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An alternative approach to the Green's function of a relativistic spinless charged particle in an external uniform electromagnetic field

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Abstract. We use a variable transformation in order to obtain the Feynman propagator for a relativistic spinless particle in an external uniform electromagnetic field in terms of a four dimensional harmonic oscillator. We also show how to get the same result in the path integral formalism.

I. Introduction

Since the first elementary particle accelerators were constructed the study of the behaviour of charged particles in the presence of external electromagnetic fields began to attract the attention of physicists. In particular, many Green's functions with prescribed external electromagnetic fields have been calculated by different techniques [1–5]. Unfortunately, exact results can be obtained only in few cases.

In this paper we obtain the exact Green's function for a spinless charged particle in an external uniform electromagnetic field using an alternative approach.

Suggested by the nonrelativistic case – where the problem of a charged particle in a constant magnetic field can be reduced to that of a bidimensional harmonic oscillator [6–7] – we find an adequate variable transformation that reduces our problem to the case of a four dimensional harmonic oscillator. Although this problem has already been solved by others methods (see for example Ref. 1), we think our technique provides a very short and elegant way of obtaining the propagator. Besides, this approach may be used in more complex problems. We also obtain the same result in the path integral formalism.

II. Variable transformations in the differential equation

The Green's function for a spinless charged particle in the presence of an external electromagnetic potential satisfies the following equation

$$(\pi^2 - m^2)G(x, x') = \delta(x - x') \tag{1}$$

where

$$\pi_\mu = p_\mu - eA_\mu \tag{2}$$

The function $G(x, x')$ can be thought as the matrix element $\langle x| G |x'\rangle$ of the Green's operator

$$G = (\pi^2 - m^2)^{-1} = -i\int_0^\infty e^{i(\pi^2 - m^2)s}\, ds \tag{3}$$

From equation (3) we see that a possible integral representation for the Green function is

$$G(x, x') = -i\int_0^\infty e^{-im^2s}\Delta(x, x'; s)\, ds \tag{4}$$

where one may impose $\Delta = 0$ for $s < 0$ without changing anything, so that $\Delta(x, x'; s)$ satisfies a Schroedinger type equation in four dimensions

$$\{i\,\partial s + \pi^2\}\Delta(x, x'; s) = \delta(s)\,\delta(x - x') \tag{5}$$

The parameter s plays the role of the time. This leads to the differential equation

$$\{i\,\partial_s - \partial^2 + 2ieA_\mu\,\partial^\mu + e^2A^2\}\Delta(x, x'; s) = \delta(s)\,\delta(x - x') \tag{6}$$

where we used $\partial_\mu A_\mu = 0$. From

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial\nu A_\mu \tag{7}$$

we see that for a constant field strength, A_μ can be written in the form

$$A_\mu = -\tfrac{1}{2}F_{\mu\nu}x^\nu \tag{8}$$

Inserting equation (8) into equation (6) we get

$$\left\{i\,\partial_s - \partial^2 - ieF_{\mu\nu}x^\nu\,\partial^\mu + \frac{e^2}{4}F_{\mu\nu}F^\mu\sigma x^\nu x^\sigma\right\}\Delta(x, x'; s) = \delta(s)\,\delta(x - x') \tag{9}$$

Now, we make the following variable transformation

$$\tau = s \tag{10.a}$$

$$x^\mu = \Lambda^\mu + y^\nu \tag{10.b}$$

Therefore

$$i\partial_s = i\frac{\partial\tau}{\partial s}\frac{\partial}{\partial\tau} + i\frac{\partial y^\mu}{\partial s}\frac{\partial}{\partial y^\mu} = i\frac{\partial}{\partial\tau} + i\Lambda^{-1\mu} + x^\nu\frac{\partial}{\partial y^\mu} \tag{11.a}$$

where the overdot indicates differentiation with respect to s.

$$\frac{\partial}{\partial x^\sigma}=\frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial}{\partial y^\mu}=\Lambda^{-1\mu}{}_\sigma\frac{\partial}{\partial y^\mu} \tag{11.b}$$

The linear term of equation (9) takes the form

$$-ieF^\mu_\nu x^\nu\frac{\partial}{\partial x^\mu}=-ieF^\mu_\nu\Lambda^\nu_\rho\Lambda^{-1\delta}_\mu y^\rho\frac{\partial}{\partial y^\delta} \tag{12}$$

