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A variational proof of the existence of a bound state in a relativistic quantum model with weak coupling

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Abstract. The two-particle bound states of $\mathcal{P}(\varphi)_2$ models with weak coupling are studied using a new method of variational type proposed by Glimm, Jaffe and Spencer. It proves the existence of such states if the Schrödinger equation of the non-relativistic limit has bound states. The binding energies found are the same as those obtained by the Bethe–Salpeter equation. Moreover zero-time approximations to eigenspaces are given. In the present work, we limit ourselves to the case where the interaction polynomial $\mathcal{P}$ is even and has a non-zero fourth degree monomial.

Introduction

The problem of finding the bound states of a Quantum Field Model is generally studied by searching for poles in functions occurring in the Bethe–Salpeter equation; this program is well presented in [a]. In the framework of the Constructive Field Theory, this method has been discussed by [b] and studied by [c] and [d]. It has been successfully applied to the $\mathcal{P}(\varphi)_2$ models with weak coupling [e].

In a lecture at Erice (1973), Glimm, Jaffe and Spencer [b, p. 175–7] proposed another way of finding bound states, combining variational methods with perturbation calculations (called here the 'variational perturbation method'). It was developed by Perreaux [f], who showed that it leads to a solution of a Schrödinger equation; then there is a bound state if the non relativistic limit of the studied model has one. We present here an extended version of this work, already announced in the literature [g].

In a first paper, we treat the case of the two-particle bound states for a $\mathcal{P}(\varphi)_2$ model where $\mathcal{P}$ is even and has a non zero fourth degree, i.e.:

$$\mathcal{P}(x) = \sum_{n=0}^{\mathcal{N}} a_{2n} x^{2n}, \qquad a_4 \neq 0$$

with $\mathcal{N} \in \mathbb{N}$, $\mathcal{N} \geqslant 2$, $a_n \in \mathbb{R}\ \forall n$ and $a_{2\mathcal{N}} > 0$. Because a_0 will play no role, we can choose it such that $\mathcal{P}$ is positive.

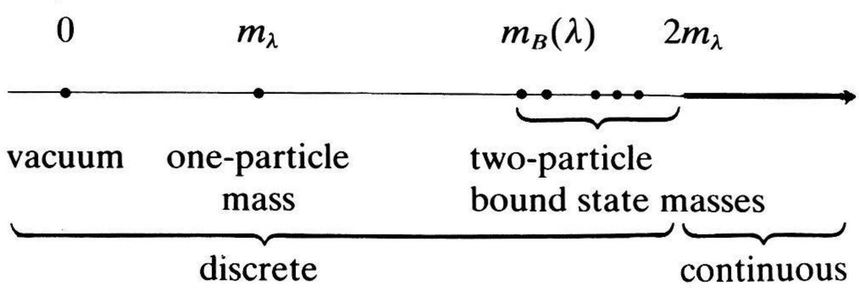
In §1 we present the variational perturbation method, and in §2, as application, the main result of this paper (§2.1 describes the $\mathscr{P}(\varphi)_2$ models; §2.2 presents a quantum mechanical model; §2.3 gives the Theorem, which connects some Rayleigh quotients of two different models; §2.4 comments on the Theorem and §2.5 deduces from it a criterion for the existence of bound states). The proof of the Theorem lies in §3 (§3.1 gives an outline of the proof, which presents the main ideas; it contains three Propositions, whose proofs, more technical, are in §3.2).

§1. The variational perturbation method

Let us describe the background in which this method works.

The weakly coupled $\mathscr{P}(\varphi)_2$ theory associates to any positive polynomial $\mathscr{P}$ and parameters $m_0 > 0$ and $\lambda \geq 0$, with λ/m_0^2 sufficiently small, a Wightman model of state space $\mathscr{H}_\lambda$, hamiltonian H_λ, momentum $\vec{P}_\lambda$, vacuum Ω_λ and field φ_λ ($\mathscr{P}$ and m_0 will be fixed and λ will move; therefore there is a λ index everywhere).

We know by [d] that for even $\mathscr{P}$, the spectrum of the mass operator M_λ, given by $M_\lambda^2 = H_\lambda^2 - \vec{P}_\lambda^2$, is:



with: $m_\lambda \to m_0$ and $m_B(\lambda) \to 2m_0$ if $\lambda \to 0$
and: $2m_\lambda$ is not an eigenvalue, for all $\lambda \geq 0$

The spectrum between m_λ and $2m_\lambda$ is discrete or empty, depending on $\mathscr{P}$, and the eigenvalues are interpreted as the masses of two-particle bound states.

Here we are interested in the mass $m_B(\lambda)$ of the 'biboson', the lightest bound state (if it exists) and we shall calculate it by perturbation expansion in λ near $\lambda = 0$.

The ordinary perturbation theory of an eigenvalue cannot be used here, because of two difficulties:

1° for all values of λ, the operator M_λ acts on another Hilbert space $\mathscr{H}_\lambda$
2° the 0th order is singular, because there is no eigenvalue if $\lambda = 0$.

The variational perturbation method is adapted to this situation. If bound states exist, m_B will be calculated with the formula:

$$m_B(\lambda)^2 = \inf_{0 \neq \psi \in \mathscr{D}_\lambda} \frac{(\psi \mid M_\lambda^2 \psi)_{\mathscr{H}_\lambda}}{\|\psi\|^2_{\mathscr{H}_\lambda}} \qquad \lambda \geq 0$$

where $\mathscr{D}_\lambda$ is the intersection of the domain of M_λ with the subspace of $\mathscr{H}_\lambda$ perpendicular to the vacuum and the one-particle states.

The minimized quantity is the Rayleigh quotient of ψ for M_λ^2, and will be noted as $RQ_\lambda(\psi)$. It has the interesting property that, for each $\psi \in \mathscr{D}_\lambda$, it is an upper bound for $m_B(\lambda)^2$.

One of the difficulties in the calculation of the minimum comes from the too large freedom of choice in $\mathscr{D}_\lambda$; Glimm, Jaffe and Spencer [b] propose replacing $\mathscr{D}_\lambda$ by the set of zero-time vectors in $\mathscr{D}_\lambda$, which we will note as $\mathscr{D}_\lambda^0$. They prove that this is not a restriction (another proof of this fact will be given in [h]).

The perturbation calculation of $RQ_\lambda(\psi_\lambda)$, where ψ_λ is a vector of $\mathscr{D}_\lambda^0$ for each $\lambda \geq 0$, begins with $RQ_0(\psi_0)$, which is greater than $(2m_0)^2$ for all ψ_0. The others terms of the expansion can be negative, but are smaller (they are $0(\lambda)$). To approach the minimum, we must compromise. Glimm, Jaffe and Spencer [b] propose introducing the fact that we are looking for bound states. For a two-particle bound state, there exists a typical compact in the relative configuration space, in which the wave function is essentially localized for all time (see Ruelle's definition of bound states in [i]).

Suppose now that $\mathscr{H}_\lambda$, for $\lambda > 0$, has a bound state and denote by $\delta(\lambda)^{-1}$ the volume of the characteristic compact. As $\lambda \to 0$, we have seen that $m_B(\lambda)$ goes continuously to $2m_0$, which is not an eigenvalue; the localization of the bound state must then disappear, and we expect that $\delta(\lambda) \to 0$. In what follows, this continuity property of δ will be assumed as an ansatz. Glimm, Jaffe and Spencer [b] propose to introduce the parameter $\delta(\lambda)$ by scaling some well chosen vectors of $\mathscr{D}_\lambda^0$, such that the first term of the perturbation calculation behaves as:

$$RQ_\lambda(\psi_{\lambda,\delta})\big|_{\lambda=0} = (2m_0)^2 + 0(\delta)$$

With a good choice of $\delta(\lambda)$, the $0(\delta)$ term (which is positive) can be cancelled by a negative part of the perturbation serie. In this way, we can hope to approach the minimum of $RQ_\lambda(\psi)$.

The choice of scaled vectors and the choice of the continuity of $\delta(\lambda)$ have the unpleasant consequence that we do not minimize on a dense set, like $\mathscr{D}_\lambda^0$; thus the result can only be an upper bound for $m_B(\lambda)^2$. However, if this result is smaller than $(2m_\lambda)^2$, then M_λ^2 has some spectrum below $(2m_\lambda)^2$ and, given the knowledge of the spectrum pointed out above, we see that bound states exist.

We resume here the discussion of the drawbacks of this method:

1° Some information about the spectrum of M_λ is necessary (in fact, we must know that the spectrum below $2m_\lambda$ is discrete); thus the Bethe–Salpeter method is needed for this point (but only for it). For this work, we use [d].

2° We obtain only an upper bound for $m_B(\lambda)$.

§2. The main result

First we present two classes of models, then the central Theorem, followed by a Corollary.

2.1. *The $\mathscr{P}(\varphi)_2$ models with weak coupling*

These models are non-trivial examples of Wightman's quantum field theory. Physically, they describe a relativistic quantum world populated with spinless, chargeless, massive, all identical bosons living in a space of one dimension.

The physical units are chosen such that $c = \hbar = 1$.

2.1.1. *The definition.* For $m_0 > 0$, $\lambda \geqslant 0$ and a positive polynomial $\mathscr{P}$, we consider the probability space (Q, Σ, μ_λ), where $Q = \mathscr{S}'(\mathbb{R}^2)$, Σ is the borelian σ-algebra of Q (given the weak topology) and μ_λ is a measure on Σ defined by:

$$\mu_\lambda(B) = \lim_{\Lambda \to \mathbb{R}^2} \lim_{g \to \delta} \mu_{\lambda,\Lambda,g}(B)$$

for all $B \in \Sigma$, with:

$$d\mu_{\lambda,\Lambda,g}(q) = \frac{1}{Z_{\lambda,\Lambda,g}} \exp\left(-\lambda \int_\Lambda d^2x :\mathscr{P}(q(x \cdot g)):\right) d\mu_0(q)$$

for all $q \in Q$, where μ_0 is the measure such that

$$\int_Q d\mu_0(q) \exp(iq(f)) = \exp -\frac{1}{2}\left(f \,\middle|\, \frac{1}{-\Delta + m_0^2} f\right)_{L^2}$$

for all $f \in \mathscr{S}(\mathbb{R}^2)$; $g \in \mathscr{S}(\mathbb{R}^2)$ and $x \cdot g(y) = g(y - x)$ for all $x, y \in \mathbb{R}^2$; :: denote the Wick polynomials and $Z_{\lambda,\Lambda,g}$ is the normalisation factor.

The two limits must be taken in the following sense:

$g \to \delta$ (the '*UV* limit') in $\mathscr{S}'(\mathbb{R}^2)$ (δ is the Dirac distribution)
$\Lambda \to \mathbb{R}^2$ (the 'thermodynamic limit'): we replace Λ by $\Lambda_n = B(0, n)$, the ball in $\mathbb{R}^2$ of centrum 0 and radius n, and we take $n \to \infty$

These two limits exist provided λ/m_0^2 is sufficiently small (see [j] for the *UV* limit, [k] for the thermodynamic limit; the formulation involving limits of measures is given in [l]).

We will keep m_0 and $\mathscr{P}$ constant, and we will move λ; we will denote by $\underline{\lambda}$ (depending on m_0 and $\mathscr{P}$) the maximal, positive value of λ for which the thermodynamic limit exists. From now on, we suppose that $\lambda \in [0, \underline{\lambda}]$.

The measure μ_λ obtained in this way satisfies all 'probabilistic axioms for quantum field theory' (see [l]). Thus, having chosen an euclidean time axis, we can construct a model of the Wightman theory, with state space $\mathscr{H}_\lambda$, hamiltonian H_λ, momentum $\vec{P}_\lambda$, vacuum Ω_λ, field φ_λ, such that its Schwinger distributions $S_{n,\lambda}$ are the moments of μ_λ, ([l]).

Central in this construction is a bounded, dense-range, operator $W_\lambda : L^2(Q, \mu_\lambda) \to \mathscr{H}_\lambda$ which we will use.

These models are called '$\mathscr{P}(\varphi)_2$ models'; they are the most simple examples of Wightman theory for which the scattering is not trivial ([m], [n]).

2.1.2. *Zero-time Wick vectors.* For all $f \in \mathcal{S}(\mathbb{R}^2)$ let us consider the random variable:

$$\phi_f : Q \ni q \to q(f)$$

ϕ_f belongs to $L^p(Q, \mu_\lambda)$ for all $1 \leq p < \infty$. Let us choose an euclidean time axis: each $x \in \mathbb{R}^2$ is written as $(\mathring{x}, \vec{x})$, where $\mathring{x}$ is an euclidean time and $\vec{x}$ is the component with respect to a perpendicular axis. It is possible to make sense of ϕ_f for $f(x) = \delta(\mathring{x})g(\vec{x})$, with $g = \mathcal{S}(\mathbb{R})$, in $L^2(Q, \mu_\lambda)$ for each λ (see [o]). This random variable will be denoted by $\phi_\lambda(g)$. For all $n \in \mathbb{N}$ we can also define ϕ^n_λ, formally given by:

$$\phi^n_\lambda(h) = \int_{\mathbb{R}^h} d\vec{x}_1 \cdots d\vec{x}_n h(\vec{x}_1, \ldots, \vec{x}_n) : \phi_\lambda(\vec{x}_1) \cdots \phi_\lambda(\vec{x}_n): \qquad h \in \mathcal{S}(\mathbb{R}^n)$$

where $\phi_\lambda(\vec{x})$ stands for $\phi_\lambda(\vec{x} \cdot \delta)$. Applying W_λ give vectors of $\mathcal{H}_\lambda : W_\lambda \phi^n_\lambda(h)$, called here 'zero-time Wick vectors'. It is proven in [o] that these vectors lie in the domain of H_λ, and that for two of them ξ_λ, η_λ, the scalar products: $(\xi_\lambda \mid \eta_\lambda)_{\mathcal{H}_\lambda}$ an $(\xi_\lambda \mid H^2_\lambda \eta_\lambda)_{\mathcal{H}_\lambda}$, as functions of λ, are $C^\infty([0, \underline{\lambda}])$, provided that their test functions belong to some normed spaces $\mathcal{B}p_{n,2}$ (for suitable $n \in \mathbb{N}$), containing $\mathcal{S}(\mathbb{R}^n)$.

2.1.3. *Particle structure.* If $E_\lambda(h, \vec{p})$ is the conjointly spectral measure of H_λ and $\vec{P}_\lambda$, the mass operator is:

$$M_\lambda = \int_{V_+} \sqrt{h^2 - \vec{p}^2}\, dE_\lambda(h, \vec{p})$$

where $V_+ = \{(h, \vec{p}) \in \mathbb{R}^2 \mid h \geq 0, h^2 - \vec{p}^2 \geq 0\}$ is the support of $dE_\lambda(h, \vec{p})$.

The domain of M_λ, called $D_\lambda(M_\lambda)$, is that one for which the integral exists. M_λ is a positive, self adjoint operator.

The spectrum of M_λ, already discussed in §1, is composed of a discrete part (from 0 to $2m_\lambda$, not inclusive) and a continuous part (from $2m_\lambda$, inclusive). The eigenspaces of the discrete spectrum are (E_λ is now the spectral measure for M_λ):

$E_\lambda(0)\mathcal{H}_\lambda = \{c\Omega_\lambda \mid c \in \mathbb{C}\}$ = the vacuum
$E_\lambda(m_\lambda)\mathcal{H}_\lambda$, which carries an irreductible representation of the Poincaré group, identified with the one-particle states, of mass m_λ

and eventually:

$E_\lambda(m_B(\lambda))\mathcal{H}_\lambda$: the 'two'-particle bound state subspace.

