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Frequency doubling of incoherent microwave radiation in a cold magnetized plasma

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Abstract. The parametric interaction of electromagnetic waves in a cold magnetized plasma is investigated as a possible source of the transfer of wave energy between fundamental and harmonic spectra of incoherent electromagnetic waves. The feasibility of the non-linear coupling process of extraordinary waves in terms of compatibility of the dispersion relation with the resonance condition, as well as the regimes of plasma density, magnetic field and temperature and the required range of wave energy are considered. The rate of wave coupling is evaluated numerically and the maximum energy transfer is estimated if the correlation length is taken into account. The effects of damping processes by gyroresonance, by Bremsstrahlung and the effect of collisions on the dispersion relation are presented.

Within a narrow range of the ratio of plasma- to cyclotron-frequency ($\frac{4}{3} < \omega_p/\omega_c^e < \frac{5}{3}$) an efficient coupling of incoherent extraordinary waves is possible and the damping processes are sufficiently small over a considerable range of density, temperature and magnetic field as to allow for generation and propagation of harmonic radiation.

1. Introduction

Evidence of occasional harmonic radiation at microwave frequencies during solar flares has been observed recently by Stähli et al. (1987). The generation of harmonic emission with a two to one ratio of the observed frequency bands is difficult to explain by plasma radiation or by gyroradiation or by electron cyclotron maser instability, the processes otherwise widely accepted for the explanation of various spectral properties of microwave radiation originating in eruptions within the coronal plasma. Non-linear interaction between electromagnetic waves can produce energy and momentum conversion between fundamental and harmonic spectra even for incoherent radiation as will be shown in this paper.

Beyond the application to the mentioned astrophysical problem it may be concluded that the efficiency of this process is sufficient to cause spectral redistribution of the energy of electromagnetic radiation in various kinds and regimes of plasma where the relations between particle density, magnetic field and temperature are favourable and the damping processes sufficiently small. These facts stimulated the author to resume a conception which was developed

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many years ago, (Schanda 1972 and 1973), but then never applied because of lacking observational evidence.

The basic concepts and various applications of non-linear wave-wave coupling has been treated in several texts (Tsytovich 1970, Davidson 1972, Akhiezer et al. 1975, Weiland and Wilhelmsson 1977, Melrose 1986). The applications of parametric coupling of three waves discussed in these texts usually involve at least one longitudinal mode, and a majority of treatments is limited to exclusively coherent waves and in a substantial portion of presentations the wave interaction without external magnetic field is dealt with. Either with these suppositions or with specific conditions of non-collinear propagation directions of the participating waves by Stenflo (1973) and Etievant et al. (1968) resonant interaction has been treated.

In a turbulent plasma non-linear terms appear due to spatial gradients if the turbulence exceeds a critical limit (Tsytovich and Stenflo 1973). In connection with the explanation of solar type III bursts in the decimeter- and meter-wave range the non-linear coupling of any wave mode with particle drift through a cold magnetized plasma has been derived in a general approach and applied to the transformation of electrostatic waves by Chin (1972).

Also among the more recent work on non-linear wave coupling in a magnetized plasma the considerations of the electrostatic waves are dominating (Sharma and Shukla 1983, Dysthe et al. 1985) and the application envisaged is plasma heating e.g. by Stefan and Krall (1985).

In connection with microwave solar burst radiation the occurrence of harmonics of the local cyclotron-frequency in very short spikes (milliseconds) is explained by the electron cyclotron maser process according to Melrose and Dulk (1982). However, in this case the fundamental radiation is missing in the observed spectrum because of re-absorption within the source region. Essentially the same idea of a loss-cone instability has been extended by Wu et al. (1985) to a hollow-beam distribution and to relativistic electron energies. The resulting synchrotron-maser instability is shown to have several peaks of the growth rate closely spaced in the spectrum at about twice the plasma frequency.

All these processes cannot explain a frequency ratio of two to one of the observed radiation spectrum.

Before entering the analysis the crucial question has to be discussed now: Can the radiative energy of a solar microwave burst be sufficient in order to cause any non-linear interaction along its propagation path in the surrounding coronal plasma? As an established criterion (Hasegawa 1975) may be taken that non-linear effects become significant if the energy density W of the wave becomes larger than the thermal energy κT per Debye volume (energy density at thermal equilibrium)

$$W > n_0 \kappa T / (n_0 \lambda_D^3) \tag{1}$$

Boltzmann's constant $\kappa = 1.38\,10^{-23}$ Ws/K, and electron density n_0, temperature T and the Debye length λ_D are taken in SI units. The densities in the lower corona are typically $n_0 \leq 10^{17}\,\mathrm{m}^{-3}$ and the temperatures in the vicinity of flaring

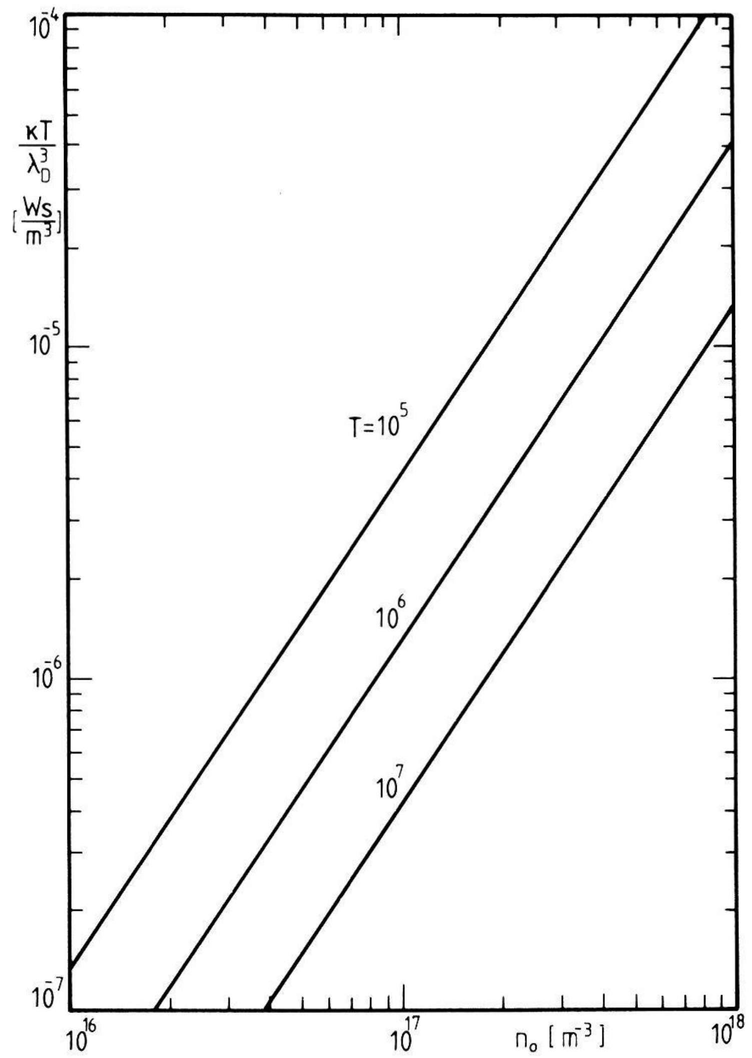

Figure 1
Energy density at thermal equilibrium over the ranges of density and temperature relevant for this study.

regions are $T \geq 10^6$ K. In Fig. 1 the domain of $\kappa T/\lambda_D^3$ according to (1) is presented for the domains of temperatures and densities which are relevant to the wave interaction process in the coronal active regions and which can be used to test the critical distance over which (1) is fulfilled. Microwave bursts for which harmonic features have been observed exhibit flux densities at the earth distance typically several hundred solar flux units (several 10^{-20} Watt m^{-2} Hz^{-1}) over a bandwidth of at least a few hundred Megahertz. This would yield a wave energy density of more than 10^{-6} Ws/m^3 within a range of some 25 km of a "point-like" source of the radiation. Hence within this distance condition (1) is fulfilled; this provides sufficient space for an effective wave interaction.

2. Non-linear interaction of incoherent wave spectra

In a previous paper (Schanda, 1987) the non-linear interaction of electromagnetic waves in cold magnetized plasma has been treated. In particular for extraordinary waves (wave vector and polarization perpendicular to the external static magnetic field) the rate of transfer of wave energy between incoherent spectra has there been derived. The main issues and rationales of this previous investigation will now briefly be summarized in order to facilitate the discussion in the subsequent sections.

Anticipating the results of Section 5 of this paper we may assume the plasma to be cold and collisionless and its dynamical behaviour be dominated by the external magnetic field. Within the physical space of the interaction region the static components of the magnetic field and of the plasma density are supposed homogeneous.