Before obtaining expressions for the other terms of equation (9) we choose $\Lambda(s)$ in order that the second term of the RHS of formula (11.a) is cancelled by the linear term given by (12), that is

$$i\dot{\Lambda}^{-1\mu}_\nu\Lambda^\nu_\rho y^\rho\frac{\partial}{\partial y^\mu}=ieF^\alpha_\upsilon\Lambda^{-1\mu}_\nu\Lambda^\nu_\rho y^\rho\frac{\partial}{\partial y^\mu} \tag{13}$$

which gives

$$\dot{\Lambda}^{-1\mu}_\nu=eF^\alpha_\nu\Lambda^{-1\mu}_\alpha \tag{14}$$

Introducing the following matricial notation

$$\begin{aligned}(\Lambda)_{\mu\nu}&=\Lambda^\mu_\nu(g^{-1})_{\mu\nu}=g^\mu_\nu\\(F)_{\mu\nu}&=F^\mu_\nu(x)_\mu=x^\mu\end{aligned} \tag{15}$$

we write equation (14) in the form

$$\dot{\Lambda}^{-1}=e\Lambda^{-1}F \tag{16}$$

Integrating the last equation we obtain

$$\dot{\Lambda}^{-1}(\tau)=\exp[eF\tau]\Lambda^{-1}(0) \tag{17}$$

where $\Lambda^{-1}(0)$ is an arbitrary integration constant matrix to be chosen conveniently. The previous transformation has the interesting properties

$$\tilde{\Lambda}(\tau)g\Lambda(\tau)=\tilde{\Lambda}(0)g\Lambda(0) \tag{18.a}$$

$$\tilde{\Lambda}(\tau)gF^2\Lambda(\tau)=\tilde{\Lambda}(0)gF^2\Lambda(0) \tag{18.b}$$

Now we choose $\Lambda(0)$ in such a way that

$$\tilde{\Lambda}(0)g\Lambda(0)=g \tag{19.a}$$

and

$$e^2\tilde{\Lambda}(0)gF^2\Lambda(0)=-g\Omega^2 \tag{19.b}$$

where the RHS of (19.b) is a diagonal matrix. Looking to equations (18.a) and (19.a) we see that $\Lambda(\tau)$ is a Lorentz transformation. Thus, $\det\Lambda=1$, and using that ∂^2 is a invariant under a Lorentz transformation we can write the differential equation (9) in the matrix notation as

$$\{i\,\partial\tau-\partial^2-\tfrac{1}{4}\tilde{y}g\Omega^2y\}\tilde{\Delta}(y,y';\tau)=\delta(\tau)\,\delta(y-y') \tag{20}$$

We note that this equation is the same as that of four dimensional harmonic oscillator. Therefore we have for $\tilde{\Delta}(y, y', \tau)$ a product of four harmonic oscillator propagators

$$\tilde{\Delta}(y, y'; \tau) = \left(\frac{1}{4\pi i}\right)^{1/2}\left(\frac{-1}{4\pi i}\right)\left\{\det\left[\frac{\Omega}{\text{sen }\Omega\tau}\right]\right\}^{1/2} \times \exp + \frac{i}{4}\{\tilde{y}'g(\Omega \text{ cotan } \Omega\tau)y' + \tilde{y}g(\Omega \text{ cotan } \Omega\tau)y - 2\tilde{y}g(\Omega \text{ cosec } \Omega\tau)y'\} \quad (21)$$

where Ω is a diagonal matrix with distinct elements, since we have different eigenvalues for F (see Appendix a). In other words, we have an anisotropic four dimensional oscillator. Coming back to the original coordinates we write

$$\Delta(x, x'; s) = i\left(\frac{1}{4\pi}\right)^2\left\{\det\left[\frac{\Omega}{\text{sen }\Omega s}\right]\right\}^{1/2} \times \exp\frac{i}{4}\{\tilde{x}'\tilde{\Lambda}(0)g\Lambda(0)\Lambda^{-1}(\Omega\ \omega \text{ cotan } \Omega s)\Lambda(0)x' + \tilde{x}\tilde{\Lambda}(0)g\Lambda(0)\Lambda^{-1}_{(0)}(\Omega \text{ cotan } \Omega s)\Lambda(0)x \cdot 2\tilde{x}\tilde{\Lambda}(0)g\Lambda(0)\Lambda^{-1}(0) \times (\Omega \text{ cotan } \Omega s)\Lambda(0)x'\} \quad (22)$$