(we do not present more than one bound state, which never occur in the $\mathcal{P}(\varphi)_2$ models, see [e]).

We will use the fact that $\lambda \to m_\lambda$ is $C^\infty([0, \underline{\lambda}])$, proved in [n].

2.1.4. *Even models.* In the case where the interaction polynomial $\mathcal{P}$ is even, the space states can be written as a direct sum: $\mathcal{H}_\lambda = \mathcal{H}^e_\lambda \oplus \mathcal{H}^o_\lambda$, where $\mathcal{H}^e_\lambda$ (and

$\mathcal{H}_\lambda^o$) is spanned by the $W_\lambda(\phi(f))^n$ with n even (n odd) and describe the states of an even number of particles (odd number, respectively). For the proofs, see ([o, §3.3]).

2.2. *The non-relativistic model*

We consider a quantum, non relativistic model describing two particles of mass m_0, moving in a one-dimensional space. The state space is $\mathcal{H} = L^2(\mathbb{R}^2)$ and the hamiltonian is given by the formal expression:

$$H^{NR}\left(\frac{\lambda}{m_0^2}\right) = \frac{1}{2m_0}(\vec{p}_1^2 + \vec{p}_2^2) + 3\lambda\frac{a_4}{m_0^2}\delta(\vec{x}_1 - \vec{x}_2)$$

where $\vec{p}_j = (1/i)\partial_{\vec{x}_j}$ and $\vec{x}_j$ is position variable of the jth particle, $j = 1$ or 2. Recall that $a_4 \neq 0$ by hypothesis. Here, $\lambda \geqslant 0$ must not necessarily be small. λ/m_0^2 is a dimensionless quantity. This formal hamiltonian can be given a precise meaning as a self-adjoint operator [p].

The spectrum of $H_{\text{rel}}^{NR}(\lambda/m_0^2)$, the hamiltonian for the relative problem, acting on $L^2(\mathbb{R})$, is the following, for all $\lambda > 0$:

$$\{\mu \mid \mu \geq 0\} \quad \text{if} \quad a_4 > 0$$

$$\{-E\} \cup \{\mu \mid \mu \geq 0\} \quad \text{if} \quad a_4 < 0$$

where $E = \alpha^2/m_0$ and $\alpha = -3a_4\lambda/2m_0$.

In the last case, the eigenspace corresponding to $-E$ has dimension one and is spanned by f given by:

$$f(\vec{x}) = \sqrt{\alpha}\, e^{-|\vec{x}|\alpha} \quad \text{or} \quad \tilde{f}(\vec{p}) = \sqrt{\frac{2\pi}{\alpha}}\frac{1}{\left(\frac{\vec{p}}{\alpha}\right)^2 + 1}$$

Note that f spreads out as $\lambda \to 0$.

2.3. *The theorem*

There is a relation between the two models of §2.1 and §2.2. Dimock [q] has proven that the two-particle scattering amplitude and the two-particle binding energies, for a $\mathcal{P}(\varphi)_2$ model with $\mathcal{P}$ even, converge as the velocity of light goes to infinity to the corresponding objects for the non-relativistic model of §2.2. Moreover, the binding energy of the two-particle bound state of the $\mathcal{P}(\varphi)_2$ model calculated at first perturbation order by [e] and that of the non relativistic model are the same.

The variational perturbation method gives another connection: there exist zero-time vectors which relate Rayleigh quotients of the two models, at first order of perturbation. Let us define, for $f \in L^1(\mathbb{R}^2)$, $f \neq 0$, the quotient:

$$q_1(f) = \frac{\int d\vec{P}\, d\vec{p}\, |\tilde{f}_K(\vec{P}, \vec{p})|^2\, \omega(\vec{p})^{5/2}\omega(\vec{P})^3}{m_0^{11/2}\, \|f\|_{L^2}^2}$$

Table I
Vectors appearing in the theorem

We use the notations of §2.1.
For $f \in \mathscr{B}p_{2,2}$ and $\lambda \in (0, \underline{\lambda}]$:

$$\psi_\lambda(f) = (1 - E_\lambda(0))W_\lambda\left(\phi_\lambda^2(g) + \lambda \sum_{n=2}^{\mathcal{N}} \phi_\lambda^{2n}(f_{2n})\right)$$

where

$$\tilde{f}_n(\vec{k}_1, \ldots, \vec{k}_n) = -\frac{a_n}{(2\pi)^{n-1}} \sum_{j=1}^{n} \frac{\tilde{f}\left(\vec{k}_j, \sum_{i\neq j} \vec{k}_i\right) + \tilde{f}\left(\sum_{i\neq j} \vec{k}_i, \vec{k}_j\right)}{\left(\sum_{i\neq j} \omega(\vec{k}_i)\right)^2 - \left(\sum_{i\neq j} \vec{k}_i\right)^2 - m_0^2}$$

and $g = U(\lambda/m_0^2)f$. The action of the scaling operator $U(\delta)$, $\delta \in (0, 1]$, is:

$$\widetilde{U(\delta)f}(\vec{k}_1, \vec{k}_2) = \sqrt{\frac{\Omega_\lambda(\vec{P}, \vec{p})}{\delta\omega_\lambda(\vec{p})}}\, \tilde{f}_K\left(\vec{P}, \frac{1}{\delta}\vec{p}\right) \qquad \forall \vec{k}_1, \vec{k}_2 \in \mathbb{R}$$

where $\vec{P}, \vec{p}$ are related to $\vec{k}_1, \vec{k}_2$ by:

$\vec{P} = \vec{k}_1 + \vec{k}_2$

$\vec{p}^2 = \frac{1}{4}((\vec{k}_1 - \vec{k}_2)^2 - (\omega_\lambda(\vec{k}_1) - \omega_\lambda(\vec{k}_2))^2)$

sign of $\vec{p}$ = sign of $\vec{k}_1 - \vec{k}_2$

f_K is related to f by the Kepler change of variables

$$f_K((\vec{x}_1 + \vec{x}_2)/2, \vec{x}_1 - \vec{x}_2) = (f(\vec{x}_1, \vec{x}_2) + f(\vec{x}_2, \vec{x}_1))/2$$

The above definitions use the functions:

$\omega_\lambda : \mathbb{R} \ni \vec{k} \mapsto \sqrt{\vec{k}^2 + m_\lambda^2}$

$\Omega_\lambda : \mathbb{R}^2 \ni (\vec{P}, \vec{p}) \mapsto \sqrt{\vec{P}^2 + 4\omega_\lambda(\vec{p})^2}$

whenever it exists. Here $\omega : \vec{k} \to \sqrt{\vec{k}^2 + m_0^2}$, and f_k is related to f by $f_k((x_1 + x_2)/2, x_1 - x_2) = (f(x_1, x_2) + f(x_2, x_1))/2$

Theorem. *There exist $K_1, K_2 \in (0, \infty)$, depending only on m_0 and $\mathscr{P}$, such that for all $f \in L^1(\mathbb{R}^2)$, $f \neq 0$, with $q_1(f) < \infty$ and for all $\lambda \in (0, \underline{\lambda}]$ with $\lambda < (K_2 q_1(f))^{-1}$ there exists a vector:*

$$\psi_\lambda(f) \in (1 - E_\lambda(0) - E_\lambda(m_\lambda))\mathscr{H}_\lambda \cap D_\lambda(M_\lambda)$$

linear combination of zero-time Wick vectors, given in Table I, satisfying

$$e^{-i\vec{s}\vec{P}_\lambda}\psi_\lambda(f) = \psi_\lambda(f(\,.\, - \vec{s}, \,.\, - \vec{s})) \qquad \forall \vec{s} \in \mathbb{R} \tag{1}$$

$$\frac{(\psi_\lambda(f) \mid M_\lambda^2 \psi_\lambda(f))_{\mathscr{H}_\lambda}}{\|\psi_\lambda(f)\|_{\mathscr{H}_\lambda}^2} = 4m_\lambda^2 + 4m_0\left(\frac{\lambda}{m_0^2}\right)^2 \frac{(f \mid H_{\text{rel}}^{NR}(1)f)_{\mathscr{H}}}{\|f\|_{\mathscr{H}}^2} + \lambda^{5/2}\mathscr{R}_1(\lambda, f) \tag{2}$$

where $\mathscr{R}_1$ satisfies:

$$|\mathscr{R}_1(\lambda, f)| < K_1 q_1(f)^4[1 - \lambda K_2 q_1(f)]^{-1}$$

The proof will be given in §3.

Remark. The set of functions f for which the rest $\mathcal{R}_1$ is bounded for some $\lambda > 0$ in not empty: take $R > 1$ and define

$$Q_R = \{f \in L^1(\mathbb{R}^2) | f \neq 0, q_1(f) \leq R\}$$

$$\lambda_R = \min\left\{\underline{\lambda}, \frac{1}{2K_2R}\right\}$$

Note that $\lambda_R > 0$, and for R big enough, $Q_R \neq \emptyset$.

Then for all $f \in Q_R$ and $\lambda \in (0, \lambda_R]$, the Theorem asserts that:

$$|\mathcal{R}_1(\lambda, f)| < 2K_1R^4$$

2.4. *Comments on the theorem*

2.4.1. *Two-particle states.* From (1), $\vec{P}_\lambda \psi_\lambda(f) = \psi_\lambda(\vec{P}f)$, where $\vec{P}$ is the total momentum in $\mathcal{K}$. Moreover the relative momentum of f is scaled by ψ_λ so as to be $0(\lambda)$ (see Table I). Then $\psi_\lambda(f)$ can be interpreted as a two-particle state of low energy.

2.4.2. *Effective non-relativistic limit.* In the Theorem, the velocity of the light is a constant $c = 1$. The effective non-relativistic limit comes from the small coupling and from the small relative momentum due to the scaling, i.e., from the hypothesis:

$$\frac{\vec{P}_{\text{rel}}}{m_0} \approx \frac{\lambda}{m_0^2} \ll 1$$

Nevertheless, the relation in the Theorem between M^2, H_{rel}^{NR}, and m, is that of a standard $1/c$ development: restoring the c constant and defining the hamiltonian

$$H = 2mc + \frac{1}{4m}\vec{P}^2 + H_{\text{rel}}^{NR}$$

where $\vec{P}$ is the total momentum, we have:

$$M^2c^4 = H^2 - c^2\vec{P}^2 = c^4\left(4m^2 + \frac{1}{c^2}4mH_{\text{rel}}^{NR} + O\left(\frac{1}{c^4}\right)\right)$$

2.4.3. *Energy domain of interest.* The formula of the Theorem has an interest only for finding bound states, i.e., in the domain of energy in the centrum of mass less than $2m_\lambda$. The reason is the following.

The Theorem gives a relation between M_λ^2 and $\lambda^2 H_{\text{rel}}^{NR}(1)$, but the formula of §2.4.2 holds for M_λ^2 and $H_{\text{rel}}^{NR}(\lambda/m_0^2)$. We can correct this mistake by defining for all functions f, a scaled function f_λ by $\tilde{f}_{\lambda,K}(\vec{P}, \vec{p}) = \delta^{-1/2}\tilde{f}_K(\vec{P}, \vec{p}/\delta)$, with $\delta = \lambda/m_0^2$, which gives:

$$\left(\frac{\lambda}{m_0^2}\right)^2 \frac{(f \mid H_{\text{rel}}^{NR}(1)f)_{\mathcal{K}}}{\|f\|_{\mathcal{K}}^2} = \frac{\left(f_\lambda \mid H_{\text{rel}}^{NR}\left(\frac{\lambda}{m_0^2}\right)f_\lambda\right)_{\mathcal{K}}}{\|f_\lambda\|_{\mathcal{K}}^2}$$

Then the formula of the Theorem can be written as:

$$\frac{(\psi_\lambda(f)\mid M_\lambda^2\psi_\lambda(f))_{\mathscr{H}_\lambda}}{\|\psi_\lambda(f)\|^2_{\mathscr{H}_\lambda}} = 4m_\lambda + 4m_0 \frac{\left(f_\lambda \mid H_{\text{rel}}^{NR}\left(\frac{\lambda}{m_0^2}\right)f_\lambda\right)_{\mathscr{H}}}{\|f_\lambda\|^2_{\mathscr{H}}} + \lambda^{5/2}\mathscr{R}_1(\lambda, f)$$

Now the two Rayleigh quotients concern scaled vectors; remember that this scaling reflects the existence of some characteristic compact, which must be for a bound state, but which would not have any physical explanation for a scattering state.

2.5. *Existence of bound state*

2.5.1. *Minimum of $RQ_\lambda(\psi_\lambda(f))$.* The minimum of this quantity, when f varies, is given by the minimum of the r.h.s of the formula of the Theorem. The minimum of the Rayleigh quotient of $H_{\text{rel}}^{NR}(1)$ is the bottom of his spectrum, given in §2.2. Suppose that $H_{\text{rel}}^{NR}(1)$ has a negative eigenvalue $-\mathscr{E}$ (i.e.: suppose $a_4 < 0$) and write the Theorem for functions f given by:

$$\tilde{f}_K(\vec{P}, \vec{p}) = \tilde{g}(\vec{P})\tilde{h}(\vec{p})$$

where $g \in \mathscr{S}(\mathbb{R})$, $g \neq 0$, and h is the eigenfunction of $H_{\text{rel}}^{NR}(1)$ (see §2.2). Obviously:

$$q_1(f) = q_2(g)$$

is well defined. We have then proved:

Corollary.
Hypothesis: $H_{\text{rel}}^{NR}(1)$ has an eigenvalue of energy $-\mathscr{E} < 0$
Conclusion: for all $g \in \mathscr{S}(\mathbb{R})$, $g \neq 0$, with $q_2(g) < \infty$, for all $\lambda \in (0, \underline{\lambda}]$ with $\lambda < (K_2 q_2(g))^{-1}$ there exists a vector:

$$\Psi_\lambda(g) \in (1 - E_\lambda(0) - E_\lambda(m_\lambda))\mathscr{H}_\lambda \cap D_\lambda(M_\lambda)$$

linear combination of zero-time Wick vectors, given in Table II, satisfying:

$$e^{-i\vec{s}\vec{P}_\lambda}\Psi_\lambda(g) = \Psi_\lambda(g(\,.\,-\vec{s})) \quad \textit{for all} \quad \vec{s} \in \mathbb{R} \tag{1}$$

$$\frac{(\Psi_\lambda(g)\mid M_\lambda^2\Psi_\lambda(g))_{\mathscr{H}_\lambda}}{\|\Psi_\lambda(g)\|^2_{\mathscr{H}_\lambda}} = 4m_\lambda^2 - 4m_0\left(\frac{\lambda}{m_0^2}\right)^2\mathscr{E} + \lambda^{5/2}\mathscr{R}_2(\lambda, g) \tag{2}$$

where $\mathscr{R}_2$ satisfies:

$$|\mathscr{R}_2(\lambda, g)| < K_1 q_2(g)^4[1 - \lambda K_2 q_2(g)]^{-1}$$

2.5.2. *Bound state and eigenvector.* It follows from the particle structure of $\mathscr{P}(\varphi)_2$ models, §2.1.3, that the Corollary predicts the existence of two-particle bound state if the hypothesis is satisfied (i.e.: if $a_4 < 0$); its mass $m_B(\lambda)$ is

Table II
Eigenvectors of the bound state

For $g \in \mathcal{S}(\mathbb{R})$ and $\lambda \in (0, \underline{\lambda}]$, the vector appearing in the Corollary of §2.3.1 is, with the notations of Table I:

$$\Psi_\lambda(g) = \psi_\lambda(f)$$

where $\tilde{f}_k(\vec{P}, \vec{p}) = \tilde{g}(\vec{P})\tilde{h}(\vec{p})$ and

$$\tilde{h}(\vec{p}) = \sqrt{2\pi}\frac{\beta^{3/2}}{\vec{p}^2 + \beta^2} \qquad \beta = -\tfrac{3}{2}a_4 m_0$$

bounded by:

$$m_B(\lambda) \leqslant \left(4m_\lambda^2 - 4m_0\left(\frac{\lambda}{m_0^2}\right)^2 \mathscr{E} + O(\lambda^{5/2})\right)^{1/2} = 2m_\lambda - \lambda^2 E' + O(\lambda^{5/2})$$

where $E' = 9a_4^2/4m_0^3$.

