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Unbounded Perturbations of Boson Equilibrium States in their GNS–Representations

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Abstract

An arbitrary quantum system in a KMS state on a C^*- (or W^*-) algebra is coupled to the Bose field in thermal equilibrium. The treated interaction is of the form $\sum_{k=1}^{m}\left(B_k \otimes a(f_k) + B_k^* \otimes a^*(f_k)\right)$, where the B_k are elements of the C^*- (or W^*-) algebra and the $a(f_k)$, $a^*(f_k)$ are (smeared) annihilation and creation operators of the bosons. Perturbation theoretical methods are used in the GNS-representation of the composite non-interacting system. With some additional conditions for the boson field and if the C^*–algebra is finite dimensional, the perturbed equilibrium state is shown to be Fock-normal and the associated density operator is calculated by a Dyson expansion in Fock space.

1 Introduction

The basic idea of a perturbation expansion in the GNS-representation is easily understood in the finite dimensional case. Let $\mathcal{H}$ be a finite dimensional Hilbert space and K and P two selfadjoint operators on $\mathcal{H}$. For some $\beta > 0$ (the inverse temperature) let us define by

$$\omega(X) := \frac{\mathrm{tr}(\mathrm{e}^{-\beta K} X)}{\mathrm{tr}(\mathrm{e}^{-\beta K})} \qquad \text{and} \qquad \omega^P(X) := \frac{\mathrm{tr}(\mathrm{e}^{-\beta(K+P)} X)}{\mathrm{tr}(\mathrm{e}^{-\beta(K+P)})}$$

two Gibbs equilibrium states on $\mathcal{L}(\mathcal{H})$. Because of the trace property one can express the perturbed state ω^P in terms of the state ω

$$\omega^P(X) = \frac{\omega(\mathrm{e}^{\frac{\beta}{2}K}\mathrm{e}^{-\frac{\beta}{2}(K+P)} X \mathrm{e}^{-\frac{\beta}{2}(K+P)}\mathrm{e}^{\frac{\beta}{2}K})}{\omega(\mathrm{e}^{-\beta(K+P)}\mathrm{e}^{\beta K})}. \tag{1}$$

By an usual Dyson expansion one gets

$$\mathrm{e}^{-\frac{\beta}{2}(K+P)}\mathrm{e}^{\frac{\beta}{2}K} = \\ = \mathbb{1} + \sum_{n=1}^{\infty}(-1)^n \int_{0\le t_1\le\cdots\le t_n\le\frac{\beta}{2}} dt_1\cdots dt_n\, \mathrm{e}^{-t_1K} P\, \mathrm{e}^{-(t_2-t_1)K} P \,\ldots\, \mathrm{e}^{-(t_n-t_{n-1})K} P\, \mathrm{e}^{t_nK}.$$

If $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ and $(\Pi_{\omega^P}, \mathcal{H}_{\omega^P}, \Omega_{\omega^P})$ denote the associated GNS-representations, because of equation (1) we may choose Ω_{ω^P} such that

$$\Omega_{\omega^P} \quad = \quad \frac{\Pi_\omega(\mathrm{e}^{-\frac{\beta}{2}(K+P)}\mathrm{e}^{\frac{\beta}{2}K})\Omega_\omega}{\left\|\Pi_\omega(\mathrm{e}^{-\frac{\beta}{2}(K+P)}\mathrm{e}^{\frac{\beta}{2}K})\Omega_\omega\right\|} .$$

In this way one can deduce an expansion of the cyclic vector Ω_{ω^P} of the perturbed state ω^P. The same result is obtained by renormalizing the operator $\Pi_\omega(K)$.

Of course in the finite dimensional case the series are well defined and uniformly convergent. In the much more general case of a C^*– or W^*–algebra, in which the perturbation operator P is an arbitrary element of the algebra, one needs the KMS property of the unperturbed state ω and the renormalized unperturbed hamiltonian to get the necessary estimations (cf. [4, Theorem 5.4.4]).

In the present work are considered unbounded perturbation operators, where the unboundedness arises from the annihilation and creation operators of the bosons. More precisely, a quantum system, given by a KMS state on a C^*– or W^*–algebra $\mathcal{A}$, is coupled to a boson field system, described by a quasi-free gauge-invariant KMS state on the Weyl algebra $\mathcal{W}(E)$ not including condensation phenomena, and by the interaction

$$P \quad = \quad \sum_{k=1}^{m} \Big(B_k \otimes a_b(f_k) \; + \; B_k^* \otimes a_b^*(f_k)\Big) \; , \tag{2}$$

where m is finite, the B_k are elements of $\mathcal{A}$ and the $a_b(f_k)$, $a_b^*(f_k)$ are annihilation and creation operators associated with the GNS-representation of the boson state. The main part of this paper is devoted to the convergence of the corresponding perturbation expansion of the cyclic vector.

As far as we know, there exists no rigorous result for this general class of models. However, the special case of the spin-boson model ($\mathcal{A} := \mathsf{M}_2$ the 2×2-matices, $E := \{f \in \mathsf{L}^2(\mathbb{R}) \mid \int \frac{|f(k)|^2}{|k|} dk < \infty\}$, $m = 1$ and $B_1 = B_1^*$) is discussed in [7] using methods different from ours.

The results of the present investigation apply to a very wide class of models. Of special interest are material lattice systems coupled to a radiation field (Bose system) in thermal equilibrium. The boson spectrum may be choosen to be discrete or continuous. The testfunctions f_k in the interaction operator P (cf. (2)) represent the coupling constants to the different modes of the radiation field. If the boson spectrum is continuous the f_k can have some kind of singular infrared behaviour (for more details, see Section 2), which may lead to some interesting phenomena. With the present perturbation theoretical methods a treatment of the temperature states of the spin-boson model (a two-level atom interacting with the radiation field) is done in [10]. Further applications we have in mind are the Dicke model and the Josephson junction weakly coupled to the microwave radiation, where both the lattice system and the boson field are taken in the infinite volume limit. Observe that in the thermodynamic limit of the material system the B_k should remain bounded.

This kind of limit behaviour of the B_k is then a kind of weak coupling. In an extensive coupling the perturbation expansion would not converge and would leave the equilibrium representation of the free system.

In Section 2 the considered class of boson fields is introduced, which in Section 3 are coupled to the KMS quantum systems. In [8] a restricted class of models has been discussed. There, an equilibrium state of the interacting system has been defined by a density operator in Fock space, for which the trace-class property was derived by a Dyson expansion also with the perturbation term P but in the Fock representation. The connection of this Fock-normal temperature state and the one of Section 3 in the GNS-representation is given in Section 4. In Section 5 all the proofs are done, which generalizes arguments in the proof of [4, Theorem 5.4.4] and some of [2].

2 The Bose System

Let us consider the Weyl algebra $\mathcal{W}(E)$ over the complex pre-Hilbert space E, generated by Weyl operators $W(f)$, $f \in E$ [4, p. 20]. Let $S \geq 0$ be a selfadjoint operator on the completion $\overline{E}$ of E, not having zero as its eigenvalue and satisfying for some $\beta > 0$ (the inverse temperature)

$$\mathrm{e}^{itS}(E) \subseteq E \quad \forall t \in \mathbb{R} \qquad \text{and} \qquad E \subseteq \mathcal{D}(T_\beta^{\frac{1}{2}}) \,, \tag{3}$$

where $T_\beta := \mathrm{e}^{-\beta S}(\mathbb{1} - \mathrm{e}^{-\beta S})^{-1}$. We define the boson equilibrium state ω_b on $\mathcal{W}(E)$ by

$$\omega_b(W(f)) = \exp\left\{-\frac{1}{4}\|f\|^2 - \frac{1}{2}\left\|T_\beta^{\frac{1}{2}} f\right\|^2\right\} \qquad \forall f \in E \,. \tag{4}$$

ω_b is gauge-invariant and quasi-free. The field, annihilation and creation operators corresponding to the GNS-representation $(\Pi_b, \mathcal{H}_b, \Omega_b)$ of ω_b are denoted by $\Phi_b(f)$, $a_b(f)$ and $a_b^*(f)$, respectively, where $f \in E$ (see e.g. [4, p. 25]).

For each $t \in \mathbb{R}$ there is a *–automorphism τ_t^b on $\mathcal{W}(E)$ (Bogoliubov transformation), such that $\tau_t^b(W(f)) = W(\mathrm{e}^{itS} f)\ \forall f \in E$. Of course $\tau_s^b \circ \tau_t^b = \tau_{s+t}^b$ and $\omega_b \circ \tau_t^b = \omega_b$ for all $s, t \in \mathbb{R}$. Consequently, by the uniqueness of the GNS-representation up to unitary equivalence [3, Corollary 2.3.17] there exist unitary operators $U_t \in \mathcal{L}(\mathcal{H}_b)$, such that $\Pi_b(\tau_t^b(Y)) = U_t \Pi_b(Y) U_t^*$ and $U_t \Omega_b = \Omega_b$. One easily deduces that $(U_t)_{t\in\mathbb{R}}$ is a strongly-continuous unitary group whose infinitesimal generator we denote by G_b, that is $U_t = \mathrm{e}^{itG_b}$. Clearly G_b is renormalized: $G_b \Omega_b = 0$.

In the GNS-representation of ω_b we set $\mathcal{M}_b := (\Pi_b(\mathcal{W}(E)))''$ ($''$ denotes the bicommutant) for the associated von Neumann algebra and extend ω_b and τ_t^b to $\mathcal{M}_b$ by setting $\omega_b(Y) := \langle \Omega_b, Y\Omega_b \rangle$ and $\tau_t^b(Y) := U_t Y U_t^*$ for each $Y \in \mathcal{M}_b$. The extended ω_b is a (τ^b, β)–KMS state on $\mathcal{M}_b$. In Section 3 our interest only concerns ω_b considered as a state on $\mathcal{M}_b$.

