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# PION-PION SCATTERING*

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## ABSTRACT

We review exact results for pion-pion scattering in the framework of axiomatic field theory and their phenomenological applications.

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⋆ Dedicated to the sixtieth birth aniversary of Gérard Wanders.

## Contents

## I. Special features of pion-pion scattering

Pion-pion scattering may be considered the simplest strong interaction process. Particularly fruitful exact results are therefore obtained from the principles of axiomatic field theory. They include Martin's impressive results on axiomatic analyticity domains and absolute bounds on $\pi\pi$ amplitudes in terms of pion-mass alone. The analyticity domains imply physical region partial wave equations. An elegant idea of Gérard Wanders, the totally crossing symmetric variables ('Wanders variables'), led to the improvement of some of these results and to the practically useful 'Wanders sum rule' for scattering lengths. We review here the exact results for $\pi\pi$ scattering in the axiomatic field theory framework and some of their phenomenological applications.

Theoretical results for pion-pion scattering are much more powerful than for the other processes because of the following special simplicities in the $\pi\pi$ case.

(*a*) Pions are spinless.

(*b*) On neglecting the mass difference between the pions, and neglecting all electromagnetic effects (i.e., effects of zero mass particles), the $\pi^+$, $\pi^-$, $\pi^0$ have a common mass lower than the mass of any other particle. Assuming also the experimental fact that pions have no bound states, the next lowest mass state with the $G$-parity of one pion is the three-pion state, and the next lowest mass state with the $G$-parity of two pions is the four-pion state. These facts lead to simple and powerful results on the $\pi\pi$ analyticity domain. In particular there are no unphysical cuts and no anomalous thresholds.

(*c*) The $\pi\pi$ amplitudes have three channel crossing symmetry and simple unitarity relations. We have,

$$\underset{\sim}{F}(s,t,u) = C_{st}\underset{\sim}{F}(t,s,u) = C_{su}\underset{\sim}{F}(u,t,s) = C_{tu}\underset{\sim}{F}(s,u,t), \tag{I.1}$$

where,

$$\underset{\sim}{F}(s,t,u) = \begin{bmatrix} F^0(s,t,u) \\ F^1(s,t,u) \\ F^2(s,t,u) \end{bmatrix}, \tag{I.2}$$

with $F^I(s,t,u)$ being the amplitude with iso-spin $I$ in the $s$-channel, and the crossing matrices being given by,

$$C_{st} = \begin{bmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{bmatrix}, \quad C_{tu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$C_{su} = \begin{bmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}. \qquad (I.3)$$

In terms of the partial wave expansion,

$$F^I(s,t,u) = \frac{\sqrt{s}}{2k} \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell^I(s) P_\ell\left(1 + \frac{t}{2k^2}\right), \qquad (I.4)$$

(units $m_\pi = 1$), unitarity implies,

$$\begin{aligned} Im\ a_\ell^I(s) &= |a_\ell^I(s)|^2, \quad 4 \le s \le 16 \\ Im\ a_\ell^I(s) &\ge |a_\ell^I(s)|^2, \quad 16 \le s \le \infty. \end{aligned} \qquad (I.5)$$

Thus, if the $S$-wave scattering lengths $a_0^I$ exist, then,

$$a_0^I \equiv \lim_{k \to 0} \frac{\delta_0^I(s)}{k} = F^I(4,0,0), \qquad (I.6)$$

where $\delta_\ell^I(s)$ are the phase-shifts for angular momentum $\ell$ and iso-spin $I$.

These simplicities have been well exploited by Martin and others to derive a wealth of exact results for $\pi\pi$ scattering [Reviews: M2-M7, B4, M12-M14,P4,S1,W1,Y2]. In contrast accurate experimental information on the $\pi\pi$ amplitudes is difficult to obtain, most of it being obtained indirectly through extrapolation of the $\pi N \to \pi\pi N$ data [Recent phase shift analyses: E1, H2-H4,F1,C6,M11]. Theoretical exact results on $\pi\pi$ scattering are therefore rather useful in providing tests of the "data" and in their analysis (sec. III.4). We want to mention also an objective which is at the moment purely theoretical, inspired by the work of Atkinson (A1, A2, K1). They have developed a procedure of building crossing symmetric $\pi\pi$ amplitudes obeying the unitarity conditions (I.5) starting from the assumption of Mandelstam representation (not established in the axiomatic framework).

This procedure probably comes as close to a theory of $\pi\pi$ scattering as possible without including in detail the contributions to the unitarity condition of the inelastic channels. It is very attractive to attempt an analogous constructive procedure in the axiomatic framework. The physical region partial wave equations (sec. III) constitute such an attempt.

## II. Constraints on pion-pion partial wave amplitudes at and below threshold

We illustrate most of the results for the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude $F(s,t)$ defined by

$$F(s,t) \equiv \frac{1}{3}F^0(s,t) + \frac{2}{3}F^2(s,t). \qquad (II.1)$$

Its absorptive part $A(s,t)$ is obtained from

$$A^I(s,t) \equiv \lim_{\epsilon\to 0+} \left[F^I(s+i\epsilon,t) - F^I(s-i\epsilon,t)\right]/(2i) \qquad (II.2)$$

We denote its partial waves by

$$f_\ell(s) \equiv \frac{1}{3}f_\ell^0(s) + \frac{2}{3}f_\ell^2(s), \quad f_\ell^I(s) \equiv \frac{\sqrt{s}}{2k}a_\ell^I(s) \qquad (II.3)$$

### II.1. The Froissart-Gribov formula and threshold behaviour

Martin proved that $F(s,t)$ for $t$ inside a domain which includes $|t| < 4$ as well as the region $t \in (-28,4)$ of the real axis must obey twice subtracted dispersion relations. In particular for the $\pi^0\pi^0 \to \pi^0\pi^0$ case.

$$F(s,t) = c(t) + \frac{1}{\pi}\int_4^\infty ds' \frac{A(s',t)}{s'^2}\left(\frac{s^2}{s'-s} + \frac{u^2}{s'-u}\right), \text{ for } |t| < 4. \qquad (II.4)$$