We compare with the literature. Using the Bethe–Salpeter method, [e] have found that there is a bound state if and only if $a_4 < 0$ (under the hypothesis $a_4 \neq 0$); this bound state is unique, and its mass is exactly equal to the bound given above, at first perturbation order.

So the variational perturbation method gives here a better result that what was expected; we have actually reached the bottom of the spectrum of M_λ^2 on $(1 - E_\lambda(0) - E_\lambda(m_\lambda))\mathscr{H}_\lambda$. Then the subspace $\{\Psi_\lambda(f) \mid f \in \mathcal{S}(\mathbb{R})\}$ of zero-time Wick vectors can be seen as an approximation of the real eigenspace.

§3. Toward the proof of the theorem

We give first, in §3.1, an outline of the proof, containing three Propositions, the demonstration of which, being more technical, will follow in §3.2.

3.1. *Outline of the proof*

We will use the following method: we try to find a class of vectors (which will play the role of an ansatz), using heuristic arguments, and then we calculate rigorously the Rayleigh quotient RQ_λ for this class. In order to minimize this result, we modify the initial class of vectors, using heuristic arguments again, and calculate RQ_λ for this new class, and so on, until the theorem is proved.

3.1.1. *First class of vectors.* All the classes of vectors that we will consider will be composed only of zero-time Wick vectors of even degree. The reasons for this, which are heuristic, are the following.

1) The 'zero-time' is suggested by the structure of the quantum mechanical model of §2.2, which we try to approach.

2) The 'zero-time Wick vectors' describe well the particle structure of the free model (i.e.: the model with $\lambda = 0$).
3) The Wick vectors of 'even degree' span the states of an even number of particles (see §2.1.4). Therefore these vectors are orthogonal to one-particle states.

We will do perturbation calculations, so it is natural to begin with the $\lambda = 0$ case, where the answer of our problem is well known. The two-particle states are spanned by the set: $\{W_0\phi_0^2(f) \mid f \in \mathcal{B}p_{2,2}\}$, and contain no bound state ($\mathcal{B}p_{n,m}$ are the normed spaces introduced in [o, §3.3]) in order to have vectors in the domain of the hamiltonian). Consequently, the first class of vectors we will consider is:

$$\mathcal{C}_\lambda^1 = \{\psi_\lambda^1(f) = (1 - E_\lambda(0))W_\lambda\phi_\lambda^2(f) \mid f \in \mathcal{B}p_{2,2}\}$$

with $\lambda \in [0, \underline{\lambda}]$.

3.1.2. *Rigorous calculation with $\mathcal{C}_\lambda^1$.*

Definition. For $f \in \mathcal{B}p_{2,2}$, $f \neq 0$, let us denote:

$$q_3(f) = \left(\frac{bp_{2,2}(f)}{m_0 bp_{2,0}(f)}\right)^2$$

where $bp_{n,m}$ is the norm of $\mathcal{B}p_{n,m}$ (see [o, §3.3]).

Proposition 1. *There exist $K_3, K_4 \in (0, \infty)$, depending only on m_0 and $\mathcal{P}$, such that: and for all $f \in \mathcal{B}p_{2,2}$, $f \neq 0$ and for all $\lambda \in [0, \underline{\lambda}]$ with $\lambda < K_4^{-1}$*

$$\frac{(\psi_\lambda^1(f) \mid M_\lambda^2\psi_\lambda^1(f))_{\mathcal{H}_\lambda}}{\|\psi_\lambda^1(f)\|_{\mathcal{H}_\lambda}^2} = \mathcal{M}_\lambda^2(f) + \lambda I_1(f) + \lambda^2(I_2(f) + T(f)) + \lambda^3\mathcal{R}_3(\lambda, f)$$

where $\mathcal{M}$, I_1, I_2, T are given in Table III, and $\mathcal{R}_3$ satisfies:

$$|\mathcal{R}_3(\lambda, f)| < K_3 q_3(f)(1 - \lambda K_4)^{-1}$$

The proof, rather technical, is given in §3.2.1.

3.1.3. *Second class of vectors.* We are interested in the minimum of the result of Proposition 1. The first term, $\mathcal{M}_\lambda^2(f)$, is positive and its minimum is $(2m_\lambda)^2$. We note that this value cannot be reached by any function f. The second term, $\lambda I_1(f)$, is smaller (because of the λ factor) but can be negative (if $a_4 < 0$).

To approach the minimum, we use the scaling of Glimm, Jaffe and Spencer, discussed in §1. We replace then the relative position variable $\vec{x}$ in f by $\delta\vec{x}$, where δ is a positive and small parameter. In a first time, we keep δ fixed, and expand the result of Proposition 1 in it.

In momentum space, this scaling replaces the quotient $\vec{p}/m_0$, where $\vec{p}$ is the relative momentum, by $\delta\vec{p}/m_0$ in all kernels of the scalar products involved in the Rayleigh quotient. Thus the δ expansion leads to the non-relativistic limit.

Table III
Expressions appearing in Proposition 1

For $f \in \mathscr{B}p_{2,2}$, $f \neq 0$, we define f_s by:

$$f_s(x, y) = (f(\vec{x}, \vec{y}) + f(\vec{y}, \vec{x}))/2 \qquad \forall \vec{x}, \vec{y} \in \mathbb{R}$$

Then:

$$\mathscr{M}_\lambda^2(f) = \frac{\int_{\mathbb{R}^2} \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega_\lambda(\vec{k}_i)} \right) |f_s(\vec{k}_1, \vec{k}_2)|^2 \left((\omega_\lambda(\vec{k}_1) + \omega_\lambda(\vec{k}_2))^2 - (\vec{k}_1 + \vec{k}_2)^2 \right)}{\int_{\mathbb{R}^2} \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega_\lambda(\vec{k}_i)} \right) |f_s(\vec{k}_1, \vec{k}_2)|^2}$$

where

$$\omega_\lambda : \mathbb{R} \ni \vec{k} \mapsto \sqrt{\vec{k}^2 + m_\lambda^2}$$

For the other terms, we note:

$$\mathscr{N}(f) = 2 \int_{\mathbb{R}^2} \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) |f_s(\vec{k}_1, \vec{k}_2)|^2$$

where $\omega = \omega_0$. Then:

$$I_1(f) = -\frac{4!\, a_4}{\pi} \mathscr{N}(f)^{-1} \int_{\mathbb{R}^4} \left(\prod_{i=1}^{4} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) f_s(\vec{k}_1, \vec{k}_2) f_s(\vec{k}_3, \vec{k}_4) \frac{\delta\left(\sum_{i=1}^{4} \vec{k}_i \right)}{\sum_{i=1}^{4} \omega(\vec{k}_i)}$$
$$\times \left(-(\omega(\vec{k}_1) + \omega(\vec{k}_2)(\omega(\vec{k}_3) + \omega(\vec{k}_4)) - (\vec{k}_1 + \vec{k}_2)^2 - \mathscr{M}_0^2(f) \right)$$

$$T(f) = 4\mathscr{N}(f)^{-1} \sum_{n=2}^{\mathscr{N}} \frac{2n(2n)!\,(a_{2n})^2}{(2\pi)^{2(n-1)}} \int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^{2n} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) \left| f_s\left(\vec{k}_1, \sum_{i=2}^{2n} \vec{k}_i \right) \right|^2$$
$$\times \frac{\left(\sum_{i=1}^{2n} \omega(\vec{k}_i) \right)^2 - \left(\sum_{i=1}^{2n} \vec{k}_i \right)^2 - \mathscr{M}_0^2(f)}{\left[\left(\sum_{i=2}^{2n} \omega(\vec{k}_i) \right)^2 - \left(\sum_{i=2}^{2n} \vec{k}_i \right)^2 - m_0^2 \right]^2}$$

$$I_2(f) = \left[(I_1(f) + \mathscr{M}_1^2(f)) \mathscr{N}(f)^{-1} \frac{4!\, a_4}{\pi} \int_{\mathbb{R}^4} \left(\prod_{i=1}^{4} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) f_s(\vec{k}_1, \vec{k}_2) f(\vec{k}_3, \vec{k}_4) \frac{\delta\left(\sum_{i=1}^{4} \vec{k}_i \right)}{\sum_{i=1}^{4} \omega(\vec{k}_i)} \right.$$
$$\left. + (\partial_\lambda m_\lambda^2|_{\lambda=0}) I_1(f) \mathscr{N}(f)^{-1} \int_{\mathbb{R}^2} \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) |f(\vec{k}_1, \vec{k}_2)|^2 \sum_{i=1}^{2} \omega(\vec{k}_i)^{-2} \right]$$
$$+ (2\pi)^{-6} \mathscr{N}(f)^{-1} \int_{\mathbb{R}^8} \left(\prod_{i=1}^{4} \frac{d^2 k_i}{k_i^2 + m_0^2} \right) f_s(\vec{k}_1, \vec{k}_2) f_s(\vec{k}_3, \vec{k}_4) \delta^{(2)}\left(\sum_{i=1}^{4} k_i \right)$$
$$\times \left(-(k_1 + k_2)^2 - \mathscr{M}_0^2(f) \right) \sum\nolimits_2^4 (k_1, \ldots, k_4)$$

where $\mathscr{M}_1^2(f) = \partial_\lambda \mathscr{M}_\lambda^2(f)|_{\lambda=0}$, and Σ_2^4 is presented in Appendix II.

We choose $\vec{P}, \vec{p}$, total and relative momentum (Kepler decomposition) as written in Table I. Appendix I gives formulas for this change of variables. The scaling operator $U(\delta)$ is defined in Table I (the factor $\Omega_\lambda(\vec{P}, \vec{p})^{1/2}$ is necessary for finding the right hamiltonian $H_{\text{rel}}^{NR}(1)$ in the Theorem; the factor $\omega_\lambda(\vec{p})^{-1/2}$ is needed for bounding the rest of the perturbation serie).

The class of vectors we consider now is:

$$\mathscr{C}^2_{\lambda,\delta} = \{\psi^2_{\lambda,\delta}(f) = (1 - E_\lambda(0))W_\lambda\phi^2_\lambda(U(\delta)f) \mid U(\delta)f \in \mathscr{B}p_{2,2}\}$$

with $\lambda \in [0, \underline{\lambda}]$ and $\delta \in (0, 1]$. This set is not empty (lemma of §3.2.2).

3.1.4. *Rigorous calculation with* $\mathscr{C}^2_\lambda$.

Definition. For $f \in \mathscr{B}p_{2,2}$, $f \neq 0$, we note:

$$q_4(f) = \frac{\int d\vec{P}\, d\vec{p}\, |\tilde{f}_K(\vec{P}, \vec{p})|^2 \omega(\vec{p})^{5/2} \omega(\vec{P})^2}{m_0^{9/2} \|f\|^2_{L^2}}$$

whenever it exists.

Proposition 2. *There exist $K_5, K_6 \in (0, \infty)$ depending only on m_0 and $\mathscr{P}$, such that: for all $f \in \mathscr{B}p_{2,2}$, $f \neq 0$, with $q_4(f) < \infty$, for all $\lambda \in [0, \underline{\lambda}]$, $\delta \in (0, 1]$ with* $\dfrac{\lambda}{m_0^2} + \delta < (K_6 q_4(f))^{-1}$

$$\frac{(\psi^2_{\lambda,\delta}(f) \mid M^2_\lambda \psi^2_{\lambda,\delta}(f))_{\mathscr{H}_\lambda}}{\|\psi^2_{\lambda,\delta}(f)\|^2_{\mathscr{H}_\lambda}}$$

$$= 4m_\lambda^2 + \frac{4\delta^2 \int d\vec{P}\, d\vec{p}\, |\tilde{f}_K(\vec{P}, \vec{p})|^2 \vec{p}^2 + \dfrac{3!\, a_4}{m_0 \pi} \lambda\delta \int d\vec{P} \left| \int d\vec{p} \tilde{f}_K(\vec{P}, \vec{p}) \right|^2}{\|f\|^2_{L^2}}$$

$$+ \lambda^2 \tau(U(\delta)f) + \sum_{i=0}^{5} \left(\frac{\lambda}{m_0^2}\right)^{i/2} \delta^{(5-i)/2} \mathscr{R}^i_4(\lambda, \delta, f)$$

where τ is given in Table IV and $\mathscr{R}^i_4$ satisfies, $\forall 0 \leq i \leq 5$:

$$|\mathscr{R}^i_4(\lambda, \delta, f)| < K_5 q_4(f)^3 \left[1 - K_6\left(\frac{\lambda}{m_0^2} + \delta\right) q_4(f)\right]^{-1}$$

The proof will be given in §3.2.3.

The fractional powers of δ and λ, arising in front of $\mathscr{R}^i_4$, are necessary so that the exponent of $\omega(\vec{p})$ in $q_4(f)$ be as small as possible, in order to admit functions $\tilde{f}$ which are weakly decreasing as $|\vec{P}| \to \infty$, such as the eigenfunctions of H^{NR}_{rel} of §2.2.

Table IV
Expression appearing in Proposition 2

With the notation of Table III:

$$\tau(f) = 4\mathscr{N}(f)^{-1} \sum_{n=2}^{\mathscr{N}} \frac{(2n)!\,(a_{2n})^2}{(2\pi)^{2(n-1)}} \int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^{2n} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)$$

$$\times \left| \sum_{i=1}^{2n} \frac{\tilde{f}_s\left(\vec{k}_i, \sum_{j\neq i} \vec{k}_j\right)}{\left(\sum_{j\neq i} \omega(\vec{k}_j)\right)^2 - \left(\sum_{j\neq i} \vec{k}_j\right)^2 - m_0^2} \right|^2 \left[\left(\sum_{i=1}^{2n} \omega(\vec{k}_i)\right)^2 - \left(\sum_{i=1}^{2n} \vec{k}_i\right) - \mathscr{M}_0^2(f)\right]$$

3.1.5. *Comments on Proposition* 2. We follow Glimm, Jaffe and Spencer [b] to find the minimum of the result of Proposition 2, in varying δ. As announced (§1), we accept that δ depend on λ in such a way that $\delta \to 0$ if $\lambda \to 0$. We then take $\delta = (\lambda/m_0^2)^\kappa$, a dimensionless quantity, with $\kappa > 0$.