We verify straightforwardly that

$$\begin{aligned} &\Lambda(0)g\Lambda^{-1}(0) = g \\ &\Lambda^{-1}(0)\Omega\Lambda(0) = ieF \\ &\Lambda^{-1}(0)[\text{cotan } \Omega s]\Lambda(0) = -i \coth(eFs) \\ &\Lambda^{-1}(0)[\text{cosec } \Omega s]\Lambda(0) = -i \text{ cosech}(eFs) \end{aligned} \quad (23)$$

Using the previous results we obtain

$$\Delta(x, x'; s) = -i(4\pi)^2\left[\det\left(\frac{\Omega}{\text{sen }\Omega s}\right)\right]^{1/2} \times \exp\frac{i}{4}\{\tilde{x}'g[eF\coth(eFs)]x' + \tilde{x}g[eF\coth(eFs)]x - 2\tilde{x}geF[\coth(eFs) - 1]x'\} \quad (24)$$

Equation (19.b) is a similarity transformation and therefore the eigenvalues of Ω are the same of F. This fact allows us to write the pre-exponential factor in terms of the eigenvalues of Ω, (see appendix), this is:

$$\Delta(x, x'; s) = i(4\pi)^{-2}\frac{e^2 I_2}{\text{Im}\cosh(eZs)} \times \exp\frac{i}{4}\{\tilde{x}'geF[\coth eFs]x' + \tilde{x}geF[\coth(eFs)]x - 2\tilde{x}geF[\coth(eFs) - 1]x'\} \quad (25)$$

where

$$I_2 = \vec{E} \cdot \vec{B} \quad (26)$$

and

$$z^2 = (\vec{B} + i\vec{E})^2 \quad (27)$$

The last term in the expression (25) can be written in the following way

$$2\tilde{x}geFx' = -4e \int_{P\,-x'}^{x} A^{\mu}(y)\, dy_{\mu} \tag{28}$$

where P means integration along the straight line joining the points x and x'.

Thus equation (25) can be written in the form

$$\Delta(x, x'; s) = i(4\pi)^{-2} \frac{e^2 I_2}{Im \cosh(eZs)} \exp \frac{i}{4} \Big\{ \tilde{x}' geF[\coth(eFs)]x' + \tilde{x}geF[\coth(eFs)]x - 2\tilde{x}geF[\coth(eFs)]x' - 4e \int_{P\,x'}^{x} A_{\mu}(y)\, dy^{\mu} \Big\} \tag{29}$$

We have been working with Schwinger gauge. However, if we want to write the Green's function in another one the prescription will be given by

$$\Delta'(x, x') = \exp ie[\eta(x, x_0) - \eta(x', x_0)]\Delta'(x, x') \tag{30}$$

where $\Delta'(x, x')$ is the Green's function in an arbitrary gauge and $\eta(x, x')$ is the gauge function defined by

$$A_{\mu}(x) = A'_{\mu}(x) + \partial_{\mu}\eta(x, x') \tag{31}$$

In (31) $A'_{\mu}(x)$ is the potential vector in an arbitrary gauge. In our case, $\eta(x, x')$ is given by

$$\eta(x, x_0) = -\int_{P\,x_0}^{x} A'_{\mu}(y)\, dy^{\mu} \tag{32}$$

Using expressions (4), (29), (32), and (30), we obtain the desired result for the propagator in agreement with Schwinger [1].

III. Path integral

We can take the expression (4) for the propagator, but in play of solving the differential equation we follow an alternative way. Since equation (5) is analogous to the Schrodinger equation, then $\Delta(x, x'; s)$ can be written in the Lagrangian form, this is.