Now let us take a look on the testfunctions f_k, $k \in \{1, \ldots, m\}$, appearing in the interaction operator P (cf. the equations (2) and (6)). By (3) and [12, Theorem 4], E is a core of $T_\beta^{\frac{1}{2}}$ and hence ω_b (regarded as a state on $\mathcal{W}(E)$) can be canonically extended to a state on $\mathcal{W}(\mathcal{D}(T_\beta^{\frac{1}{2}}))$ fulfilling (4) for each $f \in \mathcal{D}(T_\beta^{\frac{1}{2}})$ and having the same GNS-representation (cf. [9]). Hence the operators $\Phi_b(f)$, $a_b(f)$ and $a_b^*(f)$ are also well defined for each $f \in \mathcal{D}(T_\beta^{\frac{1}{2}})$ and in the perturbation operator P we can take the coupling testfunctions f_k as elements from $\mathcal{D}(T_\beta^{\frac{1}{2}})$. Indeed, as is seen from the estimations in Section 5, $f_k \in \mathcal{D}(T_\beta^{\frac{1}{2}})$ is the largest possible choice for the coupling modes f_k. This choice allows some kind of singular infrared behaviour of the coupling constants f_k, which will be explained in the physically relevant example of a radiation field. There one has $S := \sqrt{-\Delta}$ with the Laplacian Δ in the whole euclidean space $\mathbb{R}^n$. Since $\mathcal{D}(T_\beta^{\frac{1}{2}}) = \mathcal{D}(S^{-\frac{1}{2}})$ for $\beta > 0$, in momentum space the condition $f_k \in \mathcal{D}(T_\beta^{\frac{1}{2}})$ is equivalent to $\int \frac{|\widehat{f_k}(\mathbf{p})|^2}{|\mathbf{p}|} d\mathbf{p} < \infty$ ($\widehat{g}$ denotes the Fourier transform of $g \in \mathrm{L}^2(\mathbb{R}^n)$), which characterizes the admissible infrared behaviour of the coupling contants $\widehat{f_k}(\mathbf{p})$. But this condition includes e.g. the infrared divergence $\int \frac{|\widehat{f_k}(\mathbf{p})|^2}{|\mathbf{p}|^2} d\mathbf{p} = \infty$, which may give rise to some infrared collective phenomena, an example of which is given in [1], where the chirality of molecules is discussed.

3 The Interacting System in Thermal Equilibrium

Let $\beta > 0$ be the inverse temperature and ω_a a (τ^a, β)–KMS state of the C^*– or W^*–dynamical system $(\mathcal{A}, \tau^a)$. ω_a is assumed to be normal in the W^*–case. Denote by the same symbols the extensions of ω_a and τ^a to the weak closure $\mathcal{M}_a := (\Pi_a(\mathcal{A}))''$ in the GNS-representation $(\Pi_a, \mathcal{H}_a, \Omega_a)$ of ω_a. Then $\omega_a(X) = \langle \Omega_a, X\Omega_a \rangle$ is a (τ^a, β)–KMS state on $\mathcal{M}_a$ (see [3, Theorem 2.4.24] and [4, Corollary 5.3.4]). Since ω_a is τ^a–invariant, by the same arguments as in Section 2 there exists a selfadjoint operator A_a in $\mathcal{H}_a$ such that

$$A_a \, \Omega_a \;=\; 0 \qquad \text{and} \qquad \tau_t^a(X) \;=\; \mathrm{e}^{itA_a} X \, \mathrm{e}^{-itA_a} \quad \forall t \in \mathbb{R} \;\; \forall X \in \mathcal{M}_a \, .$$

Now we couple this general KMS system to the boson field of Section 2. The free hamiltonian of the composite system is given in the tensor product Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ by

$$K \;=\; A_a \otimes \mathbb{1} \;+\; \mathbb{1} \otimes G_b \tag{5}$$

and the interaction operator is of the form

$$P \;=\; \sum_{k=1}^{m} \Big(B_k \otimes a_b(f_k) \;+\; B_k^* \otimes a_b^*(f_k) \Big) \;, \tag{6}$$

where $B_1, \ldots, B_m \in \mathcal{M}_a$ and $f_1, \ldots, f_m \in \mathcal{D}(T_\beta^{\frac{1}{2}})$ and $m \in \mathbb{N}$.

The temperature state ω of the total non-interacting system is given with $\Omega := \Omega_a \otimes \Omega_b$ by

$$\omega(Z) \;=\; \omega_a \otimes \omega_b(Z) \;=\; \langle \Omega_a \otimes \Omega_b, Z \, \Omega_a \otimes \Omega_b \rangle \;=\; \langle \Omega, Z \, \Omega \rangle \qquad \forall Z \in \mathcal{M}_a \overline{\otimes} \mathcal{M}_b \, .$$

For calculating the perturbed equilibrium state ω^p we have to perform a perturbation expansion of Ω with the operator in (6). Despite the unboundedness of P this expansion is convergent, which is the main result of this work.

Theorem 3.1 *In the above situation, the interacting hamiltonian $H := K + P$ is essentially selfadjoint. Further, $\Omega \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H})$ and the perturbation expansion*

$$\Omega^p := \mathrm{e}^{-\frac{\beta}{2}H}\,\Omega = \mathrm{e}^{-\frac{\beta}{2}H}\,\mathrm{e}^{\frac{\beta}{2}K}\,\Omega =$$
$$= \Omega + \sum_{n=1}^{\infty}(-1)^n \int_{0\le t_1\le\cdots\le t_n\le\frac{\beta}{2}} dt_1\cdots dt_n\ \mathrm{e}^{-t_1K}P\,\mathrm{e}^{-(t_2-t_1)K}P\ \cdots\ \mathrm{e}^{-(t_n-t_{n-1})K}P\,\Omega$$

is converging with respect to the norm of $\mathcal{H}_a\otimes\mathcal{H}_b$.

Proof: Using the explicite GNS-representation of ω_b as stated in the beginning of Section 5, the assertion follows from Theorem 5.2(i) and Theorem 5.4. □

Now we define the temperature state ω^p of the interacting system by

$$\omega^p(Z) := \frac{1}{\|\Omega^p\|^2}\langle\Omega^p, Z\,\Omega^p\rangle \qquad \forall Z \in \mathcal{M}_a\overline{\otimes}\mathcal{M}_b\,. \tag{7}$$

Of course ω^p also can be regarded as a state on the C^*–algebra $\mathcal{A}\otimes_{\min}\mathcal{W}(E)$, the injective C^*–tensor product. We mention that $\mathcal{M}_a\overline{\otimes}\mathcal{M}_b$ is the weak closure of the representation $\Pi_a\otimes\Pi_b(\mathcal{A}\otimes_{\min}\mathcal{W}(E))$. In a subsequent work the KMS property of ω^p will be derived.

4 The Finite Dimensional Case

In this section we consider some special cases of the KMS system and the boson field.

As the (τ^a,β)–KMS system of Section 3 we consider a finite quantum system given on the finite dimensional Hilbert space $\mathcal{H}$ with the selfadjoint operator $\widetilde{A}\in\mathcal{L}(\mathcal{H})$ by

$$\tau^a_t(X) = \mathrm{e}^{it\widetilde{A}}X\,\mathrm{e}^{-it\widetilde{A}}\,, \quad \omega_a(X) = \frac{\mathrm{tr}(X\exp\{-\beta\widetilde{A}\})}{\mathrm{tr}(\exp\{-\beta\widetilde{A}\})} \qquad \forall X\in\mathcal{L}(\mathcal{H}) =: \mathcal{A}\,.$$

The operator A_a of Section 3 then is given by the renormalization of $\Pi_a(\widetilde{A})$ using the modular conjugation J_a of $(\mathcal{M}_a,\Omega_a)$: $A_a = \Pi_a(\widetilde{A}) - J_a\Pi_a(\widetilde{A})J_a$ (see [3, Section 2.5]). One has $\Pi_a(\tau^a_t(X)) = \mathrm{e}^{itA_a}\Pi_a(X)\,\mathrm{e}^{-itA_a}$, $t\in\mathbb{R}$, $X\in\mathcal{L}(\mathcal{H})$. Of course $\mathcal{M}_a = \Pi_a(\mathcal{L}(\mathcal{H}))$.

Our boson system we restrict to the one in the recent publication [8]: The one-boson hamiltonian S is a strictly positive and selfadjoint operator in the one-particle Hilbert space $E := \mathcal{K}$, such that $\mathrm{e}^{-\beta S}$ is trace-class. If $G := \mathrm{d}\Gamma(S)$ denotes its second quantization in the Fock space $\mathcal{F}_+(\mathcal{K})$, it follows that $\mathrm{e}^{-\beta G}$ is trace-class on $\mathcal{F}_+(\mathcal{K})$ and the boson equilibrium state ω_b of formula (4) is given in the Fock representation $\Pi_{\mathcal{F}}$ of $\mathcal{W}(\mathcal{K})$ by

$$\omega_b(Y) = \frac{\mathrm{tr}(\Pi_{\mathcal{F}}(Y)\,\exp\{-\beta G\})}{\mathrm{tr}(\exp\{-\beta G\})} \qquad \forall Y\in\mathcal{W}(\mathcal{K})\,.$$

It is well known that in this case the Fock representation $\Pi_{\mathcal{F}}$ and the GNS-representation Π_b of $\mathcal{W}(\mathcal{K})$ are quasi-equivalent. That is, there exists a unique W^*–isomorphism α from $(\Pi_{\mathcal{F}}(\mathcal{W}(\mathcal{K})))'' = \mathcal{L}(\mathcal{F}_+(\mathcal{K}))$ onto $\mathcal{M}_b$ such that $\alpha(\Pi_{\mathcal{F}}(Y)) = \Pi_b(Y)\ \forall Y \in \mathcal{W}(\mathcal{K})$. Since $\Pi_{\mathcal{F}}(\tau_t^b(Y)) = \mathrm{e}^{itG}\Pi_{\mathcal{F}}(Y)\,\mathrm{e}^{-itG}$, in some kind G_b is the renormalized representation of G under α.

