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ABOUT ADMISSIBLE BOUNDARY CONDITIONS FOR EULER AND PARABOLIZED NAVIER-STOKES EQUATIONS

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Dedicated to Gérard Wanders on the occasion of his 60th birthday

Abstract

We consider the parabolized approximation of Navier-Stokes equations for the two-dimensional steady flow of an incompressible or isentropic fluid. First, the equations and a perturbative method to get them are presented; then, the notion of admissible boundary conditions in the sense of Friedrichs systems of differential equations is introduced. Finally, various admissible conditions for the parabolized Navier-Stokes equations and, as a by-product, for Euler equations are exhibited.

1. Preliminaries

Parabolized Navier-Stokes (PNS) equations are used to describe the high-speed (e.g. supersonic) steady flow of a viscous compressible gas over a blunt body when there is a preferred direction, in which the component of the displacement velocity of the fluid is positive [1]. For numerical purposes, it is essential to have boundary conditions (BC) for these

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equations such that the resulting problem is well-posed; to our knowledge, this issue has never been addressed in the case of a bounded domain.

Consider the steady state Navier-Stokes equations for an incompressible fluid, without external force :

$$\begin{cases} (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0}, & (1.1a) \\ \operatorname{div}\mathbf{u} = 0, & (1.1b) \end{cases}$$

where, in cartesian coordinates $\mathbf{x} = (x,y)$, $\mathbf{u} = (u,v)$ is the displacement velocity, p the pressure and $\nu^{-1} > 0$ the Reynolds number. Throughout this paper, u will be assumed to be positive. The PNS equations are obtained from (1.1) by neglecting the diffusion in the x-direction, i.e. :

$$\begin{cases} (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\,\partial_y^2\,\mathbf{u} + \nabla p = \mathbf{0}, & (1.2a) \\ \operatorname{div}\mathbf{u} = 0, & (1.2b) \end{cases}$$

Remark 1.1 : *Like Prandtl's boundary-layer equations, the PNS equations are simplified Navier-Stokes equations, but they are valid in a larger region than the boundary-layer.*

For a compressible fluid, we restrict ourselves to the isentropic case [2], where the equation of state ($p = A\rho^\gamma$, ρ : mass density, A,γ : constants) allows to decouple the mechanical conservation laws from the thermodynamical one. The PNS equations are obtained like above by neglecting the second-order derivatives with respect to x and read

$$\begin{cases} \rho\,\mathbf{u}\cdot\nabla u + \partial_x p - \nu\,\partial_y^2 u = 0, & (1.3a) \\ \rho\,\mathbf{u}\cdot\nabla v + \partial_y p - \frac{4}{3}\,\nu\,\partial_y^2 v = 0, & (1.3b) \\ \rho\,\operatorname{div}\mathbf{u} + a^{-2}\,\mathbf{u}\cdot\nabla p = 0, & (1.3c) \\ p = a^2\rho; & (1.3d) \end{cases}$$

we have assumed for simplicity that the sound speed $a = \sqrt{p\gamma/\rho}$ is constant.

Except in the next section, we shall work in the bounded domain $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where

$$\Gamma_- = \{(x,y) \in \partial\Omega \mid x = 0, 0 < y < 1\}, \quad \Gamma_+ = \{(x,y) \in \partial\Omega \mid x = 1, 0 < y < 1\},$$
$$\Gamma_0 = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 = \{(x,y) \in \partial\Omega \mid y = 0\}, \quad \Gamma_2 = \{(x,y) \in \partial\Omega \mid y = 1\}.$$

This geometry corresponds to the flow over a flat plate lying on the positive x-axis; more general situations can be handled by replacing (x,y) by curvilinear coordinates, the type of the equations being unchanged.