We then derive the rigorous validity of the Froissart-Gribov formula for $\ell \geq 2$, and $|t| < 4$:

$$f_\ell(t) = \frac{4}{\pi(4-t)}\int_4^\infty ds' A(s',t) Q_\ell\left(\frac{2s'}{4-t} - 1\right), \ \ell \geq 2, \ |t| < 4. \qquad (II.5)$$

We know that, $A(s',t) \geq 0$, for $0 \leq t < 4$ and $Q_\ell(z) \geq 0$ for $z > 1$, $\ell > -1$. Hence,

$$f_\ell(t) \geq 0, \quad \text{for } \ell \geq 2, \ 0 < t < 4, \tag{II.6}$$

Also, for $n, \ell \geq 2$, and $0 < t < 4$,

$$\begin{aligned} f_{\ell+n}(t) &\leq f_\ell(t) Sup_{4 \leq s' \leq \infty} \frac{Q_{\ell+n}(\frac{2s'}{4-t} - 1)}{Q_\ell(\frac{2s'}{4-t} - 1)} \\ &= f_\ell(t) \frac{Q_{\ell+n}(\frac{4+t}{4-t})}{Q_\ell(\frac{4+t}{4-t})}. \end{aligned} \tag{II.7}$$

It is straightforward to show that (II.6) and (II.7) hold also with $f_\ell(t)$ replaced by $f_\ell^{I=0}(t)$ (Ref. M8) and by $f_\ell^{I=0}(t) - f_\ell^{I=2}(t)$ (Ref. R5). If the scattering lengths $\alpha_\ell^I$ exist, then from (II.6) and its generalization to $f_\ell^{I=0}(t) - f_\ell^{I=2}(t)$, we conclude that, with the definition

$$\alpha_\ell^I \equiv \lim_{k \to 0} \frac{\delta_\ell^I(k)}{k^{2\ell+1}}, \tag{II.8}$$

$$\alpha_\ell = \frac{1}{3}\alpha_\ell^{I=0} + \frac{2}{3}\alpha_\ell^{I=2} \geq 0, \quad \ell \geq 2 \tag{II.9}$$

$$\alpha_\ell^{I=0} - \alpha_\ell^{I=2} \geq 0, \quad \ell \geq 2. \tag{II.10}$$

and also (as a consequence),

$$\alpha_\ell^{I=0} \geq 0, \quad \ell \geq 2. \tag{II.11}$$

Similarly, from (II.7) and its generalization, if $\alpha_\ell^I$ exist, then,

$$\alpha_{\ell+2} \leq \alpha_\ell \frac{1}{16} \frac{(\ell+1)(\ell+2)}{(\ell+\frac{3}{2})(\ell+\frac{5}{2})}, \tag{II.12}$$

and a similar relation with $\alpha_\ell \to (\alpha_\ell^{I=0} - \alpha_\ell^{I=2})$. We see from (II.7) that for $I = 0,2$ and $0 < t < 4$, the amount of higher waves is limited by the amount of $D$-waves.

It has not been possible to establish normal threshold behaviour or the existence of $\alpha_\ell^I$ in the axiomatic framework. Martin (M9) has written down an example of an $f_\ell^I(s)$ obeying elastic unitarity for $s \neq 4$ on the elastic cut, and having an essential singularity at $s = 4$, thus ruining the threshold behaviour. We shall only record here Martin's (M8) results on how much can be said rigorously about threshold behaviour of $f_\ell(s)$.

From the dispersion relation (II.4), positivity of $A(s',t)$ for $0 \leq t < 4$ yields, for $s_1 < s < 4, 0 < t < 4$,

$$|F(s,t)| < \frac{C_1}{4-s} + C_2. \qquad (II.13)$$

Similarly, from the positivity of $A(s',t) - A(s',0)$ for $0 < t < 4$, we have, for $s_1 < s < 4$,

$$|F(s,t) - F(s,0)| < \frac{D_1}{4-s} + D_2 \qquad (II.14)$$

Due to (II.7) we have for $0 < s < 4$, $0 < t < 4$, the convergent expansion

$$F(s,t) - F(s,0) = \sum_{\ell=2}^{\infty}(2\ell+1)f_\ell(s)\left[P_\ell\left(\frac{2t}{4-s} - 1\right) - 1\right]. \qquad (II.15)$$

For $t > 4 - s_1$, we have $t > 4 - s$, and $[2t/(4-s) - 1] > 1$, and then the series is a sum of positive terms. Hence,

$$0 \leq f_\ell(s) \leq \frac{(\frac{D_1}{4-s} + D_2)}{(2\ell+1)[P_\ell(\frac{2t}{4-s} - 1) - 1]}, \quad 0 < 4 - t < s < 4. \qquad (II.16)$$

The positivity of $f_\ell^{I=0,2}(s)$ then yields,

$$\lim_{\epsilon \to 0+}\left[\frac{|f_\ell^{I=0,2}(s)|}{|(s-4)|^{\ell-1}}\right]_{s=4-\epsilon} \leq \frac{C}{(2\ell+1)}\frac{1}{t^\ell}\frac{(\ell!)^2}{(2\ell)!}, \quad 0 < t < 4. \qquad (II.17)$$

This is a rather remarkable restriction coming from the fixed-$t$ dispersion relation and the positivity of the absorptive part; but it is not enough to establish normal threshold behaviour.

### II.2. Constraints involving amplitudes at a finite number of points below threshold.

These could be useful as constraints on the extrapolation of physical region partial wave amplitudes below threshold (D1) or in the building of theoretical models for the $\pi\pi$-amplitude (G4, B9).