With the isomorphism α the von Neumann algebras $\mathcal{L}(\mathcal{H})\overline{\otimes}\mathcal{L}(\mathcal{F}_+(\mathcal{K})) = \mathcal{L}(\mathcal{H}\otimes\mathcal{F}_+(\mathcal{K}))$ and $\mathcal{M}_a\overline{\otimes}\mathcal{M}_b$ are $*$–isomorphic via the W^*–isomorphism $\gamma = \Pi_a\overline{\otimes}\alpha$.

In [8] the equilibrium state $\widetilde{\omega}^p$ of the interacting system is directly calculated in the Hilbert space $\mathcal{H}\otimes\mathcal{F}_+(\mathcal{K})$: Let be

$$\begin{aligned}\widetilde{K} &= \widetilde{A}\otimes 1\!\!1 + 1\!\!1\otimes G \\ \widetilde{P} &= \sum_{k=1}^{m}\left(\widetilde{B_k}\otimes a(f_k) + \widetilde{B_k}^*\otimes a^*(f_k)\right),\end{aligned}$$

where $B_1,\ldots,B_m \in \mathcal{L}(\mathcal{H})$, $f_1,\ldots,f_m \in \mathcal{D}(\mathrm{e}^{\frac{\beta}{2}S})$ and the $a(f_k)$, $a^*(f_k)$ are the usual Fock annihilation and creation operators. Then $\widetilde{\omega}^p$ is defined by

$$\widetilde{\omega}^p(Z) = \frac{\mathrm{tr}\Big(Z\,\exp\big\{-\beta(\widetilde{K}+\widetilde{P})\big\}\Big)}{\mathrm{tr}\Big(\exp\big\{-\beta(\widetilde{K}+\widetilde{P})\big\}\Big)} \qquad \forall Z \in \mathcal{L}(\mathcal{H}\otimes\mathcal{F}_+(\mathcal{K}))\,.$$

If $B_k := \Pi_a(\widetilde{B_k})$ one may regard P of equation (6) as the representation of $\widetilde{P}$ under γ and K of equation (5) as the renormalized representation of $\widetilde{K}$ (renormalized in the sense that $K\Omega = 0$). Thus one expects the agreement of $\widetilde{\omega}^p$ and ω^p (cf. formula (7)) via γ. That this is indeed the case is the contents of the following theorem, whose proof is given at the end of Section 5.

Theorem 4.1 *Let all be as introduced above in this section. Then it follows*

$$\widetilde{\omega}^p\circ\gamma = \omega^p\,.$$

5 Perturbation Expansions

In this section in the GNS-representation of $\omega_a\otimes\omega_b$ we calculate the perturbation of the cyclic vector and afterwards we prove Theorem 4.1. But first we establish some notations.

The Fock space over the Hilbert space $\mathcal{K}$ is given by $\mathcal{F}_+(\mathcal{K}) = \bigoplus_{n=0}^{\infty}\mathbb{P}_+(\otimes_n\mathcal{K})$ with the symmetrisation operator $\mathbb{P}_+$ and the n–fold tensor product $\otimes_n\mathcal{K}$ of $\mathcal{K}$ with itself. For the algebraic tensor product we write $\odot$. The Fock field, annihilation and creation operators are denoted by $\Phi(f)$, $a(f) = a^-(f)$ and $a^*(f) = a^+(f)$ $(f \in \mathcal{K})$ respectively, and by $\Omega_{\mathcal{F}}$ the Fock vacuum. Moreover $a^0(f) := 1\!\!1$. Also we use Segal's notation $\mathrm{d}\Gamma(R)$ for the second quantization of the selfadjoint operator R in $\mathcal{K}$.

For the proof of Theorem 3.1 we need the explicite GNS-representation of ω_b. It is given with an arbitrary antilinear involution J, satisfying $\langle Jf, Jg\rangle = \langle g, f\rangle \; \forall f, g \in \overline{E}$ by

$$\begin{aligned}\mathcal{H}_b &= \mathcal{F}_+(\overline{E}) \otimes \mathcal{F}_+(\mathcal{V}), \text{ where } \mathcal{V} = \overline{JT_\beta^{\frac{1}{2}}(E)},\\ \Omega_b &= \Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{F}},\\ \Pi_b(W(f)) &= W_{\mathcal{F}}((\mathbb{1}+T_\beta)^{\frac{1}{2}}f) \otimes W_{\mathcal{F}}(JT_\beta^{\frac{1}{2}}f) \quad \forall f \in E \text{ or } \in \mathcal{D}(T_\beta^{\frac{1}{2}}),\end{aligned}$$

where the $W_{\mathcal{F}}(f)$ are the usual Fock-Weyl operators. Especially we choose J such that $Je^{itS} = e^{-itS}J \; \forall t \in \mathbb{R}$. Then $G_b = G \otimes \mathbb{1} - \mathbb{1} \otimes G$ with $G = d\Gamma(S)$ (remark, $\mathcal{F}_+(\mathcal{V})$ is a reducing subspaces for G). For the field, annihilation and creation operators associated with ω_b one gets for each $f \in \mathcal{D}(T_\beta^{\frac{1}{2}})$

$$\begin{aligned}\Phi_b(f) &= \Phi((\mathbb{1}+T_\beta)^{\frac{1}{2}}f) \otimes \mathbb{1} + \mathbb{1} \otimes \Phi(JT_\beta^{\frac{1}{2}}f)\\ a_b(f) &= a((\mathbb{1}+T_\beta)^{\frac{1}{2}}f) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(JT_\beta^{\frac{1}{2}}f)\\ a_b^*(f) &= a^*((\mathbb{1}+T_\beta)^{\frac{1}{2}}f) \otimes \mathbb{1} + \mathbb{1} \otimes a(JT_\beta^{\frac{1}{2}}f)\,.\end{aligned}$$

Lemma 5.1 *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Let $D := \bigcap_{\alpha \geq 1} \mathcal{D}(\mathbb{1} \otimes \alpha^N \otimes \alpha^N) \subseteq \mathcal{H} \otimes \mathcal{F}_+(\mathcal{K}) \otimes \mathcal{F}_+(\mathcal{K})$, where N is the number operator in Fock space. For $C \in \mathcal{L}(\mathcal{H})$, the multiindex $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_j \in \{-,+\}$ and $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$ with $f_j, g_j \in \mathcal{K}$ define the operator*

$$Q_{f,g}^{C,\nu} := C \otimes \prod_{k=1}^{n} \left(a^{\nu_k}(f_k) \otimes \mathbb{1} + \mathbb{1} \otimes a^{-\nu_k}(g_k)\right)\,.$$

It follows that $Q_{f,g}^{C,\nu}$ is welldefined on D with $Q_{f,g}^{C,\nu}(D) \subseteq D$ and for all $\alpha, \beta \geq 1$ and all $\psi \in D$ we have the estimations

$$\left\|\left(\mathbb{1} \otimes \alpha^N \otimes \beta^N\right) Q_{f,g}^{C,\nu}\,\psi\right\| \leq \|C\| \left(2^{\frac{3}{2}}\gamma\,\delta\right)^n \sqrt{n!}\, \left\|\left(\mathbb{1} \otimes \left(\sqrt{2}\alpha\right)^N \otimes \left(\sqrt{2}\beta\right)^N\right)\psi\right\|\,,$$

where $\gamma := \max\{\|f_1\|, \ldots, \|f_n\|, \|g_1\|, \ldots, \|g_n\|\}$ and $\delta := \max\{\alpha, \beta\}$.
Moreover, if $s - \lim_{m\to\infty} C_m = C$ in $\mathcal{L}(\mathcal{H})$ and $\lim_{m\to\infty} \|f_k^m - f_k\| = 0$ and $\lim_{m\to\infty} \|g_k^m - g_k\| = 0$ for all k, then

$$\lim_{m\to\infty} \left\|\left(\mathbb{1} \otimes \alpha^N \otimes \beta^N\right)\left(Q_{f^m,g^m}^{C_m,\nu}\psi - Q_{f,g}^{C,\nu}\right)\psi\right\| = 0 \qquad \forall \psi \in D\,. \tag{8}$$

Proof: Let $\sum_i \xi_i \otimes \eta_i$ be a finite sum, such that $\xi_i \in \mathcal{D}(N^{\frac{1}{2}})$ and $\eta_i \in \mathcal{F}_+(\mathcal{K})$ with $\eta_i \perp \eta_j$ for $i \neq j$. Using the estimation (see e.g. [4, p. 9])

$$\|a^{\pm}(f)\,\psi\| \leq \|f\| \left\|(N+\mathbb{1})^{\frac{1}{2}}\psi\right\| \qquad \forall \psi \in \mathcal{D}(N^{\frac{1}{2}}) \tag{9}$$

we obtain

$$\begin{aligned}\left\|(a^{\pm}(f)\otimes \mathbb{1})\sum_i \xi_i\otimes\eta_i\right\|^2 &= \sum_i \left\|a^{\pm}(f)\xi_i\right\|^2\|\eta_i\|^2\\ &\leq \|f\|^2\sum_i\left\|(N+\mathbb{1})^{\frac12}\xi_i\right\|^2\|\eta_i\|^2\\ &= \|f\|^2\left\|\left((N+\mathbb{1})^{\frac12}\otimes\mathbb{1}\right)\sum_i\xi_i\otimes\eta_i\right\|^2 .\end{aligned}$$

Consequently

$$\left\|(a^{\pm}(f)\otimes\mathbb{1})\,\psi\right\| \;\leq\; \|f\|\left\|\left((N+\mathbb{1})^{\frac12}\otimes\mathbb{1}\right)\psi\right\| \qquad \forall\psi\in\mathcal{D}(N^{\frac12})\odot\mathcal{F}_+(\mathcal{K})\,, \tag{10}$$

and in an analogous way one deduces

$$\left\|(\mathbb{1}\otimes a^{\pm}(f))\,\psi\right\| \;\leq\; \|f\|\left\|\left(\mathbb{1}\otimes(N+\mathbb{1})^{\frac12}\right)\psi\right\| \qquad \forall\psi\in\mathcal{F}_+(\mathcal{K})\odot\mathcal{D}(N^{\frac12})\,. \tag{11}$$

