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Vanishing β-functions in $N=1$ supersymmetric gauge theories

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Abstract. Necessary and sufficient conditions for the all-order vanishing of the β-functions in $N=1$ supersymmetric gauge theories with simple gauge group are given. They contain well-known one-loop conditions and require the Yukawa coupling constants to be power series in the gauge coupling constant solving the reduction equations of Oehme and Zimmermann. A simple criterion for vanishing β-functions involving only one-loop quantities is then proposed.

1. Introduction

Many attempts have been made during the last years to obtain finite quantum field theories in four-dimensional space-time. For general theories, such a search has hardly gone beyond the one-loop approximation [1]. There is a strong indication that only supersymmetric gauge theories (SYM) can eventually be completely free of ultra-violet divergences [1], although examples of non-supersymmetric models with vanishing one-loop β-functions, i.e., without coupling constant renormalization, are known [2]. Much work [3–10] has been dedicated to the investigation of the SYM theories. The authors of Refs. [9] and [10], in particular, deal with this problem at all orders for $N=1$ SYM theories. They demand the all order vanishing of the anomalous dimensions for all fields; this ensures the vanishing of the β-functions too, hence the complete finiteness of the theory. For this purpose, they require the Yukawa coupling constants λ (self-interaction of the matter fields) to be power series in the gauge coupling

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constant g: these functions $\lambda(g)$ have to solve the condition of vanishing matter field anomalous dimensions. The authors must, however, impose some restrictions; in particular, they cannot treat theories where the number of independent anomalous dimensions exceeds that of Yukawa coupling constants. Their proof also heavily relies on the dimensional regularization which is known [11] to face difficulties in preserving supersymmetry in higher orders.

The present paper is an extension of a previous work [12] in which sufficient conditions for 'finiteness' were presented. By 'finiteness' we mean the vanishing of the β-functions – the physically relevant objects – to all orders but not necessarily of all anomalous dimensions to any order: this allows us to abandon any a priori restriction on the number of fields and couplings. The functions $\lambda(g)$ are now solutions of the reduction equations of Oehme and Zimmermann [13, 14], a necessary condition for the consistency of the theory. In order to avoid any problem with regularization, the theory is assumed to be renormalized by using the superspace renormalization scheme of Ref. [15], where it is also shown [16] that BRS invariance can be maintained at all orders of perturbation theory, provided the usual gauge anomaly is absent.

The criterion of 'finiteness' here gains precision with respect to that of Ref. [12]. Our first main result (Theorem 5.2) is that the conditions of Jones, Mezincescn, Parkes and West [4] for the one-loop and two-loop finiteness of $N = 1$ SYM theories – namely the vanishing of the gauge β-function and of the matter field anomalous dimensions at one-loop – are actually necessary and sufficient in order to have β-functions vanishing to all orders, if one completes them with the requirement that the reduction equations possess a power series solution $\lambda = \lambda(g)$. Our second main result is a set of sufficient 'finiteness' conditions relying only on one-loop quantities (Theorem 5.3): it consists of adding to the conditions of Ref. [4] a condition which ensures the existence of all-order solutions to the reduction equations.

We further show that the vanishing of the anomalies associated with all the chiral symmetries the model may have is necessary and sufficient for ensuring the compatibility of the vanishing conditions for all the one-loop anomalous dimensions of the matter fields.

Hence the 'finite' SYM theories are completely free of anomalies, of the conformal ones, i.e., the β-functions, as well as of the chiral ones. The strategy of our proof is a rigorous extension of an old formal argument [17] proposed for showing the finiteness of the $N = 4$ SYM theory. Our approach depends on the detailed structure of the supercurrent multiplet anomaly [18, 19, 15] and in particular on an explicit relation we derive, combining the β-functions, the anomalous dimensions and the axial anomalies [equation (4.14)]. The usefulness of this relation relies on the non-renormalization theorem we prove for the latter anomalies. Note that a recent paper [20] gives a 'proof' of the theorem for the anomaly of the axial current (R-current) related to the supercurrent multiplet. It uses, however, the regularization by dimensional reduction*).

*) The authors of Ref. [20] in fact claim to have a successful demonstration using this regularization, otherwise criticized [11] as being inconsistent.

In order to render the present paper self-contained and also to fill a loophole found in Ref. [12], we shall repeat part of the material presented there. Section 2 reviews general features of $N=1$ SYM models. Section 3 deals with the one-loop approximation and in particular with the relationship between vanishing axial anomalies and anomalous dimensions (Lemma 3.1). The general structure of the supercurrent anomalies and their relation with the axial anomalies are explained in Section 4. The main results mentioned in this Introduction are derived in Section 5. We apply them to an example in Section 6 and draw some general conclusions in Section 7. Appendix A gives the corrected statement and the proof of the supersymmetric non-renormalization theorem, which was formulated under too weak hypotheses in Ref. [12]. Finally, a one-loop condition for the existence of power series solutions to the reduction equations is given in Appendix B.

2. The model and its invariances

The physical field content of a general $N=1$ SYM theory [15] consists of a real gauge superfield of dimension 0, $\phi=\phi_i\tau^i$ (τ^i the generators of the gauge group G, assumed to be *simple*), and of chiral matter superfields A^R of dimension one. The upper index R labels both the field itself and the irreducible representation (irrep.) of G it belongs to. The complex conjugate field $\bar{A}_R$ transforms in the representation conjugate to R. We shall also use the multi-index notation [4]

$$A^r \equiv A^{(R,\rho)}, \tag{2.1}$$

where ρ labels the components within the irrep. R

The BRS transformations read

$$\begin{aligned} se^{\phi} &= e^{\phi}c_+ - \bar{c}_+e^{\phi}, \\ sA^{(R,\rho)} &= -c_{+i}(T^i_R)^{\rho}_{\sigma}A^{(R,\sigma)}, \\ sc_+ &= -\tfrac{1}{2}\{c_+, c_+\}, \end{aligned} \tag{2.2}$$

and are nilpotent:

$$s^2=0. \tag{2.3}$$

Here $c_+=c_{+i}\tau^i$ is the (chiral) Faddeev–Popov ghost. The Hermitian matrices T^i_R are the generators of G in the irrep. R. We omit the Lagrange multiplier and antighost fields involved in the gauge fixing of the theory.

A more general BRS transformation law preserving the nilpotency property is obtained by performing a generalized field amplitude renormalization [15], i.e., by replacing ϕ in the first line of (2.2) by

$$\mathscr{F}(\phi)=\phi+\sum_{k=2}^{\infty} a_k\phi^k, \tag{2.4}$$

where the infinite set of parameters a_k can be shown [15, 21] to be non-physical.

The most general gauge-invariant classical action is [15]*)

$$\Gamma_{inv} = -\frac{1}{128g^2}\operatorname{Tr}\int dSF^{\alpha}F_{\alpha} + \frac{1}{16}\int dV \sum_R \bar{A}_R \exp(\phi_i T^i_R)A^R + \int dSU(A) + \int d\bar{S}\bar{U}(\bar{A}), \tag{2.5}$$

with the SYM field strength F_α given in terms of the 'chiral connection' φ_α by

$$F_\alpha = \overline{DD}\varphi_\alpha, \qquad \varphi_\alpha = e^{-\phi}D_\alpha e^{\phi} \tag{2.6}$$

where ϕ is replaced by (2.4) in the general case. g is the gauge coupling constant. The two last terms in (2.5) describe the self-interaction of the matter fields in term of the chiral superpotential. With the use of notation (2.1), these terms read:

$$U(A) = \tfrac{1}{6}\lambda_{(rst)}A^r A^s A^t, \qquad \bar{U}(\bar{A}) = \tfrac{1}{6}\bar{\lambda}^{(rst)}\bar{A}_r\bar{A}_s\bar{A}_t, \tag{2.7}$$

the complex "Yukawa" coupling constants λ_{rst} being invariant symmetric tensors of G.

