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The influence of coherent drift waves on the magnetoacoustic resonance in cylindrical plasmas

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Abstract. A theory describing the influence of radially and azimuthally varying coherent density oscillations on the magnetoacoustic resonance (MAR) in cylindrical plasmas is developed. The corresponding two-dimensional wave equation is solved numerically. In some cases an analytical solution is also possible which allows to check the numerical methods and provides simple estimates of the effect. The theory is applied to model and real plasmas in the case where the density oscillations are due to coherent drift waves. It is shown that the coherent drift waves cannot account for the strong damping of the MAR as observed in many experiments, but that it is the turbulent drift waves, instead, which are responsible for this damping. The coherent drift waves can cause an appreciable shift of the resonance frequency of the MAR, however.

1. Introduction

Magnetosonic waves in axially magnetized cylindrical plasmas can be excited with the aid of r.f. oscillations in a coil which is wound around the plasma tube. The waves propagate radially into the plasma and exhibit resonant character in the radial direction with maximum amplitude of the magnetic field at the plasma center, a phenomenon known as magnetoacoustic resonance (MAR).

The anomalously low wave amplitude near the corresponding resonance frequency, which could not be explained by classical wave damping mechanisms such as resistivity and viscosity, has been observed in a number of experiments with radially inhomogeneous plasmas in the past (Krämer [1], Blackwell and Cross [2], Schneider et al. [3], Hahnekamp and Stampa [4]).

Sayasov [5], Ritz et al. [6], Sayasov and Ritz [7], and Vaucher et al. [8] succeeded in identifying the damping mechanism in such plasmas as being an effect of turbulent low frequency density fluctuations, which they identified as turbulent drift waves. Their theory explains quantitatively the observed strong damping of the MAR.

Grosse and Krämer [9] experimented with a rather quiescent plasma and observed only a weak damping of the MAR, which they tried to explain as an effect of purely coherent drift waves. In their method to solve the resulting wave equation they used a perturbation approach and disregarded relevant boundary conditions. Since the additional terms in the wave equation which account for the effect of the drift wave are not small, the applicability of a perturbation approach is rather doubtful.

In the present paper we develop a theory to describe the influence of coherent drift waves on the magnetosonic wave in radially inhomogeneous plasmas which is based on a rigorous solution of the two-dimensional wave equation with the correct boundary conditions. We show that the alteration of the MAR due to coherent drift waves cannot account for its strong damping as observed in most experiments. We discuss numerical results obtained for a homogeneous model plasma with linear radial density oscillation profile and mode number $m = 1$. As a test of the numerical procedure we also develop an approximate analytical solution valid for this model and compare its results with the ones obtained numerically. We also apply our theory to a real argon plasma with $m = 1$ which was described by Ritz et al. [6], and to a real helium plasma with $m = 6$ as described by Egger et al. [10].

2. Theory

2.1. *Derivation of the two-dimensional wave equation*

We consider an infinitely long cylindrical plasma (coordinates r, φ, z) magnetized by a magnetic field $\vec{B}_0 = B_0\hat{z}$ with the r.f. magnetic field of the MAR in the axial direction $\hat{z}$, in accordance with the way the MAR is usually excited in the experiments, namely, by a concentric cylindrical coil (which we assume to be infinitely long as well) surrounding the plasma cylinder. Assuming a time dependence of the fields of the form $\exp(-i\omega t)$, Maxwell's equations together with the proper dielectric tensor yield the wave equation for harmonic fields. For a cold plasma, we have (Skipping et al. [11])

$$\operatorname{rot}\vec{B} = -ik\overleftrightarrow{\varepsilon}\vec{E} \tag{1}$$

with the dielectric tensor $\overleftrightarrow{\varepsilon}$ being

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon & ig & 0 \\ -ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix} \tag{2}$$

where

$$\varepsilon(\vec{r}) = 1 + \frac{\omega_p^2(\vec{r})Q(r)}{Q^2(r) - \omega_{ce}^2\omega^2} \tag{3}$$

$$g(\vec{r}) = \frac{\omega_p^2(\vec{r})\omega_{ce}\omega}{Q^2(r) - \omega_{ce}^2\omega^2} \tag{4}$$

The tensor element η is not used because we only have to consider the component B_z of the magnetic field and, as follows from (1), only the component