### (a) Some simple properties of the total amplitude below threshold

Some remarkably simple properties of the total amplitude below threshold were discovered by Jin and Martin (J1). Let

$$z \equiv \left(\frac{s-u}{2}\right)^2 = \left(s - 2 + \frac{1}{2}t\right)^2 \tag{II.18}$$

Then, for $0 \le t < 4$ we have the fixed-$t$ dispersion relation

$$F(s,t) = G(z,t) = A(t) + \frac{z}{\pi} \int\limits_{z_0(t)}^{\infty} \frac{Im\ G(z',t)dz'}{z'(z'-z)}, \tag{II.19}$$

with $z_0(t) = (2 + \frac{1}{2}t)^2$, and the unitarity condition $Im\ G(z',t) \ge 0$. Hence, for $z < z_0(t)$, and $0 \le t < 4$,

$$\left.\frac{\partial^n}{\partial z^n} G(z,t)\right|_t > 0. \tag{II.20}$$

Changing variables,

$$\left.\frac{\partial^n}{\partial s^n} F(s,t)\right|_t > 0, \quad \text{for } 0 \le t < 4, \quad 2 - \frac{t}{2} < s < 4. \tag{II.21}$$

and

$$\left.\frac{\partial F(s,t)}{\partial s}\right|_t < 0, \quad \text{for } 0 \le t < 4, \quad -t < s < 2 - \frac{t}{2}. \tag{II.22}$$

Eqs. (II.21) and (II.22) remain true under any permutation of $(s,t,u)$ because of the complete symmetry of $F(s,t,u)$ in all three channels. Thus, we get information on the scattering amplitude in the triangle $s < 4$, $t < 4$, $u < 4$. In particular, we can prove that (i) the symmetry point $s = t = u = 4/3$ must be an absolute minimum of $F(s,t,u)$ inside this triangle, and (ii) inside the triangle $F$ increases along any straight line originating at the symmetry point. Another simple consequence of (II.19) is that

$$F(s,0) < F(4,0), \quad 0 < s < 4. \tag{II.23}$$

**(b) Constraints involving the $S$-wave alone.** We first state some simple results to illustrate that the constraints of unitarity, analyticity and crossing give us a fair idea of the shape of the $S$-wave below threshold, and then indicate the methods of proof, the details of which are to be found in the original papers (M8, A3, A4, B12, C1, C3, G1, G2, J1, P1).

Jin and Martin (J1) obtained the results

$$f_0(s) < f_0(4), \quad 0 < s < 4 \tag{II.24}$$

and

$$\frac{df_0(s)}{ds} > 0, \quad 2 < s < 4. \tag{II.25}$$

Martin (M8) improved (II.25) to obtain,

$$\frac{df_0(s)}{ds} > 0, \quad 1.7 \leq s < 4 \tag{II.26}$$

Auberson (A3) obtained

$$\frac{df_0(s)}{ds} < 0, \quad 0 < s < 1.127 \tag{II.27}$$

Grassberger (G1) improved this result to obtain,

$$\frac{df_0(s)}{ds} < 0, \quad 0 < s < 1.217 \tag{II.28}$$

Common (C1) has derived the important result

$$\frac{d^2 f_0(s)}{ds^2} > 0, \quad 0 < s < 1.7 \tag{II.29}$$

From (II.24) to (II.29) it follows that $f_0(s)$ has a unique minimum in the range $0 < s < 4$, located somewhere between $s = 1.217$ and $s = 1.7$. The shape of the $S$-wave thus suggested is pictured in Fig. 1.

Martin (M8) has derived a class of inequalities of the form $f_0(s_1) < f_0(s_2)$ where $0 \leq s_{1,2} \leq 4$. For example, we quote,

$$f_0(0) > f_0(3.189), \quad f_0(3.205) > f_0(0.2134) > f_0(2.9863). \tag{II.30}$$

These inequalities have been improved by Brander (B12) and by Grassberger (G1), and generalized to iso-spin combinations other than the $\pi^0\pi^0 \to \pi^0\pi^0$ by Auberson et al (A4).

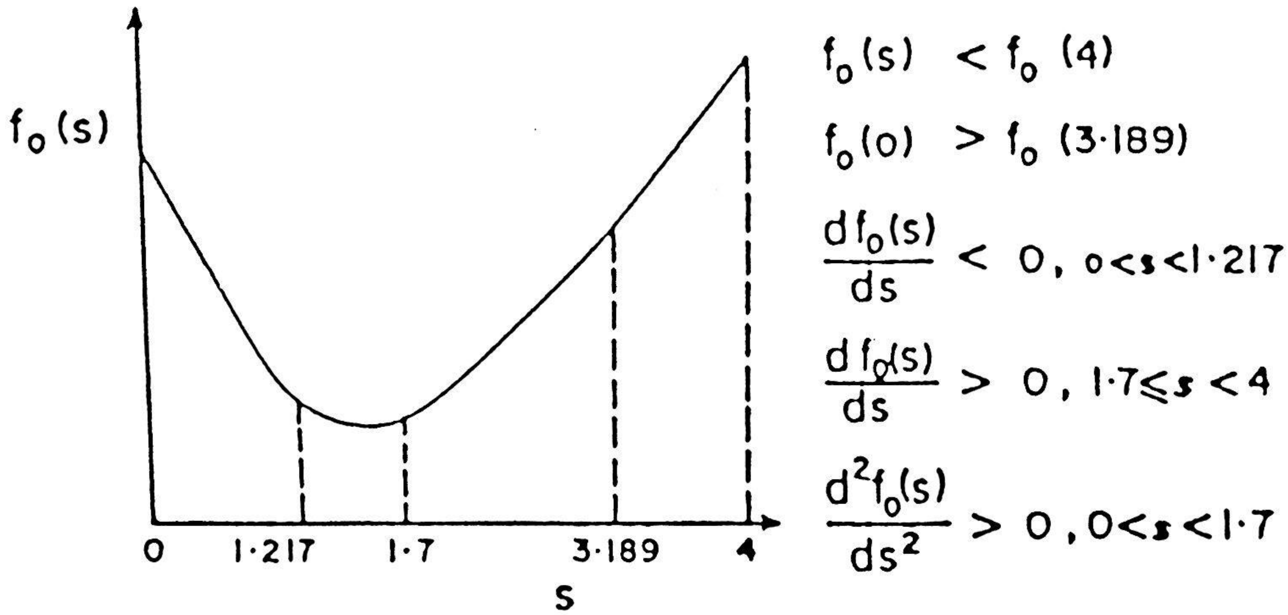


**Fig.1.** **The shape of the $\pi^0\pi^0 \to \pi^0\pi^0$ $S$-wave amplitude $f_0(s)$ below threshold suggested by the rigorous inequalities due to crossing and unitarity is illustrated.**