The continuity and the momentum equations contain products of velocity and density fluctuations caused by the electromagnetic wave fields. At sufficiently high intensity of these waves the quadratic terms of velocity and density fluctuations are no longer negligible against the linear terms and consequently cause a second order contribution $j^{(2)}$ to the current density $j = j^{(1)} + j^{(2)}$ in the wave equation.

The Fourier presentation of products of harmonic waves with frequencies ω_1 and ω_2 and corresponding wave vectors $\vec{k}_1$ and $\vec{k}_2$ leads to new waves with combination frequencies and wave vectors, ω_0, $\vec{k}_0$. We can apply this concept to the momentum equation of the electrons of a cold two-component plasma in order to find the linear and non-linear contributions to the velocity fluctuations of the electrons. For this purpose we use $\partial/\partial t = i\omega$ and $\partial/\vec{r} = -i\vec{k}$, we substitute the magnetic field of the wave $\vec{H}_k = (\vec{k} \times \vec{E}_k)/\omega\mu_0$ by its electric counterpart and set $\vec{H}_0$ for the static magnetic field

$$\vec{v}^e_{k0} = \frac{ie}{m\omega_0}[\vec{E}_{k0} + \mu_0(\vec{v}^e_{k0} \times \vec{H}_0)] + \frac{ie}{m\omega_0}\int \frac{\vec{k}_2}{\omega_2}(\vec{v}_{k1} \cdot \vec{E}_{k2})\, dk_{1,2}\, \delta_{\omega,k}$$
$$+ \frac{1}{\omega_0}\int (\vec{v}^e_{k1} \cdot \vec{k}_2)\left(\vec{v}^e_{k2} - i\frac{\vec{E}_{k2}}{\omega_2}\frac{e}{m}\right) dk_{1,2}\, \delta_{\omega,k} \tag{2}$$

with the abbreviations $dk_{1,2} = d\vec{k}_1\, d\vec{k}_2\, d\omega_1\, d\omega_2$ and

$$\delta_{\omega,k} = \delta(\omega_0 - \omega_1 - \omega_2)\, \delta(\vec{k}_0 - \vec{k}_1 - \vec{k}_2) \tag{3}$$

Equation (2) represents the first order term and two second order terms of the wave induced velocity fluctuations at ω_0, $\vec{k}_0$. Here it is assumed that for ω_0 only the sum of the wave frequencies ω_1 and ω_2 is admitted. The product of energy and momentum conservation conditions (3) teaches that besides the frequencies also the wave vectors have to 'resonate'. The linear portion of $\vec{v}^e_{k0}$ can be solved approximately by the first term r.h.s. at (2), yielding

$$(v^{e(1)}_{k0})_p = \frac{ie}{m\omega_0}(T^e_{k0})_{pq}(E_{k0})_q \tag{4}$$

where suffix notation is introduced for vectors and tensors. If the magnetic field is aligned in the z-direction of a cartesian coordinate system, the tensor assumes a simple shape

$$(T_{k0})_{pq} = \frac{1}{1-\left(\frac{\omega_c^e}{\omega_0}\right)^2} \begin{pmatrix} 1 & i\frac{\omega_c^e}{\omega_0} & 0 \\ -i\frac{\omega_c^e}{\omega_0} & 1 & 0 \\ 0 & 0 & 1-\left(\frac{\omega_c^e}{\omega_0}\right)^2 \end{pmatrix} \tag{5}$$

with the cyclotron frequency of electrons $\omega_c^e = \mu_0 H_0 e/m$. If no drift motion in the plasma is assumed ($v_0 = 0$) and if terms of orders higher than 2 are disregarded then products $n \cdot \vec{v}$ entering the current density will contain only terms as

$$n\vec{v} = n_0 \vec{v}^{(1)} + n^{(1)} \vec{v}^{(1)} + n_0 \vec{v}^{(2)} \tag{6}$$

This signifies that only mean value and first order fluctuation of density and first and second order of velocity fluctuations combine to first and second order contributions of the current density at the combination wave spectrum ω_0, $\vec{k}_0$.

In the present analysis the non-linear process is considered as weak such that the characteristic (e-folding) interaction length and interaction time are much larger than $1/\vec{k}$ and $1/\omega$ respectively. This and the additional assumption of a localized source of radiation and wave interaction along a (collinear) propagation path will permit to replace d/dt by $i\omega$ and to neglect time derivatives of the amplitudes of $\vec{E}$ and $\vec{j}$. The same assumptions allow to limit the space derivatives to the first order. For this purpose the wave field vector at any one of the spectral ranges $\vec{E}_k \equiv \vec{F}_k \cdot \exp i(\omega t - \vec{k} \cdot \vec{r})$ and the current density $\vec{j}_k \equiv \vec{J}_k \cdot \exp i(\omega t - \vec{k} \cdot \vec{r} - \varphi_k)$ can be separated into amplitude vectors (including polarization) $\vec{F}_k$, $\vec{J}_k$ and the portion of high frequency oscillation and phase φ_k respectively.

This concept used in the wave equation will yield a relation between the space derivative of the wave field $\vec{F}_{k0}$ at ω_0, $\vec{k}_0$ and the time derivative of the second order contribution to the current density $\vec{J}_{k0}$ at ω_0, $\vec{k}_0$. From the resulting wave equations it has been shown by Schanda (1987) that only transverse wave modes are interacting via the second order current density, i.e. the extraordinary and the ordinary waves for propagation perpendicular to the magnetic field as well as the left and right hand polarized waves for propagation parallel to the magnetic field.

Within the range of the assumed approximations the undisturbed (linear) wave equation can be subtracted from the one including the non-linear action. It will be shown in the next sections that the extraordinary wave is particularly useful for the generation of harmonic radiation. Therefore we concentrate in the following on this wave mode, i.e. we specify $\vec{k} \equiv k_x$, $\vec{F}_k \equiv (F_k)_y$, $\vec{H}_0 \equiv (H_0)_z$. The rate of coupling from the lower frequency waves ω_1, k_1 and ω_2, k_2 to the sum frequency ω_0, k_0 is according to the previous investigation (Schanda 1987)

$$\frac{\partial (F_{k0})_y}{\partial x} = -\frac{\mu_0 \omega}{2(k_0)_x} (J^{(2)}_{k0})_y \tag{7}$$

valid for coherent waves.

The wave spectra originating in a natural emission process are usually representing incoherent radiation. Therefore a rate equation for the transfer of intensity of incoherent waves has to be derived through an averaging procedure. The resulting rate equation (Schanda 1987) reads

$$\begin{aligned}
\left\langle \frac{\partial |F_{k0}|_y^2}{\partial x} \right\rangle &= \frac{1}{2}\left(\frac{e\varepsilon_0\mu_0\omega_p^{e2}}{m(k_0)_x}\right)^2 \\
&\times \int \frac{\left(\frac{\omega_c^e}{\omega_0}\right)^2 (k_2)^2}{\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2 \omega_1^2\omega_2^2} \cdot \frac{|F_{k2}|^2}{\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2} \left[\left(|F_{k1}|_x \pm \frac{\omega_c^e}{\omega_1}|F_{k1y}|\right)^2\right. \\
&+ \left.\left(|F_{k1}|_x \frac{\omega_c^e}{\omega_1} \pm |F_{k1}|_y\right)^2 + \left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]|F_{k1}|_z^2\right] d\omega_0\, dk_{1,2}\, \delta_{\omega,k} \\
&+ \int \frac{1}{\omega_1^2\omega_2^2} \frac{|k_2|_x^2}{\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_2}\right)^2\right]^2} \\
&\times \left[|F_{k2}|_x\left(\frac{\omega_c^{e3}}{\omega_0\omega_2^2}+\frac{\omega_c^e}{\omega_2}\right) \pm |F_{k2}|_y\left(\frac{\omega_c^{e2}}{\omega_0\omega_2}+\frac{\omega_c^{e2}}{\omega_2^2}\right)\right]^2 \\
&\times \left[|F_{k1}|_x \pm \frac{\omega_c^e}{\omega_1}|F_{k1}|_y\right]^2 d\omega_0\, dk_{1,2}\, \delta_{\omega,k} \\
&+ \int \frac{\omega_0^2}{\omega_1^4\omega_2^2} \frac{|k_1|_x^2}{\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_2}\right)^2\right]^2} \left[|F_{k1}|_x \pm \frac{\omega_c^e}{\omega_1}|F_{k1}|_y\right]^2 \\
&\times \left[\frac{\omega_c^e}{\omega_2}|F_{k2}|_x \pm |F_{k2}|_y\right]^2 d\omega_0\, dk_{1,2}\, \delta_{\omega,k}
\end{aligned} \tag{8}$$

In this equation the second order current density is taken into account as caused only by a transverse electron density wave at ω_1, $\vec{k}_1$ and a transverse electron velocity wave at ω_2, $\vec{k}_2$ with all wave vectors $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_0$ pointing into the same direction. The ions are assumed at rest with respect to the oscillations of the electrons induced by the wave field at frequencies comparable with the electron cyclotron and electron plasma frequency. The alternating signs in equation (8) take into consideration the fact that an extraordinary wave assumes elliptic polarization in a plane perpendicular to the magnetic field; the two signs signify

$(F_k)_x$ advanced (+) or retarded (−) with respect to $(F_k)_y$ according to

$$(F_k)_x/(F_k)_y = (|F_{kx}|/|F_{ky}|)\exp \pm i\frac{\pi}{2} \tag{9}$$

3. The dispersion relation of the extraordinary wave

In order to achieve the generation of a harmonic spectrum by parametric interaction along a propagation path common to the fundamental and the harmonic waves, a dispersion relation is required which yields an identical index of refraction for the fundamental and the harmonic spectra. With this requirement the condition of energy and momentum conservation (3) is satisfied for collinearly propagating waves. For the case of frequency doubling of incoherent radiation the condition of identical index of refraction has to be fulfilled over finite bandwidths centered at frequencies with a ratio of 1:2.