By multiplying out we get for the multiindex $\nu=(\nu_1,\ldots,\nu_n)$ exactly 2^n multiindices $(\mu,\lambda)(\nu)=(\mu_1,\ldots,\mu_n,\lambda_1,\ldots,\lambda_n)$ with $\lambda_k=0$ if $\mu_k\in\{-,+\}$ and $\mu_k=0$ if $\lambda_k\in\{-,+\}$, such that

$$\prod_{k=1}^n\left(a^{\nu_k}(f_k)\otimes\mathbb{1}+\mathbb{1}\otimes a^{-\nu_k}(g_k)\right) \;=\; \sum_{(\mu,\lambda)(\nu)}\prod_{k=1}^n\left(a^{\mu_k}(f_k)\otimes a^{\lambda_k}(g_k)\right)\,. \tag{12}$$

Consider now a fixed but arbitrary multiindex $(\mu,\lambda)(\nu)$. If $\#M$ denotes the cardinality of the set M, let us define $l:=l_{(\mu,\lambda)}:=\#\{\mu_k\mid\mu_k\in\{-,+\},\,k\in\{1,\ldots,n\}\}$. Then $n-l=\#\{\lambda_k\mid\lambda_k\in\{-,+\},\,k\in\{1,\ldots,n\}\}$. Let $F_m(\mathcal{K}):=\bigoplus_{n=0}^{m}\mathbb{P}_+(\otimes_n\mathcal{K})$. If $\xi\in F_m(\mathcal{K})\odot F_m(\mathcal{K})$ for some $m\in\mathbb{N}$, then $\xi=\sum_{i,j=0}^{m}\xi_{i,j}$ with $\xi_{i,j}\in\mathbb{P}_+(\otimes_i\mathcal{K})\odot\mathbb{P}_+(\otimes_j\mathcal{K})$ and for $i\neq i'$ or $j\neq j'$ we have the orthogonality of the vectors $\left(\alpha^N\otimes\beta^N\right)\prod_{k=1}^n\left(a^{\mu_k}(f_k)\otimes a^{\lambda_k}(g_k)\right)\xi_{i,j}$ and $\left(\alpha^N\otimes\beta^N\right)\prod_{k=1}^n\left(a^{\mu_k}(f_k)\otimes a^{\lambda_k}(g_k)\right)\xi_{i',j'}$. Therefore

$$\begin{aligned}&\left\|\left(\alpha^N\otimes\beta^N\right)\prod_{k=1}^n\left(a^{\mu_k}(f_k)\otimes a^{\lambda_k}(g_k)\right)\xi\right\|^2 =\\
&= \sum_{i,j=0}^m\left\|\left(\alpha^N\otimes\beta^N\right)\left(\prod_{k=1}^n a^{\mu_k}(f_k)\otimes\prod_{k=1}^n a^{\lambda_k}(g_k)\right)\xi_{i,j}\right\|^2\\
&\leq \sum_{i,j=0}^m\alpha^{2(i+l)}\beta^{2(j+n-l)}\left\|\left(\prod_{k=1}^n a^{\mu_k}(f_k)\otimes\mathbb{1}\right)\left(\mathbb{1}\otimes\prod_{k=1}^n a^{\lambda_k}(g_k)\right)\xi_{i,j}\right\|^2\\
&\overset{(10)}{\leq} \sum_{i,j=0}^m\alpha^{2(i+l)}\beta^{2(j+n-l)}(i+l)\cdots(i+1)\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|^2\right)\left\|\left(\mathbb{1}\otimes\prod_{k=1}^n a^{\lambda_k}(g_k)\right)\xi_{i,j}\right\|^2\\
&\overset{(11)}{\leq} \sum_{i,j=0}^m\alpha^{2(i+l)}\beta^{2(j+n-l)}(i+l)\cdots(i+1)\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|^2\right)\times\end{aligned}$$

$$
\begin{aligned}
& \times \ (j+n-l)\cdots(j+1)\left(\prod_{\lambda_k\in\{-,+\}}\|g_k\|^2\right)\|\xi_{i,j}\|^2 \\
\overset{(\star)}{\leq}\ & \left(\sqrt{2}\,\alpha\right)^{2l}\left(\sqrt{2}\,\beta\right)^{2(n-l)} l!\,(n-l)!\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|^2\right)\left(\prod_{\lambda_k\in\{-,+\}}\|g_k\|^2\right)\ \times \\
& \times \ \sum_{i,j=0}^{m}\left(\sqrt{2}\,\alpha\right)^{2i}\left(\sqrt{2}\,\beta\right)^{2j}\|\xi_{i,j}\|^2 \\
=\ & \left(\sqrt{2}\,\alpha\right)^{2l}\left(\sqrt{2}\,\beta\right)^{2(n-l)} l!\,(n-l)!\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|^2\right)\left(\prod_{\lambda_k\in\{-,+\}}\|g_k\|^2\right)\ \times \\
& \times \ \left\|\left(\left(\sqrt{2}\,\alpha\right)^{N}\otimes\left(\sqrt{2}\,\beta\right)^{N}\right)\xi\right\|^2, \qquad (13)
\end{aligned}
$$

where we have used the inequalities (10) and (11) and in $(\star)$ the simple estimation $(n+k)\cdots(n+1) = \frac{(n+k)!}{n!\,k!}k! = \binom{n+k}{n}k! \leq 2^{n+k}k!$. Now let $\psi = \sum_{r=0}^{p}\phi_r\otimes\xi_r$ such that $\phi_r \in \mathcal{H}$ with $\phi_r \perp \phi_{r'}$ for $r \neq r'$ and $\xi_r \in F_m(\mathcal{K})\odot F_m(\mathcal{K})$. We get

$$
\begin{aligned}
& \left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)\left(C\otimes\prod_{k=1}^{n}\left(a^{\mu_k}(f_k)\otimes a^{\lambda_k}(g_k)\right)\right)\psi\right\|^2 = \\
=\ & \left\|\left(C\otimes 1\!\!1\otimes 1\!\!1\right)\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)\left(1\!\!1\otimes\prod_{k=1}^{n}a^{\mu_k}(f_k)\otimes\prod_{k=1}^{n}a^{\lambda_k}(g_k)\right)\psi\right\|^2 \\
\leq\ & \|C\|^2\sum_{r=0}^{p}\|\phi_r\|^2\left\|\left(\alpha^N\otimes\beta^N\right)\left(\prod_{k=1}^{n}a^{\mu_k}(f_k)\otimes\prod_{k=1}^{n}a^{\lambda_k}(g_k)\right)\xi_r\right\|^2 \\
\overset{(13)}{\leq}\ & \|C\|^2\left(\sqrt{2}\,\alpha\right)^{2l}\left(\sqrt{2}\,\beta\right)^{2(n-l)} l!\,(n-l)!\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|^2\right)\left(\prod_{\lambda_k\in\{-,+\}}\|g_k\|^2\right)\ \times \\
& \times\ \left\|\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^{N}\otimes\left(\sqrt{2}\,\beta\right)^{N}\right)\psi\right\|^2 .
\end{aligned}
$$

Now use (12). Because $1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^{N}\otimes\left(\sqrt{2}\,\beta\right)^{N}$ is closed and $\bigcup_{m\in\mathbb{N}}\mathcal{H}\odot F_m(\mathcal{K})\odot F_m(\mathcal{K})$ is a core for $1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^{N}\otimes\left(\sqrt{2}\,\beta\right)^{N}$, the operator $Q_{f,g}^{C,\nu}$ is welldefined on D with

$$
\begin{aligned}
\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)Q_{f,g}^{C,\nu}\,\psi\right\| \ \leq\ & \|C\|\,(\sqrt{2}\,\delta)^n\left\|\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^{N}\otimes\left(\sqrt{2}\,\beta\right)^{N}\right)\psi\right\|\ \times \\
& \times\ \sum_{(\mu,\lambda)(\nu)}\sqrt{l!\,(n-l)!}\left(\prod_{\mu_k\in\{-,+\}}\|f_k\|\right)\left(\prod_{\lambda_k\in\{-,+\}}\|g_k\|\right) \qquad (14)
\end{aligned}
$$

for all $\psi \in D$ and all $\alpha,\beta \geq 1$. This implies $Q_{f,g}^{C,\nu}(D) \subseteq D$. Running through all the 2^n multiindices $(\mu,\lambda)(\nu)$ the same number $l = l_{(\mu,\lambda)}$ appears exactly $\binom{n}{l}$ times. Thus, using $\sum_{l=0}^{n}\binom{n}{l}\sqrt{l!\,(n-l)!} = \sqrt{n!}\sum_{l=0}^{n}\binom{n}{l}^{\frac{1}{2}} \leq \sqrt{n!}\sum_{l=0}^{n}\binom{n}{l} = 2^n\sqrt{n!}$ the stated estimation follows.