E_r and E_φ of the electric field. ω is the frequency of the magnetoacoustic wave, ω_{ce} is the electron and ω_{ci} the ion cyclotron frequency. ω_p is the plasma frequency

$$\omega_p^2(\vec{r}) = \frac{4\pi e^2 n_e(\vec{r})}{m} \tag{5}$$

$Q(r)$ is an abbreviation for the expression

$$Q(r) = \omega_{ci}\omega_{ce} - \omega[\omega + i\nu_e(r)] \tag{6}$$

where $\nu_e(r)$ is the total electron collision frequency and is given by the well-known formulas

$$\nu_e(r) = \nu_{ea}(r) + \nu_{ei}(r) \tag{7}$$

$$\nu_{ea}(r) = 4.2 \cdot 10^7 n_a \sqrt{T_{e0}(r)}\, \sigma_e \tag{8}$$

$$\nu_{ei}(r) = 2.9 \cdot 10^{-6} \frac{n_{e0}(r)}{[T_{e0}(r)]^{3/2}} \ln \Lambda(r), \tag{9}$$

where ν_{ea} is the electron-neutral collision frequency, ν_{ei} the Coulomb collision frequency, n_a the density of neutral atoms, $T_{e0}(r)$ the electron temperature distribution in eV, σ_e the collisional cross section for electrons, $n_{e0}(r)$ the undisturbed electron density distribution, and $\ln \Lambda(r)$ the Coulomb logarithm. Maxwell's equation for the electric field is

$$\operatorname{rot} \vec{E} = i\frac{\omega}{c}\vec{B} = ik\vec{B} \tag{10}$$

Writing (1) and (10) in cylindrical coordinates, substituting the electric field components E_r and E_φ computed from (1) into (10) and using the abbreviations

$$\varepsilon_1 = \frac{\varepsilon^2 - g^2}{\varepsilon}; \qquad \varepsilon_2 = \frac{\varepsilon^2 - g^2}{ig} \tag{11}$$

we find the exact two-dimensional wave equation

$$\begin{aligned} &\frac{\partial^2 B_z}{\partial r^2} + \left(\frac{1}{r} - \frac{1}{\varepsilon_1}\frac{\partial \varepsilon_1}{\partial r} - \frac{1}{r}\frac{\varepsilon_1}{\varepsilon_2^2}\frac{\partial \varepsilon_2}{\partial \varphi}\right)\frac{\partial B_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B_z}{\partial \varphi^2} \\ &\quad + \left(\frac{1}{r}\frac{\varepsilon_1}{\varepsilon_2^2}\frac{\partial \varepsilon_2}{\partial r} - \frac{1}{r^2}\frac{1}{\varepsilon_1}\frac{\partial \varepsilon_1}{\partial \varphi}\right)\frac{\partial B_z}{\partial \varphi} = -\varepsilon_1 k^2 B_z \end{aligned} \tag{12}$$

The influence of the drift wave on the magnetosonic wave is contained in the functions ε_1 and ε_2 via the electron density n_e which determines the plasma frequency (5). The electron density n_e is given as the superposition of the undisturbed density n_{e0} with the oscillating perturbation n_{e1} of the coherent drift wave. n_{e0} varies only radially whereas n_{e1} varies both radially and azimuthally. Since drift wave frequencies are usually of the order of a few tens of kHz while MAR frequencies are of the order of several MHz, we can treat the drift wave as constant in time. The azimuthal dependence of the drift wave is then of the form $\cos(m\phi)$, where m is the mode number of the drift wave mode considered.

For $Q \approx \omega_{ci}\omega_{ce}$ we can assume $|Q^2| \ll \omega_{ce}^2\omega^2$. It then follows from (3) that, if we can neglect the constant $1, \varepsilon \approx -\omega_p^2\omega_{ci}/(\omega^2\omega_{ce})$. From (4) it follows that $g \approx -\omega_p^2/(\omega\omega_{ce}) = (\omega/\omega_{ci})\varepsilon$ and thus $g^2 \gg \varepsilon^2$. From (11) it follows that $\varepsilon_1, \varepsilon_2 \sim \omega_p^2 \sim n_e$. In the following derivation it is easier to work with the exponential representation of the cos-function. Thus the expressions for the ε_j, $j = 1, 2$ can be represented in the form

$$\varepsilon_j = \varepsilon_{j0}(r)\left[1 + \frac{h(r)}{2}(e^{im\phi} + e^{-im\phi})\right] \tag{13}$$

where $\varepsilon_{j0} = \varepsilon_j$ for $h = 0$ and

$$h(r) = \frac{n_{e1}(r)}{n_{e0}(r)} \tag{14}$$

The approximations stated above have only been made in order to derive expression (13) for the angular dependence of the ε_j. For the radial dependence of the ε_{j0} we use the full expressions (11) and (3) to (9). The influence of the drift wave is contained in the ε_j according to (13), so that we can use the undisturbed electron density $n_{e0}(r)$ for $n_e(r)$ in (5).