We proceed to prove (II.24) and (II.25). Due to Bose-symmetry,

$$F_0(s) = \frac{2}{4-s} \int_0^{(4-s)/2} dt\ F(s,t,4-s-t). \tag{II.31}$$

Interchanging $s$ and $t$ in (II.22) we deduce

$$F(s,t) < F(s,0), \quad 0 \leq s < 4, \quad 0 \leq t \leq \frac{4-s}{2}. \tag{II.32}$$

Hence,

$$f_0(s) < F(s,0), \quad 0 \leq s < 4 \tag{II.33}$$

Combining this with (II.23) we have (II.24). Starting from

$$f_0(s) = 2 \int_0^{1/2} dx\ F(s, x(4-s)), \tag{II.34}$$

we have,

$$\frac{df_0(s)}{ds} = 2 \int_0^{1/2} dx \left[ \left( \frac{\partial F(s,x(4-s))}{\partial s} \right)_t - x \left( \frac{\partial F(s,x(4-s))}{\partial t} \right)_s \right] \tag{II.35}$$

Further,

$$\left( \frac{\partial F}{\partial s} \right)_s > 0 \text{ for } 0 < t < 4 \text{ and } 4 > s > 2 > 2 - \frac{t}{2},$$
$$\left( \frac{\partial F}{\partial t} \right)_s < 0 \text{ for } 0 < s < 4 \text{ and } 0 < t < \frac{4-s}{2}.$$

Hence (II.25) follows. For the remaining results we start from the fixed-$t$ dispersion relation (II.4) and project out the $S$-wave,

$$f_0(t) = c(t) + \frac{1}{\pi} \int_4^{\infty} \frac{ds'}{s'^2} \left\{ (t-4-2s') + \frac{2s'^2}{(t-4)} \ell n \left( \frac{s'+t-4}{s'} \right) \right\} A(s',t), \quad |t| < 4. \tag{II.36}$$

Thus the subtraction constant $c(t)$ in the fixed-$t$ relation can be eliminated in favour of the $S$-wave to obtain,

$$F(s,t,u) = f_0(t) + \frac{1}{\pi}\int_4^\infty ds' \left[\frac{1}{s'-s} + \frac{1}{s'+s+t-4} - \frac{2}{t-4}\ell n\left(\frac{s'+t-4}{s'}\right)\right] A(s',t)$$
$$\equiv f_0(t) + R_2(t,s), \quad |t| < 4. \tag{II.37}$$

$R_2(t,s)$ is the contribution of the partial waves with $\ell \geq 2$ in the $t$-channel to $F(s,t,u)$. Using crossing symmetry, $F(s,t,u) = F(t,s,u)$, we have,

$$f_0(s) - f_0(t) = R_2(t,s) - R_2(s,t), \text{ for } |s| < 4, \ |t| < 4; \tag{II.38}$$

and on taking derivatives,

$$\frac{df_0(s)}{ds} = \frac{d}{ds}\left[R_2(t,s) - R_2(s,t)\right], \text{ for } |s|, |t| < 4, \tag{II.39}$$

$$\frac{d^2 f_0(s)}{ds^2} = \frac{d^2}{ds^2}\left[R_2(t,s) - R_2(s,t)\right], \text{ for } |s|, |t| < 4. \tag{II.40}$$

Eqns. (II.38) - (II.40) together with the positivity of $A(s',t)$ and its derivatives with respect to $t$ for $0 \leq t < 4$ constitute the source of the relations (II.34) - (II.39). For example, from (II.38), $f_0(s) > f_0(t)$ if $R_2(t,s) > 0$ and $R_2(s,t) < 0$. This gives for example (M8)

$$f_0(0) > f_0(3.155).$$

A tighter inequality is obtained by requiring only that $R_2(t,s) > R_2(s,t)$. Thus we obtain (M8)

$$f_0(0) > f_0\ (3.189),$$

Techniques for deciding the signs of $R_2(t,s) - R_2(s,t)$ have been developed by Martin (M8).

Results from straightforward positivity of the absorptive part can sometimes be strengthened by using a relation between the absorptive parts following from (II.38).

Substituting (II.38) in

$$0 = [f(s) - f(s_1)] + [f(s_1) - f(t)] + [f(t) - f(s)]$$

we get the following relation between physical absorptive parts (R6)

$$0 = [R_2(s_1, s) - R_2(s, s_1)] + [R_2(t, s_1) - R_2(s_1, t)] + [R_2(s, t) - R_2(t, s)], \text{ for } |s|, |s_1|, |t| < 4. \tag{II.41}$$

Similar relations have also been derived by Wanders (W3), Roskies (R3), and Auberson and Khuri (A6). Grassberger (G1) has shown, for example in deriving (II.28), that results following from positivity of the absorptive part can be improved by a judicious use of (II.41). Common and Pidcok (C3) have derived very useful inequalities on the $D$-wave below threshold using (II.41). The relation (II.41) can be regarded as a crossing relation between physical absorptive parts. Such relations would be discussed further in Sec. III.

(b) **Constraints involving a few low partial waves.** We quote for illustration a few results for $\pi^0\pi^0$ scattering (M8), and a few for other iso-spin combinations (A4). For $\pi^0\pi^0$ scattering,

$$4.067 f_2(0.0341) < f_0(3.839) - f_0(0.0341) \tag{II.42}$$

$$3.061 f_2(0.0730) > f_0(3.654) - f_0(0.0730) \tag{II.43}$$

$$\begin{aligned} &1.494 f_2(0.537) - 1.623 f_2(2.363) \\ &< f_0(0.537) - f_0(2.363) < 1.510 f_2(0.537) - 1.622 f_2(2.363) \end{aligned} \tag{II.44}$$

and for other iso-spin combinations

$$\begin{aligned} &1.844 f_1^1(0.2937) + 3.765 f_1^1(2.4226) \\ &< f_0^0(0.2937) - f_0^0(2.4226) - f_0(0.2937) + f_0(2.4226). \end{aligned} \tag{II.45}$$

$$\begin{aligned} &0.6146 f_1^1(0.2937) + 2.510 f_1^1(2.4226) \\ &> f_0(2.4226) - f_0^0(0.2937) + \frac{2}{3} f_0(0.2937). \end{aligned} \tag{II.46}$$