2. Parabolized Oseen's equations

We add to eqs (1.2) a right hand side **f** which may arise from inhomogeneous BC and set

$$\mathbf{q} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \quad A_x(\mathbf{u}) = \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_y(\mathbf{u}) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \mathrm{diag}\,[1,1,0]\,; \tag{2.1}$$

then, we get the system

$$A_x(\mathbf{u})\,\partial_x\mathbf{q} + A_y(\mathbf{u})\,\partial_y\mathbf{q} - \nu\,D\,\partial_y^2\mathbf{q} = \mathbf{f}. \tag{2.2}$$

These equations can be linearized by replacing **u** in the matrices by a given velocity $\mathbf{c} = (c,d)$, $c > 0$; the same procedure applied to the Navier-Stokes equations (1.1) yields Oseen's equations

$$P(\partial_x,\partial_y)\mathbf{q} \equiv A_x(\mathbf{c})\,\partial_x\mathbf{q} + A_y(\mathbf{c})\,\partial_y\mathbf{q} - \nu\,D(\partial_x^2 + \partial_y^2)\,\mathbf{q} = \mathbf{f}. \tag{2.3}$$

We intend to show, using the method developped in [3] for the case $d \equiv 0$, that the linearized version of System (2.2) is an approximation of (2.3) when ν and $s = d/c$ are close to zero. For simplicity, we assume that **c** is constant and look for a solution of (2.3), in the domain $\Omega_\infty = \mathbb{R}_+ \times \mathbb{R}$, of the type

$$\mathbf{q}(x,y) = \sum_{k=1}^{4} \int_{\mathbb{R}} \exp\,(i\mu y + \lambda_k(\mu)x)\,\hat{\mathbf{q}}_j(\mu)d\mu + \mathbf{q}_0(x,y),$$

with $\mathbf{q}_0$ a particular solution of the inhomogeneous system; the generalized eigenvalues λ_k, such that the matrix $P(\lambda,i\mu) = \lambda A_x + i\mu\,A_y - \nu(\lambda^2 + (i\mu)^2)\,D$ is singular, determine the behavior of **q** as $x \to \infty$. The first two eigenvalues are $\lambda_1 = |\mu|$, $\lambda_2 = -\,|\mu|$ and the other ones have the asymptotic expansion

$$\lambda_3 = -\,i\mu s - \frac{\mu^2}{c}\,(1+s^2)\nu + O(\nu^2), \quad \lambda_4 = c\nu^{-1} + i\mu s + O(\nu), \quad \nu \to 0.$$

In order to get the approximation, we drop λ_4 (responsible for a divergent behavior when $x \to \infty$), we keep λ_1 (the divergence of which will be killed by a regularity condition) and λ_2, but we replace λ_3 by its asymptotic expansion up to the order ν. These new eigenvalues are the roots of the determinant of the matrix $P_s(\lambda,i\mu) = \lambda A_x + i\mu\,A_y - \nu(1+s^2)\,(i\mu)^2 D$, which is associated to the system

$$P_s(\partial_x,\partial_y)\mathbf{q} \equiv A_x(\mathbf{c})\partial_x\mathbf{q} + A_y(\mathbf{c})\partial_y\mathbf{q} - \nu(1+s^2)\, D\, \partial_y^2\mathbf{q} = \mathbf{f}. \tag{2.4}$$

Consider the problems of solving (2.3) or (2.4) with the BC $\mathbf{u}\,|_{\,x=0} = \mathbf{0}$; then, the Fourier technique of [3] allows us to prove that, for sufficiently regular data $\mathbf{f}$, both problems have a solution with unique velocities $\mathbf{u}$, resp. $\mathbf{u}_s$, which belong to $H^1(\Omega_\infty)$ * and one has the estimate

$$\| \partial_y(\mathbf{u}-\mathbf{u}_s) \|_{L^2(\Omega_\infty)^2} + \| \mathbf{u} - \mathbf{u}_s \|_{L^2(\Omega_\infty)^2} = O(\nu^2),\ \nu \to 0.$$

This result shows in what sense System (2.4) is an approximation of the Oseen equations; it remains to establish a bound for the difference $\mathbf{e} = \mathbf{u}_s - \mathbf{u}_0$ of the velocities satisfying (2.4) with $s \neq 0$ and $s = 0$. This can be done only in a finite domain, e.g. the unit square Ω. We assume that there exists a unique solution $\mathbf{q}_s \in H^1(\Omega)^3$ of (2.4) satisfying (for instance) the BC