We then substitute (13) into (12) and multiply the whole equation with the term in brackets from (13). This eliminates all angularly dependent terms in the denominators of (12).We assume that $h \ll 1$. Since $\varepsilon_1/\varepsilon_2 = (\omega/\omega_{ci})i \gg i$ we only have to retain terms in h and dh/dr which also contain the factor $\varepsilon_1/\varepsilon_2$. With the aid of the abbreviations

$$\kappa_1 = \frac{1}{\varepsilon_{10}}\frac{d\varepsilon_{10}}{dr}; \qquad \kappa_2 = \frac{1}{\varepsilon_{20}}\frac{d\varepsilon_{20}}{dr} \tag{15}$$

we can write the wave equation in the final form

$$\frac{\partial^2 B_z}{\partial r^2} + \left[\frac{1}{r} - \kappa_1 - i\frac{m}{r}\frac{\varepsilon_{10}}{\varepsilon_{20}}\frac{h}{2}(e^{im\varphi} - e^{-im\varphi})\right]\frac{\partial B_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B_z}{\partial \varphi^2} + \frac{1}{r}\frac{\varepsilon_{10}}{\varepsilon_{20}}$$
$$\times \left\{\kappa_2\left[1 + \frac{h}{2}(e^{im\varphi} + e^{-im\varphi})\right] + \frac{1}{2}\frac{dh}{dr}(e^{im\varphi} + e^{-im\varphi})\right\}\frac{\partial B_z}{\partial \varphi} + \varepsilon_{10}k^2 B_z = 0 \tag{16}$$

This equation is also valid for turbulent plasmas if the expression (3) for $\varepsilon(r)$ (without angular dependence in ω_p) is replaced by (23) given by Sayasov and Ritz [7], which contains the radial profile of the turbulent density fluctuations. This leads to new expressions for the ε_{j0}, and the angular dependence due to the coherent oscillations is then accounted for in the ε_j as in equation (13).

2.2. *Method of solution*

Since the angular dependence of $B_z(r, \varphi)$ stems from the angular dependence of the density perturbation n_{e1} of the coherent drift wave, it is natural to develop B_z in a Fourier series of harmonics of the mode number m of the drift wave where the coefficients are complex functions of the radius r:

$$B_z(r, \varphi) = \sum_{\mu=-\infty}^{\infty} C_\mu(r)e^{i\mu m\varphi} \tag{17}$$

Substitution of this Fourier series into the partial differential equation (16) and equating the coefficients of exp $(i\mu m\varphi)$ to naught for each μ leads to an infinite system of ordinary differential equations for the complex functions $C_\mu(r)$. This system can be written in a very concise form if we define the differential operators

$$L_\mu = \frac{d^2}{dr^2} + \left(\frac{1}{r} - \kappa_1\right)\frac{d}{dr} + \left(-\frac{\mu^2 m^2}{r^2} + i\frac{\mu m}{r}\frac{\varepsilon_{10}}{\varepsilon_{20}}\kappa_2 + \varepsilon_{10}k^2\right) \tag{18}$$

$$R_\mu^- = i\frac{1}{2}\frac{m}{r}\frac{\varepsilon_{10}}{\varepsilon_{20}}\left[-h\frac{d}{dr} + \mu\left(\kappa_2 h + \frac{dh}{dr}\right)\right] \tag{19}$$

$$R_\mu^+ = i\frac{1}{2}\frac{m}{r}\frac{\varepsilon_{10}}{\varepsilon_{20}}\left[h\frac{d}{dr} + \mu\left(\kappa_2 h + \frac{dh}{dr}\right)\right] \tag{20}$$

The resulting neat and compact form of the system of equations is then

$$L_\mu C_\mu + R_{\mu-1}^- C_{\mu-1} + R_{\mu+1}^+ C_{\mu+1} = 0, \tag{21}$$

where

$$\mu = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$$

In order to be able to solve the system properly, we must establish the correct boundary conditions for $B_z(r, \varphi)$ and thus for the functions $C_\mu(r)$. With the aid of Ohm's law, $\vec{j} = \sigma\vec{E}$, where σ is the conductivity, we can write Maxwell's equation for the magnetic field $\vec{B}$ in the form

$$\text{rot}\,\vec{B} = -\frac{i\omega}{c}\vec{E} + \frac{4\pi}{c}\sigma\vec{E} \tag{22}$$

In the metal of the coil which excites the magnetosonic wave we have $\sigma \gg \omega$. Thus, equation (22) reduces to