Let's prove (8). If $\phi \in \mathcal{H}$ and $\xi \in F_m(\mathcal{K})\odot F_m(\mathcal{K})$, then

$$
\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)\left(Q_{f^m,g^m}^{C_m,\nu} - Q_{f,g}^{C,\nu}\right)\phi\otimes\xi\right\| \ \leq
$$

$$\leq \ \|(C_m - C)\phi\| \left\| \left(\alpha^N \otimes \beta^N\right) \prod_{k=1}^{n} \left(a^{\nu_k}(f_k^m) \otimes 1\!\!1 + 1\!\!1 \otimes a^{-\nu_k}(g_k^m)\right) \xi \right\| \ +$$

$$+ \ \|C\phi\| \sum_{k=1}^{n} \left\| \left(\alpha^N \otimes \beta^N\right) \left(a^{\nu_1}(f_1) \otimes 1\!\!1 + 1\!\!1 \otimes a^{-\nu_1}(g_1)\right) \cdots \right.$$

$$\cdots \left(a^{\nu_k}(f_k^m - f_k) \otimes 1\!\!1 + 1\!\!1 \otimes a^{-\nu_k}(g_k^m - g_k)\right) \cdots$$

$$\left. \cdots \left(a^{\nu_n}(f_n^m) \otimes 1\!\!1 + 1\!\!1 \otimes a^{-\nu_n}(g_n^m)\right) \xi \right\| \ ,$$

which converges to zero because of (13). Thus (8) is valid for all $\psi \in \mathcal{H} \odot F_m(\mathcal{K}) \odot F_m(\mathcal{K})$ and all $m \in \mathbb{N}$. Now use the fact that $\bigcup_{m \in \mathbb{N}} \mathcal{H} \odot F_m(\mathcal{K}) \odot F_m(\mathcal{K})$ is a core for the operator $1\!\!1 \otimes \left(\sqrt{2}\,\alpha\right)^N \otimes \left(\sqrt{2}\,\beta\right)^N$, equation (14) and an $\frac{\varepsilon}{3}$–argument. □

Theorem 5.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and R a selfadjoint operator in $\mathcal{K}$ and $\mathrm{d}\Gamma(R) =: G$ the second quantization of R in $\mathcal{F}_+(\mathcal{K})$. Let A be a selfadjoint operator in $\mathcal{H}$, $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$ and $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{K}$. Furthermore let*

$$K \ = \ A \otimes 1\!\!1 \otimes 1\!\!1 + 1\!\!1 \otimes (G \otimes 1\!\!1 - 1\!\!1 \otimes G) \qquad \text{and}$$

$$P \ = \ \sum_{k=1}^{m} \left(B_k \otimes \left(a(f_k) \otimes 1\!\!1 + 1\!\!1 \otimes a^*(g_k)\right) \ + \ B_k^* \otimes \left(a^*(f_k) \otimes 1\!\!1 + 1\!\!1 \otimes a(g_k)\right) \right)$$

be operators in $\mathcal{H} \otimes \mathcal{F}_+(\mathcal{K}) \otimes \mathcal{F}_+(\mathcal{K})$. Moreover let $D := \bigcap_{\alpha \geq 1} \mathcal{D}(1\!\!1 \otimes \alpha^N \otimes \alpha^N)$. It follows:

(i) *The operator $H := K + P$ is essentially selfadjoint. $D \bigcap \mathcal{D}(K)$ is a core for H.*

(ii) *Define $U_t := \mathrm{e}^{itK}$ and for $t \in \mathbb{R}$ and $\vec{t} \in \mathbb{R}^n$*

$$F_t^{(n)}(\vec{t}) \ := \ U_{t-t_1} P U_{t_1-t_2} P U_{t_2-t_3} P \cdots U_{t_{n-1}-t_n} P U_{t_n} \, .$$

Then $D \subseteq \mathcal{D}(F_t^{(n)}(\vec{t}))$ and $\mathbb{R}^n \ni \vec{t} \mapsto F_t^{(n)}(\vec{t})\psi$ is continuous for each $\psi \in D$. If $U_t^{(n)}\psi := i^n \int_{t_1=0}^{t} dt_1 \int_{t_2=0}^{t_1} dt_2 \ldots \int_{t_n=0}^{t_{n-1}} dt_n \, F_t^{(n)}(\vec{t})\psi$, then

$$\mathrm{e}^{itH}\psi \ = \ U_t\psi \ + \ \sum_{n=1}^{\infty} U_t^{(n)}\psi \qquad \forall \psi \in D \ \forall t \in \mathbb{R} \, ,$$

where the series converges in norm. Moreover $U_t^{(n)}(D) \subseteq D$ and $\mathrm{e}^{itH}(D) \subseteq D$.

(iii) *For each $\xi, \eta \in \mathcal{K}$ we have $(1\!\!1 \otimes W_{\mathcal{F}}(\xi) \otimes W_{\mathcal{F}}(\eta))(D) \subseteq D$. Defining $\tau_t(X) := \mathrm{e}^{itK} X \mathrm{e}^{-itK}$ and $\tau_t^p(X) := \mathrm{e}^{itH} X \mathrm{e}^{-itH}$ for $X = P$ or $X \in \mathcal{L}(\mathcal{H} \otimes \mathcal{F}_+(\mathcal{K}) \otimes \mathcal{F}_+(\mathcal{K}))$ one has for each $\psi \in D$ and each $C \in \mathcal{L}(\mathcal{H})$*

$$\tau_t^p(C \otimes W_{\mathcal{F}}(\xi) \otimes W_{\mathcal{F}}(\eta))\, \psi \ =$$

$$= \ \tau_t(C \otimes W_{\mathcal{F}}(\xi) \otimes W_{\mathcal{F}}(\eta))\, \psi \ +$$

$$+ \sum_{n=1}^{\infty} i^n \int_{t_1=0}^{t} dt_1 \cdots \int_{t_n=0}^{t_{n-1}} dt_n \, [\tau_{t_n}(P), [\ldots [\tau_{t_1}(P), \tau_t(C \otimes W_{\mathcal{F}}(\xi) \otimes W_{\mathcal{F}}(\eta))] \ldots]]\, \psi.$$

Moreover it is for $\psi \in D$

$$\left\|[\tau_{t_n}(P),[\ldots[\tau_{t_1}(P),\tau_t(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))]\ldots]]\,\psi\right\| \leq$$
$$\leq (2^{\frac{9}{2}}\,m\,b\,\gamma)^n\,\sqrt{n!}\,\|C\|\,q(\xi,\eta)\left\|\left(1\!\!1\otimes(2^{\frac{3}{2}})^N\otimes(2^{\frac{3}{2}})^N\right)\psi\right\|,$$

where $b := \max\{\|B_1\|,\ldots,\|B_1\|\}$, $\gamma := \max\{\|f_1\|,\ldots,\|f_1\|,\|g_1\|,\ldots,\|g_1\|\}$ *and* $q(\xi,\eta) := \sum\limits_{k=0}^{\infty}\frac{1}{\sqrt{k!}}\left(2^{\frac{7}{2}}\max\{\|\xi\|,\|\eta\|\}\right)^k$.

Proof: For a better survey we first prove the Theorem for $m=1$ and later on we consider the general case. Thus let $P = B\otimes(a(f)\otimes 1\!\!1 + 1\!\!1\otimes a^*(g)) + B^*\otimes(a^*(f)\otimes 1\!\!1 + 1\!\!1\otimes a(g))$. We set $B^+ := B^*$ and $B^- := B$.

(i) and (ii). Using the well known relation $\mathrm{e}^{itG}a^{\pm}(f)\,\mathrm{e}^{-itG} = a^{\pm}(\mathrm{e}^{itR}f)$ and setting $B_t^{\pm} := \mathrm{e}^{-itA}B^{\pm}\mathrm{e}^{itA}$ we get by multiplying out 2^n multiindices $\nu = (\nu_1,\ldots,\nu_n)$ with $\nu_k \in \{-,+\}$ such that

$$F_t^{(n)}(\vec{t}) = U_t\sum_{\nu}\left(\prod_{k=1}^{n}B_{t_k}^{\nu_k}\right)\otimes\left(\prod_{k=1}^{n}\left(a^{\nu_k}(\mathrm{e}^{-it_kR}f)\otimes 1\!\!1 + 1\!\!1\otimes a^{-\nu_k}(\mathrm{e}^{it_kR}g)\right)\right). \quad (15)$$

From Lemma 5.1 and $\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)U_t = U_t\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)$ follows the continuity of $\mathbb{R}^n \ni \vec{t} \mapsto \left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)F_t^{(n)}(\vec{t})\psi$ and the estimation

$$\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)F_t^{(n)}(\vec{t})\psi\right\| \leq \left(2^{\frac{5}{2}}\|B\|\gamma\delta\right)^n\sqrt{n!}\left\|\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^N\otimes\left(\sqrt{2}\,\beta\right)^N\right)\psi\right\|, \quad (16)$$

Thus $U_t^{(n)}$ is well defined on D and because of $\int_{t_1=0}^{t}dt_1\int_{t_2=0}^{t_1}dt_2\ldots\int_{t_n=0}^{t_{n-1}}dt_n = \frac{t^n}{n!}$ one obtains the estimation

$$\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)U_t^{(n)}\psi\right\| \leq \frac{\left(2^{\frac{5}{2}}\,\|B\|\,\gamma\,\delta\,|t|\right)^n}{\sqrt{n!}}\left\|\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^N\otimes\left(\sqrt{2}\,\beta\right)^N\right)\psi\right\|, \quad (17)$$

from which $U_t^{(n)}(D) \subseteq D$ follows. Since $\sum\limits_{n=1}^{\infty}\frac{\kappa^n}{\sqrt{n!}} < \infty$ for $\kappa > 0$ for $\psi \in D$ the expression $V_t\psi := U_t\psi + \sum\limits_{n=1}^{\infty}U_t^{(n)}\psi$ is well defined and by similar arguments as above we conclude $V_t(D) \subseteq D$. Now calculating with convergent power series, one easily shows that $U_t^{(n)*} \supseteq U_{-t}^{(n)}$ and $\sum\limits_{k=0}^{n}U_s^{(n-k)}U_t^{(k)} = U_{s+t}^{(n)}\ \forall s,t\in\mathbb{R}$. Consequently $\langle V_s\varphi, V_t\psi\rangle = \langle\varphi, V_{t-s}\psi\rangle\ \forall\varphi,\psi\in D$. Since $V_0 \subseteq 1\!\!1$ each V_t extends to an unitary. Using (17) with $\alpha=\beta=1$ and an $\frac{\varepsilon}{3}$-argument it is immediate to check that $(V_t)_{t\in\mathbb{R}}$ is a strongly continuous unitary group. Now similar to [5, p. 69] one obtains from (17) that $\left\|\frac{1}{t}(V_t\psi-\psi) - \frac{1}{t}(U_t\psi-\psi) - iP\psi\right\| \to 0$ for $t\to 0$ for each $\psi\in D$. If $V_t =: \mathrm{e}^{itH}$, then the above calculation implies $H\psi = (K+P)\psi\ \forall\psi\in\mathcal{D}(K)\bigcap D$ and $\mathcal{D}(H)\bigcap D = \mathcal{D}(K)\bigcap D$. Since $M := \bigcup\limits_{m=1}^{\infty}\bigoplus\limits_{k=0}^{m}\mathbb{P}_+(\odot_k\mathcal{D}(R))$ is a core for G, $\mathcal{D}(A)\odot M\odot M \subset D$ is a core for K. Because of $V_t(\mathcal{D}(H)\bigcap D) \subseteq \mathcal{D}(H)\bigcap D\ \forall t\in\mathbb{R}$ (i) follows from [11, Theorem VIII.10].