In the result of Proposition 2, the first interesting term (with involves the classical kinetic energy in the centrum of mass system, $\vec{p}^2/m_0$) is positive, and of order $\delta^2 = \lambda^{2\kappa}$. The second term (which is responsible for the interaction) is negative if and only if $a_4 < 0$. It is of order $\lambda\delta = \lambda^{\kappa+1}$. The third term, $\lambda^2\tau$ (coming from the cut of the two point function, see Appendix II), is positive and of order λ^2. The minimum in varying κ is obtained when the negative term dominates the positive terms, that is when

$$\kappa + 1 \geqslant 2\kappa$$
$$\kappa + 1 \geqslant 2$$

The only solution of this system is $\kappa = 1$. We then put $\delta = \lambda/m_0^2$ in Proposition 2, which gives:

Corollary. *There exist $K_7, K_8 \in (0, \infty)$, depending only on m_0 and $\mathcal{P}$, such that for all $f \in \mathcal{B}p_{2,2}$, $f \neq 0$, with $q_4(f) < \infty$, for all $\lambda \in (0, \underline{\lambda}]$ with $\lambda < (K_8 q_4(f))^{-1}$*

$$\frac{(\psi^2_{\lambda,\delta}(f) \mid M^2_\lambda \psi^2_{\lambda,\delta}(f))_{\mathcal{H}_\lambda}}{\|\psi^2_{\lambda,\delta}(f)\|^2_{\mathcal{H}_\lambda}} = 4m^2_\lambda + 4m_0\left(\frac{\lambda}{m_0^2}\right)^2 \frac{(f \mid H^{NR}_{\text{rel}}(1)f)_{\mathcal{K}}}{\|f\|^2_{\mathcal{K}}}$$
$$+ \lambda^2\tau(g) + \lambda^{5/2}\mathcal{R}_5(\lambda, f)$$

where $g = U(\delta)f$ and $\delta = \lambda/m_0^2$; $\mathcal{R}_5$ satisfies:

$$|\mathcal{R}_5(\lambda, f)| < K_7 q_4(f)^3[1 - \lambda K_8 q_4(f)]^{-1}$$

Remember that H^{NR}_{rel} is given in §2.2.

The appearance of the classical hamiltonian in the above RQ_λ was first found by Perreaux [f].

3.1.6. *Third class of vectors.* In the result of the Corollary, the term $\lambda^2\tau(g)$, which is always positive, prevents reaching the minimum announced in the Theorem. This term comes from the cut of the two point function (see Appendix II), and it is well known that it disappears if we project $\psi^2_{\lambda,\delta}(f)$ into the subspace of mass $M_\lambda \leqslant (2 + \frac{1}{2})m_\lambda$. However, we will not follow these considerations. In order to cancel the undesirable term we modify our class of vectors in the most general way: we add to $\psi^2_{\lambda,\delta}$ any linear combination of zero-time Wick vectors of even degree. To preserve the interesting terms in the result of the Corollary, we multiply this new vector by λ. Then our new class is:

$$\mathcal{C}^3_\lambda = \left\{\psi^3_\lambda(f, \vec{f}) = (1 - E_\lambda(0)W_\lambda\left(\phi^2_\lambda(g) + \lambda \sum_{n=2}^{\infty} \phi^{2n}_\lambda(f_{2n})\right) \mid f \in \mathcal{B}p_{2,2}, f_{2n} \in \mathcal{B}p_{2n,2} \forall n\right\}$$

with $\lambda \in (0, \underline{\lambda}]$ and with the notations: $g = U(\lambda/m_0^2)f$ and $\vec{f} = (f_4, f_6, \ldots)$. Note that we have not added to ϕ_λ^2 vectors of degree zero in the fields (which would be eliminated by $1-E_\lambda(0)$) nor of degree two (because we have already given a λ-dependence in the degree two term).

We introduce functions which will play an important role:

Definition. For all $n \in \mathbb{N}$, $n>2$ and $f \in \mathscr{B}p_{2,2}$, we call $F_{f,n}$ the functions:

$$\tilde{F}_{f,n}: \mathbb{R}^n \ni (\vec{k}_1, \ldots, \vec{k}_n) \mapsto -\frac{a_n}{(2\pi)^{n-1}} \sum_{j=1}^{n} \frac{\tilde{f}\left(\vec{k}_j, \sum_{i\neq j} \vec{k}_i\right) + \tilde{f}\left(\sum_{i\neq j} \vec{k}_i, \vec{k}_j\right)}{\left(\sum_{i\neq j} \omega(\vec{k}_i)\right)^2 - \left(\sum_{i\neq j} \vec{k}_i\right)^2 - m_0^2}$$

where the a_i are the coefficients of the interactive polynomial $\mathscr{P}$.

3.1.7. *Rigorous calculation with* $\mathscr{C}_\lambda^3$.

Definition 1. For $f \in \mathscr{B}p_{2,2}$, $f\neq 0$, and $\vec{f} = (f_4, f_6, \ldots)$ with $f_{2i} \in \mathscr{B}p_{2i,2}$, $\forall i \geq 2$ let us define:

$$q_5(f, \vec{f}) = q_4(f) + \frac{\sum_{i=2}^{\infty} (bp_{2i,2}(f_{2i}))^2}{m_0^2 \|f\|_{L^2}^2}$$

whenever it exists.

Definition 2. For $n \in \mathbb{N}^*$ and $f \in \mathscr{B}p_{2,2}$, $f\neq 0$, we introduce the measure on $\mathbb{R}^n$:

$$\vec{k}_1, \ldots, \vec{k}_n \mapsto d\sigma_{f,n}(\vec{k}_1, \ldots, \vec{k}_n) = \frac{n!}{2}\left(\prod_{i=1}^{n} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\left[\left(\sum_{i=1}^{n} \omega(\vec{k}_i)\right)^2 - \left(\sum_{i=1}^{n} \vec{k}_i\right)^2 - \mathscr{M}_0^2(f)\right]$$

Proposition 3. *There exist $K_9, K_{10} \in (0, \infty)$, depending only on m_0 and $\mathscr{P}$, such that, for all $f \in \mathscr{B}p_{2,2}$, $f \neq 0$, $\vec{f} = (f_4, f_6, \ldots)$, $f_{2i} \in \mathscr{B}p_{2i,2} \forall i \geq 2$, with $q_5(f, \vec{f}) < \infty$ for all $\lambda \in (0, \underline{\lambda}]$ with $\lambda < (K_{10} q_5(f, \vec{f}))^{-1}$, we have:*

$$\frac{(\psi_\lambda^3(f,\vec{f}) \mid M_\lambda^2 \psi_\lambda^3(f,\vec{f}))_{\mathscr{H}_\lambda}}{\|\psi_\lambda^3(f,\vec{f})\|_{\mathscr{H}_\lambda}^2} = 4m_\lambda^2 + 4m_0\left(\frac{\lambda}{m_0^2}\right)^2 \frac{(f \mid H_{\text{rel}}^{NR}(1) f)_{\mathscr{H}}}{\|f\|_{\mathscr{H}}^2}$$

$$+ \lambda^2 \sum_{n=2}^{\infty} \frac{\int d\sigma_{g,2n}(k)\, |\tilde{f}_{2n}(k) - \tilde{F}_{g,2n}(k)|^2}{\int \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right) |\tilde{g}(\vec{k}_1, \vec{k}_2)|^2}$$

$$+ \lambda^{5/2} \mathscr{R}_6(\lambda, f, \vec{f})$$

where $g = U(\lambda/m_0^2)f$, *and* $\mathcal{R}_6$ *satisfies*:

$$|\mathcal{R}_6(\lambda, f, \vec{f})| < K_9 q_5(f, \vec{f})^4[1 - \lambda K_{10} q_5(f, \vec{f})]^{-1}$$

The proof of Proposition 3 is given in §3.2.4.

3.1.8. *Proof of the theorem.* The minimum of the result of Proposition 3, when $\vec{f}$ is varying, is given by the choice:

$$f_{2n} = F_{U(\lambda/m_0^2)f,\, 2n} \quad \text{for } n = 2, 3, \ldots, \mathcal{N}$$
$$f_{2n} = 0 \quad \text{otherwise}$$

We must control $bp_{2n,2}(f_{2n})$ for all n; from the lemma of §3.2.5 and then with the lemma of §3.2.2, the existence of $K \in (0, \infty)$ follows, with:

$$q_5(f, \vec{f}) \leq K q_1(f).$$

Inserting these results in Proposition 3 gives the Theorem.

3.2. *Proofs of the three Propositions*

3.2.1. *Proof of Proposition 1*

1st step: The scalar products are written as Schwinger functions. For all $s \in \mathbb{R}^2$, $\mathring{s} \geq 0$ and $\lambda \in [0, \underline{\lambda}]$, we define:

$$\chi(\lambda, s, f) = (\psi_\lambda^1(f) \mid e^{i\vec{s}\vec{P}_\lambda - \mathring{s}H_\lambda}\psi_\lambda^1(f))_{\mathcal{H}_\lambda}$$
$$= (\phi_\lambda^2(f) \mid T(s)\phi_\lambda^2(f))_{L^2(Q,\mu_\lambda)}$$

where T is the translation operator in $L^2(Q, \mu_\lambda)$. It follows from [o, theorem of §4.3] that:

$$RQ_\lambda(\psi_\lambda^1(f)) = \frac{\Delta_s \chi(\lambda, s, f)|_{s=0}}{\chi(\lambda, 0, f)}$$

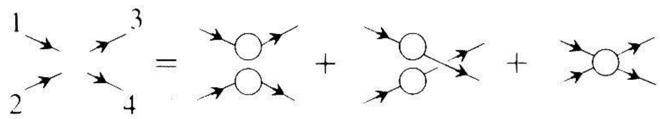
for all $f \in \mathcal{B}p_{2,2}$, $f \neq 0$. To make evident the f-bilinearity and the action of $T(s)$, we write:

$$\chi(\lambda, s, f)$$
$$= \int_{\mathbb{R}^4} \left(\prod_{i=1}^4 d\vec{x}_i\right) f_s(\vec{x}_1, \vec{x}_2) f_s(\vec{x}_3, \vec{x}_4) s_{2,2,\lambda}^{WT}((0, \vec{x}_1), (0, \vec{x}_2), (\mathring{s}, \vec{x}_3 + \vec{s}), (\mathring{s}, \vec{x}_4 + \vec{s}))$$

where f_s is the symmetrisation of f (see Table III) and $s_{2,2,\lambda}^{WT}$ is given, for even $\mathcal{P}(\varphi)_2$ models, by:

$$s_{2,2,\lambda}^{WT}(x_1, \ldots, x_4) = s_{2,\lambda}^T(x_1, x_3)s_{2,\lambda}^T(x_2, x_4) + s_{2,\lambda}^T(x_1, x_4)s_{2,\lambda}^T(x_2, x_3) + s_{4,\lambda}^T(x_1, \ldots, x_4)$$

This identity can be represented graphically by:



2nd step: Perturbation developments of the scalar products. We read in

Appendix II the perturbation development of $s^T_{2,\lambda}$ and $s^T_{4,\lambda}$, and we obtain:

$$\chi(\lambda, s, f) = 2\int_{\mathbb{R}^2}\left(\prod_{i=1}^{2}\frac{d\vec{k}_i}{2\omega_\lambda(\vec{k}_i)}\right)|\vec{f}_s(\vec{k}_1, \vec{k}_2)|^2 e^{i\vec{s}(\vec{k}_1+\vec{k}_2)}\exp\left[-|\mathring{s}|\sum_{i=1}^{2}\omega_\lambda(\vec{k}_i)\right]$$

$$+\lambda^2\frac{2}{\pi}\int_{\mathbb{R}^3}\frac{d\vec{k}_1}{2\omega_\lambda(\vec{k}_1)}\frac{d^2k_2}{(k_2^2+m_0^2)^2}|\tilde{f}_s(\vec{k}_1, \vec{k}_2)|^2 T(k_2^2)e^{i\vec{s}(\vec{k}_1+\vec{k}_2)}e^{i\mathring{s}\mathring{k}_2-|\mathring{s}|\omega(\vec{k}_1)}$$

$$+\frac{\lambda}{(2\pi)^2}\int_{\mathbb{R}^8}\left(\prod_{i=1}^{4}\frac{d^2k_i}{k_i^2+m_0^2}\right)\tilde{f}_s(\vec{k}_1, \vec{k}_2)\tilde{f}(\vec{k}_3, \vec{k}_4)\delta^{(2)}(k_1+\cdots+k_4)e^{is(k_1+k_2)}$$

$$\times\left(-\frac{4!\,a_4}{(2\pi)^2}+\lambda\sum\nolimits_2^4(k_1, \ldots, k_4)\right)+O(\lambda^3)$$

The laplacian Δ_s of χ, for $\mathring{s}\geqslant 0$, is given by the derivatives under the integrals. We put then $s = 0$.

3rd step: Integration over the $\mathring{k}_i$-variables. As a consequence of the choice of zero-time vectors, we need to integrate over the $\mathring{k}_i$-variables. We use the formula of the residues. We write the results as follows:

$$RQ_\lambda(\psi^1_\lambda(f)) = \frac{a_0^1+\lambda a_1^1+\lambda^2 a_2^1+\lambda^3 a_3^1}{a_0^2+\lambda a_1^2+\lambda^2 a_2^2+\lambda^3 a_3^2}$$

where for $i\in\{1, 2\}$:

$$a_0^i = 2(1+2\lambda^2 T'(-m_0^2))\int_{\mathbb{R}^2}\left(\prod_{j=1}^{2}\frac{d\vec{k}_j}{2\omega_\lambda(\vec{k}_j)}\right)|\vec{f}_s(\vec{k}_1, \vec{k}_2)|^2$$

$$\times\left(\delta_{2,i}+\delta_{1,i}\left(\left(\sum_{j=1}^{2}\omega_\lambda(\vec{k}_j)\right)^2-(\vec{k}_1+\vec{k}_2)^2\right)\right)$$

$$a_1^i = -\frac{4!\,a_4}{\pi}\int_{\mathbb{R}^4}\left(\prod_{j=1}^{4}\frac{d\vec{k}_j}{2\omega(\vec{k}_j)}\right)\vec{f}(\vec{k}_1, \vec{k}_2)\vec{f}(\vec{k}_3, \vec{k}_4)\frac{\delta(\vec{k}_1+\cdots+\vec{k}_4)}{\omega(\vec{k}_1)+\cdots+\omega(\vec{k}_4)}$$

$$\times(\delta_{2,i}-\delta_{1,i}((\omega(\vec{k}_1)+\omega(\vec{k}_2))(\omega(\vec{k}_3)+\omega(\vec{k}_4))+(\vec{k}_1+\vec{k}_2)^2))$$

$$a_2^i = 4\sum_{n=2}^{\mathcal{N}}\frac{2n(2n)!\,(a_{2n})^2}{(2\pi)^{2(n-1)}}\int_{\mathbb{R}^{2n}}\left(\prod_{j=1}^{2n}\frac{d\vec{k}_j}{2\omega(\vec{k}_j)}\right)|\vec{f}_s(\vec{k}_1, \vec{k}_2+\cdots+\vec{k}_{2n})|^2$$

$$\times\frac{\delta_{2,i}+\delta_{1,i}\left(\left(\sum_{j=1}^{2n}\omega(\vec{k}_j)\right)^2-\left(\sum_{j=1}^{2n}\vec{k}_j\right)^2\right)}{\left[\left(\sum_{j=2}^{2n}\omega(\vec{k}_j)\right)^2-\left(\sum_{j=2}^{2n}\vec{k}_j\right)^2-m_0^2\right]^2}$$

$$+\frac{1}{(2\pi)^4}\int_{\mathbb{R}^8}\left(\prod_{j=1}^{4}\frac{d^2k_j}{k_j^2+m_0^2}\right)\tilde{f}_s(\vec{k}_1, \vec{k}_2)\tilde{f}_s(\vec{k}_3, \vec{k}_4)$$

$$\times\,\delta^{(2)}(k_1+\cdots+k_4)\sum\nolimits_2^4(k_1, \ldots, k_4)(\delta_{2,i}-\delta_{1,i}(k_1+k_2)^2)$$

$\lambda^3 a_3^i$ are the remainder of the perturbation series (see step 5). In these calculations, we are permitted all operations which have a $O(\lambda^3)$ error.