$$\text{rot}\,\vec{B} = \frac{4\pi}{c}\vec{j} \tag{23}$$

Now we introduce a rectangular current sheet, one of the edges of which coincides with the axis of the cylindrical coil. The surface integral of (23) over the surface of the current sheet yields the total current which flows perpendicularly through the sheet. By Stokes' theorem we can replace the surface integral by a line integral along the edge of the sheet. The contribution to this integral along the edge which is parallel to the cylinder axis but lies far away from it is zero because the magnetic field vanishes there. The contributions from the two edges perpendicular to the cylinder axis exactly cancel because the magnitude of the fields along them are identical but the direction of integration is reversed. The only contribution which does not vanish stems from the edge which coincides with the cylinder axis. Now the current sheet can be oriented at any angle in the azimuthal direction, and the edge of the sheet inside the coil can be brought into any radial position as long as the edge remains within the coil and lies parallel to the cylinder axis without affecting the correctness of the statements made above. From this it follows that the magnetic field at the plasma radius r_0, $B_z(r_0)$, must be equal to the field B_{coil} produced by the coil (if the current in the coil is not affected by the plasma) and is independent of the azimuthal angle φ. On the axis, i.e. at $r = 0$, the azimuthal component j_φ of the current density $\vec{j}$ must vanish in order to assure the uniqueness of $\vec{j}$ and $\vec{B}$ there. Since the cylinder is assumed to be infinitely long, we have $\partial/\partial z = 0$, and the azimuthal component of (23) reduces to $-\partial B_z/\partial r = (4\pi/c)j\varphi$, and it follows that $\partial B_z/\partial r = 0$ at $r = 0$.

From the representation of $B_z(r, \varphi)$ as a Fourier series (17) it then follows that the following boundary conditions must hold for the functions $C_\mu(r)$:
In the center at $r = 0$:

$$\frac{dC_0}{dr} = 0; \qquad C_\mu(0) = 0 \qquad \text{for all } \mu \text{ except } \mu = 0 \tag{24}$$

On the plasma radius at $r = r_0$:

$$C_0(r_0) = B_{\text{coil}}; \qquad C_\mu(r_0) = 0 \qquad \text{for all } \mu \text{ except } \mu = 0 \tag{25}$$

We are interested in the behaviour of the magnetic field B_z and the functions C_μ as a function of the radius and especially of the frequency in order to be able to plot the resonance curves of the MAR. For this purpose, the magnetic field in the center, $B_z(0)$, is normalized to its value on the plasma radius $B_z(r_0)$, and this ratio is computed and plotted as a function of the frequency. This is also the way in which the MAR is usually measured in the experiments.

A computer program DRIMAR for solving the system of equations (21) with the differential operators (18)–(20) and the boundary conditions (24) and (25) for arbitrary radial profiles of the undisturbed electron density, of the density oscillation of the drift wave, of the electron temperature, and for an arbitrary drift wave mode number m has been worked out. The system can be solved for a user of selectable number of frequencies with given start frequency and frequency increment. With multiple runs of consecutively adapted start frequencies and refined increments this allows the lowest resonance frequency to be found to the desired accuracy remarkably quickly. The number of equations N considered in the system (21) must be odd, of course, and it can be selected arbitrarily up to $N = 13$. If e.g. a number of $N = 3$ is selected, then we are looking for solutions for the functions C_{-1}, C_0, and C_1, and R_{-2}^- as well as R_2^+ in the corresponding equations of (21) must be put equal to zero since we cannot compute C_{-2} and C_2 with the system restricted to 3 equations. DRIMAR is so designed as to assemble the required equations in the system (21) automatically for a given number of equations N. To specify N is all the user has to do. This allows convergence tests to be performed very conveniently.

A number of measured points of the radial distributions of the undisturbed electron density, of the density oscillation of the drift wave, and of the electron temperature are read into the program. Cubic spline functions are then used to approximate the continuous distributions. This allows the required derivatives to be performed very accurately by using the corresponding analytical expressions for the splines. DRIMAR uses the subroutine CRAPRO written by Räuchle [12] to solve the complex boundary value problem. It uses a five-point finite-difference scheme. A well documented listing of the program DRIMAR is available from one of the authors (H.A.A.).

2.3. *Analytical solution for homogeneous plasmas with linear density oscillation profile and $m = 1$*

For homogeneous plasmas with linear density oscillation profile and drift wave number $m = 1$ it is possible to find an analytical solution to the system of equations (21) for $N = 3$ equations, the results of which can be compared with the corresponding numerical solution for $N = 3$, thus providing a check for the correctness of the numerical methods used. From the approximations made in the derivation of equation (13) in subsection 2.1 it follows immediately that

$$\frac{\varepsilon_{10}}{\varepsilon_{20}} \approx i\frac{\omega}{\omega_{ci}} \gg i \tag{26}$$

In a homogeneous plasma ε and g are constants, and hence ε_{10} and ε_{20} are also constants. From (15) it then follows that

$$\kappa_1 = \kappa_2 = 0 \tag{27}$$

The profile of the density oscillations of the drift wave is assumed to be linear, i.e.

$$h(r) = \beta\frac{r}{r_0} \tag{28}$$

The system of equations (21) for the functions C_{-1}, C_0, and C_1 ($C_\mu = 0$ for $|\mu| > 1$) is

$$\begin{aligned} &L_0C_0 + R_{-1}^-C_{-1} + R_1^+C_1 = 0 \\ &L_{-1}C_{-1} + R_0^+C_0 = 0 \\ &L_1C_1 + R_0^-C_0 = 0 \end{aligned} \tag{29}$$