(iii) Because of $\left(\Phi(\xi)\otimes 1\!\!1 + 1\!\!1\otimes\Phi(\eta)\right)^k = 2^{\frac{k}{2}}\sum_\nu\left(\prod_{l=1}^k\left(a^{\nu_l}(\xi)\otimes 1\!\!1 + 1\!\!1\otimes a^{-\nu_l}(\eta)\right)\right)$ we get with Lemma 5.1 for each $\psi\in D$

$$\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)\left(1\!\!1\otimes(\Phi(\xi)\otimes 1\!\!1 + 1\!\!1\otimes\Phi(\eta))\right)^k\psi\right\| \leq$$

$$\leq (2^3\tilde{\gamma}\,\delta)^k\sqrt{k!}\,\left\|\left(1\!\!1\otimes(\sqrt{2}\alpha)^N\otimes(\sqrt{2}\beta)^N\right)\psi\right\|,$$

where $\tilde{\gamma} := \max\{\|\xi\|,\|\eta\|\}$. Therefore we have $(1\!\!1\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))(D)\subseteq D$. Further for all $s_1,\ldots,s_l,t,t_1,\ldots,t_n\in\mathbb{R}$; $l,n\in\mathbb{N}$ and $\psi\in D$ we get

$$\left\|\left(1\!\!1\otimes\alpha^N\otimes\beta^N\right)\tau_{s_1}(P)\cdots\tau_{s_l}(P)\tau_t(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))\tau_{t_1}(P)\cdots\tau_{t_n}(P)\psi\right\| \leq$$

$$\overset{(16)}{\leq} \left(2^{\frac{5}{2}}\|B\|\gamma\,\delta\right)^l\sqrt{l!}\,\Big\|\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^N\otimes\left(\sqrt{2}\,\beta\right)^N\right)\times$$

$$\times\left((e^{itA}Ce^{-itA})\otimes W_{\mathcal{F}}(e^{itR}\xi)\otimes W_{\mathcal{F}}(e^{-itR}\eta)\right)\tau_{t_1}(P)\cdots\tau_{t_n}(P)\psi\Big\|$$

$$= \left(2^{\frac{5}{2}}\|B\|\gamma\,\delta\right)^l\sqrt{l!}\,\Big\|\left((e^{itA}Ce^{-itA})\otimes 1\!\!1\otimes 1\!\!1\right)\left(1\!\!1\otimes\left(\sqrt{2}\,\alpha\right)^N\otimes\left(\sqrt{2}\,\beta\right)^N\right)\times$$

$$\times\left(1\!\!1\otimes W_{\mathcal{F}}(e^{itR}\xi)\otimes W_{\mathcal{F}}(e^{-itR}\eta)\right)\tau_{t_1}(P)\cdots\tau_{t_n}(P)\psi\Big\|$$

$$\leq \|C\|\left(2^{\frac{5}{2}}\|B\|\gamma\,\delta\right)^l\sqrt{l!}\left(\sum_{k=0}^\infty\frac{(2^{\frac{7}{2}}\tilde{\gamma}\,\delta)^k}{\sqrt{k!}}\right)\left\|\left(1\!\!1\otimes(2\alpha)^N\otimes(2\beta)^N\right)\tau_{t_1}(P)\cdots\tau_{t_n}(P)\psi\right\|$$

$$\overset{(16)}{\leq} \|C\|\left(2^{\frac{5}{2}}\|B\|\gamma\,\delta\right)^l\sqrt{l!}\left(\sum_{k=0}^\infty\frac{(2^{\frac{7}{2}}\tilde{\gamma}\,\delta)^k}{\sqrt{k!}}\right)\left(2^{\frac{7}{2}}\|B\|\gamma\,\delta\right)^n\sqrt{n!}\,\left\|\left(1\!\!1\otimes(2^{\frac{3}{2}}\alpha)^N\otimes(2^{\frac{3}{2}}\beta)^N\right)\psi\right\|$$

$$\leq \|C\|\left(2^{\frac{7}{2}}\|B\|\gamma\,\delta\right)^{n+l}\sqrt{(n+l)!}\;q(\xi,\eta)\;\left\|\left(1\!\!1\otimes(2^{\frac{3}{2}}\alpha)^N\otimes(2^{\frac{3}{2}}\beta)^N\right)\psi\right\|.$$

If we calculate the commutators in $[\tau_{t_n}(P),[\ldots[\tau_{t_1}(P),\tau_t(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))]\ldots]]$ we get 2^n summands of the form $\tau_{s_1}(P)\cdots\tau_{s_j}(P)\tau_t(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))\tau_{s_{j+1}}(P)\cdots\tau_{s_n}(P)$, where $\{t_1,\ldots,t_n\}=\{s_1,\ldots,s_n\}$ and the stated estimation follows. Calculating with convergent power series one easily checks that

$$\tau_t^p(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta))\psi = \sum_{n=0}^\infty\sum_{k=0}^n U_t^{(n-k)}\left(C\otimes W_{\mathcal{F}}(\xi)\otimes W_{\mathcal{F}}(\eta)\right)U_{-t}^{(k)}\psi\,.$$

By induction one can show, that for all $t\in\mathbb{R}$ one gets

$$\sum_{k=0}^n U_t^{(n-k)}\left(C\otimes W(f)\otimes W(g)\right)U_{-t}^{(k)}\psi =$$

$$= i^n\int_{t_1=0}^t dt_1\int_{t_2=0}^{t_1}dt_2\cdots\int_{t_n=0}^{t_{n-1}}dt_n\,[\tau_{t_n}(P),[\ldots[\tau_{t_1}(P),\tau_t(C\otimes W(f)\otimes W(g))]\ldots]]\psi\,.$$

Now let us turn to the generalization for $m>1$. By multiplying out we get in (15) a sum with $(2m)^n$ summands with the form as in Lemma 5.1 and the statements follow. □

For $\alpha>0$ and $n\in\mathbb{N}$ we define

$$D_\alpha^{(n)} := \{\vec{z}=(z_1,\ldots,z_n)\in\mathbb{C}^n \mid 0<\Im(z_1)<\Im(z_2)<\ldots<\Im(z_n)<\alpha\}\;.$$

The closure of $D_\alpha^{(n)}$ is denoted by $\overline{D_\alpha^{(n)}}$.

Lemma 5.3 *Let $\mathcal{K}$ be a Hilbert space, $R \geq 0$ a selfadjoint operator in $\mathcal{K}$, $\mathrm{d}\Gamma(R) := G$ its second quantization. It follows for each $\beta > 0$:*

(i) *For all $f = (f_1, \ldots, f_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ with $f_j \in \mathcal{K}$ and $\mu_j \in \{-, 0, +\}$ the maps*

$F_f^\mu : \overline{D_{\frac{\beta}{2}}^{(n)}} \longrightarrow \mathcal{F}_+(\mathcal{K})\,,\ \vec{z} = (z_1, \ldots, z_n) \longmapsto$

$$\mathrm{e}^{iz_1G} a^{\mu_1}(f_1)\, \mathrm{e}^{i(z_2-z_1)G} a^{\mu_2}(f_2)\, \mathrm{e}^{i(z_3-z_2)G} a^{\mu_3}(f_3) \,\cdots\, \mathrm{e}^{i(z_n-z_{n-1})G} a^{\mu_n}(f_n)\, \mathrm{e}^{-iz_nG}\, \Omega_{\mathcal{F}}$$

are holomorphic on $D_{\frac{\beta}{2}}^{(n)}$ and strongly continuous on $\overline{D_{\frac{\beta}{2}}^{(n)}}$.

(ii) *For all $g = (g_1, \ldots, g_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $g_j \in \mathcal{D}(\mathrm{e}^{\frac{\beta}{2}R})$ and $\lambda_j \in \{-, 0, +\}$ the maps*

$G_g^\lambda : \overline{D_{\frac{\beta}{2}}^{(n)}} \longrightarrow \mathcal{F}_+(\mathcal{K})\,,\ \vec{z} = (z_1, \ldots, z_n) \longmapsto$

$$\mathrm{e}^{-iz_1G} a^{\lambda_1}(g_1)\, \mathrm{e}^{-i(z_2-z_1)G} a^{\lambda_2}(g_2) \,\cdots\, \mathrm{e}^{-i(z_n-z_{n-1})G} a^{\lambda_n}(g_n)\, \mathrm{e}^{iz_nG}\, \Omega_{\mathcal{F}}$$

are holomorphic on $D_{\frac{\beta}{2}}^{(n)}$ and strongly continuous on $\overline{D_{\frac{\beta}{2}}^{(n)}}$.

If $\#M$ denotes the cardinality of the set M for all $\vec{z} \in \overline{D_{\frac{\beta}{2}}^{(n)}}$ we have the following estimations

$$\begin{aligned}
\left\|F_f^\mu(\vec{z})\right\| &\leq \sqrt{l!}\left(\prod_{\mu_k \in \{-,+\}} \|f_k\|\right) \quad \text{where } l := \#\{\mu_k \mid \mu_k \in \{-,+\},\, k = 1, \ldots, n\}, \\
\left\|G_g^\lambda(\vec{z})\right\| &\leq \sqrt{m!}\left(\prod_{\lambda_k \in \{-,+\}} \left\|\mathrm{e}^{\frac{\beta}{2}R} g_k\right\|\right) \quad \text{where } m := \#\{\lambda_k \mid \lambda_k \in \{-,+\},\, k = 1, \ldots, n\}.
\end{aligned}$$

Proof: For $\alpha \in \mathbb{R}$ let be $U_\alpha := \{z \in \mathbb{C} \mid \Im(z) > \alpha\}$.