## II.3 Constraints on integrals of partial wave amplitudes

Balachandran and Nuyts (B2) obtained necessary and sufficient conditions for crossing symmetry in the form a denumerable set of equality constraints involving integrals of the partial wave amplitudes in the region $0 \leq s \leq 4$. The remarkable thing is that each of these equations following from crossing symmetry involves only a finite number of partial waves. For example, some simple crossing relations for the $\pi^0\pi^0$ case are

$$\begin{aligned} &\int_0^4 ds f_0(s)(4-s)(3s-4) = 0. \\ &\int_0^4 ds(4-s)^2[4(s-1)f_0(s) + (4-s)f_2(s)] = 0. \end{aligned} \qquad (II.47)$$

Following Balachandran and Nuyts (B2) these equations were further elucidated by Roskies (R1,R4), Basdevant, Cohen-Tannoudji and Morel (B3), Cooper and Pennington (C5), and Pennington and Pond (P2). The crossing equations were supplemented with the positivity properties due to unitarity to obtain further inequality relations between partial waves (P6,B1); for example, for the $\pi^0\pi^0$ $S$-wave we have the relation,

$$\int_0^4 ds(4-s)^2(s-1)f_0(s) < 0. \qquad (II.48)$$

Piguet and Wanders (P6) have obtained a denumerable set of such inequalities resulting from the positivity of the absorptive parts; their set is complete in the sense that it provides necessary and sufficient conditions for the positivity of the $Im\ f_\ell(s')$, $s' \geq 4$.

Several authors (P5,A5,B8,B9,G4,I1) have constructed models for low energy pion-pion scattering starting from unitary parametrizations for the partial waves and imposing the crossing conditions on the partial wave amplitudes below threshold to fix the parameters. A symmary of these and other $\pi\pi$ models is given by Morgan (M12).

## III. Integral Equations for Physical Region Pion-Pion Partial Wave Amplitudes

The main idea (R6) is to use the axiomatic analyticity properties and high energy behaviour to derive partial wave equations expressing $Re\ f_\ell^I(s)$ for $4 \leq s \leq s_{max}$ in terms of the two $s$-wave scattering lengths and the $Im\ f_{\ell'}^{I'}(s')$ for $s' \geq 4$, $\ell' = 0, 1, 2, \cdots, \infty$, $I' = 0, 1, 2$. The unitarity condition $Im\ a_\ell^I(s) = \eta_\ell^I(s)|a_\ell^I(s)|^2$, with $\eta_\ell^I(s)$ given, then yields integral equations to determine the $a_\ell^I(s)$ for $4 \leq s \leq s_{max}$ with $\eta_\ell^I(s)$, the two scattering lengths $a_0^I$, and the $Im\ f_\ell^{I'}(s')$ for $s' \geq s_{max}$ as driving terms. It is desirable to have $s_{max}$ as large as possible. The first partial wave equations based on fixed-$t$ dispersion relations (R6) have $s_{max} = 60$. The idea of using dispersion relations on general algebraic curves in the Mandelstam variables $s, t$ and in the Wanders variables $x, y$ (W2) was then exploited (A6,A7,H1,M1). Using dispersion relations on straight lines in the $x - y$ plane Mahoux, Roy and Wanders (M1) have obtained partial wave equations valid upto $s_{max} = 125.31$. Using hyperbolae in the $x-y$ plane Auberson and Epele (A7) prove the existence of partial wave equations upto $s_{max} = 164.7$. There seems to be no fundamental obstacle to reaching $s_{max} = \infty$; we would then obtain an axiomatic frame work to construct the $\pi\pi$ complitude similar to the Atkinson-frame work (A1,A2,K1) based on the Mandelstam representation.

### III.1. Integral equation based on fixed momentum transfer dispersion relations

With the matrix notation (I.1)-(I.3) the fixed-$t$ dispersion relations for $|t| < 4$ and for $-28 < t < 4$, given by Jin and Martin (J2,M10) take the form

$$\begin{aligned} \underset{\sim}{F}(s,t) =& C_{st}\left[\underset{\sim}{C}(t) + (s-u)\underset{\sim}{D}(t)\right] + \\ &+ \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\left(\frac{s^2+}{s'-s} + \frac{u^2}{s'-u}C_{su}\right)\underset{\sim}{A}(s',t), \end{aligned} \qquad (III.1)$$

where, the absorptive parts $A^I(s',t)$ are defined by

$$A^I(s',t) = \frac{\sqrt{s'}}{sk'}\sum_{\ell=0}^{\infty}(2\ell+1)Im\ a_\ell^I(s')P_\ell\left(1+\frac{t}{2k'^2}\right), \qquad (III.2)$$

within the large Lehmann-Martin ellipse (L1,M10), and the subtraction constants $\underset{\sim}{C}(t)$, $\underset{\sim}{D}(t)$ are of the form

$$\underset{\sim}{C}(t) = \begin{bmatrix} C^0(t) \\ 0 \\ C^2(t) \end{bmatrix}, \quad \underset{\sim}{D}(t) = \begin{bmatrix} 0 \\ d^{(1)}(t) \\ 0 \end{bmatrix} \qquad (III.3)$$

due to crossing symmetry. The main trick needed in obtaining partial wave equations is the elimination of the unknown $t$-dependence of $\underset{\sim}{C}(t)$ and $\underset{\sim}{D}(t)$. Using crossing symmetry at a fixed value of $s$, say $s = 0$, we substitute Eq. (III.1) into the equation

$$\underset{\sim}{F}(0,t) = C_{st}\underset{\sim}{F}(t,0) \tag{III.4}$$

to express $\underset{\sim}{C}(t)$ and $\underset{\sim}{D}(t)$ in terms of $\underset{\sim}{C}(0)$ and $\underset{\sim}{D}(0)$. Eliminating $\underset{\sim}{C}(0)$ and $\underset{\sim}{D}(0)$ in favour of the $S$-wave scattering lengths defined by (I.6) we obtain,

$$\underset{\sim}{F}(s,t) = \frac{1}{4}g_1(s,t)\underset{\sim}{a}_0 + \int_4^\infty ds' \left[g_2(s,t,s')\underset{\sim}{A}(s',0) + g_3(s,t,s')\underset{\sim}{A}(s',t)\right], \tag{III.5}$$