We are only interested in the normalized resonance curve here, i.e. $|B_z(0)|/|B_z(r_0)|$ as a function of frequency, which is identical to $|C_0(0)|/|C_0(r_0)|$ due to the boundary conditions (24) and (25). This means that we can reduce the system (29) to two equations by defining

$$C^* = C_{-1} - C_1 \tag{30}$$

The new system of equations is then

$$\begin{aligned} &L_0C_0 + R_1C^* = 0 \\ &L_1C^* - 2R_0C_0 = 0 \end{aligned} \tag{31}$$

with the differential operators

$$L_\mu = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\mu^2 m^2}{r^2} + \varepsilon_{10}k^2; \qquad \mu = 0, 1 \tag{32}$$

$$R_0 = m\alpha \frac{d}{dr} \tag{33}$$

$$R_1 = m\alpha\left(\frac{d}{dr} + \frac{1}{r}\right) \tag{34}$$

with

$$\alpha = \frac{1}{2}\frac{\omega}{\omega_{ci}}\frac{\beta}{r_0} \tag{35}$$

Now we try the assumption of the following particular solutions:

$$\begin{aligned} &C_0 = aJ_0(\chi r) \\ &C^* = bJ_1(\chi r) \end{aligned} \tag{36}$$

J_0 and J_1 are the Bessel functions of the first kind of the order zero and one, respectively. From (32) we see that $L_0J_0(\varepsilon_{10}k^2r) = 0$. But the parameter in the argument of J_0 in (36) is χ. For $J_0(\chi r)$ we have $(L_0 + \chi^2 - \varepsilon_{10}k^2)J_0(\chi r) = 0$. From this it follows that $L_0J_0(\chi r) = (\varepsilon_{10}k^2 - \chi^2)J_0(\chi r)$. Analogously for L_1.

$$\begin{aligned} &L_0J_0(\chi r) = (\varepsilon_{10}k^2 - \chi^2)J_0(\chi r) \\ &L_1J_1(\chi r) = (\varepsilon_{10}k^2 - \chi^2)J_1(\chi r) \end{aligned} \tag{37}$$

We further need $R_0J_0(\chi r)$ and $R_1J_1(\chi r)$. With the aid of well-known relations between J_0 and J_1 we find the following equations, the validity of which is restricted to the case $m = 1$:

$$R_0J_0(\chi r) = -\alpha\chi J_1(\chi r) \tag{38}$$

$$R_1J_1(\chi r) = \alpha\chi J_0(\chi r) \tag{39}$$

Now we substitute the expressions (36) for C_0 and C^* into the system (31) and replace the operators on the Bessel functions with the expressions (37), (38), and (39). Then we get the following system of algebraic equations for the constants a and b:

$$\begin{aligned} &(\varepsilon_{10}k^2 - \chi^2)a + \alpha\chi b = 0 \\ &2\alpha\chi a + (\varepsilon_{10}k^2 - \chi^2)b = 0 \end{aligned} \tag{40}$$

The condition required for this system of equations to have a non-trivial solution is that its determinant vanishes:

$$(\varepsilon_{10}k^2 - \chi^2)^2 - 2\alpha^2\chi^2 = 0 \tag{41}$$

This condition is fulfilled if

$$\chi_{1,2} = \pm\frac{\alpha}{\sqrt{2}} + \sqrt{\frac{\alpha^2}{2} + \varepsilon_{10}k^2} \tag{42}$$

From the second equation in the system (40) we find the following relation between the two possible values of a and b corresponding to the two possible values of χ:

$$b_{1,2} = -\frac{2\alpha\chi_{1,2}}{\varepsilon_{10}k^2 - \chi_{1,2}^2}a_{1,2} \tag{43}$$

Here and in the following formulas we make use of the simplification

$$\frac{\chi_{1,2}}{\varepsilon_{10}k^2 - \chi_{1,2}^2} = \mp\frac{1}{\sqrt{2}\,\alpha} \tag{44}$$

The solutions C_0 and C^* now have the form

$$C_0(r) = a_1J_0(\chi_1 r) + a_2J_0(\chi_2 r) \tag{45}$$

$$C^*(r) = \sqrt{2}\,[a_1J_1(\chi_1 r) - a_2J_1(\chi_2 r)] \tag{46}$$

From the boundary condition $C^*(r_0) = 0$ we then find

$$a_1 = AJ_1(\chi_2 r_0) \tag{47}$$

$$a_2 = AJ_1(\chi_1 r_0) \tag{48}$$

The value of the constant A follows from the boundary condition $C_0(r_0) = B_{\text{coil}}$ to be

$$A = \frac{B_{\text{coil}}}{J_1(\chi_2 r_0)J_0(\chi_1 r_0) + J_1(\chi_1 r_0)J_0(\chi_2 r_0)} \tag{49}$$