(i) It is $\|\mathrm{e}^{izG}\| = 1\ \forall z \in \overline{U_0}$ and e^{izG} leaves $\mathbb{P}_+(\otimes_n \mathcal{K})$ invariant. Moreover $\frac{d}{dz}\mathrm{e}^{izG}\phi = iG\,\mathrm{e}^{izG}\phi\ \forall z \in U_0\ \forall \phi \in \mathcal{F}_+(\mathcal{K})$. Thus, using (9) one easily checks the assumptions of [6, Lemma 3.6] and F_f^μ is partial differentiable. Now use Hartogs' theorem (see e.g. [13, p.65]). The strong continuity of $\vec{z} \in \overline{D_{\frac{\beta}{2}}^{(n)}} \longmapsto F_f^\mu(\vec{z})$ follows immediately from that of $z \in \overline{U_0} \mapsto \mathrm{e}^{izG}$ and (9).

(ii) For $f \in \mathcal{D}(\mathrm{e}^{\frac{\beta}{2}R})$ and $z \in \mathbb{C}$ with $\Im(z) \in [0, \frac{\beta}{2}]$ one has $\mathrm{e}^{-izG}a^*(f)\,\mathrm{e}^{izG} = a^*(\mathrm{e}^{-izR}f)$ and $\mathrm{e}^{-izG}a(f)\,\mathrm{e}^{izG} = a(\mathrm{e}^{-i\bar{z}R}f)$. Consequently

$$G_g^\lambda(\vec{z}) \;=\; a^{\lambda_1}(\mathrm{e}^{-iz_1^{\lambda_1}R}g_1)\, a^{\lambda_2}(\mathrm{e}^{-iz_2^{\lambda_2}R}g_2) \,\cdots\, a^{\lambda_n}(\mathrm{e}^{-iz_n^{\lambda_n}R}g_n)\,\Omega_{\mathcal{F}}\,, \tag{18}$$

where for $z \in \mathbb{C}$ is defined $z^\nu := \bar{z}$ if $\nu = -$, $z^\nu := 0$ if $\nu = 0$ and $z^\nu := z$ if $\nu = +$. But the vector $a^{\lambda_{k+1}}(\mathrm{e}^{-iz_{k+1}^{\lambda_{k+1}}R}g_{k+1}) \cdots a^{\lambda_n}(\mathrm{e}^{-iz_n^{\lambda_n}R}g_n)\,\Omega_{\mathcal{F}}$ has the form $\mathbb{P}_+(\xi_1 \otimes \cdots \otimes \xi_l)$ for some $l \leq n-k+1$ and $\xi_p \in \mathcal{K}$. We get

$$a^{\lambda_k}(\mathrm{e}^{-iz_k^{\lambda_k}R}g_k)\, a^{\lambda_{k+1}}(\mathrm{e}^{-iz_{k+1}^{\lambda_{k+1}}R}g_{k+1})\cdots a^{\lambda_n}(\mathrm{e}^{-iz_n^{\lambda_n}R}g_n)\,\Omega_{\mathcal{F}} =$$

$$= \begin{cases} \dfrac{1}{\sqrt{l}}\displaystyle\sum_{i=1}^{l}\langle \mathrm{e}^{-i\overline{z_k}R}g_k,\xi_i\rangle\, \mathbb{P}_+(\xi_1\otimes\cdots\otimes\xi_{i-1}\otimes\xi_{i+1}\otimes\cdots\otimes\xi_l) & \text{if} \quad \lambda_k = - \\ \mathbb{P}_+(\xi_1\otimes\cdots\otimes\xi_l) & \text{if} \quad \lambda_k = 0 \\ \sqrt{l+1}\,\mathbb{P}_+((\mathrm{e}^{-iz_kR}g_k)\otimes\xi_1\otimes\cdots\otimes\xi_l) & \text{if} \quad \lambda_k = + \ . \end{cases}$$

Since $\frac{d}{dw}\mathrm{e}^{iwR}f = iR\mathrm{e}^{iwR}f\ \forall w \in U_{-\frac{\beta}{2}}\ \forall f \in \mathcal{D}(\mathrm{e}^{\frac{\beta}{2}R})$ the kth partial derivative $\frac{\partial}{\partial z_k}G_g^{\lambda}(\vec{z})$ exists and by Hartogs' theorem G_g^{λ} is holomorphic on $D_{\frac{\beta}{2}}^{(n)}$. The strong continuity of G_g^{λ} on $\overline{D_{\frac{\beta}{2}}^{(n)}}$ follows from (18) with (9). The stated estimations are immediate by use of (9). □

Theorem 5.4 *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $R \geq 0$ be a selfadjoint operator in $\mathcal{K}$ and $G := \mathrm{d}\Gamma(R)$ its second quantization in $\mathcal{F}_+(\mathcal{K})$. Further let $(\mathcal{M}_a, \tau^a)$ be a W^*-dynamical system acting on $\mathcal{H}$ such that $\tau_t^a(X) := \mathrm{e}^{itA}X\,\mathrm{e}^{-itA}$ with the selfadjoint operator A in $\mathcal{H}$, and $\Omega_a \in \mathcal{H}$ a normalized cyclic vector with $A\Omega_a = 0$, such that the associated vector state $\omega_a(X) := \langle\Omega_a, X\,\Omega_a\rangle$ is a (τ^a,β)-KMS state on $\mathcal{M}_a$ for some $\beta > 0$. For $B_1,\ldots,B_m \in \mathcal{M}_a$ and $f_1,\ldots,f_m \in \mathcal{K}$ and $g_1,\ldots,g_m \in \mathcal{D}(\mathrm{e}^{\frac{\beta}{2}R})$ let*

$$K \ := \ A\otimes 1\!\!1\otimes 1\!\!1 \ + \ 1\!\!1\otimes(G\otimes 1\!\!1 \ - \ 1\!\!1\otimes G) \quad \textit{and}$$

$$P \ := \ \sum_{k=1}^{m}\Big(B_k\otimes\big(a(f_k)\otimes 1\!\!1 + 1\!\!1\otimes a^*(g_k)\big) \ + \ B_k^*\otimes\big(a^*(f_k)\otimes 1\!\!1 + 1\!\!1\otimes a(g_k)\big)\Big) \ ,$$

be operators in $\mathcal{H}\otimes\mathcal{F}_+(\mathcal{K})\otimes\mathcal{F}_+(\mathcal{K})$. Moreover for $\vec{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n$ let

$$\mathcal{P}_n(\vec{z}) \ := \ \mathrm{e}^{iz_1K}P\,\mathrm{e}^{i(z_2-z_1)K}P\,\mathrm{e}^{i(z_3-z_2)K}P\ \ldots\ \mathrm{e}^{i(z_n-z_{n-1})K}P\,\mathrm{e}^{-iz_nK} \ .$$

The following statements are valid:

(i) *$\Omega := \Omega_a\otimes\Omega_{\mathcal{F}}\otimes\Omega_{\mathcal{F}} \in \mathcal{D}(\mathcal{P}_n(\vec{z}))$ for all $\vec{z} \in \overline{D_{\frac{\beta}{2}}^{(n)}}$ and all $n \in \mathbb{N}$. Moreover the function $\vec{z} \in \overline{D_{\frac{\beta}{2}}^{(n)}} \mapsto \mathcal{P}_n(\vec{z})\,\Omega$ is holomorphic on $D_{\frac{\beta}{2}}^{(n)}$ and strongly continuous and bounded on $\overline{D_{\frac{\beta}{2}}^{(n)}}$ with*

$$\sup\Big\{\|\mathcal{P}_n(\vec{z})\,\Omega\| \mid \vec{z}\in\overline{D_{\frac{\beta}{2}}^{(n)}}\Big\} \ \leq \ (2\,m\,b(\gamma+\gamma'))^n\,\sqrt{n!}\,,$$

where $b := \max\{\|B_1\|,\ldots,\|B_n\|\}$, $\gamma := \max\{\|f_1\|,\ldots,\|f_n\|\}$ and $\gamma' := \max\{\|\mathrm{e}^{\frac{\beta}{2}R}g_1\|,\ldots,\|\mathrm{e}^{\frac{\beta}{2}R}g_n\|\}$.

(ii) *$\Omega \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H})$, where $H := K + P$. The vector $\Omega^p := \mathrm{e}^{-\frac{\beta}{2}H}\Omega$ is given by the strongly convergent perturbation expansion*

$$\Omega^p \ = \ \Omega \ + \ \sum_{n=1}^{\infty}(-1)^n\int_{0\leq t_1\leq\cdots\leq t_n\leq\frac{\beta}{2}} dt_1\cdots dt_n\ \mathcal{P}_n(it_1,\ldots,it_n)\,\Omega \ .$$

Proof: For a better survey we prove this theorem in the case $m = 1$. The generalization for $m > 1$ is similar to Theorem 5.2. Thus let $P = B \otimes (a(f) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(g)) + B^* \otimes (a^*(f) \otimes \mathbb{1} + \mathbb{1} \otimes a(g))$. Again we set $B^+ := B^*$ and $B^- := B$.