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_- \cup \Gamma_0, \quad p = 0 \quad \text{on } \Gamma_+ \tag{2.5}$$

(this holds if $\mathbf{f}$ is regular enough [4]). From the first two equations (2.4), we get an equation for $\mathbf{e}$; taking the dot product of this latter with $\mathbf{e}$, integrating over Ω, performing some integrations by parts and using the equation for $p_s - p_0$ obtained from the last equation (2.4), yields, with the help of the Cauchy-Schwartz inequality :

$$\| \partial_y\mathbf{e} \|_{L^2(\Omega)^2} \leq s^2 \| \partial_y\mathbf{u}_s \|_{L^2(\Omega)^2} ;$$

finally, Poincaré's inequality $\| \mathbf{e} \|_{L^2(\Omega)^2} \leq \alpha \| \partial_y\mathbf{e} \|_{L^2(\Omega)^2}$, $\alpha > 0$, implies that

$$\| \partial_y(\mathbf{u}_s - \mathbf{u}_0) \|_{L^2(\Omega)^2} + \| \mathbf{u}_s - \mathbf{u}_0 \|_{L^2(\Omega)^2} = O(s^2), \quad s \to 0.$$

Consequently, the linear PNS equations (2.4) with $s = 0$ can be viewed, for $s^2 = o(1)$ (e.g. $s = O(\nu)$) as an approximation of Oseen's equations (2.3).

* For an open domain $\Omega \subset \mathbb{R}^2$, $H^1(\Omega)$ denotes the Sobolev space of functions $\Omega \to \mathbb{R}$ which, together with their first-order derivatives, are in $L^2(\Omega)$.

3. Admissible boundary conditions

Admissible BC were first introduced within the theory of linear first-order differential equations system [5], e.g. in our case :

$$\mathcal{A}\mathbf{q}(\mathbf{x}) \equiv \mathrm{A_x}(\mathbf{x})\partial_x\mathbf{q}(\mathbf{x}) + \mathrm{A_y}(\mathbf{x})\partial_y\mathbf{q}(\mathbf{x}) + \mathrm{K}(\mathbf{x})\mathbf{q}(\mathbf{x}) = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \Omega, \tag{3.1}$$

where the unknown $\mathbf{q}$ and the r.h.s. $\mathbf{f}$ take values in $\mathbb{R}^p$ and the matrix-valued functions A_x, A_y *are assumed to be symmetric.* With the help of a matrix M defined on $\partial\Omega$ and of the matrix

$$\mathrm{B}(\mathbf{x}) = (n_x A_x + n_y A_y)(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \tag{3.2}$$

($\mathbf{n}$: outward unit normal to $\partial\Omega$), homogeneous Dirichlet BC for (3.1) are laid down by requiring that

$$\mathbf{q}(\mathbf{x}) \in \mathrm{Ker}\ (\mathrm{B-M})(\mathbf{x}),\ \mathbf{x} \in \partial\Omega. \tag{3.3}$$

The BC are called *admissible* iff

$$\begin{cases} \cdot \text{ the matrix } M + M^t \text{ is positive semi-definite } (M+M^t \geq 0) \text{ on } \partial\Omega, & (3.4a) \\ \cdot\ \mathrm{Ker}(B-M) + \mathrm{Ker}(B+M) = \mathbb{R}^p \text{ on } \partial\Omega. & (3.4b) \end{cases}$$

Example : *The matrix* $M = \sqrt{B^2}$ *generates admissible BC.*

Finally, the system (3.1) is said to be *positive* iff the matrix $C = K + K^t - \partial_x A_x - \partial_y A_y$ is positive definite in Ω.

We quote hereafter, in an informal way, the basic results of [5]. *A symmetric positive system with admissible BC* has at least one solution $\mathbf{q} \in L^2(\Omega)^p$, i.e.

$$\int_\Omega \mathbf{q}^t \mathcal{A}^*\varphi\, dx = \int_\Omega \mathbf{f}^t \varphi\, dx \quad \forall\ \varphi \in C^1(\bar{\Omega})^p \text{ with } \varphi \in \mathrm{Ker}(B+M^t) \text{ on } \partial\Omega,$$
$$\mathbf{f} \in L^2(\Omega)^p,$$

where $\mathcal{A}^*$ is the formal adjoint of $\mathcal{A}$; moreover, if $\mathbf{q}$ is regular enough (e.g. in $H^1(\Omega)^p$), the solution is unique and satisfies the BC in the sense of traces. These results led us to consider admissible BC for (linearized) PNS equations.