The dispersion relation of the high frequency extraordinary wave in a cold, collisionless magnetized plasma is given, e.g. (Hasegawa 1975), by

$$N^2 - 1 = \frac{\omega_p^2}{\omega_c^i\omega_c^e - \omega^2\left(1 + \dfrac{\omega_c^{e2}}{\omega_p^2 + \omega_c^i\omega_c^e - \omega^2}\right)} \tag{10}$$

with $N = c\,|k|/\omega$ the index of refraction and where in the plasma frequency ω_p only the electrons are taken into account. The condition of equal values of N for simultaneously a fundamental ω_f and the harmonic frequency $\omega_0 = 2\omega_f$ can be satisfied by equating (10) for ω_f and $2\omega_f$. The solution for the fundamental frequency is

$$\omega_f^2 = \frac{\omega_a^2}{8}\left[5 \pm \sqrt{9 - 16\frac{\omega_c^{e2}}{\omega_a^2}}\right] \tag{11}$$

with the abbreviation $\omega_a^2 = \omega_p^2 + \omega_c^i\omega_c^e$. The requirement of ω_f being real imposes a lower bound on $\omega_a/\omega_c^e \geq \frac{4}{3}$. The refractive index corresponding to $\omega_a/\omega_c^e = \frac{4}{3}$ is $N = 0.6$. The alternate signs in (11) cause branching of the solutions of ω_f for increasing ω_a/ω_c^e. If the positive sign is taken, N_+ increases asymptically towards unity for $\omega_a/\omega_c^e \rightarrow \infty$. This means that the phase velocities of the interacting fundamental and harmonic waves are always larger than the vacuum velocity. If the negative sign is taken in (11) then – besides the lower bound $\omega_a/\omega_c^e \geq \frac{4}{3}$ – also an upper bound $\omega_a/\omega_c^e \leq \sqrt{2}$ appears due to the requirement of a real index of refraction. The value of N_- decreases from 0.6 to zero for ω_a/ω_c^e increasing between $\frac{4}{3}$ and $\sqrt{2}$. The solution (11) for a frequency pair ω_f and $2\omega_f$ with equal index of refraction is feasible over a wide range of values $\frac{4}{3} \leq \omega_a/\omega_c^e < \infty$ if the + sign is taken, resulting in reasonable values of N_+. The solution with the − sign has two disadvantages with regard to wave coupling: the range of validity in terms of the ratio of plasma to cyclotron-frequency is extremely narrow ($\frac{4}{3} \leq \omega_a/\omega_c^e \leq \sqrt{2}$) and N_- is very low. This causes strongly variable and high-valued phase velocities not compatible with the requirements for wave coupling over a distance of many wavelengths and over a finite bandwidth under the conditions of a real plasma.

The solution ω_f/ω_c^e (with the + sign in the bracket expression of (11)) is traced in Fig. 2 over a reasonable range of ω_p/ω_c^e-values. Also shown in Fig. 2 are the areas (hatched) forbidden for the propagation of extraordinary waves according to the cold plasma concept (Krall and Trivelpiece 1973) below the line of the lower cut-off frequency ω_l

$$\frac{\omega_l}{\omega_c^e} = \frac{1}{2}\left[\sqrt{1+\left(\frac{2\omega_p}{\omega_c^e}\right)^2} - 1\right] \tag{12a}$$

and between the lines of the upper cut-off frequency ω_u

$$\frac{\omega_u}{\omega_c^e} = \frac{1}{2}\left[\sqrt{1+\left(\frac{2\omega_p}{\omega_c^e}\right)^2} + 1\right] \tag{12b}$$

and the hybrid frequency ω_h

$$\frac{\omega_h}{\omega_c^e} = \sqrt{\left(\frac{\omega_p}{\omega_c^e}\right)^2 + 1} \tag{12c}$$

The solution of (11), the fundamental frequency ω_f, is well between the forbidden regions over the whole range $\omega_p/\omega_c^e > \frac{4}{3}$ and the harmonic frequency $2\omega_f$ is well above the forbidden regions. The effects of final bandwidth about ω_f and $2\omega_f$ as well as the effect of the spectral width of the gyroresonance due to

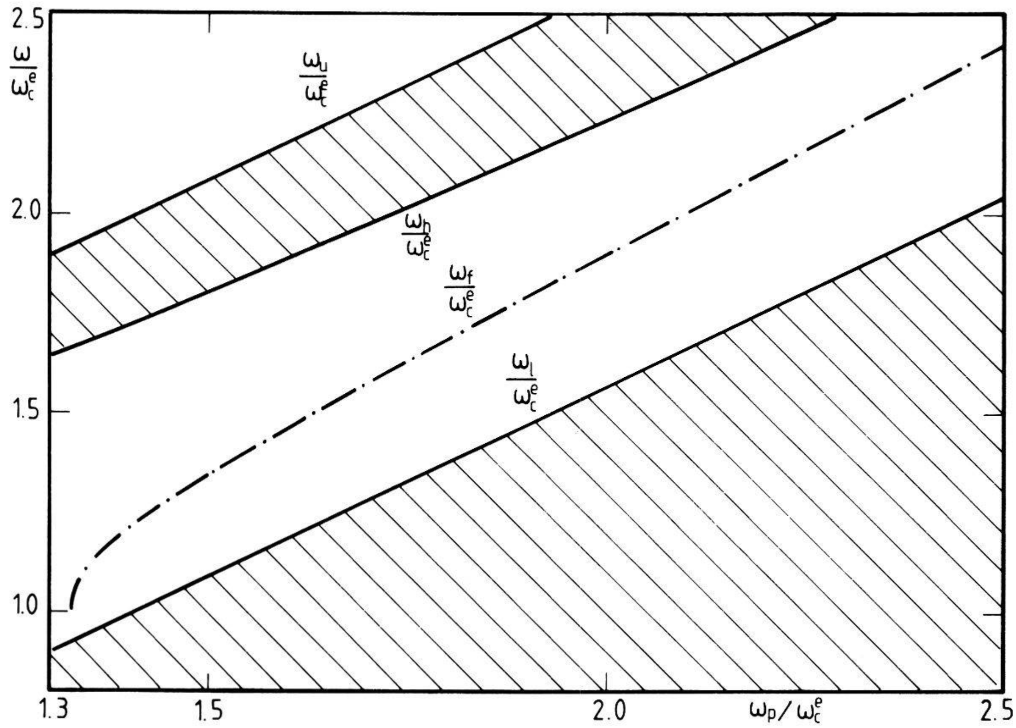

Figure 2
The fundamental frequency ω_f according to equation (11) and the hybrid frequency ω_h, the upper and the lower cut-off frequencies ω_u, ω_l of the extraordinary wave as a function of the plasma frequency, all normalized to the electron cyclotron frequency ω_c^e.