4th step: Division of the Rayleigh quotient. We will use the identity:

$$\frac{\sum_{i=0}^{3}\lambda^i c_i}{\sum_{i=0}^{3}\lambda^i d_i} = c_0' + \lambda(c_1' - c_0'd_1') + \lambda^2\{c_2' - c_0'd_2' - d_1'(c_1' - c_0 d_1')\}$$

$$+ \lambda^3 \frac{\sum_{i=0}^{2}\lambda^i P_i(c_0', \ldots, c_3', d_1', \ldots, d_3')}{1 + \sum_{i=1}^{3}\lambda^i d_i'}$$

where $c_i' = c_i/d_0\ \forall 0 \leqslant i \leqslant 3$, $d_j' = d_j/d_0\ \forall 1 \leqslant j \leqslant 3$ and P_i are polynomials of degree one in c_i' and of degree three in d_j'.

This identity holds for all $c_i, d_i \in \mathbb{C}$ provided that the denominators do not vanish. It is purely algebraic. c_i, d_i can also be functions of λ. With c_i and d_i replaced by a_i^1, a_i^2 as above, the first term of the division gives $\mathcal{M}_\lambda^2(f)$, and the other terms, after complete perturbation expansion, give $\lambda I_1(f) + \lambda^2(I_2(f) + T(f))$, listed in Table III.

5th step: Bound on the neglected terms. First we estimate the remainder: $\lambda^3\mathcal{R}'$ of the perturbation series of $RQ_\lambda(\psi_\lambda^1(f))$ calculated up to order λ^2 (which supposes that we have also developed $\lambda \mapsto \mathcal{M}_\lambda^2(f)$). We use the λ^3-term of the algebraic formula for the quotient of the step 4, where d_0 is now replaced by:

$$\|\psi_0^1(f)\|_{\mathcal{H}_0}^2 = \chi(0, 0, f) = bp_{2,0}(f)^2$$

Then it follows from [o, theorem 4.3] that the d_i' are bounded by constants, and the c_i' are $O(q_3(f))$. The denominator is now greater than $1 - \lambda C_1$, for some $C_1 \in (0, \infty)$, and so we have proved that

$$\mathcal{R}' = O(q_3(f)(1 - \lambda C_1)^{-1})$$

We must now take into account the difference between $\mathcal{M}_\lambda^2(f)$ and its development up to order λ^2. This can be done in the same way as for $\mathcal{R}'$, because $\mathcal{M}_\lambda^2(f)$ is itself the $RQ_\lambda(\psi_\lambda^1(f))$ of an even $\mathcal{P}(\varphi)_2$ model: the free model with single mass m_λ. Then this difference is also $0(q_\lambda(f)(1 - \lambda C_2)^{-1})$ for some $C_2 \in (0, \infty)$, where $q_\lambda(f)$ is just $q_3(f)$ in which the function ω has been replaced by ω_λ. But we can substitute ω_λ by ω in each estimation, because there exist $C_3, C_4 \in (0, \infty)$ with:

$$C_3 \leqslant \frac{\omega(\vec{k})}{\omega_\lambda(\vec{k})} \leqslant C_4$$

for all $\vec{k} \in \mathbb{R}$ and $\lambda \in [0, \underline{\lambda}]$. This is due to the fact that $m_0 > 0$ and that $\lambda \mapsto m_\lambda$ is C^∞, ([n]). We can now easily see that $\mathcal{R}_3$ is bounded as claimed. ■

3.2.2. *The norm of $U(\delta)f$–functions.* We must control the $bp_{2,2}$ norm of the $U(\delta)f$-functions.

Lemma. *There exists* $K_{11} \in (0, \infty)$ *such that*

$$bp_{2,\alpha}(U(\delta)f)^2 < K_{11} \int d\vec{P}\, d\vec{p}\, |\tilde{f}_K(\vec{P}, \vec{p})|^2\, \Omega(\vec{P}, \vec{p})^\alpha$$

for all $\delta \in (0, 1]$, $\alpha \geq 0$ *and* $f \in \mathscr{B}p_{2,2}$ *such that the r.h.s. exists.*

We have used the notation: $\Omega(\vec{P}, \vec{p})^2 = \vec{P}^2 + 4\omega(\vec{p})^2$. The Lemma asserts that there exist many functions f with $U(\delta)f \in \mathscr{B}p_{2,2}$.

Proof. The $bp_{2,2}$ norm ([o, §3.3.2]) of $U(\delta)f$ is given by (see the Jacobian of $\vec{k}_1, \vec{k}_2 \to \vec{P}, \vec{p}$ in Appendix I):

$$bp_{2,\alpha}(U(\delta)f)^2 = \int d\vec{P}\, d\vec{p}\, |\tilde{f}_K(\vec{P}, \vec{p})|^2 \left(\prod_{i=1}^{2} \frac{\omega_\lambda(\vec{k}_{i,\delta})}{\omega(\vec{k}_{i,\delta})}\right)\left(\sum_{i=1}^{2} \omega(\vec{k}_{i,\delta})\right)^\alpha \omega_\lambda(\delta\vec{p})^{-2}$$

where $\vec{k}_{i,\delta} = \vec{k}_i(\vec{P}, \delta\vec{p})$. From Appendix I: $(\sum_{i=1}^2 \omega_\lambda(\vec{k}_{i,\delta}))^2 = \vec{P}^2 + 4\delta^2\vec{p}^2 + 4m_\lambda^2 = \Omega_\lambda(\vec{P}, \delta\vec{p})^2$, which is $0(\Omega(\vec{P}, \vec{p})^2)$ and we have just seen (5th step of §4.2.1) that $\dfrac{\omega_\lambda}{\omega}$ is bounded. Obviously, $\omega_\lambda(\delta\vec{p})^{-2}$ is bounded, too. ■

3.2.3. *Proof of Proposition 2.* From Proposition 1:

$$RQ_\lambda(\psi^2_{\lambda,\delta}(f)) = \mathscr{M}^2_\lambda(g) + \lambda I_1(g) + \lambda^2(I_2(g) + T(g)) + \lambda^3\mathscr{R}_3(\lambda, g)$$

with $g = U(\delta)f$. We must expand all these terms in δ. We will find two sorts of terms:

1) $\int dP\, dp\, |\tilde{f}_K(P, p)|^2\, \mathscr{N}(P, p)$

2) $\int dP\, dp_1\, dp_2 \tilde{f}_K(-P, p_1)\tilde{f}_K(P, p_2)\mathscr{N}'(P_1, p_1, p_2)$

with $\mathscr{N}$, $\mathscr{N}'$ continuous functions. (We omit the arrows on P, p_i, k_j henceforth.) The first objects, which we will call 'kinetic', do not vanish if $\delta \to 0$; the second ones, called 'interactive', are $0(\delta)$. The scaling makes the interactive terms smaller.

1st step: Expansion of the denominators (kinetic type). With the use of Appendix I, the denominator of $\mathscr{M}^2_\lambda(g)$ can be written as:

$$2\int\left(\prod_{i=1}^{2} \frac{dk_i}{2\omega_\lambda(k_i)}\right) |\widetilde{U(\delta)f}(k_1, k_2)|^2 = \int dP\, dp\, |\tilde{f}_K(P, p)|^2\, \omega_\lambda(\delta p)^{-2}$$

With the algebraic identity:

$$\frac{1}{A+B} = \frac{1}{A}\left(1 - \frac{B}{A}\frac{1}{1+\dfrac{B}{A}}\right)$$

and with the estimates:

$$\omega_\lambda(\delta p)^{-2}=m_\lambda^{-2}+O\left(\frac{\delta\,|p|}{m_\lambda}\right)=m_0^{-2}+O(\lambda)+O\left(\frac{\delta\,|p|}{m_0}\right)$$

we find that

$$\left[\int dP\,dp\,|\tilde f_K(P,p)|^2\,\omega_2(\delta p)^{-2}\right]^{-1}=m_0^2\left[\int dP\,dp\,|\tilde f_K(P,p)|^2\right]^{-1}$$
$$\times\left(1+\frac{\lambda}{m_0^2}r_1(\lambda,\delta,f)+\delta r_2(\lambda,\delta,f)\right)$$

where, for $i\in\{1,2\}$:

$$|r_i(\lambda,\delta,f)|<C_1q_4(f)\left[1-\left(\delta+\frac{\lambda}{m_0^2}\right)C_2q_4(f)\right]^{-1}$$

for some C_1, $C_2\in(0,\infty)$. If we do not need to expand this denominator, but only to control it, we use the algebraic inequality: $(A+B)^{-1}\leqslant A^{-1}(1-|B|/A)^{-1}$ for all $A>|B|$, to write:

$$\left[\int dP\,dp\,|\tilde f_K(p,p)|^2\,\omega_\lambda(\delta p)^{-2}\right]^{-1}$$
$$<m_0^2\left[\int dP\,dp\,|\tilde f_K(P,p)|^2\right]^{-1}\left[1-\left(\delta+\frac{\lambda}{m_0^2}\right)C_2q_4(f)\right]^{-1}$$

The denominators of $I_{1,2}$, T and q_3 contain ω instead of ω_λ. From:

$$\frac{k^2+m_\lambda^2}{k^2+m_0^2}=1+\frac{m_\lambda^2-m_0^2}{k^2+m_0^2}\Rightarrow\frac{\omega_\lambda(k)}{\omega(k)}=1+O(\lambda)\qquad\forall k\in\mathbb{R}$$

it follows that they are bounded by the same expression as the denominator of $\mathcal{M}_\lambda^2(g)$. (Will will suppose that the constants C_1 and C_2 are big enough to include this case.)

Remark. We will often use the above argument to replace ω_λ by ω, and Ω_λ by Ω, when it is suitable, which gives an $O(\lambda)$-error.

2nd step: Expansion of $\mathcal{M}_\lambda^2(g)$ (kinetic type). With the formula of Appendix I, we obtain:

$$\mathcal{M}_\lambda^2(g)=4m_\lambda^2+4\delta^2\,\frac{\int dP\,dp\,|\tilde f_K(P,p)|^2\,p^2\omega_\lambda(\delta p)^{-2}}{\int dP\,dp\,|\tilde f_K(P,p)|^2\,\omega_\lambda(\delta p)^{-2}}$$

With the estimate:

$$\omega_\lambda(\delta p)^{-2}=m_\lambda^{-2}+O\left(\sqrt{\frac{\delta\,|p|}{m_\lambda}}\right)=m_0^{-2}+O\left(\left(\sqrt{\delta}+\frac{\lambda}{m_0^2}\right)\sqrt{\omega(p)}\right)$$

and with the 1st step we find:

$$\mathcal{M}_\lambda^2(g) = 4m_\lambda^2 + 4\delta^2 \frac{\int dP\, dp\, |\tilde{f}_K(P, p)|^2\, p^2}{\int dP\, dp\, |\tilde{f}_K(P, p)|^2} + \delta^{5/2} r_3(\lambda, \delta, f) + \delta^2 \sqrt{\frac{\lambda}{m_0^2}}\, r_4(\lambda, \delta, f)$$

where, for $i \in \{3, 4\}$:

$$|r_i(\lambda, \delta, f)| < C_3 q_4(f)^2 \left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_2 q_4(f)\right]^{-1}$$

for some $C_3 \in (0, \infty)$. To control $\mathcal{M}_\lambda^2(g)$, or $\mathcal{M}_\lambda^2(g) - 4m_\lambda^2$, we use that $\omega_\lambda(\delta p)^{-2}$ is bounded to find:

$$|\mathcal{M}_\lambda^2(g) - 4m_\lambda^2| < C_4 \delta^2 q_4(f) \left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_2 q_4(f)\right]^{-1}$$

with $C_4 \in (0, \infty)$. We have also to control $\partial_\lambda \mathcal{M}_\lambda^2(g)$; deriving $\mathcal{M}_\lambda^2$, and using:

$$\frac{\int dP\, dp\, |\tilde{f}_K(P, p)|^2\, m_\lambda \omega_\lambda(\delta p)^{-3}}{\int dP\, dp\, |\tilde{f}_K(P, p)|^2\, \omega_\lambda(\delta p)^{-2}} \leqslant 1$$

we find a similar bound as for $\mathcal{M}_\lambda^2(g)$ (we will suppose C_2 and C_4 big enough to include this case).

3rd step: Expansion of $I_1(g)$ (interactive term). With the formula of Appendix I, we find:

$$\lambda I_1(g) = \lambda \delta \frac{4!\, a_4}{4\pi} \frac{\int dP\, dp_1\, dp_2 \tilde{f}_K(-P, p_1) \tilde{f}_K(P, p_2) F_1(P, \delta p_1, \delta p_2)}{m_0^3 \int dP\, dp\, |\tilde{f}_K(P, p)|^2\, \omega_\lambda(\delta p)^{-2}}$$

where

$$F_1(P, p_1, p_2) = m_0^3 \frac{\Omega(P, p_1,)\Omega(P, p_2) + \Omega(P, 0)^2 + \mathcal{M}_0^2(g) - 4m_0^2}{(\Omega(P, p_1) + \Omega(P, p_2))\omega(p_1)^{3/2}\omega(p_2)^{3/2}\sqrt{\Omega(P, p_1)\Omega(P, p_2)}} (1 + 0(\lambda))$$

A little effort gives the expansion of F_1

$$F_1(P, \delta p_1, \delta p_2) = 1 + O(\lambda) + O(\sqrt{\delta \omega(p_1)\omega(p_2)}) + O(\mathcal{M}_0^2(g) - 4m_0^2)$$

so we find:

$$\lambda I_1(g) = \lambda\delta \frac{4!\, a_4}{4\pi m_0} \frac{\int dP\, dp_1\, dp_2 \tilde{f}_K(-P, p_1)\tilde{f}_K(P, p_2)}{\int dP\, dp\, |\tilde{f}_K(P, p)|^2} + \frac{\lambda}{m_0^2}\delta^{3/2} r_5(\lambda, \delta, f)$$

$$+ \delta\left(\frac{\lambda}{m_0^2}\right)^{3/2} r_6(\lambda, \delta, f)$$

where, for $i = 5$ or 6:

$$|r_i(\lambda, \delta, f)| < C_4 \frac{\int dP\, dp_1\, dp_2\, |\tilde{f}_K(P, p_1)\tilde{f}_K(P, p_2)|\, \sqrt{\omega(p_1)\omega(p_2)}}{\int dP\, dp\, |\tilde{f}_K(P, p)|^2}$$

$$\times q_4(f)\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_2 q_4(f)\right]^{-2}$$

for some $C_4 \in (0, \infty)$. We use: $[1 - A]^{-2} \leqslant [1 - 2A]^{-1}\ \forall A < \frac{1}{2}$, and the Cauchy–Schwarz inequality for the p_1 and p_2 variables:

$$\int dP\, dp_1\, dp_2\, |\tilde{f}_K(P, p_1)\tilde{f}_K(P, p_2)|\, \sqrt{\omega(p_1)\omega(p_2)} \leqslant C_5 \int dP\, dp\, |\tilde{f}_K(P, p)|^2\ \omega(p)^{5/2}$$

where $C_5 = \int_{\mathbb{R}} dp\omega(p)^{-3/2}$, to find that there exist $C_6, C_7 \in (0, \infty)$ with, for $i \in \{5, 6\}$:

$$|r_i(\lambda, \delta, f)| < C_6 q_4(f)^2\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_7 q_4(f)\right]^{-1}$$

We remark that $\delta^{-1}\, |I_4(g)|$ itself admits also the same bound (we suppose here C_6 and C_7 big enough) because

$$F_1(P, \delta p_1, \delta p_2) = O\left(q_4(f)\left[1 - C_2\left(\frac{\lambda}{m_0^2} + \delta\right) q_4(f)\right]^{-1}\right).$$

4th step: Expansion of $I_2(g)$ (interactive term). The first term of $\lambda^2 I_2(g)$ contains terms already discussed in steps 2 and 3; then we find easily that it is

$$O\left(\lambda^2 \delta q_4(f)^3\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_7 q_4(f)\right]^{-1}\right].$$

For the 2nd term of $\lambda^2 I_2(g)$, we use that the Σ_2^4 function is bounded (Appendix II). Then this term is bounded by a constant, time an expression which looks like $I_1(g)$, except that $\tilde{f}$ is replaced by $|\tilde{f}|$; it is bounded then in the same way.