Beyond supersymmetry and BRS invariance, the massless action (2.5) is invariant under the R-transformations [22, 15]

$$\delta_R\psi = i(n_\psi + \theta^\alpha\,\partial_{\theta^\alpha} - \bar{\theta}^{\dot{\alpha}}\,\partial_{\bar{\theta}^{\dot{\alpha}}})\psi, \tag{2.8}$$

with the R-weights being respectively

$$n_\psi = 0,\ -\tfrac{2}{3}(\tfrac{2}{3}),\ 0(0) \quad \text{for} \quad \psi = \phi,\ A(\bar{A}),\ c_+(\bar{c}_+).$$

The theory is, in general, also invariant under a (possibly empty) set of chiral transformations.

$$\delta_a\phi = \delta_a c_+ = 0, \qquad \delta_a A^R = ie_{aS}^{\ R}A^S, \qquad \delta_a\bar{A}_R = -i\bar{A}_S e_{aR}^{\ S}, \tag{2.9}$$

where the chiral charge matrices e_a are Hermitian. These transformations commute with the BRS transformations (hence $e_{aS}^{\ R} = 0$ if irrep. $R \neq$ irrep. S), and with supersymmetry.

The classical action (2.5) is invariant under (2.9) if and only if the Yukawa coupling constants obey the constraints

$$\forall_a\colon\ \lambda_{rsu}e_{at}^{\ u} + \text{cycl. perm.}\ (r, s, t) = 0, \tag{2.10}$$

with the notation

$$M_t^u = \delta_\tau^\mu M_T^U. \tag{2.11}$$

The quantum theory in loop expansion, described by the vertex functional

*) $dV = d^4x d^4\theta = d^4x\, DD\bar{D}\bar{D}$, $dS = d^4x d^2\theta = d^4x DD$.

$\Gamma(\phi, A, \ldots) = \Gamma_{\text{class}} + O(\hbar)$, can be shown to preserve all the invariances listed above, up to soft breakings induced by supersymmetric masses (which we add to the action (2.5) in order to avoid infra-red difficulties [15]). Supersymmetry is explicit (and exact) due to the use of a superspace subtraction scheme [15]. BRS invariance is expressed by the Slavnov identity*)

$$\mathcal{S}(\Gamma) \sim 0 \tag{2.12}$$

which holds (up to soft breakings: this is the meaning of the symbol $\sim$) provided the representation of the matter fields A^R is chosen to be anomaly free [15, 16]:

$$a \equiv \sum_R a(R) = 0, \tag{2.13}$$

$a(R)$ being the 'anomaly index' of the irrep. R; these indices are tabulated, e.g., in Ref. [23]. R-invariance (2.8) and the chiral invariances (2.9) are expressed by the Ward identities

$$W_R\Gamma \equiv -i \sum_{\psi=\phi,A,c_+} \int \delta_R\psi \frac{\delta\Gamma}{\delta\psi} \sim 0, \tag{2.14}$$

$$\begin{aligned} W_a\Gamma &\equiv -i \sum_{\psi} \int \delta_a\psi \frac{\delta\Gamma}{\delta\psi} \\ &\equiv \sum_{R,S} e_{aS}^{\ R}\left[\int dSA^S \frac{\delta}{\delta A^R} - \int d\bar{S}\bar{A}_R \frac{\delta}{\delta \bar{A}_S}\right]\Gamma \sim 0 \end{aligned} \tag{2.15}$$

holding at all orders [15], up to soft breakings, too. The operators W_a generate the Lie algebra $\mathscr{C}$ associated to the infinitesimal chiral transformations (2.9), with the commutation relations

$$[W_a, W_b] = W_c, \tag{2.16}$$

W_c having the charge matrix $e_c = -[e_a, e_b]$. We shall denote by W_{0a} a basis of the centre $\mathscr{C}_0$ of the algebra $\mathscr{C}$:

$$[W_{0a}, W_b] = 0 \quad \text{for any} \quad W_b \in \mathscr{C}, \tag{2.17}$$

and by e_{0a} the corresponding charge matrices.

Let us close this section by recalling the Callan–Symanzik equation [15] fulfilled by the vertex functional Γ, up to soft mass insertions:

$$\begin{aligned} C\Gamma \equiv [&m\,\partial_m + \beta_g\,\partial_g + \beta_{rst}\partial_{\lambda_{rst}} \\ &+ \bar{\beta}^{rst}\,\partial_{\bar{\lambda}^{rst}} - \gamma_\phi \mathcal{N}_\phi - \gamma_R^S \mathcal{N}_S^R - \gamma_k\,\partial_{a_k}]\Gamma \sim 0. \end{aligned} \tag{2.18}$$

where $m\,\partial_m$ (with summation over all mass parameters of the theory) is the

*) We shall not give the explicit form of the (non-linear) Slavnov functional operator $\mathcal{S}$; it involves external superfields coupled to the BRS variations of the different fields of the theory (see for instance Ref. [12]).

scaling operator. The counting operators $\mathcal{N}$ are*)

$$\mathcal{N}_\phi = \mathrm{Tr}\int dV\phi\delta_\phi, \tag{2.19}$$

$$\mathcal{N}^R_S = \int dSA^R\delta_{A^S} + \int d\bar{S}\bar{A}_S\delta_{\bar{A}_R}. \tag{2.20}$$

Due to the reality of Γ, the gauge beta-function β_g, the anomalous dimension γ_ϕ and the coefficients γ_k – which describe the generalized amplitude renormalization (2.4) – are real. The Yukawa beta-functions β_λ and $\beta_{\bar\lambda}$ are the complex conjugates of each other, and the matrix γ^R_S of matter field anomalous dimensions is Hermitian. Note the absence of an anomalous dimension term for the ghost c_+: we are using a particular renormalization scheme with the effect that its anomalous dimension vanishes [12].

In fact, due to the chiral invariances (2.9) and (2.15), only combinations of the counting operators (2.20) which commute with the Ward identity operators W_a can occur in the Callan–Symanzik equation. They have the form

$$\mathcal{N} = g^R_S\mathcal{N}^S_R, \tag{2.21}$$

where the Hermitian matrix g commutes with all matrices e_a of (2.9). A convenient choice for a basis of such counting operators is realized by

$$\mathcal{N}_{0a} = e_{0a}{}^R_S\mathcal{N}^S_R, \tag{2.22}$$

$$\mathcal{N}_{1K} = f_{1K}{}^R_S\mathcal{N}^S_R. \tag{2.23}$$

Here the matrices e_{0a} are the charge matrices of the centre of the algebra of chiral transformations W_a [see equations (2.9), (2.15)–(2.17)]. The operators $\mathcal{N}_{1K}$, with f_{1K} Hermitian and commuting with all e_a complete the basis. Let us note for later use that the $\mathcal{N}_{0a}$ form a basis for the counting operators commuting with all chiral symmetries W_a and annihilating the superpotential (2.7):

$$\mathcal{N}_{0a}U(A) = 0. \tag{2.24}$$

It follows that the chiral field polynomials

$$\mathcal{N}_{1K}U(A) = 3\lambda_{rsu}f_{1K}{}^u_t A^r A^s A^t, \tag{2.25}$$

[with the notation (2.11) for $M = f_{1K}$] are linearly independent, and the invariant symmetric tensors

$$T^K_{(rst)} \equiv \lambda_{rsu}f_{1K}{}^u_t + \text{cycl. perm.}\ (r, s, t) \tag{2.26}$$

are therefore independent.

In the basis (2.22) and (2.23) the Callan–Symanzik equation now reads

$$\begin{aligned} C\Gamma = [&m\,\partial_m + \beta_g\,\partial_g + \beta_{rst}\,\partial_{\lambda_{rst}} + \bar\beta^{rst}\,\partial_{\bar\lambda^{rst}} \\ &- \gamma_\phi\mathcal{N}_\phi - \gamma_{0a}\mathcal{N}_{0a} - \gamma_{1K}\mathcal{N}_{1K} - \gamma_k\,\partial_{a_k}]\Gamma \sim 0. \end{aligned} \tag{2.27}$$

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*) We neglect contributions from the external fields, antighost fields, etc., cf. Refs. [15, 12].