4. The incompressible case

It is quite usual to infer BC for nonlinear equations from those of their linear version (see [6] for instance). Hence, we consider the linearized PNS system (2.4) with $s = 0$ in the domain Ω; introducing the unknowns $q = \partial_y u$ and $r = \partial_y v$, it takes the standard form (3.1) with

$$\mathbf{q} = \begin{pmatrix} \mathbf{u} \\ p \\ q \\ r \end{pmatrix}, \qquad A_x = \left(\begin{array}{c|c} \overline{A}_x & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right), \qquad A_y = \left(\begin{array}{c|c} \overline{A}_y & \begin{matrix} -\nu & 0 \\ 0 & -\nu \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \end{matrix} & \mathbf{O} \end{array}\right),$$

$$K = \operatorname{diag}\,[0,0,0,\nu,\nu], \tag{4.1}$$

where $\overline{A}_x$ and $\overline{A}_y$ are the matrices for the corresponding Euler system ($\nu=0$), defined by (2.1) with $\mathbf{c(x)}$ in place of $\mathbf{u}$. It is easy to check that equations (3.1) and (4.1) yield a symmetric positive system if div $\mathbf{c} < 0$; unfortunately, this is unphysical since $\mathbf{c}$ must mimic $\mathbf{u}$. However, it is shown in [4] that most admissible BC in the present case lead to the same results we would obtain if the system were positive.

It is worthwhile noticing that, since the matrix B (3.2) is given by

$$B = \left(\begin{array}{c|c} \overline{B} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right) = \left(\begin{array}{c|c} n_x\overline{A}_x & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right) \quad \text{on } \Gamma_- \cup \Gamma_+,$$

we obtain, from admissible BC for the Euler system generated with the matrix $\overline{M}$, admissible BC for the PNS system with the help of

$$M = \left(\begin{array}{c|c} \overline{M} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right) \quad \text{on } \Gamma_- \cup \Gamma_+; \tag{4.2}$$

of course, written down in function of u,v,p these BC coincide.

In order to describe the physical situation of the flow over a flat plate, we make the following hypotheses on the given velocity $\mathbf{c}$:

$$\mathbf{c} \in C^1(\overline{\Omega})^2, \quad \operatorname{div} \mathbf{c} = 0 \text{ in } \Omega, \tag{4.3a}$$

$$\mathbf{c}\cdot\mathbf{n} = 0 \text{ on } \Gamma_0, \quad \mathbf{c}\cdot\mathbf{n} > 0 \text{ on } \Gamma_+, \quad \mathbf{c}\cdot\mathbf{n} < 0 \text{ on } \Gamma_- . \tag{4.3b}$$

First, we want to show that the standard BC (2.5) are admissible. On Γ_2, one has $B = A_y$ and the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & -\nu & 0 \\ 0 & 0 & 1 & 0 & -\nu \\ 0 & -1 & 0 & 0 & 0 \\ \nu & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 \end{pmatrix} \tag{4.4}$$

generates the admissible BC $\mathbf{u} = \mathbf{0}$; on Γ_1, since $B = - A_y$, we get the same BC by replacing M by -M. By setting

$$\bar{M} = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in (4.2), we get the admissible BC $\mathbf{u} = \mathbf{0}$ on Γ_- and also $p = 0$ on Γ_+ . We remark that the condition $\mathbf{u} = \mathbf{0}$ on Γ_0 does not depend from the hypothesis $\mathbf{c}\cdot\mathbf{n} = 0$ there.

