d(r)_q} \approx -\left[k^2(F_k)_p + 2ik_q\frac{\partial(F_k)_p}{\partial(r)_q}\right] \exp i(\omega t - \vec{k}\cdot\vec{r}) \tag{23}$$

and

$$\frac{d}{d(r)_q}\frac{d(E_k)_p}{d(r)_p} \approx -\left[(k)_q(k)_p(F_k)_p + i(k)_p\frac{\partial(F_k)_p}{\partial(r)_q} + i(k)_q\frac{\partial(F_k)_p}{\partial(r)_p}\right] \exp i(\omega t - \vec{k}\cdot\vec{r}) \tag{24}$$

with this kind and degree of approximation the spatial portion of the wave equation (1) becomes (we suppress the oscillatory portion)

$$\left\{-\left[k^2 + 2i(k)_p\frac{\partial}{\partial(r)_p}\right]\delta_{pq} + (k)_p(k)_q + i(k)_q\frac{\partial}{\partial(r)_p} + i(k)_p\frac{\partial}{\partial(r)_q}\right\}(F_k)_q \tag{25}$$

In the special case where polarization vector and wave vector are parallel, the terms k^2 and $(k)_p(k)_q$ of the linear wave equation as well as the terms due to the non-linear action $(\vec{k}\cdot\partial F_k/\partial\vec{r})$ cancel each other and, within the accuracy of our approximation, as in the linear wave equation in cold plasma only a non-propagating plasma oscillation develops. The time derivatives at the l.h.s. of (1), having the temporal quasi stationarity in mind, are given as in the linear wave equation by

$$\mu_0\varepsilon_0(\varepsilon)_{rs}\omega_k^2(F_k)_s \tag{26}$$

with the dielectric tensor

$$(\varepsilon_k)_{rs} = (\delta)_{rs} - \frac{\omega_p^{i2}}{\omega_k^2}(T^i_k)_{rs} - \frac{\omega_p^{e2}}{\omega_k^2}(T^e_k)_{rs} \tag{27}$$

where $(T^i_k)_{rs}$ and $(T^e_k)_{rs}$ for ions and electrons are according to (15). When putting (25) and (26) in the wave equation (1), the cases with different mutual orientations between static magnetic field, wave vector, and polarization of the electric wave field have to be separated. We can specify the magnetic field to be aligned with the z-axis leading to the simplified version (16) of the tensor (15) and, with some loss of generality, consider two special directions of wave propagation, parallel and perpendicular (say along the x-axis) to the magnetic field. For the sake of lucidity we put $\vec{j}^{(2)}_k = \vec{J}^{(2)}_k \exp i(\omega t - \vec{k}\cdot\vec{r})$, where polarization and phase are contained in the "amplitude" $\vec{J}^{(2)}_k$ while the oscillatory portion is separated.

The case of propagation perpendicular to the magnetic field $(\vec{H}_0 \equiv (H_0)_z, \vec{k} \equiv (k)_x)$ the wave equation written out in components yields

$$\begin{aligned}
&\mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{xq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_x\\
&-(k_0)_x^2(F_{k_0})_y - i2(k_0)_x\frac{\partial(F_{k_0})_y}{\partial x} + \mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{yq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_y\\
&-(k_0)_x^2(F_{k_0})_z - i2(k_0)_x\frac{\partial(F_{k_0})_z}{\partial x} + \mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{zq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_z
\end{aligned} \tag{28}$$

In each line of equation (28), the subscript q of the dielectric tensor and of the wave amplitude vector has to assume successively all values x, y, z of the three spatial components of the wave field. According to the discussion after equation (25), not only the term $(k_0)_x^2\,(F_{k_0})_x$ of the undisturbed wave equation vanishes for $\vec{k}_0$ parallel to $\vec{F}_{k_0}$, but also the term containing the rate of wave coupling, $d(F_{k_0})_x/dx$, i.e. there is no direct coupling to a longitudinal wave. This is a consequence of the limitation to first order derivatives in (23) and (24). However, also from a physics point of view this seems reasonable, because the longitudinal wave cannot propagate synchroneously neither to the driving fundamental nor to the harmonic electromagnetic wave.

This means there remain only the ordinary electromagnetic wave (third line of (28)) and the extraordinary wave (second line) for which non-linear wave coupling via the second order portion of the current density is predicted by (28).