3. The one-loop problem

The one-loop β-functions and anomalous dimensions of the matter fields are [4]

$$\beta_g^{(1)} = \frac{g^3}{4(4\pi)^2}\left[\sum_R T(R) - 3C_2(G)\right], \tag{3.1}$$

$$\beta_{rst}^{(1)} = \lambda_{rsu}\gamma^{(1)u}_{\ \ t} + \text{cycl. perm. } (r, s, t), \tag{3.2}$$

$$\begin{aligned}\gamma^{(1)r}_{\ \ s} &\equiv \gamma^{(1)R}_{\ \ S}\delta^\rho_\sigma \\ &= \frac{1}{2\pi^2}[\bar\lambda^{ruv}\lambda_{suv} - \tfrac{1}{16}g^2C_2(R)\delta^r_s] \\ &\equiv K[\bar\lambda^{ruv}\lambda_{suv} - \alpha C_2(R)\delta^r_s].\end{aligned} \tag{3.3}$$

where the Dynkin index $T(R)$ and the Casimir eigenvalue $C_2(R)$ of the irrep. R are defined by

$$\begin{aligned}&\mathrm{Tr}\,(T^i_RT^j_R) = \delta^{ij}T(R),\\ &(T^i_RT^i_R)^\rho_\sigma = \delta^\rho_\sigma C_2(R),\\ &C_2(G) = C_2(\text{adj.}) = T(\text{adj.}),\end{aligned} \tag{3.4}$$

and are related by the identity

$$d(G)T(R) = d(R)C_2(R), \tag{3.5}$$

$d(G)$ and $d(R)$ being the dimensions of the gauge group and of the irrep. R, respectively.

We shall see in Section 5 that the vanishing of the β-functions to all orders requires that the one-loop anomalous dimensions (3.3) vanish too. This last condition, however, is in general stronger than the vanishing of the β-functions (3.2), since there may be more γ's than β's. Thus, the equations*)

$$\gamma^{(1)R}_{\ \ S}(\lambda, g) = 0 \tag{3.6}$$

may overdetermine the solution $\lambda = \lambda(g)$. Let us look for conditions ensuring the compatibility of these equations. They are provided by the following

Lemma 3.1. *The equations* (3.6) *are compatible if and only if the conditions*

$$x_{0a} \equiv \sum_R e_{0a\,R}^{\ \ R}T(R) = 0 \tag{3.7}$$

hold. The charge matrices e_{0a} here correspond to the Ward identity operators W_{0a} generating the centre of the algebra (2.16) *of chiral symmetries.*

Remark. The quantities x_{0a} are the coefficients of the anomalies of the

*) These, together with the condition $\beta_g^{(1)} = 0$, are the one-loop conditions of Ref. [4].

(classically conserved) axial currents associated to the symmetries W_{0a}. They will be later shown (Appendix A) to be not renormalized. It will also be proved in Section 5, Lemma 5.1, that the conditions (3.6) are necessary for having vanishing β-functions at all orders.

Proof. Let us begin by proving the sufficiency: we show that, under condition (3.7), the equations $\beta^{(1)}_{rst} = 0$ – which are compatible since their number equals the number of unknowns λ_{rst} – imply the vanishing of all $\gamma^{(1)R}_{\ S}$. Thus, let us assume $\beta^{(1)}_{rst}$ to be zero. Multiplying (3.2) with $\bar{\lambda}^{rst}$ and using the expressions (3.3) for the one-loop anomalous dimensions yields

$$\begin{aligned} 0 &= [\gamma^{(1)r}_{\ u} + \alpha\delta^r_u C_2(R)]\gamma^{(1)u}_{\ r} \\ &= \sum_{R,U} d(R)\gamma^{(1)R}_{\ U}\gamma^{(1)U}_{\ R} + \alpha \sum_R d(R) C_2(R)\gamma^{(1)R}_{\ R} \\ &= \sum_{R,U} d(R)\,|\gamma^{(1)R}_{\ U}|^2 + \alpha d(G)\sum_R T(R)\gamma^{(1)R}_{\ R}, \end{aligned} \tag{3.8}$$

where use has been made of the Hermiticity of $\gamma^{(1)R}_{\ S}$ and of the relation (3.5). On the other hand, let us insert in $\beta^{(1)}_{rst}$ (3.2) the expression

$$\gamma^{(1)R}_{\ S} = \gamma^{(1)}_{0a} e_{0a\,S}^{\ R} + \gamma^{(1)}_{1K} f_{1K\,S}^{\ R} \tag{3.9}$$

deduced by comparing the two forms (2.18) and (2.27) of the Callan–Symanzik equation. The contributions of the $\gamma^{(1)}_{0a}$ drop out because of the chiral invariance conditions (2.10) for the Yukawa coupling constants and we are left with

$$0 = T^K_{rst}\gamma^{(1)}_{1K}, \tag{3.10}$$

where the tensors T^K, given by (2.26), are independent. Thus,

$$\begin{aligned} &\gamma^{(1)}_{1K} = 0, \\ &\gamma^{(1)R}_{\ S} = \gamma^{(1)}_{0a} e_{0a\,S}^{\ R}, \end{aligned} \tag{3.11}$$

and we get

$$\sum_R T(R)\gamma^{(1)R}_{\ R} = \sum_a \gamma^{(1)}_{0a}\sum_R e_{0a\,R}^{\ R} T(R). \tag{3.12}$$

Here the right-hand side vanishes due to (3.7), hence equation (3.8) reduces to

$$\sum_{R,U} d(R)\,|\gamma^{(1)R}_{\ U}|^2 = 0, \tag{3.13}$$

which means the vanishing of all $\gamma^{(1)R}_{\ S}$ and ends the proof of the sufficiency of conditions (3.7).

In order to show their necessity, let us multiply the chiral invariance conditions (2.10) by $\bar{\lambda}^{rst}$. Using the expression (3.3) for $\gamma^{(1)}$, we get in the same way as we obtained equation (3.8),

$$\sum_{R,U} d(R)\gamma^{(1)R}_{\ U} e_{a\,R}^{\ U} + \alpha d(G)\sum_R T(R) e_{a\,R}^{\ R} = 0. \tag{3.14}$$

The compatibility of equations (3.6) then implies

$$\sum_R T(R) e_{aR}^{\ R} = 0. \tag{3.15}$$

For the special case $e_a = e_{0a}$ these are conditions (3.7).

From (3.14) follows also the

Corollary 3.2. *The vanishing of the one-loop anomalous dimensions of the matter fields implies the conditions* (3.7) *of Lemma* 3.1.