$$\Delta(x, x'; s) = \int \mathscr{D}[x(s)] \exp -i\Big\{\int_0^s L\, ds\Big\} \tag{33}$$

where L is the Lagrangian given by

$$L = \tfrac{1}{4}\dot{x}_{\mu}\dot{x}^{\mu} + e\dot{x}_{\mu}A^{\mu} \tag{34}$$

Using the same gauge as in the previous section we rewrite the Lagrangian in the form

$$L = \tfrac{1}{4}\dot{x}^{\mu}g_{\mu\nu}\dot{x}^{\nu} + \frac{e}{4}\dot{x}^{\mu}g_{\mu\nu}F^{\mu}_{\sigma}x^{\sigma} + \frac{e}{4}F^{\mu}_{\sigma}x^{\sigma}g_{\mu\nu}\dot{x}^{\nu} \tag{35}$$

or in matricial notation

$$L = \tfrac{1}{4}\tilde{\dot{x}}g\dot{x} + \frac{e}{4}\tilde{\dot{x}}gFx - \frac{e}{4}\tilde{x}gF\dot{x} \tag{36}$$

The latter form involves the fact that

$$\tilde{g}F = \tilde{F}g = -gF \tag{37}$$

If we make the transformation

$$x = \Lambda(s)y \tag{38}$$

where

$$\Lambda(s) = \Lambda(0)\exp(-eFs) \tag{39}$$

we have

$$\dot{x} = \Lambda(0)(-eF)\exp(-eFs)y + \Lambda(0)\exp(-eFs)\dot{y} \tag{40.a}$$

$$\tilde{\dot{x}} = \tilde{y}(-e\tilde{F})\exp(-e\tilde{F}s)\tilde{\Lambda}(0) + \tilde{\dot{y}}\exp(-e\tilde{F}s)\tilde{\Lambda}(0) \tag{40.b}$$

Substituting these expressions in (36) and using the properties (19.a), (19.b) and (37) we obtain

$$L = \tfrac{1}{4}(\tilde{\dot{y}}g\dot{y} - \tilde{y}g\Omega^2 y) \tag{41}$$

that is formally the Lagrangian for the four dimensional harmonic oscillator.

Thus as in the earlier case the propagator is that of the four dimensional harmonic oscillator, given by the expression (21).

Hereafter the same procedure is used to come back to the original coordinates. It is obvious that we reobtain the same result as that of the previous section.

IV. Conclusions

We solved explicitly the problem of a spinless charged particle in an external uniform electromagnetic field. We obtained the exact propagator by using two different approaches, the path integral and the differential equation one.

Although we treated an old problem, we made a variable transformation in the differential equation for the propagator, a technique which is not very common in literature. We also showed that the same variable transformation in the path integral formalism leads to the exact result, as expected.

We feel that in both cases, our solution is much simpler than others (see for example Ref. 1). Besides, this method may also be used in more complex problems.

A possible extension of this method is to work with the Schrodinger pictures operators and make the transformations of these operators. A physical example of this can be found in Ref. 8.

V. Appendix

It was convenient to write the pre-exponential factor in the expression (25) in terms of the eigenvalues for. Here we develop explicitly this term.

Defining the Lorentz invariants:

$$2I_1 = (*F^2 - F^2) = \tfrac{1}{2}(B^2 - E^2) \tag{A.1}$$

and

$$I_2 = *F*F = \vec{E} \cdot \vec{B} \tag{A.2}$$

where $*F$ is the dual of F, we have

$$F*F\psi = F' I_2 \psi \tag{A.3}$$

and

$$(*F^2 - F^2)\psi = \left[\left(\frac{I_2}{F'}\right)^2 - F'^2\right]\psi = 2I,\ \psi \tag{A.4}$$

where F' is the eigenvalue for F.

Thus we obtain the following equation

$$(F')^4 + 2I_1(F')^2 - I_2^2 = 0 \tag{A.5}$$

whose solution is

$$F'^2 = -I_1 \pm \sqrt{I_1^2 + I_2^2} \tag{A.6}$$

Then we have for Ω^2:

$$\Omega^2 = e^2[I_1 \mp \sqrt{I_1^2 + I_2^2}] \tag{A.7}$$

that is

$$\begin{aligned} \Omega &= e\{\pm i[-I_1 + \sqrt{I_1^2 + I_2^2}]^{1/2}\} \\ \Omega &= e\{\pm [I_1 + \sqrt{I_1^2 + I_2^2}]^{1/2}\} \end{aligned} \tag{A.8}$$

In this manner

$$(\det \Omega)^{1/2} = \pm i e^2 I_2 \tag{A.9}$$

and

$$[\det \operatorname{sen} \Omega_s]^{1/2} = -i \operatorname{Im} \cosh(eZs) \tag{A.10}$$

where

$$z^2 = (\vec{B} + i\vec{E})^2 \tag{A.11}$$

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