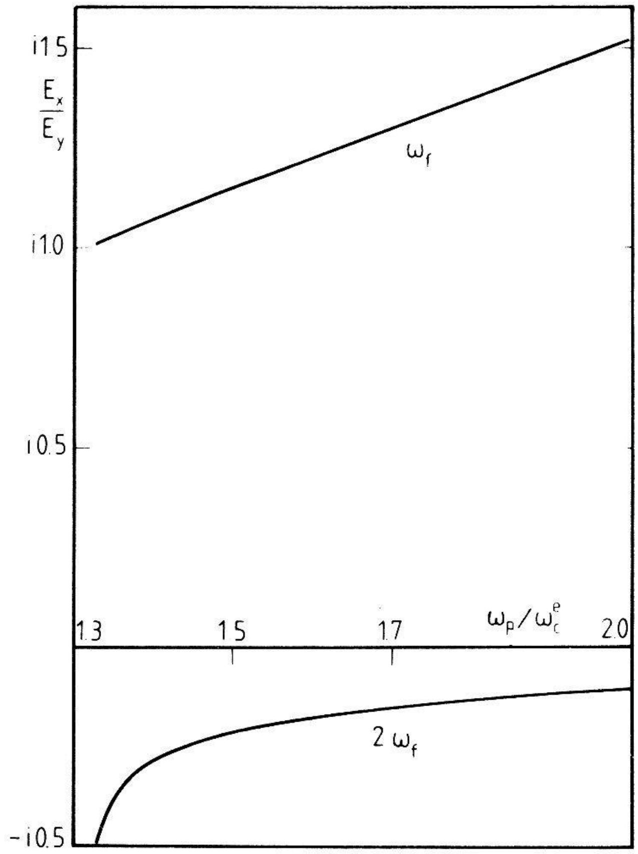

Figure 3
Polarization of the fundamental (ω_f) and the harmonic wave ($2\omega_f$) over the relevant range of ω_p/ω_c^e.

collisions on the propagation behaviour of these wave spectra will be discussed in Sections 4 and 6 respectively.

On the assumption of weak non-linear interaction the linear regime of the dispersion relation (10) can be used for an approximate computation of the polarization behaviour of the fundamental ω_f, and the harmonic frequency $2\omega_f$, of extraordinary waves. An initially transverse electromagnetic wave with propagation and polarization vectors perpendicular to the static magnetic field will acquire elliptic polarization with a longitudinal (along the wave vector) component of the electric wave field. Within the concept of cold collisionless plasma an extraordinary wave in an orthogonal arrangement as $\vec{k} \equiv k_x$, $\vec{H}_0 \equiv H_{0z}$ and initial polarization $\vec{E} \equiv E_y$ will assume a elliptic polarization given by

$$\frac{E_x}{E_y} = i \frac{(N^2 - 1)\dfrac{\omega}{\omega_c^e}}{\left(\dfrac{\omega}{\omega_c^e}\right)^2 - \left(\dfrac{\omega_a}{\omega_c^e}\right)^2} \tag{13}$$

where the effect of ions is neglected.

Figure 3 presents the polarizations of the fundamental and harmonic frequencies as derived from (13) for $\omega = \omega_f$ and $\omega = 2\omega_f$ where ω_f is taken from (11). The fundamental, starting with circular polarization at $\omega_f \approx \omega_c^e$, produces a relatively increasing longitudinal component, i.e. it develops towards the behaviour of a plasma wave the more the frequency ω_f approaches ω_p. The harmonic on the contrary assumes asymptotically the property of a transverse wave as the effect by the magnetic field decreases (for ω_f sufficiently larger than ω_c^e). The phases of the longitudinal components of fundamental and harmonic are opposite, i.e. the circularly polarized portions of the waves are rotating in opposite sense. This means that only a portion of the wave amplitude (and energy) corresponding to the transversely polarized components can effectively participate in the parametric interaction process.

4. Growth rate of the harmonic radiation

The growth rate of incoherent radiation at the harmonic frequency due to non-linear wave-wave interaction has formally been derived by Schanda (1987) and a representative result of this derivation is reproduced here by expression (8). In the present consideration of high frequency electromagnetic waves with frequencies in the order of magnitude or higher than the upper hybrid frequency we will neglect the contribution by the ions. Therefore in (8) only three terms according to the contribution by the electrons are presented. The integrations over $\vec{k}_1$, $\vec{k}_2$, ω_1, ω_2 have to be performed simultaneously satisfying the conditions $\delta(\omega_0 - \omega_1 - \omega_2)\,\delta(\vec{k}_0 - \vec{k}_1 - \vec{k}_2)$. The following simplifying assumptions will be made for the numerical computation:

- All wave vectors $\vec{k}_0$, $\vec{k}_1$, $\vec{k}_2$ are pointing into the same direction perpendicular to the static magnetic field and only extraordinary mode with polarization vectors perpendicular to the static magnetic field is considered.
- The intensity of the fundamental radiation is only weakly dependent on frequency within the (narrow) bandwidth $\Delta\omega_0 \approx \Delta\omega_1 \approx \Delta\omega_2$ over which the integration is performed.
- The energy and momentum conservations are satisfied for harmonic generation ($\omega_0 = \omega_1 + \omega_2$) within the bandwidth $\pm\Delta\omega$ if simultaneously $\omega_1 = \omega_f + d\omega$ and $\omega_2 = \omega_f - d\omega$, with maximum of $d\omega$ being $\Delta\omega$, are assumed.
- The intensity of the radiation is supposedly bound to values such as to allow the approximate use of the linear dispersion relation.
- The full elliptic polarization will be replaced by the transverse (y) component as the only one which is rersponsible for the coupling process.

Equation (8) can be separated into a common factor

$$V \equiv \frac{1}{2}\left(\frac{e}{m}\frac{\omega_p^2}{c^2 k_0}\right)^2 \approx 1.91\,10^{-12}\,\frac{\omega_p^4}{k_0^2} \tag{14}$$

and three integrals. The assumptions of weak frequency dependence of the intensity and of narrow bandwidth reduce the Fourier transform of the wave fields to a simple multiplication $E(r, t) = E(\omega, k)\,\Delta\omega\,\Delta k$. Hence the replacement of $dk_{1,2}$ by $\Delta k_1\,\Delta k_2\,\Delta\omega_1\,\Delta\omega_2$ will be permitted and we can substitute the integrations by multiplications. Further multiplications of both sides of equation (8) by $(\Delta k_0)^2$ $(\Delta\omega_0)^2$ converts the spectral components of the wave amplitudes modulus squared $|F_k|^2$, (unit of F_k is Volt · second) into the space and time dependent field strength (modulus squared) $|E(r, t)|^2$, with unit of $E(r, t)$ Volt per meter. Finally there remains the integration over $d\omega_0$. The three integrals read

$$I_1 = \int_{\Delta\omega_0} \frac{\left(\frac{\omega_c^e}{\omega_0}\right)^2 k_2^2\left[1+\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]}{\omega_1^2\omega_2^2\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2} |E_2|^2\,|E_1|^2\,\delta_{\omega,k}\,d\omega_0$$

$$I_2 = \int_{\Delta\omega_0} \frac{k_2^2\left(\frac{\omega_c^{e2}}{\omega_0\omega_2}+\frac{\omega_c^{e2}}{\omega_2^2}\right)^2\left(\frac{\omega_c^e}{\omega_1}\right)^2}{\omega_1^2\omega_2^2\left[1-\left(\frac{\omega_c^e}{\omega_0}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_2}\right)^2\right)^2} |E_2|^2\,|E_1|^2\,\delta_{\omega,k}\,d\omega_0$$

$$I_3 = \int_{\Delta\omega_0} \frac{\omega_0^2 k_1^2\left(\frac{\omega_c^e}{\omega_1}\right)^2}{\omega_1^4\omega_2^2\left[1-\left(\frac{\omega_c^e}{\omega_1}\right)^2\right]^2\left[1-\left(\frac{\omega_c^e}{\omega_2}\right)^2\right]^2} |E_2|^2\,|E_1|^2\,\delta_{\omega,k}\,d\omega_0 \tag{15}$$

Consequently the growth rate of the wave field squared, proportional to intensity of the harmonic radiation (frequency ω_0, wave-number k_0), can be expressed by

$$\left\langle\frac{d\,|E_0|^2}{dx}\right\rangle = V(I_1+I_2+I_3)\,|E_1|^2\,|E_2|^2 \tag{16}$$

In Fig. 4 the term $V(I_1+I_2+I_3)$ as a function of ω_p/ω_c^e is presented for two values of the cyclotron frequency ω_c^e (corresponding to static magnetic fields of 568 and 1137 Gauss respectively) and for two different bandwidths $\Delta\omega_0$. The pole at $\omega_p/\omega_c^e = \frac{4}{3}$ results from the fact that the solution of the dispersion relation (11) tends to $\omega_f \sim \omega_1 \sim \omega_2 \to \omega_c^e$.