We have then found that, for some $C_8 \in (0, \infty)$:

$$\lambda^2\, |I_2(g)| \leqslant \lambda^2 \sqrt{\delta}\, C_8 q_4(f)^3\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right) C_7 q_4(f)\right]^{-1}$$

5th step: Expansion of $T(g)$. We will not expand $T(U(\delta)f)$ in δ, but only obtain:

$$\lambda^2 T(U(\delta)f) = \lambda^2 \tau(U(\delta)f) + O(\lambda^2\delta)$$

In the integral of the numerator of $T(f)$, we see $|F(\ )|^2$, where

$$F(k_1; k_2, \ldots, k_{2n}) = \tilde{f}\left(k_1, \sum_{i=2}^{2n} k_i\right)\left[\left(\sum_{i=2}^{2n} \omega(k_i)\right)^2 - \left(\sum_{i=2}^{2n} k_i\right)^2 - m_0^2\right]^{-1}$$

which is not symmetric in $k_1, \ldots, k_{2n}$ (but the symmetrization is actually done by the integration); we do not make any modification in replacing $|F(\ )|^2$ by its symmetrized one, which we write as follows:

$$\frac{1}{2n}\sum_{i=1}^{2n} |F(k_i; k_1, \ldots, \check{k}_i, \ldots, k_{2n})|^2 = \frac{1}{2n}\left|\sum_{i=1}^{2n} F(k_i; k_1, \ldots, \check{k}_i, \ldots, k_{2n})\right|^2$$
$$-\frac{1}{2n}\sum_{i\neq j,=1}^{2n} F(k_i; k_1, \ldots, \check{k}_i, \ldots, k_{2n})F(k_j; k_1, \ldots, \check{k}_j, \ldots k_{2n})$$

(notation: $.\check{.}.$ stands for 'omitted'). We replace all this in $T(f)$; the first term gives just $\tau(f)$; the numerator of the second one gives the interactive term:

$$-\int\left(\prod_{i=1}^{4}\frac{dk_i}{2\omega(k_i)}\right)\tilde{f}(k_1, k_2)\tilde{f}(k_3, k_4)\delta(k_4 + \cdots + k_4)F_2(k_1, \ldots, k_4)$$

where

$$F_2(k_1, \ldots, k_4) = 8\omega(k_2)\omega(k_4)\sum_{n=2}^{\mathcal{N}}\frac{[(2n)!\, a_{2n}]^2}{(2n-2)!\,(2\pi)^{2(n-1)}}\int\left(\prod_{i=1}^{2n-2}\frac{dp_i}{2\omega(p_i)}\right)$$
$$\times\left.\frac{\delta(\sum p_i + k_2 + k_3)[(\eta + \omega(k_1) + \omega(k_3))^2 - (k_1 + k_2)^2 - \mathcal{M}_0^2(g)]}{[(\eta + \omega(k_3))^2 - \omega(k_2)^2][(\eta + \omega(k_1))^2 - \omega(k_4)^2]}\right|_{\eta = \sum\omega(p_i)}$$

We use two identities:

$$\frac{\eta + \omega_1 + \omega_3}{(\eta + \omega_3)^2 - \omega_2^2} = \frac{1}{\eta + \omega_2 + \omega_3} + \frac{\omega_1 + \omega_2}{(\eta + \omega_3)^2 - \omega_2^2}$$
$$\frac{\eta + \omega_1 + \omega_3}{(\eta + \omega_1)^2 - \omega_4^2} = \frac{1}{\eta + \omega_1 + \omega_4} + \frac{\omega_3 + \omega_4}{(\eta + \omega_1)^2 - \omega_4^2}$$

(with $\omega_i = \omega(k_i)$, $i = 1, \ldots, 4$) and the fact that

$$\int\left(\prod\frac{dp_i}{2\omega(p_i)}\right)\delta\left(\sum p_i + k\right)$$

is bounded to find:

$$|F_2(k_1, \ldots, k_4)| < C_9(m_0^2 + \mathcal{M}_0^2(g))(\omega_1 + \omega_2)^2(\omega_3 + \omega_4)^2$$

for some $C_9 \in (0, \infty)$. Then the numerator of $|T(g) - \tau(g)|$ can be bounded (with the formulas of Appendix I) by:

$$\delta C_{10}(m_0^2 + \mathcal{M}_0^2(g)) \int dP\, dp_1\, dp_2\, |\tilde{f}_K(P, p_1)\tilde{f}_K(P, p_2)|\, F_3(P, \delta p_1)F_3(P, \delta p_2)$$

for some $C_{10} \in (0, \infty)$, and with:

$$F_3(P, p) = \frac{\Omega(P, p)^2}{\omega(p)^{3/2}\sqrt{\Omega(P, p)}}(1 + O(\lambda))$$

Obviously $F_3(P, \delta p) = O(\Omega(P, 0)^{3/2})$. Using the Cauchy–Schwarz inequality as in the step 3, and the result of step 1, we find:

$$|T(g) - \tau(g)| < \delta C_{11} q_4(f)^2\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right)C_7 q_4(f)\right]^{-1}$$

for some $C_{11} \in (0, \infty)$.

6th step: Bound on $\mathcal{R}_3(\lambda, g)$. The first step and the Lemma 3.2.2 condense to:

$$q_3(g) = O\left(q_4(f)\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right)C_2 q_4(f)\right]^{-1}\right)$$

Then, using the inequality for $\mathcal{R}_3$ of the Proposition 1, we find:

$$|\mathcal{R}_3(\lambda, g)| < C_{12} q_4(f)\left[1 - \left(\delta + \frac{\lambda}{m_0^2}\right)C_{13} q_4(f)\right]^{-1}$$

for some $C_{12}, C_{13} \in (0, \infty)$. ■

3.2.4. *Proof of Proposition 3*

1st step: New contributions to the Rayleigh quotient. Here we drop the scaling. We follow exactly the proof of Proposition 1. We introduce the scalar products:

$$\chi(\lambda, s, f, \vec{f}) = 2\lambda \operatorname{Re} \sum_{n=2}^{\infty} (\psi_\lambda^1(f) \mid \hat{T}_\lambda(s) W_\lambda \phi_\lambda^{2n}(f_{2n}))_{\mathcal{H}_\lambda}$$
$$+ \lambda^2 \sum_{n,m=2}^{\infty} (W_\lambda \phi_\lambda^{2n}(f_{2n}) \mid \hat{T}_\lambda(s) W_\lambda \phi_\lambda^{2m}(f_{2m}))_{\mathcal{H}_\lambda}$$

and we must add $\Delta_s \chi(\lambda, s, f, \vec{f})|_{\lambda=0}$ to the numerator of $RQ_\lambda(\psi_\lambda^1(f))$ and $\chi(\lambda, 0, f, \vec{f})$ to his denominator.

The first term is decomposed according to a Wick formula ([o], §2.2.2) which gives, in symbolic notation:

$$:\phi_1^2:\, :\phi_2^n: = :\phi_1^2\phi_2^n: + 2nc(1-2) :\phi_1\phi_2^{n-1}: + n(n-1)c(1-2)^2 :\phi_2^{n-2}:$$

This gives three terms.

$\chi(\lambda, s, f, \vec{f})$ can be written with Schwinger functions $\int :\phi^k: d\mu_\lambda$, the perturbation expansion of which is given in Appendix B. We obtain:

$$\chi(\lambda, s, f, \vec{f}) = -2\lambda^2 \operatorname{Re} \sum_{n=2}^{\infty} \Theta_n$$
$$+ \lambda^2 \sum_{n=2}^{\infty} (2n)! \int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^{2n} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) |\tilde{f}_{2n}(\vec{k}_1, \ldots, \vec{k}_{2n})|^2$$
$$\times \exp\left(i\vec{s} \sum_{j=1}^{n} \vec{k}_j - |\mathring{s}| \sum_{j=1}^{n} \omega(\vec{k}_j) \right)$$
$$+ \lambda^3 r_1(\lambda, s, f, \vec{f})$$

where

$$\Theta_n = \frac{(2n+2)!\, a_{2n+2}}{(2\pi)^{3n+1}} \int_{\mathbb{R}^{4n+4}} \left(\prod_{i=1}^{2n+2} \frac{d^2k_i}{k_i^2 + m_0^2} \right)$$
$$\times \tilde{g}(\vec{k}_1, \vec{k}_2) \tilde{f}_{2n}(\vec{k}_3, \ldots, \vec{k}_{2n+2}) \delta^{(2)}\left(\sum_{i=1}^{2n+2} k_i \right) e^{is(k_1+k_2)}$$
$$+ \frac{4n(2n)!\, a_{2n}}{(2n)^{3n-2}} \int_{\mathbb{R}^{4n+1}} \frac{d\vec{k}_1}{2\omega(\vec{k}_1)} \left(\prod_{i=2}^{2n+1} \frac{d^2k_i}{k_i^2 + m_0^2} \right)$$
$$\times \tilde{g}(-\vec{k}_1, \vec{k}_2) \tilde{f}_{2n}(\vec{k}_1, \vec{k}_3, \ldots, \vec{k}_{2n+1}) \delta^{(2)}\left(\sum_{i=2}^{2n+1} k_i \right) e^{i\vec{s}(\vec{k}_1+\vec{k}_2)} e^{i\mathring{s}\mathring{k}_2 - |\mathring{s}|\omega(\vec{k}_1)}$$
$$+ \frac{(2n)!\, a_{2n-2}}{(2\pi)^{3n-5}} \int_{\mathbb{R}^{4n-2}} \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)} \right) \left(\prod_{i=3}^{2n} \frac{d^2k_i}{k_i^2 + m_0^2} \right) \tilde{g}(-\vec{k}_1, -\vec{k}_2) \tilde{f}_{2n}(\vec{k}_1, \ldots, \vec{k}_{2n})$$
$$\times \delta^{(2)}\left(\sum_{i=3}^{2n} k_i \right) \exp\left(i\vec{s}(\vec{k}_1 + \vec{k}_2) - |\mathring{s}| \sum_{j=1}^{2} \omega(\vec{k}_j) \right)$$

$\lambda^2 r_1$ is the remainder of the serie and $g = U(\lambda/m_0^2)f$. The Laplacian Δ_s of χ is given by the derivatives under the integrals. We put then $s = 0$. We integrate over the $\mathring{k}_j$-variables, using the formula of the residues. We do the division in the Rayleigh quotient with the algebraic formula of the 4th step. Then the new contribution in $RQ(\psi_\lambda^3(f, \vec{f}))$ due to $\vec{f}$ is:

$$\frac{\lambda^2}{2 \int \left(\prod_{i=1}^{2} \frac{d\vec{k}_i}{2\omega_\lambda(\vec{k}_i)} \right) |\tilde{g}(\vec{k}_1, \vec{k}_2)|^2}$$
$$\times \left\{ 2 \operatorname{Re} \sum_{n=2}^{\infty} (\Xi_n^1 + \Xi_n^2 + \Xi_n^3) + \sum_{n=2}^{\infty} 2 \int_{\mathbb{R}^{2n}} d\sigma_{g,2n}(k)\, |\tilde{f}_{2n}(k)|^2 \right\} + \lambda^3 r_2(\lambda, f, \vec{f})$$

where:

$$\Xi_n^1 = -\frac{2(2n+2)!\,a_{2n+2}}{(2\pi)^n}\int\left(\prod_{i=1}^{2n+2}\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\bar{g}(\vec{k}_1,\vec{k}_2)\bar{f}_{2n}(\vec{k}_3,\ldots,\vec{k}_{2n})$$

$$\times\frac{\delta\left(\sum_{i=1}^{2n+2}\vec{k}_i\right)}{\sum_{i=1}^{2n+2}\omega(\vec{k}_i)}\left(\left(\sum_{i=1}^{2}\omega(\vec{k}_i)\right)\left(\sum_{i=3}^{2n+2}\omega(\vec{k}_i)\right)-(\vec{k}_1+\vec{k}_2)^2-\mathcal{M}_0^2(g)\right)$$

$$\Xi_n^2 = -\frac{8n(2n)!\,a_{2n}}{(2\pi)^{n-1}}\int\left(\prod_{i=1}^{2n+1}\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\bar{g}(-\vec{k}_1,\vec{k}_2)\bar{f}_{2n}(\vec{k}_1,\vec{k}_3,\ldots,\vec{k}_{2n+1})$$

$$\times\frac{\delta\left(\sum_{i=2}^{2n+1}\vec{k}_i\right)}{\sum_{i=2}^{2n+1}\omega(\vec{k}_i)}\left(\omega(\vec{k}_1)^2-\omega(\vec{k}_2)\sum_{i=3}^{2n+1}\omega(\vec{k}_i)-(\vec{k}_1+\vec{k}_2)^2-\mathcal{M}_0^2(g)\right)$$

$$\Xi_n^3 = -\frac{2(2n)!\,a_{2n-2}}{(2\pi)^{n-2}}\int\left(\prod_{i=1}^{2n}\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\bar{g}(-\vec{k}_1,-\vec{k}_2)\bar{f}_{2n}(\vec{k}_1,\ldots,\vec{k}_{2n})$$

$$\times\frac{\delta\left(\sum_{i=3}^{2n}\vec{k}_i\right)}{\sum_{i=3}^{2n}\omega(\vec{k}_i)}\left(\left(\sum_{i=1}^{2}\omega(\vec{k}_i)\right)^2-(\vec{k}_1+\vec{k}_2)^2-\mathcal{M}_0^2(g)\right)$$

Here $\lambda^3 r_2$ is the total remainder (including that one of Proposition 1). From [o, Theorem of §3.3] it follows that all c_i' and d_j' of the algebraic formula of the 4th step (see proof of Proposition 1) are bounded by a constant time $\bar{q}_5(g,\vec{f})$, where:

$$\bar{q}_5(g,\vec{f}) = \frac{bp_{2,2}(g)^2+\sum_n bp_{2n,2}(f_{2n})^2}{bp_{2,0}(g)^2}$$