The final formula for $B_N = |B_z(0)|/|B_z(r_0)| = |C_0(0)|/|C_0(r_0)|$ is then

$$B_N = \frac{J_1(\chi_2 r_0) + J_1(\chi_1 r_0)}{J_1(\chi_2 r_0)J_0(\chi_1 r_0) + J_1(\chi_1 r_0)J_0(\chi_2 r_0)} \tag{50}$$

The resonance frequency in the real approximation (which does not involve damping) is given by the root of the denominator of (50):

$$J_1(\chi_2 r_0)J_0(\chi_1 r_0) + J_1(\chi_1 r_0)J_0(\chi_2 r_0) = 0 \tag{51}$$

This defines a transcendental equation for the resonance frequency $\omega = \omega_0$ since χ_1 and χ_2 depend on χ by virtue of (35) and (42).

In the case of small β it is possible to derive analytically explicit estimates for $|B_N|$ and the resonance frequency ω_0 when the $\chi_{1,2}$ are allowed to be complex.

$\chi_{1,2}$ can then be written in the form

$$\chi_{1,2} \approx \sqrt{\varepsilon_{10} k^2} \pm \frac{\alpha}{\sqrt{2}} \tag{52}$$

With the aid of (26) given by Sayasov and Ritz [7], we have for $\chi_0 = (\varepsilon_{10} k^2)^{1/2}$

$$\chi_0 \approx \frac{\omega}{c_A} \left(1 + i \frac{\omega \nu_e}{2\omega_h^2}\right) \tag{53}$$

where c_A is the Alfvén velocity and $\omega_h = (\omega_{ce}\omega_{ci})^{1/2}$ is the lower hybrid frequency. We have $\omega \nu_e \ll \omega_h^2$. Taylor expansion in α in the denominator of (50) then yields the typical resonance formula

$$|B_N(\omega)| \approx \frac{1}{y_0 J_1(y_0) \sqrt{\left(\frac{\omega - \omega_0'}{\omega_0}\right)^2 + \gamma^2}} \tag{54}$$

where ω_0 is the resonance frequency of the MAR when the density oscillations are absent, i.e. for $\beta = 0$, defined by $J_0(\omega_0 r_0 / c_A) = 0$ where $\omega_0 r_0 / c_A = y_0 \approx 2.4$, the first root of $J_0(x)$. The damping term γ is given by

$$\gamma = \frac{\omega_0 \nu_e}{2\omega_h^2} \tag{55}$$

ω_0' is the resonance frequency with the density oscillations present, i.e. for $\beta > 0$, and is given by

$$\omega_0' \approx \omega_0 \left[1 - \frac{1}{8} \left(\frac{\omega_0 \beta}{\omega_{ci}}\right)^2 \frac{1}{y_0^2}\right] \tag{56}$$

3. Numerical results and discussion

We compute the effect of the coherent drift wave on the MAR in three different plasmas:

1. A model plasma with linear density oscillation profile and $m = 1$.
2. A real, turbulent argon plasma with $m = 1$ which was described by Ritz et al. [6].
3. A real helium plasma with $m = 6$ as described by Egger et al. [10].

We are mainly interested in the behaviour of $B_N = |B_z(0)| / |B_z(r_0)|$ as a function of frequency, i.e. the resonance curve of the MAR.

3.1. *Model plasma with linear density oscillation profile and $m = 1$*

The undisturbed plasma (helium) is assumed to be homogeneous and to have the following parameters: $B_0 = 1000$ Gauss, $r_0 = 5$ cm, $n_{e0} = 1.0 \cdot 10^{12}$ cm^{-3}, $n_a =$