The feasibility of substantial transfer of wave energy depends on the distance along the wave path over which the conditions of energy and momentum conservation (3) are sufficiently satisfied for a finite bandwidth. In order to establish a criterion for this we make use of the correlation length. Let us assume the energy conservation to be fulfilled for any deviation from ω_f within the bandwidth $\Delta\omega$. For the case of frequency doubling this can be formulated by

$$\delta\omega = \omega_0 - \omega_1 - \omega_2 = \omega_0 - \omega_f\left(1-\frac{\Delta\omega}{\omega_f}\right) - \omega_f\left(1+\frac{\Delta\omega}{\omega_f}\right) = 0 \tag{17}$$

According to the non-linear dispersion relation (10), however, the corresponding balance of wave numbers will deviate from zero for deviations of ω_1 and ω_2 from ω_f.

$$\delta k = k_0(\omega_0) - k_1(\omega_1) - k_2(\omega_2) \neq 0 \tag{18}$$

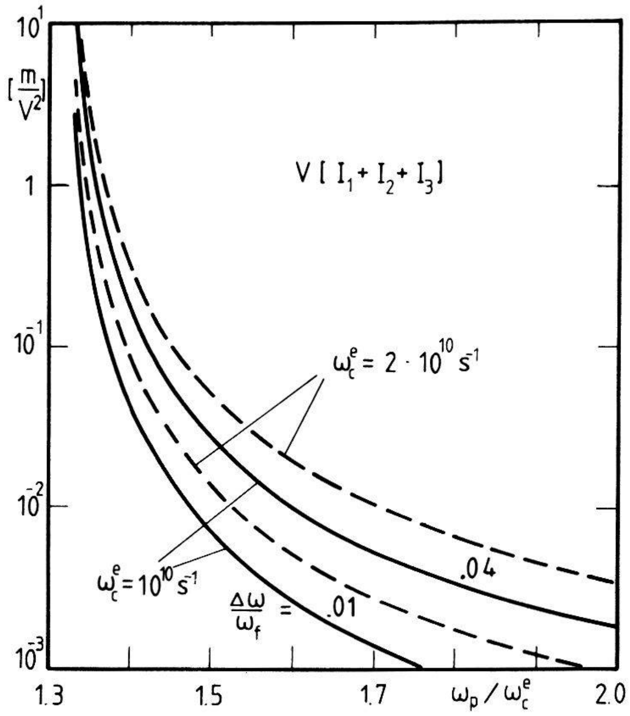

Figure 4
Growth rate of the wave field squared of the harmonic radiation, as expressed by the function $V(I_1 + I_2 + I_3)$ of equation (16), for different cyclotron frequencies and different bandwidths.

The correlation length ΔX_k can be defined by the reciprocal value of this deviation δk.

This definition

$$\Delta X_k \equiv \frac{1}{\delta k} = \left[2\pi\left(\frac{1}{\lambda_0(\omega_0)} - \frac{1}{\lambda_1(\omega_f - \Delta\omega)} - \frac{1}{\lambda_2(\omega_f + \Delta\omega)}\right)\right]^{-1} \tag{19}$$

will be sufficiently rigorous for a conservative estimation of the path length over which efficient wave interaction can take place.

In Fig. 5 the correlation lengths according to equation (19) are presented for extraordinary waves obeying dispersion relation (10) with ω_f computed from (11). The pole of ΔX_k at $\omega_p/\omega_c^e \approx 1.48$ is a consequence of an inflexion of the dispersion relation at this point. The increase of wave energy, $\langle \Delta |E_0|^2 \rangle$, at the harmonic frequency ω_0 over the correlation length ΔX_k follows from (16) as

$$\frac{\langle \Delta |E_0|^2 \rangle}{|E_1|^2 |E_2|^2} = V(I_1 + I_2 + I_3)\, \Delta X_k \tag{20}$$

when normalized to $|E_1|^2 |E_2|^2$. Figure 6 is a presentation of expression (20) as a function of the ratio of plasma to electron cyclotron frequency for two different bandwidths and the assumption $|E_1| = |E_2|$. In a real plasma the inhomogeneities will cause slight fluctuations of the value ω_p/ω_c^e, hence the extreme correlation lengths indicated by the poles in Fig. 5 cannot become effective. The density gradient in the vicinity of an active region may be assumed as $(1/n_0)\, dn_0/dx \ll 10^{-3}\,\mathrm{m}^{-1}$. The gradients of temperature and magnetic field will be larger but

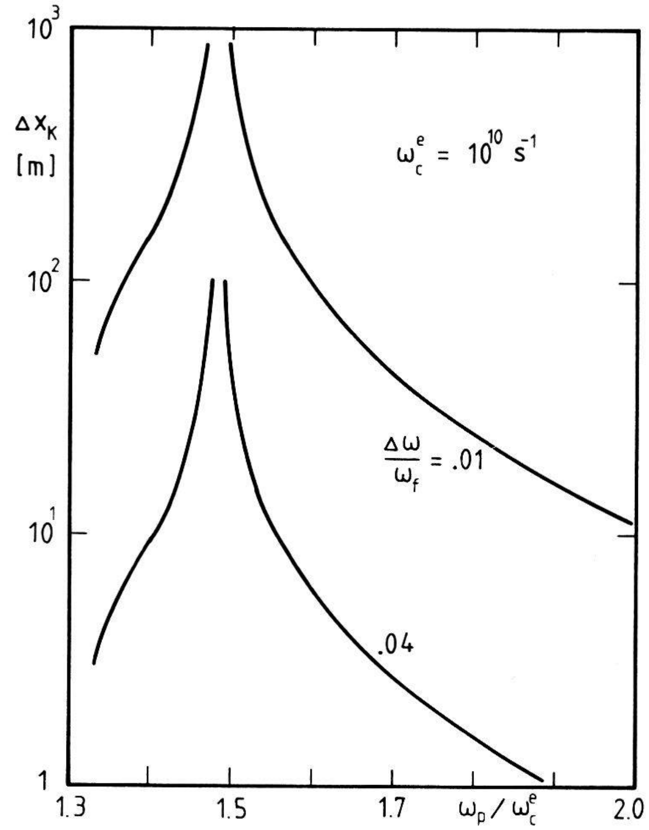

Figure 5
Correlation length ΔX_k as defined by equation (19) for $\omega_c^e = 10^{10}\,\mathrm{s}^{-1}$ and two bandwidths normalized to the fundamental frequency ω_f.

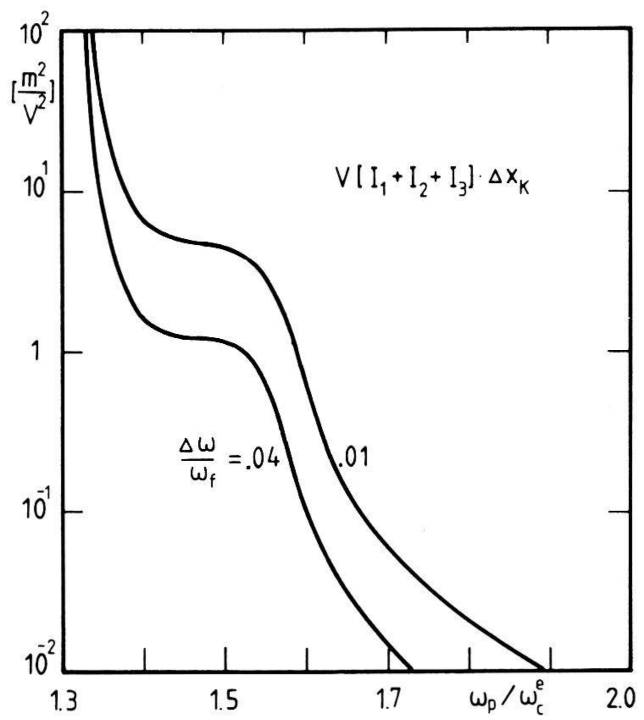

Figure 6
Change of wave field squared of the harmonic radiation normalized to the fourth power of the wave field of the fundamental radiation, as obtained over the correlation length ΔX_k.

certainly smaller than e.g. $(1/T)\,dT/dx \ll 10^{-2}\,\mathrm{m}^{-1}$. Therefore the wave dispersion does not vary substantially over a distance of e.g. 10^2 meter. For these reasons, in Fig. 6, the poles of the function (20) have been smoothed out with some extent of arbitrariness and the resulting shoulders have been traced in Fig. 6. The different values of the curves in Fig. 4 for different frequency ranges ($\omega_c^e = 10^{10}$ and $2 \cdot 10^{10}$) become compensated in (20) due to the different correlation lengths. Obviously, it is only due to the shape of the dispersion relation of extraordinary waves, i.e. the large correlation length for, say $\approx 1.4 < \omega_p/\omega_c^e < 1.55$, that for this wave mode a considerable coupling can take place. If the intensity of the primary (fundamental) radiation spectrum is sufficiently high (corresponding to $|E_1|^2 > 1(\mathrm{V/m})^2$) and if the radiation meets a plasma with ω_p/ω_c^e between 1.4 and 1.55 over a distance of at least its correlation length, an essentially complete transfer of wave energy from the fundamental to the harmonic can take place. The curve for the larger bandwidth is lower because the same amount of wave energy is spread over a wider frequency range, hence the energy per unit of bandwidth is lower.