4. The supercurrent anomaly

The supercurrent [18, 15, 19] is a *BRS* invariant supermultiplet containing the conserved spinor current and energy momentum tensor associated with supersymmetry and translation invariance, together with the anomalous axial current associated with R-invariance (2.14). The anomalies of the R-axial current, of the spinor current 'trace' and of the energy-momentum tensor trace belong to a chiral supermultiplet whose superfield representation is denoted by S [15, 19]. This chiral insertion*) S has dimension 3, R-weight -2 [see (2.8)] and is invariant under *BRS,* as well as under the chiral transformations (2.9). It can be expanded as [15, 19, 12]

$$\begin{aligned} S &= \beta_g L_g + \beta_{rst} L^{rst} - \gamma_\Phi L_\Phi - \gamma_k L_k - \gamma_S^R L_R^S \\ &= \beta_g L_g + \beta_{rst} L^{rst} - \gamma_\Phi L_\Phi - \gamma_k L_k - \gamma_{0a} L_{0a} - \gamma_{1K} L_{1K}. \end{aligned} \tag{4.1}$$

The coefficients β and γ are those of the Callan–Symanzik equation, either in the form (2.18) or in the form (2.27). Each set of insertions L appearing in the two expressions above forms a basis for the chiral insertions which have the dimension, R-weight and invariances of S. They are defined through the quantum action principle [24, 15] by

$$\nabla_i \Gamma \sim \int dS L_i + \int d\bar{S} \bar{L}_i, \tag{4.2}$$

for

$$\nabla_i = \partial_g,\ \partial_{\lambda_{rst}},\ \mathcal{N}_\Phi,\ \partial_{a_k},\ \mathcal{N}_R^S,\ \mathcal{N}_{0a},\ \mathcal{N}_{1k}.$$

In particular,

$$\begin{aligned} &L_R^S = A^S \delta_{A^R}, \\ &L_{0a} = e_{0a}{}_S^R L_R^S, \qquad L_{1K} = f_{1K}{}_S^R L_R^S, \end{aligned} \tag{4.3}$$

the Hermitian matrices e_{0a} and f_{1K} being defined in equations (2.22) and (2.23).

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*) An 'insertion' I is the generating functional of the (one-particle-irreducible) Green functions with the composite field operator I inserted in.

One can show [12] that

$$L_\Phi = \overline{DD}\mathscr{L}_\Phi, \tag{4.4}$$

where $\mathscr{L}_\Phi$ is *BRS* invariant and real. It has been proved [12] that any *BRS* invariant chiral insertion T of dimension 3 and R-weight -2 admits the representation

$$T \sim \overline{DD}(r_T K^0 + J_T^{inv}) + T^c, \tag{4.5}$$

where K^0 is the 'supersymmetric Chern–Simons insertion' defined in Appendix A and related to the finite insertion Trc_+^3 through the quantum extension of the classical descent equations (A.2). The coefficient r_T is gauge independent and uniquely defined. J_T^{inv} is *BRS* invariant and T^c, *BRS* invariant as well, is a 'genuinely chiral' insertion, i.e., it cannot be written as a double derivative $\overline{DD}(\ldots)$. The basis of genuinely chiral insertions with the appropriate dimension and R-weight is a quantum extension of the independent field polynomials constituting the superpotential U (2.7). One can choose the basis*)

$$\{L_{1K}, U_{0L}\} \tag{4.6}$$

with L_{1K} given by (4.3) – the L_{1K} are independent, see the remark following equation (2.25) – and with some insertions U_{0L} for completing the basis if necessary.

Let us use the representation (4.5) for the supercurrent anomaly S and for each of the L_i appearing in both right-hand sides of (4.1):

$$\begin{aligned}
&S \sim \overline{DD}[rK^0 + J^{inv}],\\
&L_g \sim \overline{DD}\left[\left(\frac{1}{128g^3} + r_g\right)K^0 + J_g^{inv}\right] + L_g^c,\\
&L^{rst} \sim \overline{DD}[r^{rst}K^0 + J^{rst,inv}] + L^{rst,c},\\
&L_k \sim \overline{DD}[r_k K^0 + J_k^{inv}] + L_k^c,\\
&L_S^R \sim \overline{DD}[r_S^R K^0 + J_S^{R\,inv}] + L_S^{R,c},\\
&L_{0a} \sim \overline{DD}[r_{0a}K^0 + J_{0a}^{inv}].
\end{aligned} \tag{4.7}$$

We have not written the corresponding representations of L_Φ and L_{1K} which are trivial due to (4.4) and the choice of (4.6) for the basis of genuinely chiral insertions. Note the absence of genuinely chiral terms for S and L_{0a}. This is due to the R-invariance (2.14) and to the identity [15, 19]

$$i\int dSS - i\int d\bar{S}\bar{S} \sim W_R\Gamma, \tag{4.8}$$

*) There is in fact also a term involving the ghost c_+: $c_+\delta_{c_+}\Gamma$ which, however, does not contribute to the Green function without external ghost lines and which is irrelevant for the present discussion.

and, for L_{0a}, to the Ward identities (2.15) which read

$$\int dSL_{0a} - \int d\bar{S}\bar{L}_{0a} \sim 0. \tag{4.9}$$

All coefficients r, r_g, etc., in (4.7) are of order $\hbar$ at least – we have explicitly displayed the zeroth order in L_g.

It is shown in Appendix A that r – the anomaly of the R-axial current – and r_{0a} – the anomalies of the axial currents associated to the chiral symmetries W_{0a}, i.e., with the centre of the algebra of all chiral symmetries – are not renormalized: they are exactly given by their one-loop contributions. The coefficients r and r_{0a} turn out [12] to be proportional respectively to the one-loop gauge β-function (3.1) and to the expression (3.7):

$$r = \frac{1}{128g^3}\beta_g^{(1)}, \tag{4.10}$$

$$r_{0a} = -\frac{1}{256(4\pi)^2}x_{0a}. \tag{4.11}$$

We also show in Appendix A that the coefficients r_k in the representation (4.7) for L_k (although renormalized contrary to the claim in Ref. [12]) are of order $\hbar^2$ at least and governed by the non-renormalized coefficients r_{0a}:

$$r_k = \sum_a t_{ka}r_{0a}, \tag{4.12}$$

where t_{ka} is of order $\hbar$ at least.

Let us come back to equations (4.7), insert them in each of the two equations (4.1) and identify the coefficients of the K^0 term. We thus get two relations:

$$r = \beta_g\left(\frac{1}{128g^3} + r_g\right) + \beta_{rst}r^{rst} - \gamma_k r_k - \gamma_S^R r_R^S, \tag{4.13}$$

$$r = \beta_g\left(\frac{1}{128g^3} + r_g\right) + \beta_{rst}r^{rst} - \gamma_k r_k - \gamma_{0a}r_{0a}. \tag{4.14}$$

The first of these equations will be useful for proving Lemma 5.1 in Section 5, whereas the second one will be crucial for proving the vanishing of the β-functions to all orders (Theorem 5.2, Section 5), due to the non-renormalization properties of r and r_{0a}.

An identity similar to equation (4.13) was proposed in Ref. [25] where the terms with coefficients r^{rst} and r_k are absent. Moreover r_g and r_R^S are claimed to be strictly one-loop. We remark that our less spectacular result takes rigorously into account all the possible renormalization effects.

5. The criteria for vanishing β-functions

Before stating the main theorems (Theorems 5.2 and 5.3), let us prove a result which yields necessary conditions for the vanishing of the β-functions to all orders:

Lemma 5.1. *Let us assume that the gauge β-function vanishes up to the two-loop order and the Yukawa β-functions at the one-loop order, i.e.,*

$$\beta_g = O(\hbar^3), \qquad \beta_{rst} = O(\hbar^2). \tag{5.1}$$

Then the following three conditions are necessarily fulfilled:

1) *The axial current of R-invariance is anomaly free:*

$$r = \frac{1}{128 g^3} \beta_g^{(1)} = 0. \tag{5.2}$$

2) *The one-loop anomalous dimensions* (3.3) *of the matter fields vanish:*

$$\gamma^{(1)R}{}_S = 0. \tag{5.3}$$

3) *The axial currents of the symmetries W_{0a} belonging to the centre of the algebra of chiral symmetries* (2.16) *are anomaly free:*

$$r_{0a} = 0. \tag{5.4}$$

Remark. The anomaly coefficients and the one-loop anomalous dimensions above are given in (4.10), (4.11) and (3.3). The third condition ensures the compatibility of the system of equations (5.3) – the second condition – due to Lemma 3.1 and relation (4.11).