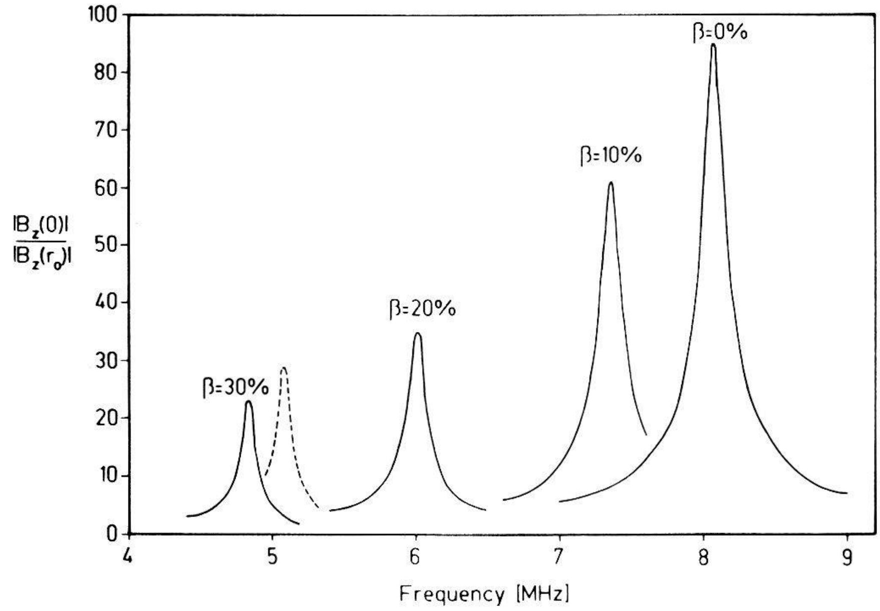

Figure 1
Resonance curves of the MAR for a model plasma with linear density oscillation profile and $m = 1$ for different maximum levels β of the density oscillation in percent of the undisturbed electron density. The dashed curve shows the approximate solution for $\beta = 30\%$ where only $N = 3$ equations are considered in the system (21).

$1.6 \cdot 10^{14}\,\mathrm{cm}^{-3}$, $T_e = 3\,\mathrm{eV}$, $T_i = 0.5\,\mathrm{eV}$. For the electron cross section we use the value $\sigma_e = 7 \cdot 10^{-16}\,\mathrm{cm}^2$ for helium given by Laborie et al. [13].

3.1.1. *Numerical solution.* Figure 1 shows the resonance curves of the MAR, i.e. B_N as a function of frequency for four different values of the maximum relative density oscillation $\beta = h(r_0) = n_{e1}(r_0)/n_{e0}(r_0)$ in percent. The curve for $\beta = 0\%$ shows the MAR when drift waves are absent. When there is a drift wave present then the resonance frequency is shifted towards lower frequencies and the MAR undergoes additional damping. These effects are the more pronounced the stronger the density oscillation of the drift wave, as can be seen in the figure from the resonance curves for $\beta = 10\%$, 20%, and 30%.

The curves shown are computed with $N = 13$. The solutions $|C_\mu(r)|$ are found to converge continuously, i.e. $|C_{\mu 1}(r)| < |C_{\mu 2}(r)|$ if $|\mu_1| > |\mu_2|$ and $\mu_1 \cdot \mu_2 > 0$, for all values of r within the plasma cylinder. The stronger the density oscillation of the drift wave, the larger the number N of equations which is required in order to solve the system (21) to any given accuracy, of course. For $\beta = 30\%$ density oscillation the resonance frequency is found to be $f_0 = (4.84 \pm 0.02)\,\mathrm{MHz}$, and $B_N(f_0) = 22.852 \pm 0.001$. If only $N = 11$ equations are used the result is indistinguishable from the one obtained with $N = 13$ down to the accuracies specified above. All this proves that our system of equations (21) on which our numerical calculations are based mathematically behaves very well and that we can be confident in the correctness of our results.

The dashed curve in Fig. 1 is obtained for $\beta = 30\%$ with the system reduced to $N = 3$ equations. The numerical results are then $f_0 = (5.08 \pm 0.02)$ MHz and

$B_N(f_0) = 29.317 \pm 0.001$. This agrees to within 5% and 30% of the exact values. For smaller values of β the agreement is even better, of course.

3.1.2. *Analytical solution.* The analytical solution for $\beta = 30\%$ based on the system (31) in which ε_{10} is replaced by its real part yields a resonance frequency of $f_0 = (5.16 \pm 0.01)$ MHz which agrees to within 7% with the exact value. The same value is obtained with the program DRIMAR when it is modified for a purely real-valued calculation. This provides a further confirmation of the correctness of the numerical methods we use.

The explicit analytical estimate for ω_0 in the case $\beta = 0\%$ is $\omega_0 = 2.4\, c_A/r_0$ which yields a value $f_0 = 8.34$ MHz instead of $f_0 = 8.08$ MHz as the exact value obtained numerically (see Fig. 1). Thus the error is only 3%. For $\beta = 10\%$ formula (56) yields $f_0 = 7.47$ MHz instead of $f_0 = 7.36$ MHz, the error being only 1.5%. The formula (54) gives $|B_N(\omega_0')| = 89$ instead of 61 or 46% too much. Thus, the explicit formulas (54) and (56) for B_N and ω_0' are useful approximations for values of $\beta < 10\%$ for this plasma.

3.2. *Real turbulent argon plasma with* $m = 1$

It is interesting to see whether the consideration of coherent drift waves, in addition to turbulent ones, in the argon plasma described by Ritz et al. [6] leads to any significant changes in the theoretical predictions of the MAR, e.g. to a frequency shift as seen in Fig. 1 for the case of a linear density oscillation profile. Here we have taken into account the actually measured radial profiles of the undisturbed electron density, of the coherent (maximum 20%) and the turbulent (maximum 10%) density fluctuations, and of the electron temperature. For $\varepsilon(r)$ we use the formula (23) given by Sayasov and Ritz [7] for an anisotropy of the turbulent fluctuations $J = 30$ [6]. The main plasma parameters are $B_0 = 2000$ Gauss, $r_0 = 4.6$ cm, $\bar{n}_{e0} = 1.5 \cdot 10^{12}\,\text{cm}^{-3}$, $\bar{T}_e = 2.5$ eV, $T_i = 0.3$ eV, and $n_a = 8.13 \cdot 10^{12}\,\text{cm}^{-3}$. In contrast to the model plasma with linear density oscillation profile described in subsection 3.1, in a real plasma the density fluctuations are concentrated in a narrow region near the plasma boundary where the plasma density displays its steepest descent. For this reason, we overestimate the effect of the coherent density oscillations in our model calculations based on a linear density oscillation profile.