where,

$$g_1(s,t) \equiv s(1 - C_{su}) + t(C_{st} - C_{su}) + 4C_{su}, \tag{III.6}$$

$$\begin{aligned} g_2(s,t,s') \equiv & C_{st}\left(\frac{1 + C_{tu}}{2} + \frac{2s + t - 4}{t - 4}\frac{1 - C_{tu}}{2}\right) \times \\ & \times \frac{1}{\pi}\cdot\frac{1}{s'^2}\left[\frac{t^2}{s' - t} + \frac{(4 - t)^2 C_{su}}{s' - (4 - t)} - \frac{4t + 4(4 - t)C_{su}}{s' - 4}\right], \end{aligned} \tag{III.7}$$

$$g_3(s,t,s') \equiv \frac{1}{\pi s'^2}\left[\frac{s^2}{s' - s} + \frac{u^2}{s' - u}C_{su} - \frac{(4 - t)^2}{s' - (4 - t)}\left\{\frac{C_{su} + 1}{2} + \frac{2s + t - 4}{t - 4}\frac{C_{su} - 1}{2}\right\}\right], \tag{III.8}$$

and

$$\underset{\sim}{a}_0 \equiv \begin{bmatrix} a_0^0 \\ 0 \\ a_0^2 \end{bmatrix} \tag{III.9}$$

From Bose-symmetry,

$$f_l^I(s) = \frac{\sqrt{s}}{2k}a_l^I(s) = \frac{1 + (-)^{l+I}}{2}\int_0^1 d(\cos\theta)P_l(\cos\theta)F^I(s, -2k^2(1 - \cos\theta)). \tag{III.10}$$

Thus, to calculate $f_l^I(s)$ we need to use (III.5) for $t$ in the interval $\left(\frac{4-s}{2}, 0\right)$, and hence need that $A^I(s',t)$ is well defined by (III.2) for all $s' \geq 4$ for $t$ in $((4 - s)/2, 0)$. The large Lehmann-Martin ellipse (M10) guarantees the convergence of (III.2) for all $s' \geq 4$ if $t$ is in the interval $(-28, 4)$. For $s = 60$, $(4-s)/2 = -28$; for $s = -4$, $(4-s)/2 = 4$. Thus, for $-4 < s < 60$, we obtain the partial wave relations (R6)

$$
\begin{aligned}
a_\ell^I(s)\frac{\sqrt{s}}{2k} = & \frac{1+(-)^{\ell+I}}{2}\left[\frac{1}{4}\int_0^1 dx P_\ell(x) \sum_{I'=0,2} g_1^{II'}\left(s, \frac{4-s}{2}(1-x)\right) a_0^{(I')} + \right. \\
& + \int_0^1 dx P_\ell(x) \int_4^\infty ds' \sqrt{\frac{s'}{s'-4}} \sum_{\ell'=0}^\infty (2\ell'+1) \sum_{I'=0,1,2} \left\{ g_2^{II'}\left(s, \frac{4-s}{2}(1-x), s'\right)\right. \\
& \times Im\ a_{\ell'}^{(I')}(s') + g_3^{II'}\left(s, \frac{4-s}{2}(1-x), s'\right) Im\ a_{\ell'}^{(I')}(s') \\
& \left.\left. \times P_{\ell'}\left(1 + \frac{(4-s)(1-x)}{(s'-4)}\right)\right\}\right],
\end{aligned}
\tag{III.11}
$$

where the $g_i^{II'}$, $i = 1,2,3$, are the matrix elements of the $g_i$ defined in Eqs. (III.6) to (III.8). We remark that Eq. (III.5) has $s-u$ crossing symmetry built in and hence complete crossing symmetry would be satisfied if $t-u$ symmetry also is satisfied. From (III.5) and $(t-u)$ crossing symmetry we obtain the crossing conditions,

$$
\begin{aligned}
\int_4^\infty ds' & \left[\left\{g_2(s,t,s') - C_{tu} g_2(s,u,s')\right\} \underset{\sim}{A}(s',0) + \right. \\
& \left. + \left\{g_3(s,t,s')\underset{\sim}{A}(s',t) - C_{tu} g_3(s,u,s')\underset{\sim}{A}(s',u)\right\}\right] = 0,
\end{aligned}
\tag{III.12}
$$

valid for $-28 < t,\ u < 4$ and for $|t|, |u| < 4$. Eqs. (III.5) and (III.12) together are necessary and sufficient conditions for complete crossing symmetry; both of them involve only physical region partial wave amplitudes and can be checked directly against experiment. Crossing relations involving physical region absorptive parts similar to Eq. (III.12) have also been obtained by Wanders (W3), Roskies and Yen (R3,Y1), Auberson and Khuri (A6), and Grassberger (G3).

Eq. (III.11) expresses $Re\ a_\ell^{(I)}(s)$ for $4 < s < 60$ in terms of $a_0^{(0)}$, $a_0^{(2)}$ and a principal value integral involving the $Im\ a_{\ell'}^{(I')}(s')$ for $\ell' = (0, \infty)$, $I' = (0,1,2)$, and $s' \geq 4$. When this value is substituted into unitarity equations,

$$
Im\ a_\ell^{(I)}(s) = \begin{cases} [Im\ a_\ell^I(s)]^2 + [Re\ a_\ell^I(s)]^2, & 4 \leq s < 16 \\ \eta_\ell^I(s)\{[Im\ a_\ell^I(s)]^2 + [Re\ a_\ell^I(s)]^2\}, & s \geq 16 \end{cases}
\tag{III.13}
$$

we obtain a set of coupled non-linear singular integral equations for $Im\ a_\ell^I(s)$ in the interval

$4 < s < 60$, with the driving terms involving $a_0^{(0)}$, $a_0^{(2)}$, $\eta_\ell^I(s)$ for $16 < s < 60$, and the $Im\ a_\ell^{(I)}(s)$ for $s \geq 60$. These integral equations resemble in a mathematical sense those studied earlier by Atkinson, Warnock and Kupsch (A1,A2,K1); mathematical questions such as existence of solutions of these equations and their uniqueness have been investigated by Pool (P7), Heemskerk (H5), and Atkinson et al (A10, A11). A rather serious difficulty is that in solving the integral equations obtained by combining (III.11) and (III.13), there seems to be no practical way to incorporate the crossing conditions (III.12). An analysis of the constraints and correlations imposed by physical region partial wave equations on the $S$- and $P$-waves has been performed by Epele and Wanders (E3).