5. Conditions for the magneto-ionic concept

In the previous section it has been shown that a cold collisionless magnetized plasma can support the parametric transfer of incoherent radiation from a sufficiently intense fundamental to the harmonic frequency in the extraordinary wave mode if the ratio of particle density to magnetic field is within narrow limits corresponding to $\frac{4}{3} < \omega_p/\omega_c^e < \frac{5}{3}$. In order to prove whether this process can become effective in a real plasma, various damping processes will be discussed in the next section.

One damping mechanism which is of kinetic nature, Landau-like wave-particle interaction, cannot become operative because of the extremely high phase velocity of the waves. Other damping mechanisms caused by particle collisions and by cyclotron motion can be treated within the fluid concept of plasma as we have already done for computing the non-linear wave interaction if the necessary conditions for this concept are satisfied:

First, the plasma parameter $g = (n_0 \lambda_D^3)^{-1}$ has to be sufficiently low in order to neglect all fluctuations upon the uniform hydrodynamic motions. Figure 7 shows the plasma parameter over ranges of densities and temperatures relevant to the present study. The strong (power $\frac{3}{2}$) dependence on temperature is evident. However, even for the lower, less important, temperature range the plasma parameter remains below 10^{-4}.

Second, the plasma beta $\beta = 2\mu_0\, n_0 \kappa T/B_0^2$ has to be sufficiently low in order to guarantee the particle motions to be dominated by the magnetic field over thermal motions. In Fig. 8 beta is presented for given ratios $\omega_p/\omega_c^e = 1.35$ and 2 corresponding to constant ratios n_0/B_0^2. Here obviously the higher temperature end of the considered range extends from the low β into the medium β domain but is still far below the high β values.

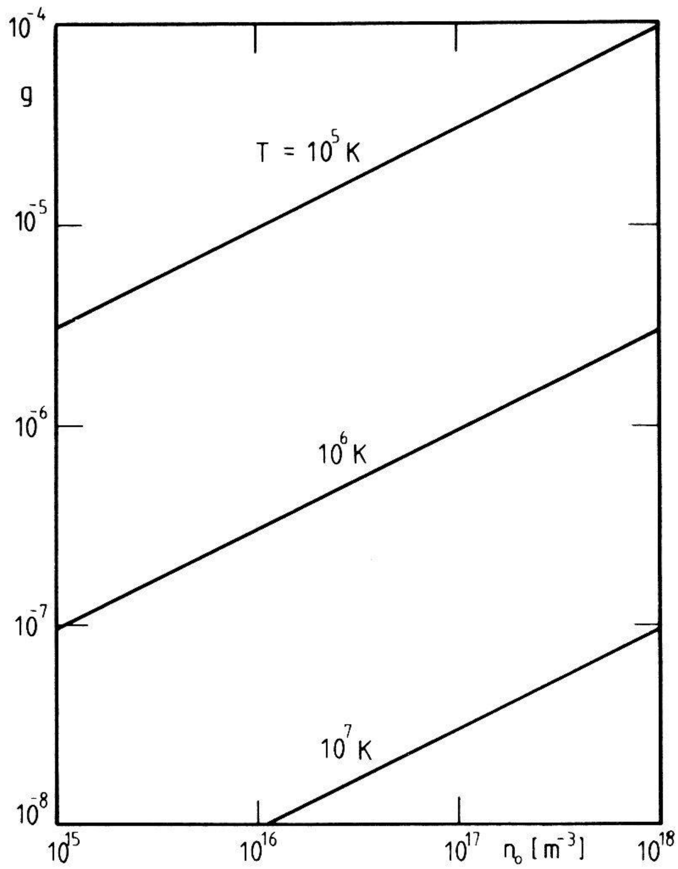

Figure 7
The plasma parameter $g = (n_0\lambda_D^3)^{-1}$ over the density and temperature range considered in this stydy.

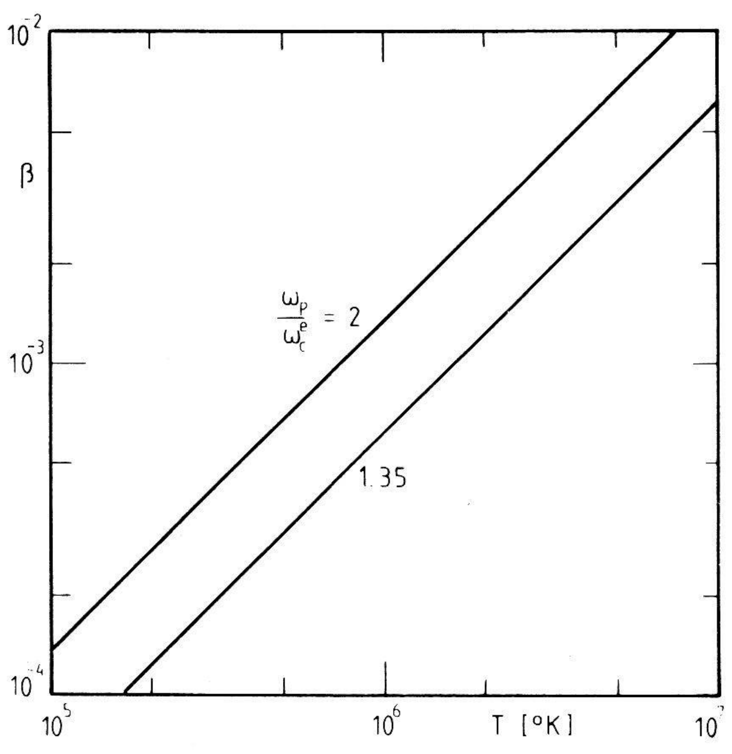

Figure 8
The plasma beta $\beta = 2\mu_0 n_0 \kappa T / B_0^2$ over a temperature range and for ratios n_0/B_0^2 as considered in this study.

From these considerations it seems well justified to use the fluid concept of plasma also for the discussion of damping processes.

6. Damping processes

(a) *The complex dispersion relation*

A very obvious damping mechanism counteracting the growth of the harmonic radiation is caused by the electron-electron collisions. We assume that the linear dispersion relation is still valid in a first-order consideration of the damping. The relation (10) has to be modified by introducing a complex frequency $\omega - i\nu$ where ν is the collision frequency according to

$$\nu \approx 0.13 \cdot 10^{-5} \frac{n_0}{T^{3/2}} \ln \Lambda \tag{20}$$

with electron density n_0 temperature T and the Coulomb logarithm $\ln \Lambda$. The reformulated dispersion relation becomes

$$(k' - ik'')^2 = \left(\frac{\omega - i\nu}{c}\right)^2 \left\{ 1 + \frac{(\omega_p/\omega_c^e)^2}{\dfrac{\omega_c^i}{\omega_c^e} - \left(\dfrac{\omega - i\nu}{\omega_c^e}\right)^2 \left[1 + \dfrac{1}{\left(\dfrac{\omega_p}{\omega_c^e}\right)^2 + \dfrac{\omega_c^i}{\omega_c^e} - \left(\dfrac{\omega - i\nu}{\omega_c^e}\right)^2} \right]} \right\} \tag{21}$$

with real k' and imaginary part k'' of the complex wave number.

In Fig. 9 lines of k' and k'' are presented as a function of the temperature T for fundamental and harmonic waves resulting from different choices of ω_c^e with $\omega_p/\omega_c^e = 1.35$. The steep decrease of k'' with increasing temperature causes only very weak damping in the expected parameter range. The real parts k' are essentially unaffected by the collision frequencies. The complex dispersion properties do not give rise to a wave damping stronger than the creation rate of the harmonic radiation if simultaneously the plasma temperature is $T \gtrsim 10^6$ K and the cyclotron frequency is $\omega_c^e \lesssim 4\,10^{10}\,\mathrm{s}^{-1}$.