Proof. The first condition is obvious and the third one follows from the second according to Corollary 3.2. Let us show the necessity of the second condition. In view of the last equality (3.8) used in the proof of Lemma 3.1, it is enough to check that

$$\sum_R T(R) \gamma^{(1)R}{}_R = 0. \tag{5.5}$$

But the latter follows from the identity (4.13) and the hypotheses (5.1), if we recall that in (4.13) the coefficients r_g and r^{rst} are of order $\hbar$ and r_k of order $\hbar^2$ [see (4.12)], and if we note that

$$r_R^S = -\frac{1}{256(4\pi)^2} \delta_R^S T(R) + O(\hbar^2), \tag{5.6}$$

as it results from a one-loop computation.

The present Lemma shows that the Yukawa and gauge coupling constants are not independent. The former must be functions of the latter,

$$\lambda_{rst} = \lambda_{rst}(g), \tag{5.7}$$

solutions of equation (5.3). These functions also solve the equations

$$\beta^{(1)}_{rst} = 0 \tag{5.8}$$

in view of (3.2).

So far so good for the one loop approximation, where the functions (5.7) are proportional to g [see (3.3)]. We now have to extend such a relationship to all orders, the functions (5.7) being formal power series in g. It is well known [13, 14] that these functions must then be solutions of the 'reduction equations'

$$\beta_{rst} = \beta_g \frac{d\lambda_{rst}}{dg}, \tag{5.9}$$

in order for the resulting theory, depending on the single coupling constant g, to be consistent. We note that the equations (5.8) are just the reduction equations at the one-loop order. But we also know from Lemma 5.1 that the stronger condition (5.3) of vanishing anomalous dimensions must in fact hold at this order. Let us thus state and prove the following.

Theorem 5.2. *The three conditions hereafter are necessary and sufficient for the β-functions of the gauge and Yukawa couplings to vanish to all orders of perturbation theory:*

(1) $\beta^{(1)}_g = 0$,

(2) $\gamma^{(1)R}{}_S = 0$,

(3) *The reduction equations* (5.9) *admit a formal power series solution which, in its lowest order, also has to be a solution of the condition* (2).

Remark. These conditions are in fact those of Ref. [4] [conditions (1) and (2)], but supplemented by a consistency requirement [condition (3)].

Proof. The necessity follows from Lemma 5.1 and from the discussion above. Let us show the sufficiency. The starting point is the identity (4.14). From condition (1) it follows that the R-current axial anomaly r vanishes [see (4.10)]. Condition (2) implies through Lemma 3.1 and its Corollary 3.2 that the quantities x_{0a} (3.7), hence the axial anomalies r_{0a} (4.11), vanish. This, in turn, ensures the vanishing of the coefficients r_k (4.12). At this stage the identity (4.14) becomes homogeneous in the β-functions. Condition (3) allows us to substitute for β_{rst} the right-hand side of the reduction equations (5.9), and we get

$$0 = \beta_g\left(\frac{1}{128g^3} + r_g + \frac{d\lambda_{rst}}{dg} r^{rst}\right). \tag{5.10}$$

The term in brackets being invertible in the perturbative sense, it results from this equation and from the reduction equations (5.9) that

$$\beta_g = 0, \qquad \beta_{rst} = 0 \tag{5.11}$$

at all orders. This concludes the proof.

The first two conditions of Theorem 5.2 are simple one-loop criteria. On the other hand, the last condition demands that the reduction equations be solvable at all orders. It is shown in Appendix B that a solution exists at all orders (and is unique) if the lowest-order solution, i.e., the solution of (5.8), is isolated and non-degenerate. We can thus state the following criterion:

Theorem 5.3 (criterion for vanishing β-functions). *Let us assume that a* SYM *gauge theory with simple gauge group obeys the following four conditions:*

(1) *It is free of gauge anomalies* [*equation* (2.13)];

(2) *The one-loop gauge β-function* (3.1) *vanishes,*

$$\beta_g^{(1)} = 0. \tag{5.12}$$

(3) *There exist solutions of the form*

$$\lambda_{rst} = \rho_{rst}\, g, \qquad \rho_{rst} \text{ complex number}, \tag{5.13}$$

to the condition of vanishing one-loop matter field anomalous dimensions (3.3),

$$\gamma^{(1)R}_{\ S} = 0. \tag{5.14}$$

(4) *The solutions* (5.13) *of* (5.14) *are isolated and non-degenerate when considered as solutions to the condition of vanishing one-loop Yukawa β-functions,*

$$\beta_{rst}^{(1)} = 0. \tag{5.15}$$

Then each of the solutions (5.13) *can be uniquely extended to a formal power series of g, giving a theory which depends on a single coupling constant – the gauge coupling g – with a β-function vanishing to all orders.*

The last theorem provides us with a simple criterion for vanishing β-functions which involves only standard one-loop computations. It can, in principle, be checked explicitly for every model at hand. However, the last condition can cause problems: the solutions of (5.14) are generally far from being isolated and non-degenerate. But it may happen that an extension of the given group of chiral symmetries W_a (2.15) yields enough supplementary constraints on the Yukawa coupling constants in order to lift the degeneracy. The use of a special renormalization scheme, based on the non-renormalization of chiral vertices, may also help to reach this goal. The example treated in the next section will show clearly how all this works in practice.

6. An $SU(6)$ model with vanishing β-functions

We consider here one of the 'two-loop finite' models of Ref. [5]. We shall show by checking the criterion given in Theorem 5.3 that it can be made 'all-loop finite' in the sense of vanishing β-functions.

This model has $SU(6)$ gauge invariance and its chiral matter fields belong to a complex representation of $SU(6)$, as one can read off from the Table. The present representation is free of the gauge anomaly (2.13) and makes the one-loop gauge β-function vanish: conditions (1) and (2) of Theorem 5.3 are fulfilled. The most general gauge invariant superpotential (2.7) is

Fields	ψ_i^α $(i = 1, \ldots, 8)$	ϕ_α^a $(a = 1, \ldots, 16)$	Λ_M	H^A
Representations	6	$\bar{6}$	$\overline{15}$	21
C_2	35/12	35/12	14/3	20/3
T	1/2	1/2	2	4

Chiral matter field representations, Casimir eigenvalues C_2 and Dynkin indices T [according to the definitions (3.4)]. The letters $\alpha = 1, \ldots, 6$; $M = 1, \ldots, 15$; $A = 1, \ldots, 21$ are representation indices. i and a are 'flavour' indices.

$$\begin{aligned} &U = U^1 + U^2, \\ &U^1 = \lambda_{(ab)} t_A^{(\alpha\beta)} \phi_\alpha^a \phi_\beta^b H^A, \\ &U^2 = \lambda^{[ij]} u_{[\alpha\beta]}^M \psi_i^\alpha \psi_j^\beta \Lambda_M + \lambda_3 v^{(MNP)} \Lambda_M \Lambda_N \Lambda_P, \end{aligned} \tag{6.1}$$

where t, u and v are $SU(6)$ invariant tensors normalized by

$$\begin{aligned} &t_A^{\alpha\beta} \bar{t}_{\alpha\beta}^B = 2\delta_A^B, \\ &u_{\alpha\beta}^M \bar{u}_N^{\alpha\beta} = 2\delta_N^M, \\ &v^{MNP} \bar{v}_{MNQ} = \tfrac{1}{18}\delta_Q^P. \end{aligned} \tag{6.2}$$

The superpotential is invariant under the chiral transformations [see (2.9)]

$$\begin{aligned} &\delta_1 \phi^a = i\phi^a, \qquad \delta_1 H = -2iH, \\ &\delta_1 \psi_i = 0, \qquad \delta_1 \Lambda = 0, \end{aligned} \tag{6.3}$$

with vanishing anomaly $r_{01} = 0$ [see (4.11) and (3.7)]. Hence from Lemma 3.1, the one-loop matter field anomalous dimensions can consistently be set to zero, thus the third condition of Theorem 5.3 is satisfied. The equations are

$$\begin{aligned} &\gamma_{\phi\ b}^{(1)a} = 28x(L_b^a - \alpha\delta_b^a) = 0, \\ &\gamma_H^{(1)} = 4x(L_a^a - 16\alpha) = 0, \\ &\gamma_{\psi\ i}^{(1)j} = 5x(4K_i^j - 7\alpha\delta_i^j) = 0, \\ &\gamma_\Lambda^{(1)} = 2x(2K_i^i + 9\,|\lambda_3|^2 - 28\alpha) = 0, \end{aligned} \tag{6.4}$$

where

$$L_b^a = \bar{\lambda}^{ac}\lambda_{cb}, \qquad K_i^j = \bar{\lambda}_{ik}\lambda^{kj}, \tag{6.5}$$