Then there exist $C_1, C_2 \in (0,\infty)$ with:

$$|r_2(\lambda,f,\vec{f})| < C_1\bar{q}_5(g,\vec{f})^4[1-\lambda C_2\bar{q}_5(g,\vec{f})]^{-1}$$

2nd step: Expansion in $\delta=\lambda/m_0^2$. We follow the proof of Proposition 2, using the inequalities established there. With $g=U(\delta)f$ and with the help of Appendix I, Ξ_n^1 can be written as:

$$\Xi_n^1 = C_3\sqrt{\delta}\int d\vec{P}\,d\vec{p}\bar{f}_K(\vec{P},\vec{p})\int\left(\prod_{i=1}^{2n}\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\bar{f}_{2n}(\vec{k}_1,\ldots,\vec{k}_{2n})\delta\left(\vec{P}+\sum_{i=1}^{2n}\vec{k}_i\right)$$

$$\times F_1(\vec{P},\delta\vec{p},\vec{k}_1,\ldots,\vec{k}_{2n})$$

where $C_3\in(0,\infty)$ and:

$$F_1(\vec{P},\vec{p},\vec{k}_1,\ldots,\vec{k}_n) = \frac{\left(\Omega(\vec{P},\vec{p})+\sum_{i=1}^{n}\omega(\vec{k}_i)\right)^2-\left(\vec{P}+\sum_{i=1}^{n}\vec{k}_i\right)^2-\mathcal{M}_0^2(g)}{\left(\Omega(\vec{P},\vec{p})+\sum_{i=1}^{n}\omega(\vec{k}_i)\right)\omega(\vec{p})\sqrt{\Omega(\vec{P},\vec{p})}}$$

Obviously:

$$|F_1(\vec{P}, \vec{p}, \vec{k}_1, \ldots, \vec{k}_n)| = O\Big(\Omega(\vec{P}, 0) + \sum_{i=1}^{n} \omega(\vec{k}_i) + \mathcal{M}_0^2(g)\Big)$$

We use three times the Cauchy–Schwarz inequality: the first time for

$$\int d\vec{p}\, |\tilde{f}(\vec{P}, \vec{p})| \leq \text{cste}\left[\int d\vec{p}\, |\tilde{f}(\vec{P}, \vec{p})|^2\, \omega(\vec{p})^2\right]^{1/2},$$

the second time for:

$$\int \Big(\prod_{i=1}^{2n} \frac{d\vec{k}_i}{\omega(\vec{k}_i)}\Big) |\tilde{f}_{2n}(\vec{k}_1, \ldots, \vec{k}_{2n})|\, \delta\Big(\vec{P} + \sum_{i=1}^{2n} \vec{k}_i\Big) \sum_{i=1}^{2n} \omega(\vec{k}_i)$$
$$\leq \text{cste}\left[\int \Big(\prod_{i=1}^{2n} \frac{d\vec{k}_i}{\omega(\vec{k}_i)}\Big) |\tilde{f}_{2n}(\vec{k}_1, \ldots, \vec{k}_{2n})|^2\, \delta\Big(\vec{P} + \sum_{i=1}^{2n} \vec{k}_i\Big)\Big(\sum_{i=1}^{2n} \omega(\vec{k}_i)\Big)^2\right]^{1/2}$$

and the third time on the dP-integration, to find:

$$|\Xi_n^1| < C_4\sqrt{\delta}\left[\int dP\, dp\, |\tilde{f}_K(P, p)|^2\, \omega(P)^2 \omega(p)^2\right]^{1/2} bp_{2n,2}(f_{2n})\mathcal{M}_0^2(g)$$

with

$$C_4 \in (0, \infty)$$

Ξ_n^2 is written as a sum of two pieces, using the identity (with symbolic notation):

$$\frac{\omega_1^2 - \omega_2(\omega_3 + \cdots + \omega_{2n+1}) - (\vec{k}_1 + \vec{k}_2)^2 - \mathcal{M}^2}{\omega_2(\omega_2 + \cdots + \omega_{2n+1})}$$
$$= -\frac{(\omega_1 + \omega_3 + \cdots + \omega_{2n+1})^2 - (k_1 + k_2)^2 - \mathcal{M}^2}{(\omega_3 + \cdots + \omega_{2n+1})^2 - \omega_2^2}$$
$$+ \frac{(\omega_3 + \cdots + \omega_{2n+1})[(\omega_1 + \omega_2)^2 - (k_1 + k_2)^2 - \mathcal{M}^2]}{\omega_2[(\omega_3 + \cdots + \omega_{2n+1})^2 - \omega_2^2]}$$

the first term gives the following contribution to Ξ_n^2:

$$-2\int_{\mathbb{R}^{2n}} d\sigma_{g,2n}(k)\overline{\tilde{F}_{g,2n}(k)}\tilde{f}_{2n}(k)$$

and the second one contributes to:

$$C_5 \int \Big(\prod_{i=1}^{2} \frac{dk_i}{2\omega(k_i)}\Big)\tilde{g}(-k_1, k_2)\left[\Big(\sum_{i=1}^{2} \omega(k_i)\Big)^2 - (k_1 + k_2)^2 - \mathcal{M}_0^2(g)\right]$$
$$\times \int \Big(\prod_{i=3}^{2n+1} \frac{dk_i}{2\omega(k_i)}\Big)\tilde{f}_{2n}(k_1, k_3, \ldots, k_{2n+1})\, \frac{\delta\Big(\sum_{i=2}^{2n+1} k_i\Big) \sum_{i=3}^{2n+1} \omega(k_i)}{\Big(\sum_{i=3}^{2n+1} \omega(k_i)\Big)^2 - \omega(k_2)^2}$$

with $C_5 \in (0, \infty)$, which is bounded by:

$$C_6\Big[\int\Big(\prod_{i=1}^{2}\frac{dk_i}{\omega(k_i)}\Big)|\tilde{g}(k_1,k_2)|^2\,[(\omega(k_1)+\omega(k_2))^2-(k_1+k_2)^2-\mathcal{M}_0^2(g)]^2\Big]^{1/2}$$
$$\times\Big[\int\Big(\prod_{i=1}^{2n+1}\frac{dk_i}{\omega(k_i)}\Big)|\tilde{f}_{2n}(k_1,k_3,\ldots,k_{2n+1})|^2$$
$$\times\,\delta(k_2+\cdots+k_{2n+1})\Big(\sum_{i=3}^{2n+1}\omega(k_i)\Big)^2\Big]^{1/2}$$

with $C_6 \in (0, \infty)$. For the first factor, we use $g = U(\delta)f$, and:

$$\Big[\int dP\,dp\,|\tilde{f}_K(P,p)|^2\frac{\delta^4p^4}{\omega_\lambda(\delta p)^2}\Big]^{1/2} = O\Big(\delta\sqrt{\int dP\,dp\,|\tilde{f}_K(P,p)|^2\,\omega(p)^2}\Big)$$

(The factor $\omega_\lambda^{-1/2}$ in the definition of $U(\delta)$ of Table I is here essential.) The second factor, after integration over k_2 using the δ function, is $O(bp_{2n,2}(f_{2n}))$.

Ξ_n^3 can be treated as the second term of Ξ_n^2, and be bounded in the same way. Remember that Ξ_n^i must be divided by

$$2\int\Big(\prod_{i=1}^{2} dk_i/2\omega_\lambda(k_i)\Big)|\tilde{g}(k_1,k_2)|^2$$

which is $\frac{1}{4}bp_{2,0}(g)^2(1+O(\lambda))$. We bound $bp_{2,2}(g)$ with the lemma of §3.2.2. Then we obtain: $\bar{q}_5(g,\vec{f}) = O(q_5(f,\vec{f}))$.

3rd step: We collect the results. The addition of the vectors $\lambda\sum\phi^{2n}(f_{2n})$ gives the following contribution to the Rayleigh quotient:

$$\frac{\lambda^2}{\frac{1}{4}bp_{2,0}(g)^2}\sum_{n=2}^{\infty}\Big\{-2\,\mathrm{Re}\int_{\mathbb{R}^{2n}} d\sigma_{g,2n}(k)\bar{\tilde{F}}_{g,2n}(k)\tilde{f}_{2n}(k)$$
$$+\int_{\mathbb{R}^{2n}} d\sigma_{g,2n}(k)\,|\tilde{f}_{2n}(k)|^2\Big\} + \lambda^{5/2}r_3(\lambda,f,\vec{f})$$

where

$$|r_3(\lambda,f,\vec{f})| < C_7q_5(f,\vec{f})^4[1-\lambda C_8q_5(f,\vec{f})]^{-1}$$

for some $C_7, C_8 \in (0, \infty)$. If we add $\tau(f)$ of Table IV, which can be written as

$$\tau(f) = \frac{\lambda^2}{\frac{1}{4}bp_{2,0}(g)^2}\sum_{n=2}^{\mathcal{N}}\int_{\mathbb{R}^{2n}} d\sigma_{g,2n}(k)\,|\tilde{F}_{g,2n}(k)|^2$$

we obtain the sum of perfect squares announced. ■

3.2.5. *Bound on $bp_{2n,2}(F_{f,2n})$*

Lemma. *For all $n \in \mathbb{N}$, $n \geq 2$, there exist $K \in (0, \infty)$ such that for all $f \in \mathcal{B}p_{2,5/2}$: $bp_{2n,2}(F_{f,2n}) < Kbp_{2,5/2}(f)$*

Proof. We read the bp-norm in [o, §3.3.2]. In the definition of $F_{f,2n}$, there is a sum, which we must square. We use the inequality (in symbolic notation):

$$\left|\sum_{1\leq i,j\leq 2n} F_iF_j\right| \leq C\sum |F_i|^2,$$

where $C\in(0,\infty)$, to write:

$$bp_{2n,2}(F_{g,2n})^2 \leq C_1\int\left(\prod_{i=1}^{m}\frac{dk_i}{\omega(k_i)}\right)\left\{\underset{k_1,\ldots,k_m}{\mathrm{Sym}}\frac{|\tilde{f}_s(k_1,k_2+\cdots+k_m)|^2}{\left[\left(\sum_{i=2}^{m}\omega(k_i)\right)^2-\left(\sum_{i=2}^{m}k_i\right)^2-m_0^2\right]^2}\right\}$$
$$\times\sum_{\substack{j=0\\ \text{even}}}^{m-1}\left(\sum_{i=j+1}^{m}\omega(k_i)\right)^2\sum_{p\in\mathcal{P}p_j}\prod_{I\in p}\delta\left(\sum_{i\in I}k_i\right)$$

with $C_1\in(0,\infty)$ and $m=2n$. $\mathrm{Sym}_{k_1,\ldots,k_m}$ is the symmetrisation operator with respect to $k_1,\ldots,k_m$. Note that the δ functions concern only an even number of variables k_j, $\mathcal{P}p_j$ being the set of even partitions of $\{1,\ldots,j\}$ (see [o, §3.3.2]).

In the factor in $\{\ldots\}$, a variable is distinguished (noted 'k_1'). The development of $bp(F)$ gives two kinds of terms, depending on:

1) the δ functions do not concern the distinguished variable
2) the distinguished variable enters in a δ function.

Using that, for all $m\in\mathbb{N}^*$, $\int_{\mathbb{R}^m}(\prod_{1\leq i\leq m}dk_i/\omega(k_i))\delta(\sum_{1\leq i\leq m}k_i+Q)$ is bounded for all $Q\in\mathbb{R}$, each term of the type 1) is bounded by:

$$C_2\int\left(\prod_{i=1}^{r}\frac{dk_i}{\omega(k_i)}\right)\frac{|\tilde{f}_s(k_1,k_2+\cdots+k_r)|^2}{\left[\left(\sum_{i=2}^{r}\omega(k_i)+(2n-r)m_0\right)^2-\left(\sum_{i=2}^{r}k_i\right)-m_0^2\right]^2}\left(\sum_{i=1}^{r}\omega(k_i)\right)^2$$

for some $C_2\in(0,\infty)$ and $1<r\leq 2n$ ($r=1$ is excluded because r is even). This can be written as:

$$C_2\int\left(\prod_{i=1}^{2}\frac{dk_i}{\omega(k_i)}\right)|\tilde{f}_s(k_1,k_2)|^2$$
$$\times\int\left(\prod_{i=1}^{r-1}\frac{dp_i}{\omega(p_i)}\right)\frac{\delta\left(k_2-\sum_{i=1}^{r-1}p_i\right)\omega(k_2)(\omega(k_1)+\eta)^2}{[(\eta+\alpha)^2-\omega(k_2)^2]^2}\Bigg|_{\eta=\sum_{i=1}^{r-1}\omega(p_i)}$$

where $\alpha=(2n-r)m_0$. Because of the δ function, the denominator causes no difficulty. Using the equality (in symbolic notation):

$$\frac{\eta}{\eta+\alpha-\omega_2}=1+\frac{\omega_2-\alpha}{\eta+\alpha-\omega_2}$$

the integral over $p_1,\ldots,p_{r-1}$ is bounded by

$$(\omega_1+\omega_2)^2\omega_2\int\left(\prod_{1\leq i\leq r-1}dp_i/\omega(p_i)\right)\times\delta\left(k_2-\sum_{1\leq i\leq r-1}p_i\right)$$

time a constant. With the standard estimate, for all $m \in \mathbb{N}^*$:

$$\int_{\mathbb{R}^m} \left(\prod_{i=1}^{m} \frac{dk_i}{\omega(k_i)}\right) \delta\left(\sum_{i=1}^{m} k_i + Q\right) \leq C_3 \omega(Q)^{-1/2}$$

for some $C_3 \in (0, \infty)$, for all $Q \in \mathbb{R}$ (which can be proved using [r]), we obtain that the sum of the terms of type 1) is bounded by:

$$C_4 \int \left(\prod_{i=1}^{2} \frac{dk_i}{\omega(k_i)}\right) |\tilde{f}(k_1, k_2)|^2 (\omega(k_1) + \omega(k_2))^{5/2}$$

with $C_4 \in (0, \infty)$, in agreement with the claimed result.

Each term of the type 2) can be bounded by a constant time:

$$\int \left(\prod_{i=1}^{u} \frac{dk_i}{\omega(ki)}\right) \frac{|\tilde{f}_s(k_1, k_2 + \cdots + k_u)|^2}{\left[\left(\sum_{i=2}^{u} \omega(k_i) + (2n-t)m_0^2\right)^2 - \left(\sum_{i=2}^{u} k_i\right)^2 - m_0^2\right]^2}$$
$$\times \left(\sum_{i=2}^{t} \omega(k_i)\right)^2 \delta\left(k_1 + \sum_{i=t+1}^{u} k_i\right)$$

for some $2 \leq t \leq 2n-1$ and $1 + t \leq u \leq 2n$. This can be written as:

$$\int \left(\prod_{i=1}^{2} \frac{dk_i}{\omega(k_i)}\right) |\tilde{f}_s(k_1, k_2)|^2 \int \left(\prod_{i=1}^{t-1} \frac{dp_i}{\omega(p_i)}\right) \frac{\delta\left(k_1 + k_2 - \sum_{i=1}^{t-1} p_i\right) \omega(k_2) \eta^2}{[(\eta + \beta)^2 - (k_1 + k_2)^2 - m_0^2]^2} \Bigg|_{\eta = \sum_{i=1}^{t-1} \omega(p_i)}$$

where $\beta(2n-u)m_0$. Using the same tool as for the type 1), we find that the integral over $p_1, \ldots, p_{t-1}$ can be bounded by a constant time $\omega(k_2)\psi(k_1 + k_2)^{3/2} \leq 4(\omega(k_4) + \omega(k_2))^{5/2}$. Thus we obtain the same estimations as for 1). ■

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Appendix I. Relativistic kinematics of two free particles

We are interested in the physical system of two free particles of mass m, in one space dimension.