Other simple admissible conditions for the PNS system are for instance :

$$\begin{cases} c\ u + p = 0,\ v = 0 \text{ on } \Gamma_- , & (4.5a) \\ u = 0 \text{ on } \Gamma_+ , & (4.5b) \end{cases}$$

with

$$\bar{M} = \begin{pmatrix} c & 0 & 1 \\ 0 & c & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ in eq. (4.2) ;}$$

$$u = 0, \quad -p + \nu\, \partial_y v = 0 \text{ on } \Gamma_0, \tag{4.6}$$

with the matrix M (4.4) on Γ_1 and - M on Γ_2. This latter BC is a zero-strain condition, frequently used for the Navier-Stokes equations.

It is also interesting to look at the BC given by the choice $M = \sqrt{B^2}$; an easy computation of the eigenvalues and eigenvectors of B yields :

$$\begin{cases} u + \frac{1}{2} (\sqrt{c^2+4} - c)\ p = 0,\ v = 0 \text{ on } \Gamma_- , & (4.7a) \\ u - \frac{1}{2} (c + \sqrt{c^2+4})\ p = 0 \text{ on } \Gamma_+ . & (4.7b) \end{cases}$$

5. The isentropic case

This section deals with the linearized version of System (1.3) obtained by adding a r.h.s. due to inhomogeneous BC, setting $\rho = 1$ and replacing $\mathbf{u}\cdot\nabla$ by $\mathbf{c}\cdot\nabla$, where $\mathbf{c} = (c,d)$ satisfies (4.3); the underlying physics is the isentropic flow of a *weakly* *compressible fluid*, the density of which ($\rho = 1 + \rho_1$, $\rho_1 << 1$) is almost constant. The resulting system can be put into the standard form (3.1), with the new unknowns $q = \partial_y u$, $r = \partial_y v$, by setting

$$\mathbf{q} = \begin{pmatrix} \mathbf{u} \\ p \\ q \\ r \end{pmatrix}, \; A_x = \left(\begin{array}{c|c} \bar{A}_x & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right), \; A_y = \left(\begin{array}{ccc|cc} & \bar{A}_y & & -\nu & 0 \\ & & & 0 & -4/3\,\nu \\ & & & 0 & 0 \\ \hline -\nu & 0 & 0 & & \\ 0 & -4/3\,\nu & 0 & & \mathbf{O} \end{array}\right),$$

$$K = \text{diag}\,[0,0,0,\nu,\, 4/3\,\nu], \tag{5.1}$$

where the matrices

$$\bar{A}_x = \begin{pmatrix} c & 0 & 1 \\ 0 & c & 0 \\ 1 & 0 & c\,a^{-2} \end{pmatrix}, \qquad \bar{A}_y = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 1 \\ 0 & 1 & d\,a^{-2} \end{pmatrix}$$

are those of the corresponding Euler problem. Here again, admissible BC for the Euler system on $\Gamma_- \cup \Gamma_+$, defined with the help of a matrix $\bar{M}$, are also admissible for the PNS system and given by the matrix M (4.2).

Remark 5.1 : *As far as BC are concerned, the assumption of weak compressibility does not play any role.*

Compared to the incompressible case, the compressible problem has a new feature : there are two regions in the flow; with regard to the PNS equations, one must distinguish between the "supersonic" zone in which $c > a$ and the "subsonic" zone where $c < a$. With the change of variables $\mathbf{q} = \tilde{\mathbf{q}} \exp(\alpha x)$, we get from (3.1) the equivalent system

$$A_x\, \partial_x \tilde{\mathbf{q}} + A_y\, \partial_y \tilde{\mathbf{q}} + (K + \alpha A_x)\, \tilde{\mathbf{q}} = \exp(-\alpha x)\, \mathbf{f}. \tag{5.3}$$

The following conditions are sufficient to insure that the symmetric system (5.3), (5.1) and (5.2) is positive :

(i) <u>For $c > a$</u> : $\alpha > 0$ if $\text{div}\,\mathbf{c} = 0$; $\alpha = \dfrac{\max \text{div}\,\mathbf{c}}{\min\,(c-a)}$ if $\text{div}\,\mathbf{c} \neq 0$.