For wave propagation parallel to the magnetic field $(\vec{H}_0 \equiv (H_0)_z, \vec{k} \equiv (k)_z)$, again the terms containing spatial derivatives of the longitudinal mode $(\vec{F}_k \equiv (F_k)_z)$ vanish,

$$\begin{aligned}
&-(k_0)_z^2(F_{k_0})_x - i2(k_0)_z\frac{\partial(F_{k_0})_x}{\partial z} + \mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{xq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_x\\
&-(k_0)_z^2(F_{k_0})_y - i2(k_0)_z\frac{\partial(F_{k_0})_y}{\partial z} + \mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{yq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_y\\
&\mu_0\varepsilon_0\omega_0^2(\varepsilon_{k_0})_{zq}(F_{k_0})_q = i\omega_0\mu_0(J^{(2)}_{k_0})_z
\end{aligned} \tag{29}$$

According to the assumption of weak non-linear interaction $(|\partial F_k/\partial r| \ll kF_k)$ it will, in a first order approximation, be permitted to subtract the undisturbed (linear) wave equation from (28) and (29) respectively.

With this we arrive at the approximate solution for the rate of coupling of waves propagating perpendicular to the magnetic field ($\vec{H}_0 = (H_0)_z$, $\vec{k}_0 \equiv (k_0)_x$), i.e. of extraordinary and ordinary wave respectively

$$\frac{\partial (F_{k_0})_y}{\partial x} = -\frac{\omega_0 \mu_0}{2(k_0)_x} (J_{k_0}^{(2)})_y \tag{30a}$$

$$\frac{\partial (F_{k_0})_z}{\partial x} = -\frac{\omega_0 \mu_0}{2(k_0)_x} (J_{k_0}^{(2)})_z \tag{30b}$$

and of waves propagating along the magnetic field ($\vec{k}_0 \equiv (k_0)_z$).

$$\frac{\partial (F_{k_0})_x}{\partial z} = -\frac{\omega_0 \mu_0}{2(k_0)_z} (J_{k_0}^{(2)})_x \tag{31a}$$

$$\frac{\partial (F_{k_0})_y}{\partial z} = -\frac{\omega_0 \mu_0}{2(k_0)_z} (J_{k_0}^{(2)})_y \tag{31b}$$

Close inspection of the current density (22) and the tensor (15) shows that the number of coupling equations (30), (31) will further be reduced if $J_k^{(2)}$ vanishes for particular mutual directions of magnetic field, wave vector and polarization. Besides there is another necessary condition to be satisfied before wave coupling happens; the dispersion relations of the participating waves have to be such as to allow for interaction according to (30) or (31) over a sufficiently long distance R. This means that over a narrow but finite bandwidth $\Delta\omega$ the conditions of energy and momentum conservation (8), (9) have to be satisfied sufficiently accurate, i.e. $\delta k < 1/R$, where $\delta k = k_0 - k_1 - k_2$ is the maximum deviation from (9) over the given bandwidth $\Delta\omega$. At another place it will be shown that for frequency doubling, i.e. when $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$ within the limits of a narrow bandwidth, the extraordinary wave exhibits a very favorable dispersion relation. Therefore in the next sections exclusively this wave will be considered.

Moreover, for (30) and (31) to be applicable, the interacting waves have to be coherent waves with fixed mutual phase relations. The coherence length $c/\Delta\omega$ has to be much larger than the minimum distance R for substantial wave coupling. In the considerations so far it has been supposed that the interacting waves are coherent. In the following, incoherent wave spectra are admitted for which – besides (8) and (9) – no mutual phase coherence exists.

4. Non-linear interaction between incoherent wave spectra

In nature high intensity radiation usually originates from mechanisms which are inherently non-coherent, e.g. from the beam of uncorrelated electrons accelerated to relativistic velocities during a solar flare. Only in very special conditions a certain degree of coherence is created, e.g. by the maser effect driven by electrons in a magnetic loss cone instability.

For high intensity of incoherent radiation, we assume a high radiance continuously distributed over a substantial bandwidth. The large bandwidth and the generation by uncorrelated processes are the causes for completely uncorrelated phases within the radiation.

We suppose a frequency bandwidth $\Delta\omega$ and a corresponding range of wave numbers Δk over which the radiance of at least one wave is sufficient to initiate a noticeable non-linear interaction and within which the dispersion relation of the interacting wave modes allows to satisfy the conditions of energy and momentum conservation (8) and (9) respectively. The reciprocal bandwidth $1/\Delta\omega = \tau$ and $1/\Delta k = \rho$ may be considered as correlation time and correlation length respectively. The wave interaction has to be treated as a random process if the characteristic time T and length R for wave coupling (e.g. e-folding) are much larger than the correlation time and correlation length

$$T \gg \tau, \qquad R \gg \rho \tag{32}$$

Nevertheless, the condition of energy and momentum conservation (8) and (9) have to be satisfied over a duration and length larger than T and R respectively. Hence the dispersion relation has to allow for

$$\frac{1}{\delta\omega} \gg T, \qquad \frac{1}{\delta k} \gg R \tag{33}$$

($\delta\omega = \omega_0 - \omega_1 - \omega_2$) for combinations of spectral portions within the bandwidth $\Delta\omega$.