α is proportional to the square of the gauge coupling constant g and x is a numerical constant. The solutions are

$$\lambda_{(ab)} = \sqrt{\alpha}\, l_{(ab)}, \qquad \lambda^{[ij]} = \sqrt{\frac{7\alpha}{4}}\, k^{[ij]}, \qquad \lambda_3 = 0, \tag{6.6}$$

where (l_{ab}, k^{ij}) is any solution of

$$L^a_b = \delta^a_b, \qquad K^j_i = \delta^j_i. \tag{6.7}$$

We see that the last condition of Theorem 5.3 is *not* fulfilled, and this for two reasons. First, the solutions are not isolated: they form a continuous family parametrized by the complex numbers l_{ab}, k^{ij} solutions of (6.7). Second, the value $\lambda_3 = 0$ is a double, hence degenerate, root of the equations [see (3.2)]

$$\begin{aligned} &\beta^{(1)}_{\lambda_3} = 3\lambda_3\gamma^{(1)}_\Lambda = 0, \\ &\beta^{(1)ij} = \lambda^{ik}\gamma^{(1)j}_{\psi\ k} - \lambda^{jk}\gamma^{(1)i}_{\psi\ k} + \lambda^{ij}\gamma^{(1)}_\Lambda = 0. \end{aligned} \tag{6.8}$$

But there is a way out. Let us pick out an element of the family (6.6) by choosing an arbitrary solution (l_{ab}, k^{ij}) of (6.7). Then the superpotential

$$U_{(l,k)} = U^1_{(l)} + U^2_{(k)}, \tag{6.9}$$

obtained by replacing in (6.1) λ_{ab}, λ^{ij} and λ_3 by l_{ab}, k^{ij} and 0, is invariant under the three chiral symmetries.

$$\delta_2\psi_i = i\psi_i, \qquad \delta_2\Lambda = -2i\Lambda, \qquad (\phi, H \text{ invariant}) \tag{6.10}$$

$$\delta_{(l)}\phi^a = ie_{(l)}{}^a_b\phi^b, \qquad (H, \psi, \Lambda \text{ invariant}) \tag{6.11}$$

$$\delta_{(k)}\psi_i = ie_{(k)}{}^j_i\psi_j, \qquad (\phi, H, \Lambda \text{ invariant}) \tag{6.12}$$

provided the matrices $e_{(l)}$ and $e_{(k)}$ are constrained by [see (2.10)]

$$\begin{aligned} &l_{ac}e_{(l)}{}^c_b + l_{bc}e_{(l)}{}^c_a = 0, \\ &k^{il}e_{(k)}{}^j_l - k^{jl}e_{(k)}{}^i_l = 0. \end{aligned} \tag{6.13}$$

Conversely, keeping the choice of (l, k) as a solution of (6.7), we find that these chiral symmetries fix the superpotential up to two complex coupling constants:

$$U = \lambda_1 U^1_{(l)} + \lambda_2 U^2_{(k)}, \tag{6.14}$$

i.e.,

$$\lambda_{ab} = \lambda_1 l_{ab}, \qquad \lambda^{ij} = \lambda_2 k^{ij}, \qquad \lambda_3 = 0. \tag{6.15}$$

The system of equations for vanishing anomalous dimensions is still compatible*) and one finds

$$\begin{aligned} &\lambda_I = \rho_I e^{i\varphi_I}, \qquad \varphi_I \text{ arbitrary}, \quad (I = 1, 2), \\ &\rho_1^2 = \alpha, \qquad \rho_2^2 = \tfrac{7}{4}\alpha. \end{aligned} \tag{6.16}$$

Unluckily, this is again a continuous family of solutions parametrized by the phases left undetermined in (6.16): the last condition of Theorem 5.3 is still not satisfied. One can, however, fix by hand these phases to be zero if the

*) One can check that the anomalies of the chiral symmetries (6.10)–(6.12) are zero.

corresponding β-functions identically vanish:

$$\beta_{\varphi_I} = \mathrm{Im}\left(\frac{\beta_{\lambda_I}}{\lambda_I}\right) = 0. \tag{6.17}$$

It is easy to see that (6.17) is achieved if the renormalization scheme used to define the theory – prior to reduction – preserves for all orders the one-loop relations (3.2) between the matter β- and γ-functions**). In the present case – with chiral symmetries (6.10)–(6.12) and superpotential (6.14) – these relations read

$$\frac{\beta_{\lambda_1}}{\lambda_1} = 2\gamma_\Phi + \gamma_H, \qquad \frac{\beta_{\lambda_2}}{\lambda_2} = 2\gamma_\Psi + \gamma_\Lambda. \tag{6.18}$$

We used the fact that the chiral symmetries imply

$$\gamma_{\Phi}{}^{a}_{b} = \gamma_\Phi \delta^a_b, \qquad \gamma_{\Psi}{}^{j}_{i} = \gamma_\Psi \delta^j_i, \tag{6.19}$$

and substituted this in (6.4). Equations (6.17) are now seen to hold due to the reality of the expressions (6.18).

After having set to zero by hand the phases φ_I in (6.16), we get a unique solution of the one-loop problem: the last condition of Theorem 5.3 is now satisfied and its conclusion then follows.

7. Conclusions and outlook

i) The criterion given in Theorem 5.3 for all-order 'finiteness', i.e., for vanishing β-functions, is specially simple since it only involves standard one-loop quantities. Its conditions are sufficient. They are also necessary, condition (4) excepted. This last condition – existence of isolated and non-degenerate solutions to the one-loop problem – ensures the existence of power series solutions to all orders. If condition (4) is not met, this is not guaranteed but still possible, and such solutions are then to be characterized by additional requirements. Section 6 actually shows that such a solution exists for the model considered there, although condition (4) is violated when one starts with the most general interaction: this is the solution we got after reducing the dimension of the coupling constant space through the imposition of additional chiral invariances and the use of a particular renormalization scheme.

In general, one can expect the procedure for getting 'finite' theories from theories obeying the first three conditions of Theorem 5.3 to have two steps. Reduce first the number of independent Yukawa coupling constants by means of new symmetries and/or the use of a particular renormalization scheme, until the fourth condition is met. Then solve iteratively the reduction equations (5.9),

**) This scheme consists of replacing the normalization conditions on the vertex functions defining the Yukawa coupling constants, by the requirement of the absence of counterterms cubic in the chiral fields. This is consistent since the corresponding vertex graphs are ultra-violet finite [26, 15].

starting with a lowest-order solution for which the matter field anomalous dimensions vanish.