How can we obtain partial wave equations valid in a larger interval than Eq. (III.11)? We see from the Mandelstam $s-t-u$ diagram (Fig. 2) that for $s = 60$, $t = -28$, the fixed-$t$ straight line touches the forbidden region in which the partial wave expansion (III.2) is not known to converge at the point $s = u = 16$, $t = -28$. (Even if we assume Mandelstam analyticity for $A^I(s', t)$ we only increase $s_{max}$ from 60 to 68). Substantial increase in $s_{max}$ would require the use of dispersion relations on curves (other than fixed-$t$ lines) which carefully avoid the forbidden regions in the $s-t$ plane.

## III.2. Integral equation based on dispersion relations incorporating explicit three channel crossing symmetry

Mahoux, Roy and Wanders (M1) obtain dispersion relations with explicit crossing symmetry in all three channels and derive partial wave relations valid in the interval $-28 < s < 125.31$. The crossing conditions on $F^I(s,t,u)$ are equivalent to the total symmetry in $s,t,u$ of the amplitudes $G_k(s,t,u)$ $(k = 0,1,2)$ defined as follows (R2):

$$G_0(s,t,u) \equiv F^{\pi^0\pi^0\to\pi^0\pi^0}(s,t,u) = \frac{1}{3}F^0(s,t,u) + \frac{2}{3}F^2(s,t,u) \qquad (III.14)$$

$$G_1(s,t,u) \equiv \frac{F^1(s,t,u)}{t-u} + \frac{F^1(t,u,s)}{u-s} + \frac{F^1(u,s,t)}{s-t} \qquad (III.15)$$

$$\begin{aligned} G_2(s,t,u) \equiv & \frac{1}{s-t}\left(\frac{F^1(s,t,u)}{t-u} - \frac{F^1(t,s,u)}{s-u}\right) + \\ & + \frac{1}{t-u}\left(\frac{F^1(t,u,s)}{u-s} - \frac{F^1(u,t,s)}{t-s}\right) + \\ & + \frac{1}{u-s}\left(\frac{F^1(u,s,t)}{s-t} - \frac{F^1(s,u,t)}{u-t}\right) \end{aligned} \qquad (III.16)$$

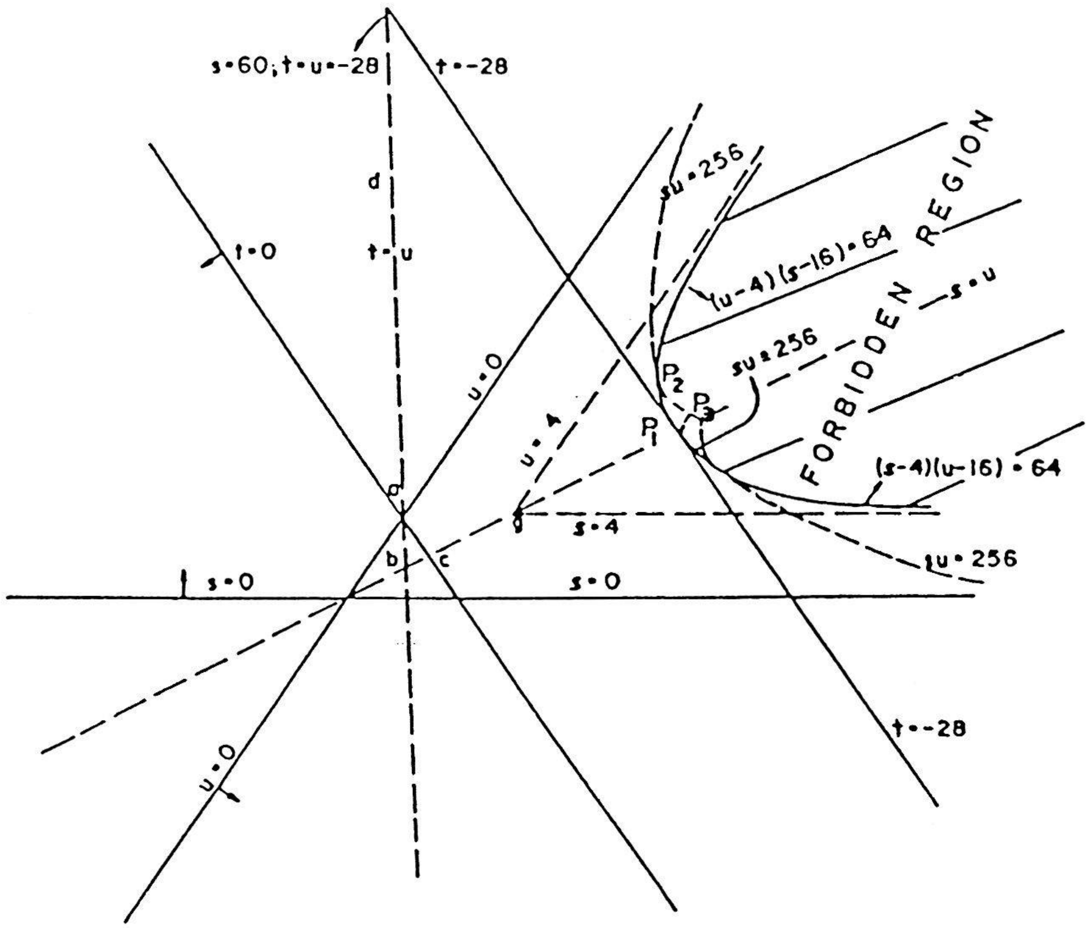


**Fig.2.** **This $s - t - u$ plot shows that fixed-$t$ straight lines with $t < -28$ intersect the 'forbidden region' where partial wave expansions for the absorptive part are not known to converge; hence the maximum value of $s$ for points lying on the $t = u$ line through which an allowed fixed-$t$ line can be drawn is $s_{max} = 60$.**

Due to the antisymmetry of $F^1(s,t,u)$ under $t \leftrightarrow u$ the denominators appearing in the definitions of $G_1$ and $G_2$ do not introduce new singularities at $t = u$, $u = s$, and $s = t$. Hence the functions $G_k(s,t,u)$ have the same analyticity properties as the amplitudes $F^I(s,t,u)$. The $F^I(s,t,u)$ can be obtained from the $G_k(s,t,u)$ through the inverse relations.