With this in mind, the wave field may be considered as averaged out over the characteristic interaction time, hence $\langle E_k \rangle = 0$. The averaging of products of two field strengths of the same spectral range

$$\begin{aligned}\langle E_k E_{k'}^* \rangle &= \langle F_k e^{i(\omega t - \vec{k}\cdot\vec{r} - \varphi_k)} F_{k'}^* e^{-i(\omega' t - \vec{k}'\cdot\vec{r} - \varphi_k')} \rangle \\ &= |F_k|\,|F_{k'}|\,\delta(\omega - \omega')\,\delta(\vec{k} - \vec{k}')\,\delta(\varphi - \varphi')\end{aligned} \tag{34}$$

yields finite intensities only for correlated field quantities.

For the purpose of computing the interaction of incoherent waves, we consider here only the case of collinearly propagating extraordinary waves. Multiplication of the equation (30a) with the complex conjugate of the wave field $(F_{k_0}^*)_y$ with wave number k_0 and putting it behind the differential operator at the l.h.s., yields

$$\frac{\partial\,|F_{k_0}|_y^2}{\partial x} = -\frac{\omega_0\mu_0}{(k_0)_x}(F_{k_0}^*)_y (J_{k_0}^{(2)})_y \tag{35}$$

At the r.h.s. of (35) $(F_{k_0}^*)_y$ can be replaced by the integral of equation (30a)

$$\frac{\partial\,|F_{k_0}|_y^2}{\partial x} = \frac{1}{2}\left(\frac{\omega_0\mu_0}{(k_0)_x}\right)^2 \int (J_{k_0}^{(2)*})_y\, dx (J_{k_0}^{(2)})_y \tag{36}$$

Because of the random character of the interacting waves, the average of (36) has to be taken.

In view of equation (22) for the non-linear portion of the current density, it is obvious that this averaging is a lengthy task. To avoid the presentation of the whole procedure, only the first term caused by the electrons (4th line of equation (22)) is shown. The simplified tensor (16) can be utilized because of the choice of $\vec{H}_0 \equiv (H_0)_z$ and $\vec{k} \equiv (k)_x$

$$\langle (T^e_{k_1})_{rs}(F_{k_1})_s(F_{k_2})_r(T^{e*}_{k'_1})_{rs}(F^*_{k'_1})_s(F^*_{k'_2})_r \rangle$$

$$= \left\langle \frac{1}{\left(1-\dfrac{\omega_c^{e2}}{\omega_1^2}\right)\left(1-\dfrac{\omega_c^{e2}}{\omega_1'^2}\right)} \begin{bmatrix} F_{k_2x} \\ F_{k_2y} \\ F_{k_2z} \end{bmatrix} \begin{bmatrix} F^*_{k'_2x} \\ F^*_{k'_2y} \\ F^*_{k'_2z} \end{bmatrix} \begin{bmatrix} F_{k_1x} + i\dfrac{\omega_c^e}{\omega_1} F_{k_1y} \\ -i\dfrac{\omega_c^e}{\omega_1} F_{k_1x} + F_{k_1y} \\ \left(1-\dfrac{\omega_c^{e2}}{\omega_1^2}\right) F_{k_1z} \end{bmatrix} \begin{bmatrix} F^*_{k'_1x} - i\dfrac{\omega_c^e}{\omega_1'} F^*_{k'_1y} \\ i\dfrac{\omega_c^e}{\omega_1'} F^*_{k'_1x} + F^*_{k'_1y} \\ \left(1-\dfrac{\omega_c^{e2}}{\omega_1'^2}\right) F^*_{k'_1z} \end{bmatrix} \right\rangle.$$

$$= \frac{1}{\left(1-\dfrac{\omega_c^{e2}}{\omega_1^2}\right)^2} [|F_{k_2x}|^2 + |F_{k_2y}|^2 + |F_{k_2z}|^2]$$

$$\cdot \left\{ \left[|F_{k_1x}| \pm \frac{\omega_c^e}{\omega_1} |F_{k_1y}| \right]^2 + \left[|F_{k_1x}| \frac{\omega_c^e}{\omega_1} \pm |F_{k_1y}| \right] + \left(1-\frac{\omega_c^{e2}}{\omega_1^2}\right)^2 |F_{k_1z}|^2 \right\} \tag{37}$$

In this averaging process, use has been made of the following relations:

According to (34) only correlated field quantities of the respective spectral ranges yield contributions.