The models with complex representations obeying the first three conditions are listed in Ref. [5]. Those with real representations may be found in Ref. [8]. That all or part of them lead to 'finite' theories is under investigation.

ii) All anomalies vanish. Indeed the β-functions are set to zero: there is no conformal anomaly. Moreover all chiral anomalies vanish too, according to Lemma 5.1. One may ask whether at least a subclass of these theories are completely finite, i.e., whether the anomalous dimensions, which are in general gauge dependent, may all vanish as well. For the gauge field anomalous dimension, this may be the case in a suitable gauge, e.g., in the background gauge [27] where the gauge field anomalous dimension and the gauge β-function are not independent. The question is anyway more relevant for the matter field anomalous dimensions due to their relation with the Yukawa β-functions. For instance, in the $N = 4$ SYM theory written in terms of $N = 1$ superfields, which fulfils our criterion [12], there is one independent anomalous dimension and one β-function in the 'matter field' sector, thus both have to vanish. But in a generic case with more anomalous dimensions than β-functions – such cases are in fact excluded in Refs. [8–10] – we do not see any way for these anomalous dimensions to vanish altogether, although they have to do so in the one-loop approximation. Let us, however, mention Ref. [7], which suggests the possibility of a renormalization scheme where this vanishing holds at all orders.

iii) We have introduced masses in order to avoid the complications of the off-shell infra-red problem [15, 28]. These masses have been taken to be supersymmetric so that the finiteness of chiral insertions, used in the proof of the non-renormalization theorem for axial anomalies, holds. But they break softly the *BRS* invariance. There exists [15, 28], however, an infra-red cut-off procedure which preserves *BRS* invariance but softly breaks supersymmetry. The cut-off is shown to be a gauge parameter, hence unphysical. One has to extend our results to these truly gauge invariant theories. An argument will be presented elsewhere [29].

iv) In the present work we have restricted ourselves to the case of simple gauge groups. For semi-simple groups, the non-renormalization theorem for axial anomalies certainly holds (see Ref. [30] for usual gauge theories). In this case there is more than one gauge coupling constant and one will presumably have to reduce them too, so that all Yukawa and gauge coupling constants will be functions of a single one. The case of a gauge with $U(1)$ factors is excluded since the corresponding gauge β-functions can never be set to zero unless the $U(1)$ coupling constants vanish.

Appendix A. The supersymmetric non-renormalization theorem for the axial anomalies

We present here a corrected proof of the non-renormalization theorem of Ref. [12]. The hypotheses are now a little stronger but this is of no concern in view of the applications we discuss at the end of this Appendix and which are needed in the text.

Let us first introduce the 'supersymmetric Chern–Simons insertions' [12] K^q, $q = 0, \ldots, 3$. Their classical approximations are the following superfield 'polynomials'

$$\begin{aligned} k^0 &= \mathrm{Tr}\,(\varphi^\alpha \overline{DD}\varphi_\alpha), \\ k^{1\dot\alpha} &= -\mathrm{Tr}\,(D^\alpha c_+ \bar D^{\dot\alpha}\varphi_\alpha + \bar D^{\dot\alpha} D^\alpha c_+ \varphi_\alpha), \\ k^2_\alpha &= \mathrm{Tr}\,(c_+ D_\alpha c_+), \\ k^3 &= \tfrac{1}{3}\,\mathrm{Tr}\, c_+^3, \end{aligned} \tag{A.1}$$

where φ_α is the chiral superconnection (2.6). The K^q are solutions of the quantum extension of the classical descent equations*)

$$\begin{aligned} sk^0 &= \bar D_{\dot\alpha} k^{1\dot\alpha}, \\ sk^{1\dot\alpha} &= (\bar D^{\dot\alpha} D^\alpha + 2D^\alpha \bar D^{\dot\alpha}) k^2_\alpha, \\ sk^2_\alpha &= D_\alpha k^3, \\ sk^3 &= 0, \end{aligned} \tag{A.2}$$

where s is the *BRS* operator (2.2). K^3 is uniquely defined as the insertion of k^3 which is finite due to the non-renormalization of chiral vertices [26, 15]. Then one can show that K^0 is uniquely defined up to a *BRS* invariant insertion and a total derivative $\bar D(\ldots)$.
We can now state and prove the general theorem:

Theorem A.1. *Let T be a BRS invariant chiral superfield insertion of dimension 3 and R-weight**) -2. Moreover let its chiral superspace integral fulfil the Callan–Symanzik equation (2.27) without anomalous dimension, i.e.*

$$C \int dS T \sim 0. \tag{A.3}$$

Then:

1) *T admits the representation*

$$T \sim \overline{DD}(rK^0 + J^{inv}) + T^c, \tag{A.4}$$

where J^{inv} and T^c are BRS invariant, T^c is genuinely chiral [i.e., $T^c \neq \overline{DD}(\ldots)$], and the coefficient r of the Chern–Simons insertion K^0 is gauge independent and uniquely defined.

2) *The coefficient r is not renormalized, i.e., only one-loop graphs contribute to it.*

Proof. The first conclusion does not depend on the hypothesis (A.3). It was

*) See Ref. [12] for more details.
**) See equation (2.8).

proved in Ref. [12] (Proposition 3). In order to prove the second conclusion, let us begin by showing *) that the condition (A.3) of the theorem implies

$$CT \sim \overline{DD}X^{inv}, \tag{A.5}$$

where X^{inv} is BRS invariant. The proof of (A.5) at all orders being iterative, it suffices to discuss the classical problem, i.e., to show that

$$\int dSU = 0 \Rightarrow U = \overline{DD}X^{inv}, \tag{A.6}$$

where U and X^{inv} are classical insertions. U admits a representation analogous to (A.4)

$$U = \overline{DD}(xk^0 + X^{inv}), \tag{A.7}$$

without a genuinely chiral term since its chiral integral vanishes by assumption. Then

$$\int dV(xk^0 + X^{inv}) = 0, \tag{A.8}$$

and the integrand must be a total superspace derivative:

$$xk^0 + X^{inv} = D^\alpha A_\alpha + \bar{D}_{\dot\alpha} B^{\dot\alpha}. \tag{A.9}$$

Applying the BRS operator to this equation and using the descent equations (A.2) yields

$$x\bar{D}_{\dot\alpha} k^{1\dot\alpha} = D^\alpha s A_\alpha + \bar{D}_{\dot\alpha} s B^{\dot\alpha}. \tag{A.10}$$

A detailed superspace analysis then shows the existence of classical insertions G^1 and $\hat{G}^1$ such that

$$\begin{aligned} sA_\alpha &= -\overline{DD}G^1_\alpha + (D\bar{D} + 2\bar{D}D)_{\alpha\dot\alpha}\hat{G}^{1\dot\alpha}, \\ xk^{1\dot\alpha} - sB^{\dot\alpha} &= -DD\hat{G}^{1\dot\alpha} + (\bar{D}D + 2D\bar{D})^{\dot\alpha\alpha}G^1_\alpha. \end{aligned} \tag{A.11}$$

Applying s again gives the equations

$$\begin{aligned} (D\bar{D} + 2\bar{D}D)_{\alpha\dot\alpha} s\hat{G}^{1\dot\alpha} &= \overline{DD} sG^{1\alpha}, \\ (\bar{D}D + 2D\bar{D})^{\dot\alpha\alpha}(xk^2_\alpha - sG^1_\alpha) &= -DDs\hat{G}^{1\dot\alpha}, \end{aligned} \tag{A.12}$$

which can be solved by

$$s\hat{G}^{1\dot\alpha} = \bar{D}^{\dot\alpha} I^2, \qquad sG^1_\alpha = -D_\alpha I^2 + xk^2_\alpha, \tag{A.13}$$

where I^2 has dimension 0.

A last application of s and of the descent equations yields

$$\bar{D}^{\dot\alpha} sI^2 = 0, \qquad -D_\alpha sI^2 + xD_\alpha k^3 = 0. \tag{A.14}$$

*) It is just here that the present proof differs from the one given in Ref. [12], the condition (A.3) here being stronger than the corresponding one there.

The first equation means that sI^2 is chiral. Being of dimension 0, it must be proportional to k^3 (A.1) which, however, is not an s-variation. Therefore, $sI^2 = 0$ and the second equation (A.14) implies the vanishing of x. Equation (A.7) then yields the desired result (A.6).