(ii) For $c < a$: $\max \frac{\operatorname{div} \mathbf{c}}{2\,(c+a)} < \alpha < \min \frac{\operatorname{div} \mathbf{c}}{2\,(c-a)}$ if $\operatorname{div} \mathbf{c} < 0$.

In the case $c < a$, $\operatorname{div} \mathbf{c} = 0$, admissible BC yield again the same existence and uniqueness results as for a positive system [4].

We have also to distinguish between two parts of the boundary, namely :

$$\Gamma_s = \{\, \mathbf{x} \in \partial\Omega \mid c(\mathbf{x}) > a \,\}, \quad \Gamma_i = \{\, \mathbf{x} \in \partial\Omega \mid c(\mathbf{x}) < a \,\}.$$

Standard BC for the PNS system are given by

$$\begin{cases} c\,u + p = 0,\ v = 0 \text{ on } \Gamma_- \cap \Gamma_i\,, & (5.4a) \\ \mathbf{u} = \mathbf{0},\ p = 0 \text{ on } \Gamma_- \cap \Gamma_s\,, & (5.4b) \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\,, & (5.4c) \\ p = 0 \text{ on } \Gamma_+ \cap \Gamma_i\,; & (5.4d) \end{cases}$$

thus, we see that on the supersonic inflow $\Gamma_- \cap \Gamma_s$, every unknown has to be prescribed, whereas on the supersonic outflow $\Gamma_+ \cap \Gamma_s$ no condition is required. The BC (5.4) are admissible, given by the following matrices :

$$\bar{M} = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & 2/c - c/a^2 \end{pmatrix} \text{ in (4.2), on } (\Gamma_- \cup \Gamma_+) \cap \Gamma_i\,,$$

$$\bar{M} = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & c/a^2 \end{pmatrix} \text{ in (4.2), on } \Gamma_- \cap \Gamma_s\,,$$

$$M = \begin{pmatrix} 0 & 0 & 0 & -\nu & 0 \\ 0 & 0 & 1 & 0 & -4/3\,\nu \\ 0 & -1 & 0 & 0 & 0 \\ \nu & 0 & 0 & 0 & 0 \\ 0 & 4/3\,\nu & 0 & 0 & 0 \end{pmatrix}, \qquad (5.5)$$

on Γ_2 and $-M$ on Γ_1.

On the horizontal boundaries, the matrices M (5.5) on Γ_1 and $-M$ on Γ_2 define the admissible zero-strain condition

$$\partial_y u = 0, \quad -p + \tfrac{4}{3}\nu\, \partial_y v = 0 \text{ on } \Gamma_0. \tag{5.6}$$

Finally, the matrix $M = \sqrt{B^2}$ generates the conditions

$$(c+\lambda_+)\, u + p = 0, \quad v = 0 \text{ on } \Gamma_- \cap \Gamma_i, \tag{5.7a}$$

(and of course $\mathbf{u} = \mathbf{0}$, $p = 0$ on $\Gamma_- \cap \Gamma_s$),

$$u + (\lambda_- - c)\, p = 0 \text{ on } \Gamma_+ \cap \Gamma_i, \tag{5.7b}$$

with $\lambda_\pm = \frac{1}{2}(c(1+a^{-2}) \pm \sqrt{c^2(1+a^{-2})^2 - 4\,(c^2a^{-2}-1)}$.

Remark 5.2 : *The BC on* $(\Gamma_- \cup \Gamma_+) \cap \Gamma_i$ *look like those for the incompressible PNS system; in this latter case* a *is infinite and* $\Gamma_s = \varnothing$. *For instance, the BC* (5.4a), (5.4d) *become* (4.5) *and* (5.4c) *is also admissible when* a *tends to infinity. However, it is very important to notice that the condition* $\mathbf{u} = \mathbf{0}$ *on* Γ_- *is not admissible. The matrix* M *defining this condition would be such that* $(0,0,1,0,0)^t \in \operatorname{Ker}(B-M)$ *and consequently its third diagonal element should be equal to* $-c\,a^{-2}$, *thus preventing* M *be positive semi-definite.*

Remark 5.3 : *Some of the boundary conditions proposed in this paper coincide with results of* [6], *where time dependent compressible Euler equations are studied.*

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