For our choice of orthogonal directions of magnetic field, wave vector, and initial wave polarization, only elliptic polarization in the x–y plane can occur

$$\frac{(F_k)_x}{(F_k)_y} = \frac{|F_{kx}|}{|F_{ky}|} e^{\pm i\varphi}$$

with phase of $(F_k)_x$ advanced (+) or retarded (−) with respect to $(F_k)_y$. Concluding from the linear case of wave propagation, φ can approximately be set equal $\pi/2$. With this the averaging of the mixed products in (37) yields, e.g.

$$-i\frac{\omega_c^e}{\omega_1} \langle (F_{k_1})_x(F^*_{k_1})_y - (F^*_{k_1})_x(F_{k_1})_y \rangle = \pm 2\frac{\omega_c^e}{\omega_1} |F_{k_1x}|\,|F_{k_1y}| \tag{38}$$

for the two orientations of the magnetic field.

The factor in front of e.g. the first electronic term (4th line in (22)) results in

$$(T^e_{k_0})_{yx}(k_2)_x(T^{e*}_{k_0})_{yx}(k'_2)_x = \frac{\dfrac{\omega_c^e}{\omega_0}(k_2)_x}{1-\left(\dfrac{\omega_c^e}{\omega_0}\right)^2} \frac{\dfrac{\omega_c^e}{\omega_0}(k'_2)_x}{1-\left(\dfrac{\omega_c^e}{\omega_0}\right)^2} \delta(k_2 - k'_2)\,\delta(\omega_2 - \omega'_2) \tag{39}$$

if the direction of the propagation of all interacting waves is along the x-axis.

In the application of the present concept, we consider high frequency electromagnetic waves in a plasma of density and magnetic field such that the electronic branches of the dispersion relation are responsible for satisfying the necessary conditions (8) and (9) of wave–wave interaction. If the density and velocity fluctuations of merely the electrons mediate the wave coupling then, for the sake of clarity, the formulation can be reduced to the contribution by the electrons. The result of the described procedure applied to equation (36) is

$$\left\langle \frac{\partial |F_{k_0}|_y^2}{\partial x} \right\rangle = \frac{1}{2}\left(\frac{e\varepsilon_0\mu_0\omega_p^{e2}}{m(k_0)_x}\right)^2$$

$$\cdot\left\{\int \frac{\left(\frac{\omega_c^e}{\omega_0}\right)^2 (k_2)^2}{\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2 \omega_1^2\omega_2^2} \cdot \frac{|F_{k_2}|^2}{\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2}\left[\left(|F_{k_1}|_x \pm \frac{\omega_c^e}{\omega_1}|F_{k_1 y}|\right)^2\right.\right.$$

$$\left. + \left(|F_{k_1}|_x \frac{\omega_c^e}{\omega_1} \pm |F_{k_1}|_y\right)^2 + \left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]|F_{k_1}|_z^2\right] d\omega_0\, dk_{1,2}\,\delta_{\omega,k}$$

$$+ \int \frac{1}{\omega_1^2\omega_2^2} \frac{|k_2|_x^2}{\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_2}\right)^2\right]^2}$$

$$\times\left[|F_{k_2}|_x\left(\frac{\omega_c^{e3}}{\omega_0\omega_2^2} + \frac{\omega_c^e}{\omega_2}\right) \pm |F_{k_2}|_y\left(\frac{\omega_c^{e2}}{\omega_0\omega_2} + \frac{\omega_c^{e2}}{\omega_2^2}\right)\right]^2$$

$$\times\left[|F_{k_1}|_x \pm \frac{\omega_c^e}{\omega_1}|F_{k_1}|_y\right]^2 d\omega_0\, dk_{1,2}\,\delta_{\omega,k}$$

$$+ \int \frac{\omega_0^2}{\omega_1^4\omega_2^2} \frac{|k_1|_x^2}{\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^2}{\omega_2}\right)^2\right]^2}\left[|F_{k_1}|_x \pm \frac{\omega_0^e}{\omega_1}|F_{k_1}|_y\right]^2$$

$$\left.\times\left[\frac{\omega_c^e}{\omega_2}|F_{k_2}|_x \pm |F_{k_2}|_y\right]^2 d\omega_0\, dk_{1,2}\,\delta_{\omega,k}\right\} \tag{40}$$

The analogous expression for the contribution by the ions contains their respective values and signs of mass, density and charge, and consequently for plasma and cyclotron frequencies. The alternating signs for the sense of polarization are reversed for the ions.