(i) For $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_k \in \{-,+\}$ it is $\Omega_G \in \mathcal{D}\left(\left(\prod_{k=1}^n e^{i(z_k - z_{k-1})A} B^{\nu_k}\right) e^{-iz_nA}\right)$ for each $\vec{z} \in \overline{D^{(n)}_{\frac{\beta}{2}}}$ (see by [2, §3]). Define $H^\nu_B(\vec{z}) := \left(\prod_{k=1}^n e^{i(z_k - z_{k-1})A} B^{\nu_k}\right) e^{-iz_nA}\, \Omega_G$ with $z_0 := 0$. By multiplying out we first get 2^n multiindices ν and then for each ν we get 2^n multiindices $(\mu, \lambda)(\nu)$ (compare the proof of Lemma 5.1) such that with the notation of Lemma 5.3

$$\mathcal{P}_n(\vec{z})\,\Omega \;=\; \sum_{\nu} \sum_{(\mu,\lambda)(\nu)} H^\nu_B(\vec{z}) \otimes F^\mu_f(\vec{z}) \otimes G^\lambda_g(\vec{z})\,. \tag{19}$$

Let $l := l_{(\mu,\lambda)}$ as in the proof of Lemma 5.1. Running for fixed ν through all the 2^n multiindices $(\mu, \lambda)(\nu)$ the same number $l = l_{(\mu,\lambda)}$ appears exactly $\binom{n}{l}$ times and by [2, §3] (compare also [4, Theorem 5.4.4]) and Lemma 5.3 one gets

$$\begin{aligned}
\left\| \sum_{(\mu,\lambda)(\nu)} H^\nu_B(\vec{z}) \otimes F^\mu_f(\vec{z}) \otimes G^\lambda_g(\vec{z}) \right\| &\le \|H^\nu_B(\vec{z})\| \sum_{(\mu,\lambda)(\nu)} \left\|F^\mu_f(\vec{z})\right\| \left\|G^\lambda_g(\vec{z})\right\| \\
&\le \|B\|^n \sum_{(\mu,\lambda)(\nu)} \sqrt{l!}\, \|f\|^l \sqrt{(n-l)!}\, \left\|e^{\frac{\beta}{2}R} g\right\|^{n-l} \\
&= \|B\|^n \sum_{l=0}^{n} \binom{n}{l} \sqrt{l!(n-l)!}\, \|f\|^l \left\|e^{\frac{\beta}{2}R} g\right\|^{n-l} \\
&\le \|B\|^n \left(\|f\| + \left\|e^{\frac{\beta}{2}R} g\right\|\right)^n \sqrt{n!}\quad,
\end{aligned}$$

where at the last inequality sign we have used $l!(n-l)! \le n!$. Because in (19) the sum $\sum_\nu$ has 2^n summands, we finally get

$$\|\mathcal{P}_n(\vec{z})\,\Omega\| \;\le\; \left(2\,\|B\| \left(\|f\| + \left\|e^{\frac{\beta}{2}R} g\right\|\right)\right)^n \sqrt{n!} \qquad \forall \vec{z} \in \overline{D^{(n)}_{\frac{\beta}{2}}}\,.$$

Now (i) follows from [2, §3], Lemma 5.3 and Weierstraß' theorem [13, p.18].

(ii) Let $D := \{z \in \mathbb{C} \mid \Re(z) \in]0,1[\,\}$ and $E_n := \left\{\vec{t} \in \mathbb{R}^n \mid 0 < t_1 < \cdots < t_n < \frac{\beta}{2}\right\}$. We have $iz\vec{t} \in D^{(n)}_{\frac{\beta}{2}}\ \forall z \in D\ \forall \vec{t} \in E_n$ and $iz\vec{t} \in \overline{D^{(n)}_{\frac{\beta}{2}}}\ \forall z \in \overline{D}\ \forall \vec{t} \in \overline{E_n}$. From (i) it follows, that the function $\vec{t} \in \overline{E_n} \mapsto \mathcal{P}_n(iz\vec{t})\,\Omega$ is strongly continuous for each $z \in \overline{D}$ with $\sup_{z \in \overline{D}} \left\|\mathcal{P}_n(iz\vec{t})\,\Omega\right\| \le c^n \sqrt{n!}$ for some $c > 0$. By (i) the function

$$\overline{D} \ni z \longmapsto \Omega^p(z) \;:=\; \Omega + \sum_{n=1}^{\infty} (-z)^n \int_{E_n} \mathcal{P}_n(iz\vec{t})\,\Omega\, d\vec{t}$$

is well defined, it is holomorphic on D and strongly continuous on $\overline{D}$. Now using $e^{itK}\Omega = \Omega$ and a suitable substitution and from Theorem 5.2 follows

$$e^{-is\frac{\beta}{2}H}\,\Omega \;=\; \Omega^p(is) \qquad \forall s \in \mathbb{R}\,. \tag{20}$$

Let $\varphi \in \bigcup_{n=1}^{\infty} \mathcal{R}(E(]-n,n])) =: \mathcal{N}$, where $E(.)$ is the spectral measure of H and $\mathcal{R}(Q)$ denotes the range of the operator Q. Then φ is an entire analytic vector for H. Therefore the function $f_\varphi : \overline{D} \to \mathbb{C}, z \mapsto \langle e^{-\overline{z}\frac{\beta}{2}H}\varphi, \Omega\rangle - \langle\varphi, \Omega^p(z)\rangle$ is holomorphic on D and continuous on $\overline{D}$. By (20) we have $f_\varphi(z) = 0$ for all $z \in \mathbb{C}$ with $\Re(z) = 0$. From the Edge of the Wedge Theorem and the continuity of f_φ we conclude $f_\varphi \equiv 0$ on $\overline{D}$. Consequently $\langle e^{-\overline{z}\frac{\beta}{2}H}\varphi, \Omega\rangle = \langle\varphi, \Omega^p(z)\rangle$ $\forall z \in \overline{D}$. Because $\mathcal{N}$ is a core for each e^{wH}, $w \in \mathbb{C}$, we have $\langle e^{-\overline{z}\frac{\beta}{2}H}\varphi, \Omega\rangle = \langle\varphi, \Omega^p(z)\rangle$ for all $\varphi \in \mathcal{D}(e^{-\overline{z}\frac{\beta}{2}H})$. Therefore one obtains $\Omega \in \mathcal{D}(e^{-z\frac{\beta}{2}H})$ and $\Omega^p(z) = e^{-z\frac{\beta}{2}H}\Omega$ for each $z \in \overline{D}$, especially $\Omega^p := \Omega^p(1) = e^{-\frac{\beta}{2}H}\Omega$. □

Proof of Theorem 4.1: For a better survey the proof is given for $m = 1$. The generalization for $m > 1$ is immediate. Thus let $\widetilde{P} = \widetilde{B}\otimes a(f) + \widetilde{B}^*\otimes a^*(f)$ with $f \in \mathcal{D}(e^{\frac{\beta}{2}S})$ and set $\widetilde{B}^- := \widetilde{B}$ and $\widetilde{B}^+ := \widetilde{B}^*$. From [8, Theorem 4.2] follows $e^{-\frac{\beta}{2}(\widetilde{K}+\widetilde{P})} = \sum_{n=0}^{\infty} U^{(n)}$ with $U^{(0)} = e^{-\frac{\beta}{2}\widetilde{K}} = e^{-\frac{\beta}{2}\widetilde{A}} \otimes e^{-\frac{\beta}{2}G}$ and

$$U^{(n)} = (-1)^n \int_{E^{(n)}_{\frac{\beta}{2}}} d^n\vec{t} \sum_\nu \left(\left(\prod_{k=1}^n e^{-t_k\widetilde{A}}\widetilde{B}^{\nu_k}e^{t_k\widetilde{A}}\right) e^{-\frac{\beta}{2}\widetilde{A}} \otimes \left(\prod_{k=1}^n a^{\nu_k}(e^{-\nu_k t_k S}f)\right) e^{-\frac{\beta}{2}G}\right),$$

where the sum $\sum_\nu$ runs over 2^n multiindizes $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_k \in \{-,+\}$. Using $\Pi_a(e^{iz\widetilde{A}}X\,e^{-iz\widetilde{A}}) = e^{izA_a}\Pi_a(X)e^{-izA_a}$ $\forall z \in \mathbb{C}$ $\forall X \in \mathcal{L}(\mathcal{H})$, the relation $a_b^\lambda(e^{-\lambda\tau S}f) = e^{-\tau G_b}a_b^\lambda(f)e^{\tau G_b}$ $\forall\tau \in \mathbb{R}$ and $\lambda \in \{-,+\}$, [8, Corollary 3.3] and a suitable substitution, one easily deduces

$$\begin{aligned}
&\mathrm{tr}\Big(U^{(m)}\left(C \otimes \Pi_{\mathcal{F}}(W)\right) U^{(n)}\Big) = \\
&= \mathrm{tr}\left(e^{-\beta\widetilde{K}}\right) \left\langle (-1)^m\int_{E^{(m)}_{\frac{\beta}{2}}} d^m\vec{s}\,\mathcal{P}_m(i\vec{s})\,\Omega\,,\ \Pi_a\otimes\Pi_b(C\otimes W)(-1)^n\int_{E^{(n)}_{\frac{\beta}{2}}} d^n\vec{t}\,\mathcal{P}_n(i\vec{t})\,\Omega\right\rangle,
\end{aligned}$$

for all $C \in \mathcal{L}(\mathcal{H})$, $W \in \mathcal{W}(\mathcal{K})$, $E^{(m)}_{\frac{\beta}{2}}$ from [8, Theorem 4.2] and $\mathcal{P}_m$ from Theorem 5.4. Consequently by these theorems

$$\mathrm{tr}\left(e^{-\beta(\widetilde{K}+\widetilde{P})}\left(C\otimes\Pi_{\mathcal{F}}(W)\right)\right) = \mathrm{tr}\left(e^{-\beta\widetilde{K}}\right)\langle\Omega^p, \Pi_a\otimes\Pi_b(C\otimes W)\,\Omega^p\rangle\,.$$

Especially for $C = \mathbb{1}$ and $W = \mathbb{1}$ one gets $\mathrm{tr}\left(e^{-\beta(\widetilde{K}+\widetilde{P})}\right) = \mathrm{tr}\left(e^{-\beta\widetilde{K}}\right)\|\Omega^p\|^2$. Now extend to all of $\mathcal{L}(\mathcal{H}\otimes\mathcal{F}_+(\mathcal{K}))$ and the theorem is proved. □

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