We now insert the representation (A.4) of T in the relation (A.5) we have just proved and thus get

$$\overline{DD}[C(rK^0) + CJ^{inv} - X^{inv}] \sim 0, \qquad CT^c \sim 0, \tag{A.15}$$

the genuinely chiral part CT^c dropping out. The term in brackets must be a total $\bar{D}$ derivative:

$$C(rK^0) + CJ^{inv} - X^{inv} \sim \bar{D}_{\dot{\alpha}} L^{\dot{\alpha}}. \tag{A.16}$$

A sequence of BRS variations and of integrations with respect to superspace differential operators, combined with the quantum descent equations, finally yields [12]

$$C(rK^3) \sim 0. \tag{A.17}$$

Then, since K^3 is finite, $CK^3 \sim 0$, and

$$O = Cr = (\beta_g\, \partial_g + \beta_{rst}\, \partial_{\lambda_{rst}} + \bar{\beta}^{rst}\, \partial_{\bar{\lambda}^{rst}})r. \tag{A.18}$$

The second equality results from r being dimensionless and gauge independent. The non-renormalization of r then follows [12] from equation (A.18).

Corollary A.2. *The coefficients r and r_{0a} of S and L_{0a} respectively in equations* (4.7) *are not renormalized. Their values are given in the text* [*equations* (4.10) *and* (4.11)].

Proof. R-invariance implies [see Eq. (4.8)]

$$\int dSS - \int d\bar{S}\bar{S} \sim 0. \tag{A.19}$$

On the other hand, the equation

$$\int dSS + \int d\bar{S}\bar{S} \sim (C - m\, \partial_m)\Gamma \tag{A.20}$$

follows [15, 19] from the relation of the Callan–Symanzik equation with the broken dilatation invariance. Hence the hypothesis (A.3) holds for S, as one can see by applying the Callan–Symanzik operator C to both equations (A.19) and (A.20), and r is not renormalized. For r_{0a} we note that [see equations (4.3) and (2.15)]

$$L_{0a} = D_{0a}\Gamma, \qquad D_{0a} = e_{0a}{}^{R}_{S} A^S \delta_{A^R},$$
$$W_{0a} = \int dSD_{0a} - \int d\bar{S}\bar{D}_{0a}. \tag{A.21}$$

Hence the differential operator D_{0a} commutes with the Callan–Symanzik operator C since W_{0a} is a symmetry. It follows that the hypothesis (A.3) holds for L_{0a}; r_{0a} is thus not renormalized.

Remarks. The coefficient r in the representation (4.7) for S is the anomaly of the axial current associated with R-invariance [12]. The coefficients r_{0a} are the anomalies of the axial currents associated with the invariances W_{0a}: the representation (4.7) for L_{0a} is nothing other than the anomalous Ward identity for the associated current which is a component of the superfield J_{0a}^{inv} (the left-hand side L_{0a} is a contact term) [31]. We have formulated the Corollary above for the anomalies r_{0a} corresponding to the centre of the algebra of chiral symmetries W_a. This is what we need in the text; in particular just these r_{0a} participate in equation (4.14) and have to vanish. This Corollary is the supersymmetric extension of the well-known Alder–Bardeen theorem for the $U(1)$ anomalies [32, 30].

The coefficients r_k in the representation (4.7) for L_k are renormalized, but they are governed by the anomalies r_{0a}. Let us recall the definition (4.2) of L_k:

$$\partial_{a_k}\Gamma \sim \int dSL_k + \int d\bar{S}\bar{L}_k. \tag{A.22}$$

a_k is a gauge parameter [15, 21], i.e.,

$$\partial_{a_k}\Gamma \sim \mathscr{B}\Delta_k, \tag{A.23}$$

where $\mathscr{B}$ denotes the quantum extension of the BRS operator s [15, 12] and Δ_k is an insertion of dimension 4 and ghost number -1. The most general form for Δ is

$$\Delta_k = \int dV\hat{\mathscr{L}}_k + t_{kS}^{\ R} \int dSY_R A^S + \text{conj.}, \tag{A.24}$$

where Y_R is the chiral external field coupled with the BRS variation of A^R [12, 15]. The chiral invariances W_a imply that the matrices t_k can be expanded in the matrices e_{0a} and f_{1K}, defined by equations (2.22) and (2.23):

$$t_{kS}^{\ R} = t_{ka}e_{0aS}^{\ \ R} + t'_{kK}f_{1KS}^{\ \ R}. \tag{A.25}$$

Moreover [12, 15]

$$\mathscr{B}(Y_R A^S) = L_R^S \tag{A.26}$$

hence we can choose, for L_k, in agreement with the definition (A.22),

$$L_k = \overline{DD}\mathscr{L}_k^{inv} + t_{ka}L_{0a} + t'_{kK}L_{1K}, \tag{A.27}$$

where $\mathscr{L}_k^{inv} = \mathscr{B}\mathscr{L}_k$. Inserting here the representation (4.7) of L_{0a} and comparing the result with the representation (4.7) of L_k, keeping in mind that L_{1K} belongs to the basis of genuinely chiral insertions, we get the result we looked for:

$$r_k = \sum_a t_{ka}r_{0a}. \tag{A.28}$$

The coefficients t are of order $\hbar$, hence r_k is of order $\hbar^2$. Moreover r_k vanishes if the axial anomalies r_{0a} vanish.

Appendix B. Reduction of coupling constants for SYM theories

We want to show that the reduction equations (5.9) admit a power series solution $\lambda_{rst}(g)$ if there is a lowest-order solution which is isolated and non-degenerate. If the gauge β-function is zero at the one-loop order – the case of interest here – the lowest-order equations are

$$\beta_\lambda^{(1)}(\lambda, g) = 0 \tag{B.1}$$

for all Yukawa coupling constants λ.

By separating the complex coupling constants λ_{rst} into their real and imaginary parts – we consider only the set of independent ones – we can assume all Yukawa coupling constants to be real and denote them by λ_i. The reduction equations read

$$\beta_i = \beta_g \frac{d\lambda_i}{dg}. \tag{B.2}$$

We shall follow Ref. [14], specializing to the structure of SYM gauge theories, for which the power series expansion of the β-functions has the form

$$\begin{aligned} \beta_i &= \sum_{n=1}^{\infty} \sum_{a=0}^{n} \sum_{k} C_i^{(n)k_1 \cdots k_{2a+1}} g^{2n-2a} \lambda_{k_1} \cdots \lambda_{k_{2a+1}} \\ &= C_i^{(1)k} g^2 \lambda_k + C_i^{(1)klm} \lambda_k \lambda_l \lambda_m + O(\hbar^2), \\ \beta_g &= g^3 \sum_{n=2}^{\infty} \sum_{a=0}^{n-1} \sum_{k} B^{(n)k_1 \cdots k_{2a}} g^{2n-2-2a} \lambda_{k_1} \cdots \lambda_{k_{2a}} \\ &= O(\hbar^2). \end{aligned} \tag{B.3}$$

The index n denotes the loop order. We have assumed β_g to vanish at order 1. Let us look for a solution of (B.2) of the form

$$\lambda_i(g) = \sum_{n=0}^{\infty} \rho_i^{(n)} g^{2n+1}. \tag{B.4}$$

At the lowest order, one finds that $\rho_i^{(0)}$ must be a solution of the equations

$$F_i(\rho^{(0)}) \equiv C_i^{(1)k} \rho_k^{(0)} + C_i^{(1)klm} \rho_k^{(0)} \rho_l^{(0)} \rho_m^{(0)} = 0, \tag{B.5}$$

which are just equations (B.1).

In higher orders we get the recurrence equations

$$M_i^k \rho_k^{(n)} = f_i, \qquad n \geq 1, \tag{B.6}$$

where the right-hand side depends only on the $\rho^{(p)}$ for $p < n$. The matrix M

depends on $\rho^{(0)}$ only:

$$M_i^k = \partial F_i(\rho^{(0)})/\partial \rho_k^{(0)}. \tag{B.7}$$

If this matrix is non-singular, i.e., if and only if the solution $\rho^{(0)}$ of equation (B.5) is isolated and not degenerate, then (B.6) determines the higher coefficients of (B.4) in terms of $\rho^{(0)}$.

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