(b) *Absorption by electron-ion encounters*

Electron-ion collisions lead to Bremsstrahlung; by the inverse process a photon of the radiation is absorbed and its energy goes into kinetic energy of the collision partners. The absorption coefficient of this process in the considered

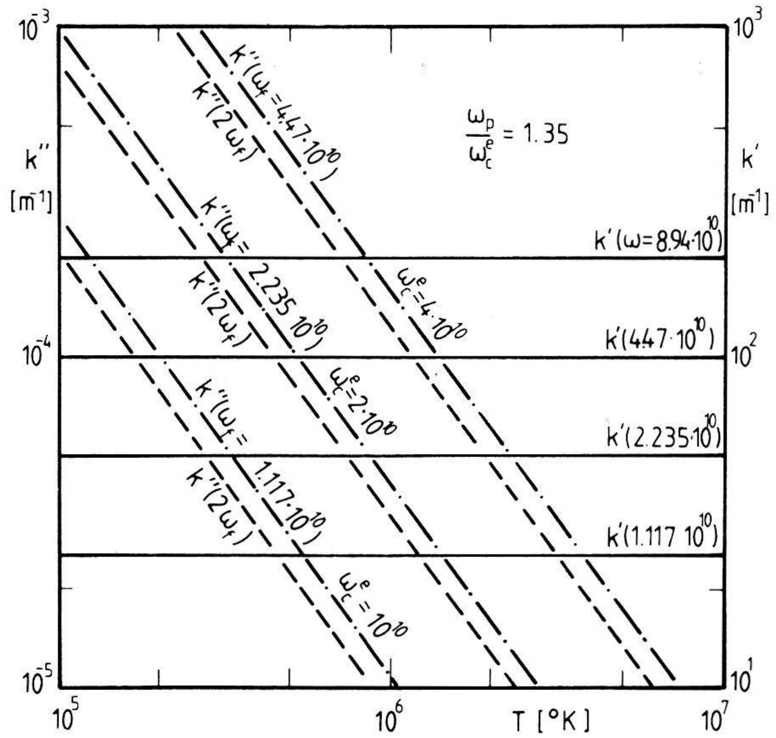

Figure 9
The real and imaginary parts of wave numbers corresponding to ω_f and $2\omega_f$ for different values of ω_c^e and fixed ratio $\omega_p/\omega_c^e = 1.35$.

range of parameters can be taken, see e.g. Bekefi (1966) p. 87, as

$$\alpha = 3.86\,10^{-11} \cdot \frac{n_0^2}{\omega^2 T^{3/2}} \left[19.56 + \ln \frac{T^{3/2}}{\omega} \right] \tag{22}$$

in units of meter^{-1}. A pure electron-proton plasma is supposed with density n_0 of each species. In Fig. 10 the absorption coefficient according to equation (22) is presented, for two plasma temperatures, as a function of frequency over a range relevant for this study. Again for $T \gtrsim 10^6$ K the absorption coefficient remains far below the creation rate of harmonic radiation.

(c) *The spectral width of the cyclotron-resonance*

The absorption process by cyclotron-resonance has two aspects which will now be discussed separately. From equation (11) and Fig. 2 can be recognized that the fundamental frequency is higher than the gyrofrequency by about 10–50% over a useful range of ω_p/ω_c^e from – say – 1.35 to 1.7. The harmonic $2\omega_f$ is higher than $2\omega_c^e$ by the same percentage, i.e. by twice as much difference in frequency. Therefore instantaneous absorption of the radiation in the region of the non-linear interaction process is only possible if the width of the gyro-resonance absorption line is very broad.

For wave propagation perpendicular to the magnetic field as supposed for extraordinary waves, there may essentially no Doppler broadening be expected. The dominant line broadening mechanism will be collisional broadening. The

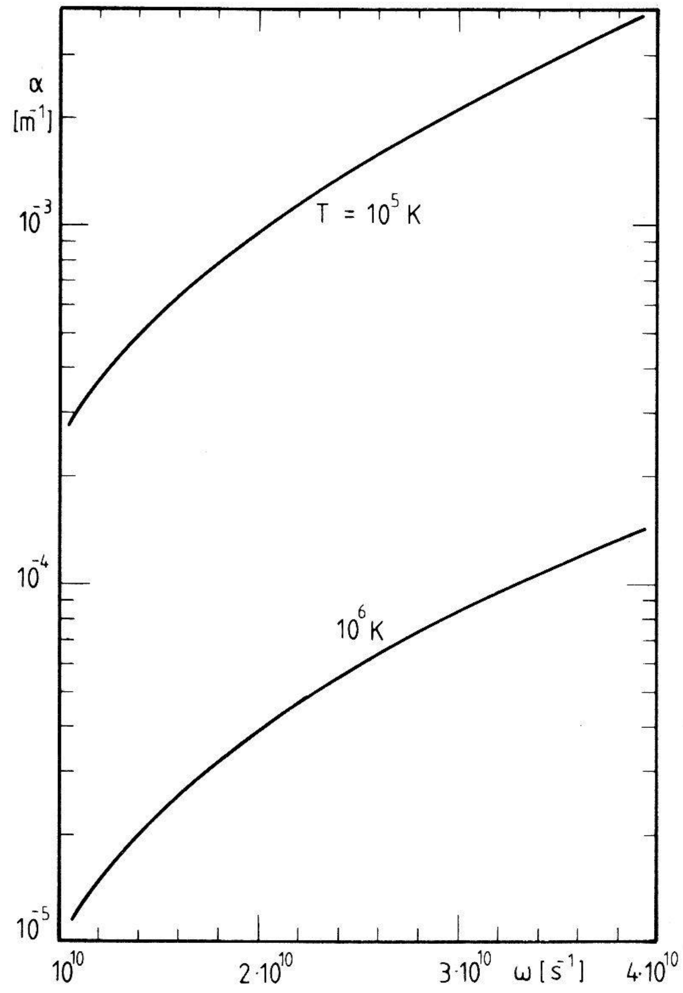

Figure 10
Absorption coefficient of inverse Bremsstrahlung according to equation (22).

absorption coefficient can be derived for propagation perpendicular to the magnetic field (Bekefi 1966, p. 186), as

$$\alpha(\omega) = \frac{1}{2}\left(\frac{\omega_p}{\omega_c^e}\right)^2 \frac{\nu/c}{\left(\frac{\omega}{\omega_c^e} - 1\right)^2 + \left(\frac{\nu}{\omega_c^e}\right)^2} \tag{23}$$

where again ν is the collision frequency and it determines the width of the gyro-resonance line. Figure 11 shows the absorption coefficient according to equation (23) as a function of the ratio ω_p/ω_c^e. Along the lines the values $\omega = \omega_f(\omega_p/\omega_c^e)$ and $\nu(\omega_p/\omega_c^e)$ have been continuously computed as a function of ω_p/ω_c^e. Magnetic field (ω_c^e) and temperature of the plasma are fixed parameters of the curves. For $\omega_c^e = 10^{10}\,\mathrm{s}^{-1}$, $T = 10^6\,\mathrm{K}$ the absorption of a wide band radiation spectrum about the fundamental ω_f is considered. The absorption coefficient at the harmonic frequency caused by the linewidth at ω_c^e is more than one order of magnitude smaller than at the fundamental frequency. Within the considered range of temperature and magnetic fields one may expect the absorption to be sufficiently low not to suppress the creation of the harmonic radiation.

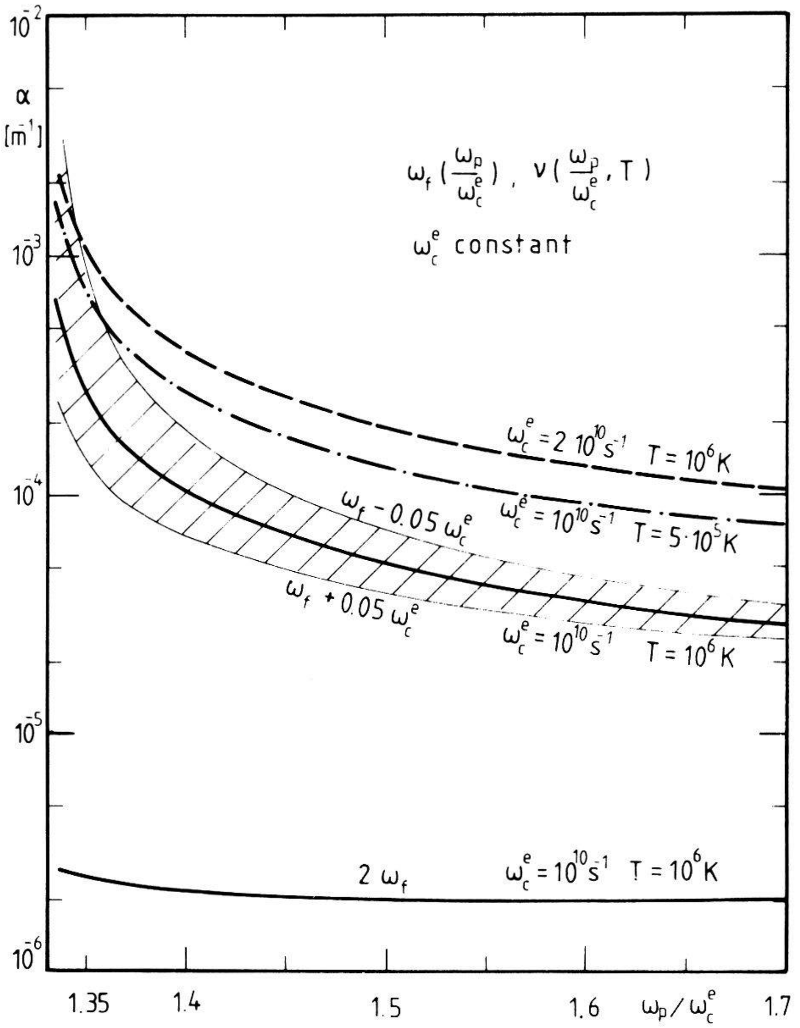

Figure 11
Absorption coefficient of fundamental and harmonic waves due to the finite width of the cyclotron-resonance. The effect on a bandwidth of $\pm 0.05\omega_c^e$ about ω_f is indicated for $\omega_c^e = 10^{10}\,\mathrm{s}^{-1}$, $T = 10^6\,\mathrm{K}$.