5. Conclusions and Application

The proposed concept of non-linear wave–wave interaction in a cold, collisionless magnetized plasma allows for detailed analysis and numerical computation of the rate of wave coupling. In a first order approximation the coupling rates of coherent wave fields for propagation of all participating waves perpendicular or parallel to the static magnetic field have been derived and from this the coupling rate of the wave intensities of incoherent, extraordinary waves have been deduced.

The present analysis did not account for any loss mechanism. For the special combinations of particle density, temperature and magnetic field at which the cold plasma concept is valid, the most probable loss mechanisms are cyclotron absorption, inverse Bremstrahlung in electron–ion encounters and electron–electron collisions. These processes set a threshold which has to be exceeded by the non-linear interaction before one wave can grow at the expense of another one.

The even more stringent condition imposed by the compatibility of the energy and momentum conservation principle with the dispersion relation of the interacting waves has already been discussed. It is difficult to identify a dispersion relation which allows to satisfy (8) and (9) for three different spectral ranges of electromagnetic waves. For a simple geometric arrangement as the interaction of extraordinary waves only frequency doubling ($\omega_1 \approx \omega_2 \approx \omega_0/2$) can be accommodated. The limitation of the present analysis to pure orthogonal arrangements of the directions of magnetic field vector, wave vector and polarization vector is a draw-back which, however, made allowance for a reasonable clarity of the presentation. Interaction of three different spectral ranges seems to be limited to more general arrangements of the above mentioned vectors.

In spite of the limitations of the presented concept, it can be applied to a variety of plasma conditions and configurations. For the case of the solar corona in the vicinity of an active region conditions are prevailing for which the present concept is applicable.

A numerical investigation has been made of the coupling rate of incoherent radiation within finite bandwidth and the correlation length has been determined over which substantial wave coupling can be achieved. A detailed analysis of the effects of the mentioned damping processes on the feasibility of the non-linear coupling has been made and the compatibility of the dispersion relation of the extraordinary waves with the resonance conditions (8) and (9) has been proved. The reduced coupling efficiency due to polarization effects has been estimated. A detailed discussion of these topics will be presented in a separate paper currently in preparation, therefore only a few indications of the orders of magnitude are given here.

The dispersion relation of extraordinary waves allows to satisfy the conditions of momentum and energy conservation for frequency doubling if $\omega_p/\omega_c^e > 4/3$.

A simplified and frequency-integrated formulation of the rate equation (40) is

$$\frac{\Delta\,|E_0|^2}{\Delta X} = C\,|E_1|^2\,|E_2|^2 \tag{41}$$

The coupling rate C attains values between $\sim 10^{-3}$ and $\sim 10^{-1}$ for $\omega_c^e \sim 10^{10}\,\mathrm{s}^{-1}$

and a bandwidth of $\sim 1\%$ of an incoherent wave spectrum. An inflexion of the dispersion curve at $\omega_p/\omega_c^e \sim 1.5$ gives rise to a correlation length of more than thousand wavelengths for the 1% bandwidth. Therefore

$$\frac{\Delta\,|E_0|^2}{|E_1|^2\,|E_2|^2} = C\,\Delta X_c \tag{42}$$

can yield values between 1 and 10 if the correlation length is taken for ΔX_c and $1.4 \lesssim \omega_p/\omega_c^e \lesssim 1.55$ is chosen. This means complete conversion of the wave energy between the lower and the higher frequencies (ω_1, ω_2 vs. ω_0) within the correlation length if unity field strength of the wave is exceeded. Comparison with wave intensities close to the solar flare regions ($|E|^2 > 10^2 (\mathrm{V/m})^2$) suggest that this process may be considered feasible.

Among the damping processes gyro-resonance is the most efficient one. If we assume a decreasing strength of the static magnetic field on the propagation path (which is perpendicular to the field direction) then mainly the fundamental frequency wave (ω_f) will be attenuated within an extremely narrow frequency range ($\ll 1\%$), hence over a small path length, by the second harmonic of the local gyrofrequency. This is based on the fact that at maximum coupling between fundamental and harmonic (for $4/3 < \omega_p/\omega_{c0}^e < 5/3$), the fundamental frequency is related to the gyrofrequency ω_{c0}^e of the coupling region by about $\omega_{c0}^e < \omega_f < 1.5\,\omega_{c0}^e$. All other combinations of wave frequencies and local gyrofrequency as well as electron-ion and electron–electron encounters are two or more orders of magnitude less effective in damping.

According to the fairly low loss mechanisms in the frequency ranges relevant for the wave coupling an efficient and observable creation of harmonic radiation from an intense incoherent fundamental radiation may be expected.

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