A state of this system is given by two momentum $(\vec{k}_1, \vec{k}_2) \in \mathbb{R}^2$. We introduce four vectors in the energy-momentum space:

$$k_1 = (\mathring{k}_1, \vec{k}_1) = (\omega(\vec{k}_1), \vec{k}_1)$$
$$k_2 = (\mathring{k}_2, \vec{k}_2) = (\omega(\vec{k}_2), \vec{k}_2)$$
$$K = (\mathring{K}, \vec{K}) = k_1 + k_2$$
$$k = (\mathring{k}, \vec{k}) = k_1 - k_2$$

with $\omega(\vec{k}) = \sqrt{\vec{k}^2 - m^2}$. They satisfy:

$$K^2 = 4m^2 - k^2$$
$$K \cdot k = 0$$

(with Minkowsky's metric). Note that $k^2 \leqslant 0$

First change of variables
The states of the system are parametrized by two hyperbolic angles $(\alpha, \chi) \in \mathbb{R}^2$, defined by:

$$K = \sqrt{K^2}\,(\text{ch}\,\alpha, \text{sh}\,\alpha)$$
$$\text{ch}\,\chi = \frac{k_1 \cdot k_2}{m^2}$$

sign of χ = sign of $\vec{k}_1 - \vec{k}_2$. We obtain immediately:

$$K^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = m^2 + m^2 + 2m^2\,\text{ch}\,\chi = 4m^2\left(\text{ch}\,\frac{\chi}{2}\right)^2$$
$$-k^2 = -k_1^2 - k_2^2 + 2k_1 \cdot k_2 = -m^2 - m^2 + 2m^2\,\text{ch}\,\chi = 4m^2\left(\text{sh}\,\frac{\chi}{2}\right)^2$$

Our four vectors can be written as functions of α and χ:

$$k_1 = \left(m\,\text{ch}\,\frac{\chi}{2}\,\text{ch}\,\alpha + m\,\text{sh}\,\frac{\chi}{2}\,\text{sh}\,\alpha,\ m\,\text{ch}\,\frac{\chi}{2}\,\text{sh}\,\alpha + m\,\text{sh}\,\frac{\chi}{2}\,\text{ch}\,\alpha\right)$$
$$k_2 = \left(m\,\text{ch}\,\frac{\chi}{2}\,\text{ch}\,\alpha - m\,\text{sh}\,\frac{\chi}{2}\,\text{sh}\,\alpha,\ m\,\text{ch}\,\frac{\chi}{2}\,\text{sh}\,\alpha - m\,\text{sh}\,\frac{\chi}{2}\,\text{ch}\,\alpha\right)$$
$$K = 2m\,\text{ch}\,\frac{\chi}{2}\,(\text{ch}\,\alpha, \text{sh}\,\alpha)$$
$$k = 2m\,\text{sh}\,\frac{\chi}{2}\,(\text{sh}\,\alpha, \text{ch}\,\alpha)$$

Lorentz transformation
The action of a Lorentz transformation of velocity $\beta < 1$ on a state $(\vec{k}_1, \vec{k}_2)$ is given by:

$$\vec{k}_i \rightsquigarrow \vec{k}_i' = \frac{\vec{k}_i + \beta\omega(\vec{k}_i)}{\sqrt{1-\beta^2}} \qquad i = 1, 2$$

The transformation of ω is given by $\omega(\vec{k}_i)' = \omega(\vec{k}_i')$, $i = 1, 2$. As function of α and χ, $\vec{k}_1'$ is:

$$\vec{k}_1' = \frac{m}{\sqrt{1-\beta^2}}\left(\text{ch}\,\frac{\chi}{2}\,\text{sh}\,\alpha + \text{sh}\,\frac{\chi}{2}\,\text{ch}\,\alpha + \beta\,\text{ch}\,\frac{\chi}{2}\,\text{ch}\,\alpha + \beta\,\text{sh}\,\frac{\chi}{2}\,\text{sh}\,\alpha\right)$$
$$= m\left(\text{ch}\,\frac{\chi}{2}\,\frac{\text{sh}\,\alpha + \beta\,\text{ch}\,\alpha}{\sqrt{1-\beta^2}} + \text{sh}\,\frac{\chi}{2}\,\frac{\beta\,\text{sh}\,\alpha + \text{ch}\,\alpha}{\sqrt{1-\beta^2}}\right)$$
$$= m\left(\text{ch}\,\frac{\chi}{2}\,\text{sh}\,(\alpha + \gamma) + \text{sh}\,\frac{\chi}{2}\,\text{ch}\,(\alpha + \gamma)\right)$$

where $\gamma = \arg\operatorname{th}\beta$. The same calculations of $\vec{k}_2'$ convince us of that the transformation of (α, χ) is:

$$\alpha \rightsquigarrow \alpha' = \alpha + \arg\operatorname{th}\beta$$

$$\chi \rightsquigarrow \chi' = \chi$$

This looks like a galilean transformation (for which the relative momentum is invariant, and the total momentum is shifted by a constant).

Invariant measure

The Jacobian of the change of variables $(\vec{k}_1, \vec{k}_2) \rightsquigarrow (\alpha, \chi)$ is:

$$\det\begin{pmatrix} \frac{1}{2}m\operatorname{sh}\frac{\chi}{2}\operatorname{sh}\alpha + \frac{1}{2}m\operatorname{ch}\frac{\chi}{2}\operatorname{ch}\alpha & m\operatorname{ch}\frac{\chi}{2}\operatorname{ch}\alpha + m\operatorname{sh}\frac{\chi}{2}\operatorname{sh}\alpha \\ \frac{1}{2}m\operatorname{sh}\frac{\chi}{2}\operatorname{sh}\alpha - \frac{1}{2}m\operatorname{ch}\frac{\chi}{2}\operatorname{ch}\alpha & m\operatorname{ch}\frac{\chi}{2}\operatorname{ch}\alpha - m\operatorname{sh}\frac{\chi}{2}\operatorname{sh}\alpha \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{1}{2}\omega_1 & \omega_1 \\ -\frac{1}{2}\omega_2 & \omega_2 \end{pmatrix} = \omega(\vec{k}_1)\omega(\vec{k}_2)$$

The invariant measure on the phase space is:

$$\frac{d\vec{k}_1}{\omega(\vec{k}_1)}\frac{d\vec{k}_2}{\omega(\vec{k}_2)} = d\alpha\, d\chi$$

Second change of variables

We must change the variables (α, χ) because

1° we need variables with units of momentum
2° the choice of zero-time vectors distinguishes the variable $\vec{K}$.

We define the new variables $(\vec{P}, \vec{p})$ by:

$$\vec{P} = \vec{K}$$

$$\vec{p} = m\operatorname{sh}\frac{\chi}{2}$$

The Jacobian of $(\alpha, \chi) \rightsquigarrow (\vec{P}, \vec{p})$ is given by:

$$J^{-1} = \det\left(\frac{\partial\vec{P}}{\partial\alpha}\frac{\partial\vec{p}}{\partial\chi}\right) = m\operatorname{ch}\frac{\chi}{2}m\operatorname{ch}\frac{\chi}{2}\operatorname{ch}\alpha = \tfrac{1}{2}\omega(\vec{p})\mathring{K}$$

We note $\mathring{K} = \omega(\vec{k}_1) + \omega(\vec{k}_2) = \sqrt{\vec{p}^2 + 4\omega(\vec{p})^2} = \Omega(\vec{P}, \vec{p})$
We have then the following formulas:

$$k_1 = \tfrac{1}{2}\left(\Omega(\vec{P}, \vec{p}) + \frac{\vec{P}\vec{p}}{\omega(\vec{p})}, \vec{P} + \vec{p}\,\frac{\Omega(\vec{p}, \vec{p})}{\omega(\vec{p})}\right)$$

$$k_2 = \tfrac{1}{2}\left(\Omega(\vec{P}, \vec{p}) - \frac{\vec{P}\vec{p}}{\omega(\vec{p})}, \vec{P} - \vec{p}\,\frac{\Omega(\vec{P}, \vec{p})}{\omega(\vec{p})}\right)$$

$$K = (\Omega(\vec{P}, \vec{p}), \vec{P})$$
$$k = \left(\frac{\vec{P}\vec{p}}{\omega(\vec{p})}, \vec{p}\,\frac{\Omega(\vec{P}, \vec{p})}{\omega(\vec{p})}\right)$$
$$K^2 = 4\omega(\vec{p})^2 = 4m^2 + 4\vec{p}^2$$
$$k^2 = -4\vec{p}^2$$
$$\omega(\vec{k}_1)\omega(\vec{k}_2) = \frac{m^2\vec{P}^2 + 4\omega(\vec{p})^4}{4\omega(\vec{p})^2}$$

Appendix II. Perturbative expansion of some Schwinger functions

The perturbative expansion of the Schwinger functions will be performed by using the formula of [o, §2.1.3]. The remainder of the series will be simply noted as '$O(\lambda^n)$', and will not be discussed here. We adopt the notation $\langle F\rangle_\lambda = \int_Q F(q)\, d\mu_\lambda(q)$ for $F \in L^1(Q, \mu_\lambda)$. The truncated functions (see [o, §2.1.2]) are noted as $\langle .\, ; \ldots ; .\rangle_\lambda^T$.

Two-point function
The function $\langle \phi(x), \phi(y)\rangle^T = s_{2,\lambda}^T(x-y, 0)$ can be written as:

$$s_{2,\lambda}^T(.\,, 0) = c - \lambda\langle :\mathcal{P}''(\phi(0)):\rangle_\lambda c * c + c * c * T_\lambda$$

where $c(x) = (2\pi)^{-2}\int dx e^{ik\cdot x}/(k^2+m^2)$ and $T_\lambda(x) = \langle :\mathcal{P}'(\phi(x)):, :\mathcal{P}'(\phi(0)):\rangle_\lambda^T$ for all $x \in \mathbb{R}^2$, $x \neq 0$. The first terms of the expansions in λ are:

$$s_{2,\lambda}^T(.\,, 0) = c + (-\lambda\alpha + \beta\lambda^2)c * c + \alpha^2\lambda^2 c * c * c + \lambda^2 c * c * \tau + O(\lambda^2)$$

where $\alpha = 2a_2$, $\beta = \sum_n (n+1)(n+2)a_{n+2}a_n \int c^n$ and $\tau(x) = \sum_{n=3}^{2\mathcal{N}} (na_n)^2(n-1)!\, c(x)^{n-1}$. By Fourier transformation:

$$\begin{aligned}\tilde{s}_{2,\lambda}^T(k_1, k_2) &= \delta^{(2)}(k_1+k_2)\\ &\quad\times\left(\frac{1}{k_1^2+m_0^2} + \frac{-\lambda\alpha+\beta\lambda^2}{(k_1^2+m_0^2)^2} + \frac{\lambda^2\alpha^2}{(k_1^2+m_0^2)^3} + \frac{\lambda^2\mathcal{T}(k_1)}{(k_1^2+m_0^2)^2} + O(\lambda^3)\right)\\ &= \frac{\delta^{(2)}(k_1+k_2)}{k_1^2+m_0^2+\lambda\alpha+\lambda^2(\beta-\mathcal{T}(k_1))} + O(\lambda^3)\end{aligned}$$

where:

$$\mathcal{T}(k) = 2\pi\tilde{\tau}(k) = 2\sum_{n=3}^{2\mathcal{N}} \frac{nn!\,a_n^2}{(2\pi)^{n-2}}\int\left(\prod_{i=1}^{n-1}\frac{d\vec{p}_i}{2\omega(\vec{p}_i)}\right)\frac{\delta\left(\sum_{i=1}^{n-1}\vec{p}_i - \vec{k}\right)\sum_{i=1}^{n-1}\omega(\vec{p}_i)}{\mathring{k}^2 + \left(\sum_{i=1}^{n-1}\omega(\vec{p}_i)\right)^2}$$

is a function of k^2, analytic in the hole plane $\mathbb{C}$ except for the cut: $\{k^2 \in \mathbb{C} \mid Jmk^2 = 0, \operatorname{Re} k^2 \leq -2m_0\}$. Thus the above approximation of $\tilde{s}_{2,\lambda}^T$ has a simple pole for $k^2 = -m_\lambda^2$, near $-m_0^2$, where:

$$m_\lambda^2 = m_0^2 + \lambda\alpha + \lambda^2(\beta - \mathcal{T}(k)\,|_{k^2=-m_\lambda^2}) = m_0^2 + \lambda\alpha + \lambda^2(\beta - \mathcal{T}(k)\,|_{k^2=-m_0^2}) + O(\lambda^3)$$

Then $\tilde{s}^T_{2,\lambda}$ can be written as:

$$\tilde{s}^T_{2,\lambda}(k_1, k_2) = \delta^{(2)}(k_1 + k_2)\left(\frac{1}{k_1^2 + m_0^2} + \lambda^2 \frac{T(k_1^2)}{(k_1^2 + m_0^2)^2} + O(\lambda^3)\right)$$

where $\mathfrak{T}(k^2) = T(k) - \mathfrak{T}(k)\,|_{k^2=-m_0^2}$.

Four point function

The perturbation development of the function $s^T_{4,\lambda}(x_1, \ldots, x_4) = \langle \phi(x_1); \ldots ; \phi(x_4)\rangle^T_\lambda$ begins with:

$$\tilde{s}^T_{4,\lambda}(k_1, \ldots, k_4) = \frac{\delta^{(2)}\left(\sum_{i=1}^{4} k_i\right)}{\prod_{i=1}^{4} (k_i^2 + m_0^2)} \left[-\gamma\lambda + \lambda^2 \sum_2^4 (k_1, \ldots, k_4)\right] + O(\lambda^3)$$

where

$$\gamma = \langle :\mathcal{P}^{(\mathrm{IV})}(\phi(0)):\rangle_0 (2\pi)^{-2} = 4!\, a_4 (2\pi)^{-2}$$

and

$$\sum_2^4 (k_1, \ldots, k_4) = \partial_\lambda \langle :\mathcal{P}^{(\mathrm{IV})}(\phi(0)):\rangle_\lambda\,|_{\lambda=0} + \partial_\lambda^2 \sum_\lambda^4 (k_1, \ldots, k_4)\,|_{\lambda=0}$$

where $\sum_\lambda^4$ is introduced in [o, §2.3.3].

It follows from [o, §2.4.3] that $k_1, \ldots, k_4 \mapsto \sum_2^4 (k_1, \ldots, k_4)$ is a bounded function on the manifold defined by $\sum k_i = 0$.

The n-point functions

The Proposition 3 need the first order of $\langle :\phi(x_1) \cdots \phi(x_n):\rangle_\lambda$ for $n > 1$, which is given by:

$$\langle :\phi(x_1) \cdots \phi(x_n):\rangle_\lambda = -\lambda n!\, a_n \int dz \prod_{i=1}^{n} c(x_i - z) + O(\lambda^2)$$

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