(d) *Harmonics of the cyclotron-frequency*

The second effect of the cyclotron-resonance is the one which is exerted by the harmonics of the cyclotron-frequency. On the passage of the radiation out of the interaction region through decreasing magnetic field it is traversing zones where either the fundamental or the harmonic wave frequency coincide with harmonics of the local cyclotron-frequency. In a cold plasma considered here the linewidth of the cyclotron-resonance is very narrow. Dependent on the spatial gradient of the magnetic field, the resonant zones along the wave propagation path may be expected extremely short. However, in the first few of these resonant zones corresponding to the first few harmonics of the gyrofrequency in successive order, the abosrption may be very strong.

A numerical study has been made by applying the formulation due to Bekefi (1966), p. 201. The absorption coefficient of extraordinary waves by harmonic number m of the local cyclotron-frequency, wave propagation perpendicularly to the magnetic field, is

$$\alpha_m = \frac{\omega_p}{\omega_c c}\sqrt{2\pi}\,\frac{m^3\sqrt{m^2-x^2}}{(\beta x)^5}\,\frac{(m\beta)^{2m}}{(2m+1)!}\exp\left[-\left(\frac{m}{x}-1\right)\Big/\beta^2\right] \tag{24}$$

Maxwellian distribution of particle velocities in the plasma is supposed, therefore the normalized r.m.s. velocity $\sqrt{\bar{v}^2}/c = \sqrt{\kappa T/m_0}/c$ can be used for β; here m_0 is the restmass of the electron and κ is Boltzmann's constant. The ratio $\omega/\omega_c = x$ relates the wave frequency to the local cyclotron-frequency ω_c, which is assumed different (smaller) than the cyclotron-frequency ω_c^e in the non-linear interaction region. The absorption coefficient around the combinations: $x = \omega_f/\omega_c = 2, 3, 4$ and $x = 2\omega_f/\omega_c = 3, 4$ with $m = 2, 3, 4$ and $m = 3, 4$ respectively, has been computed. These combinations involve the lowest harmonics of the local cyclotron-frequency which can coincide with the respective waves on their way into lower magnetic field. Figure 12 shows the results with the highest values of

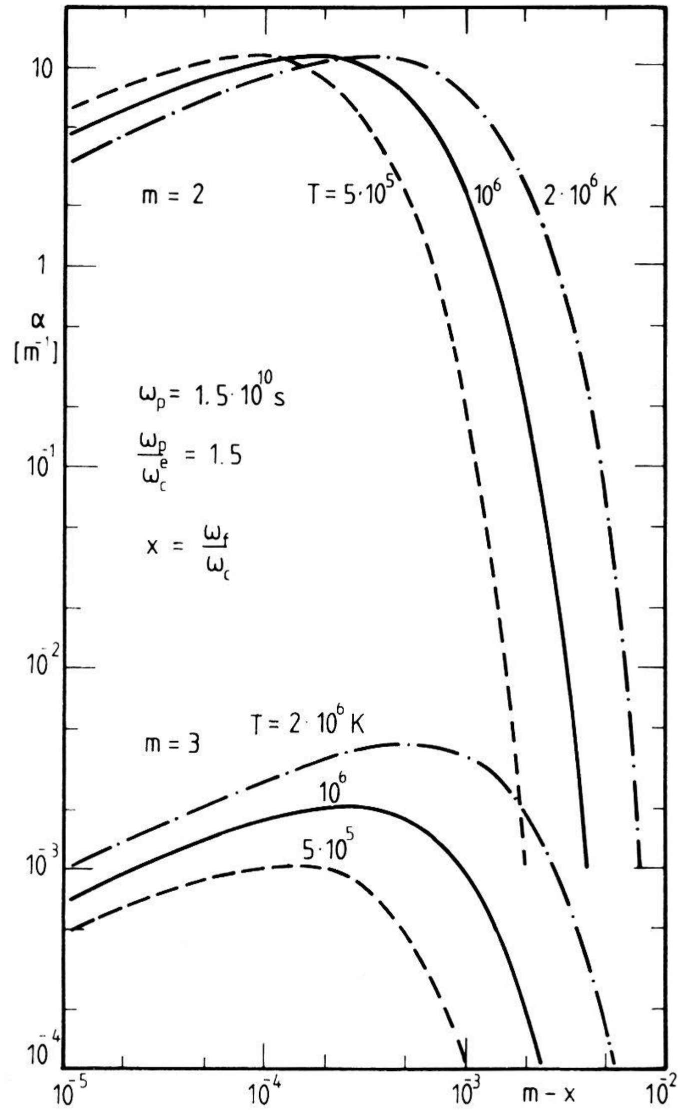

Figure 12
The absorption coefficient at about the fundamental frequency caused by the second and third harmonics of the local cyclotron frequencies ω_c. The variable on the abscissa is the difference between harmonic number of ω_c and ω_f/ω_c.

the absorption coefficient as a function of the deviation from exact resonance, $m-x$. The highest absorption coefficient met by the wave harmonic $2\omega_f$ ($x=2\omega_f/\omega_c\approx 3$, $m\approx 3$) resembles very much the lines of $x=\omega_f/\omega_c\approx 3$, $m\approx 3$ except for the fact that the absolute value is still lower by a factor of 2.

The absorption of the fundamental wave (ω_f) by the harmonic of the local cyclotron-frequency ($2\omega_c$) can indeed be very high and can even suppress this wave, i.e. transform it into thermal energy, if the resonance condition is fulfilled within one promille over a distance of a few meters. The spectral width over which this condition has to be satisfied in order to cause absorption is extremely narrow at the low temperatures considered in this study. For example a relative gradient of the magnetic field of $1/B\, dB/dx\approx 10^{-3}\,\mathrm{m}^{-1}$ is sufficient for avoiding severe absorption of the fundamental wave. The harmonic wave does not suffer at all from cyclotron-resonance absorption because for $T=10^6$ K the absorption coefficients reaches the value $10^{-3}\,\mathrm{m}^{-1}$ only within $m-x<4\cdot 10^{-4}$.

7. Conclusions

It has been shown that for incoherent radiation propagating in the extraordinary mode through a cold, low-collision, magnetized plasma, the creation of harmonic radiation by a parametric wave interaction is feasible. A narrow range of the ratio electron density to magnetic field corresponding to $1.4\lesssim \omega_p/\omega_c^e\lesssim 1.55$ can be admitted in order to simultaneously satisfy the necessary conditions of energy and momentum conservation and to yield sufficient transfer of wave energy from the fundamental to the harmonic wave. The frequency range, hence ranges of density and magnetic field as well as the range of temperatures considered here, have been taken from the microwave observations of the solar corona. From there some evidence of occasional appearance of flare radiation with a 1:2 frequency ratio has been reported. Therefore the same range of physical parameters has been taken in order to investigate damping mechansims which may counteract or even suppress the process of parametric creation of harmonic radiation. The cyclotron-resonance absorption of the waves, when propagating as extraordinary waves perpendicular to the direction of magnetic field of decreasing strength, is limited to extremely narrow frequency bands ($\Delta\omega/\omega\lesssim 0.001$) and – as a consequence of the assumed field gradient – to extremely short spatial zones. Only the fundamental wave may seriously suffer from this absorption. However, irregularities in strength and direction of the magnetic field in a natural plasma may reduce this absorption process; over the considerable distance between the region of the non-linear interaction and the absorbing region the direction of the magnetic field may have changed sufficiently as to let escape part of the radiation in the ordinary mode.

In most of the natural plasmas observational evidence of creation of harmonic radiation by the considered process will be difficult to prove due to other spectral effects. However, high intensity radiation may undergo spectral redistribution due to this parametric wave interaction.

The author considers three topics still to be studied more deeply in the near future:

- More cases of evidence of harmonic radiation from solar flares and, if possible, from plasmas with different physical parameters have to be searched for.
- The present analysis has to be extended to different ranges of the physical parameters in order to determine the limits of its applicability.
- The present and the previous study (Schanda 1987) consider strictly extraordinary waves propagating exactly perpendicularly to the magnetic field. The extension to situations, where the directions of polarization and of propagation deviate from these limiting assumption, have to be realized for the parametric interaction as well as for the damping processes.

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