The experimental results are $f_0 = 4.0$ MHz, $B_N(f_0) = 2.0$ [6]. With the classical dielectric tensor without fluctuations and coherent oscillations we find $f_0 = 4.8$ MHz and $B_N(f_0) = 33$. With the coherent oscillations alone we get $f_0 = 5.4$ MHz, $B_N(f_0) = 30$. The coherent oscillations lead to a very weak damping only. With the turbulent fluctuations alone we get $f_0 = 3.6$ MHz, $B_N(f_0) = 3.1$. Turbulent fluctuations and coherent oscillations together result in $f_0 = 3.6$ MHz, $B_N(f_0) = 2.8$. The main damping effect on the MAR clearly stems from the turbulent fluctuations, whereas the additional alteration due to the coherent oscillations is marginal. They also don't cause any significant frequency shift or amplitude reduction in this case.

3.3. *Real helium plasma with $m = 6$*

Since our numerical solution allows us to investigate the effect of coherent drift waves on the MAR for arbitrary mode numbers m, in contrast to the analytical solution which is restricted to the case $m = 1$, it is interesting to apply our theory to a plasma in which m is greater than one. Such a helium plasma with $m = 6$ was described by Egger et al. [10]. The plasma parameters are $B_0 =$ 770 Gauss, $r_0 = 4$ cm, $\bar{n}_{e0} = 2.3 \cdot 10^{12}$ cm^{-3}, $\bar{T}_e = 3.5$ eV, $T_i = 0.6$ eV, and $n_a =$ $1.6 \cdot 10^{14}$ cm^{-3}. The density oscillations are concentrated in a narrow region around $r = 3.2$ cm where they peak to 18.8% of the undisturbed density n_{e0} [14].

While the system (21) reduced to $N = 3$ equations exhibits additional damping of the MAR of a factor 2, the full solution with $N = 13$ equations reveals that the drift wave does not exert the slightest influence on the MAR et al. It is not surprising that the influence of the drift wave diminishes with larger m since the azimuthal disturbance in density which it introduces gets somewhat averaged out. The effect of every crest in the azimuthal density ripple is partly compensated by a trough. The higher m, the better this compensatory effect can work. This example shows that it may be dangerous to rely on the system of equations (21) reduced to $N = 3$ equations. It may result in wrong conclusions being drawn concerning the damping of the MAR.

4. Conclusions

In contrast to the oversimplified theoretical treatment of the influence of the coherent drift waves on the magnetoacoustic resonance (MAR) based on a perturbation approach with relevant boundary conditions disregarded, as given by Grosse and Krämer [9], we have presented a theory based on a rigorous solution of the corresponding two-dimensional wave equation with the correct boundary conditions. Since the terms describing the effect of drift waves in the wave equation for the magnetoacoustic wave (16) are multiplied by the very large ratio $\varepsilon_{10}/\varepsilon_{20}$, viz. equation (26), the perturbation approach is not applicable in this case.

For a homogeneous model plasma with linear density oscillation profile and $m = 1$ an analytical solution has been derived which has proven to be useful in order to check the correctness of the numerical methods which are used to apply our theory to real plasmas. The computer program DRIMAR which has been developed for this purpose allows the simultaneous consideration of turbulent fluctuations and coherent oscillations of the electron density.

In the case of the model plasma with linear density oscillation profile (in which our calculations overestimate the effect of the density oscillations since in a real plasma these are concentrated in a rather narrow region near the plasma boundary) the results have shown that the coherent drift waves can cause an appreciable shift of the resonant frequency but only a limited reduction in the magnetic field amplitude (by a factor <4) even for maximum oscillation levels as high as $\beta = 30\%$.

Our numerical calculations applied to real plasmas for $m = 1$ and $m = 6$ have shown that the claim held in previous papers on MAR ([6], [7], and [8]) that it is the turbulent density fluctuations which are mainly responsible for the observed strong damping of the MAR (by factors of 10–20) is correct. In these papers, coherent density oscillations were not considered. In the present paper we have taken into account both types of drift waves simultaneously where applicable, and thus we have been able to proof this long-held claim directly. The results have shown that the influence of the coherent drift waves on the MAR is very small in real plasmas and that it is smaller for larger drift wave mode numbers m.

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