The complete symmetry of the $G_k(s,t,u)$ in $s,t,u$ is exploited by Mahoux, Roy and Wanders (M1) by considering the $G_k$ as functions $G_k(x,y)$ of the Wanders variables $x,y$ (W2):

$$x \equiv -\frac{1}{16}(st+tu+us), \quad y \equiv \frac{1}{64}stu \qquad (III.17)$$

If $(s,t,u)$ corresponds to a given value of $(x,y)$, so does any permutation $P(s,t,u)$ of $(s,t,u)$; however this does not introduce any multivaluedness in $G_k(x,y)$ because $G_k(s,t,u) = G_k(P(s,t,u))$. Thus the singularities associated with the change of variables (III.17) do not appear in $G_k(x,y)$ due to crossing symmetry. The only singularities of $G_k(x,y)$ are the images of the singularities of $G_k(s,t,u)$ through the mapping $(s,t,u) \to (x,y)$. On the complex straight line

$$y = a(x - x_0) \qquad (III.18)$$

$G_k(x,y) = G_k(x, a(x-x_0))$ is a function of the single complex variable $x$. Martin (M10) has shown that $F^I(s,t,u)$ has no singularities except for the cuts $s \geq 4$, $t \geq 4$, $u \geq 4$ if any one of the three variables $s,t,u \in D$, where $D$ is a finite domain of the complex plane containing the segment $(-28,4)$ of the real axis as well as the circle of radius 4 around the origin (see page 66 of Ref. M4). Now, the straight line (III.18) is a cubic in $s,t$ plane given by

$$(s+4a)(t+4a)(u+4a) = 64a[a(a+1) - x_0]. \qquad (III.19)$$

Hence,

$$min\,(|s+4a|, |t+4a|, |u+4a|) \leq 4|a|^{1/3}|a(a+1) - x_0|^{1/3} \qquad (III.20)$$

Hence, for small enough $|a|$, say for $a \in V(x_0)$, where $V(x_0)$ is a complex neighbourhood of the origin, at least one of the three variables $s,t,u \in D$, for all points of the straight line (III.18). Hence $G_k(x, a(x-x_0))$ is analytic in the complex $x$-plane except for a cut $C(a,x_0)$ consisting of the part of the complex straight line (III.18) for which at least one of the variables $s,t,u \geq 4$. $C(a,x_0)$ connects the point $x = ax_0/(1+a)$ with the point at infinity. From the Jin-Martin high $s$, fixed-$t$ bounds (J2,M9) we have, for $a \in V(x_0)$, that $G_0(x, a(x-x_0))$ and $G_1(x, a(x-x_0))$ obey once - substracted dispersion relations, and $G_2(x, a(x-x_0))$ obeys an unsubtracted dispersion relation in $x$. In terms of the original variables $s, t, u$ we have:

$$G_k(s,t,u) - G_k(s_1,t_1,u_1)(1-\delta_{k2}) =$$
$$= \frac{1}{2i\pi}\int_4^\infty ds' discG_k(s', t'(s',a,x_0), 4-s'-t'(s',a,x_0))[s'-t'(s',a,x_0)]\times$$
$$\times [2s'+t'(s',a,x_0)-4]\left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{(1-\delta_{k2})}{(s'-s_1)(s'-t_1)(s'-u_1)}\right], \; k=0,1,2, \tag{III.21}$$

where, $a \in V(x_0)$, $(s', t', u')$ and $(s_1, t_1, u_1)$ lie on the straight line (III.18).

Equation (III.21) can actually be continued to certain values of a outside $V(x_0)$. To see this, choose $x_1 = x_0$, so that $s_1 = s_0 = 2(\sqrt{1+4x_0}+1)$, and $t_1 = 0$. $G_k(s_0, 0, 4-s_0)$ is independent of '$a$' and a holomorphic function of $s_0$ in the $s_0$-plane cut along $(-\infty, 0)$ and $(4,\infty)$; hence the right-hand side integral, for $x_0$ outside the cut $(0,\infty)$, is defined for all values of '$a$' such that $t'(s',a,x_0)$ belongs to the holomorphy domain of $(s'-t)(2s'+t-4)discG_k(s',t,4-s'-t)$ in the $t$-plane given by the large Lehmann-Martin ellipse $E(s')$, for all $s' \geq 4$. Let $\bar{V}(x_0)$ denote this set of values of $a$, i.e.

$$\bar{V}(x_0) = \{a | t'(s',a,x_0) \in E(s'), \quad s' \geq 4\}\,. \tag{III.22}$$

$\bar{V}(x_0)$ is of course larger than $V(x_0)$. Thus Eq. (III.21) with $s_1 = s_0$, $t_1 = 0$ define the analytic continuation of $G_k(s,t,u)$ to all points $(s,t,u)$ such that

$$a = -\frac{stu}{4(st+tu+us+16x_0)} \equiv a(s,t,u,x_0) \in \bar{V}(x_0). \tag{III.23}$$

We thus obtain new points of axiomatic analyticity of $F^I(s,t,u)$.

Consider the application of Eq. (III.21) to the derivation of partial wave relations valid for $-28 < s < 125.31$ (M1). Since,

$$G_0(4,0,0) = \frac{1}{3}(a_0^{(0)} + 2a_0^{(2)}), \quad G_1(4,0,0) = \frac{1}{12}(2a_0^{(0)} - 5a_0^{(2)} + 9a_1^{(1)}),$$

$$G_2(4,0,0) = -\frac{1}{32}(2a_0^{(0)} - 5a_0^{(2)} - 18a_1^{(1)}), \tag{III.24}$$

Eq. (III.21) then furnishes $G_k(s,t,u)$ in terms of $a_0^{(0)}$, $a_0^{(2)}$, $a_1^{(1)}$ and the physical absorptive

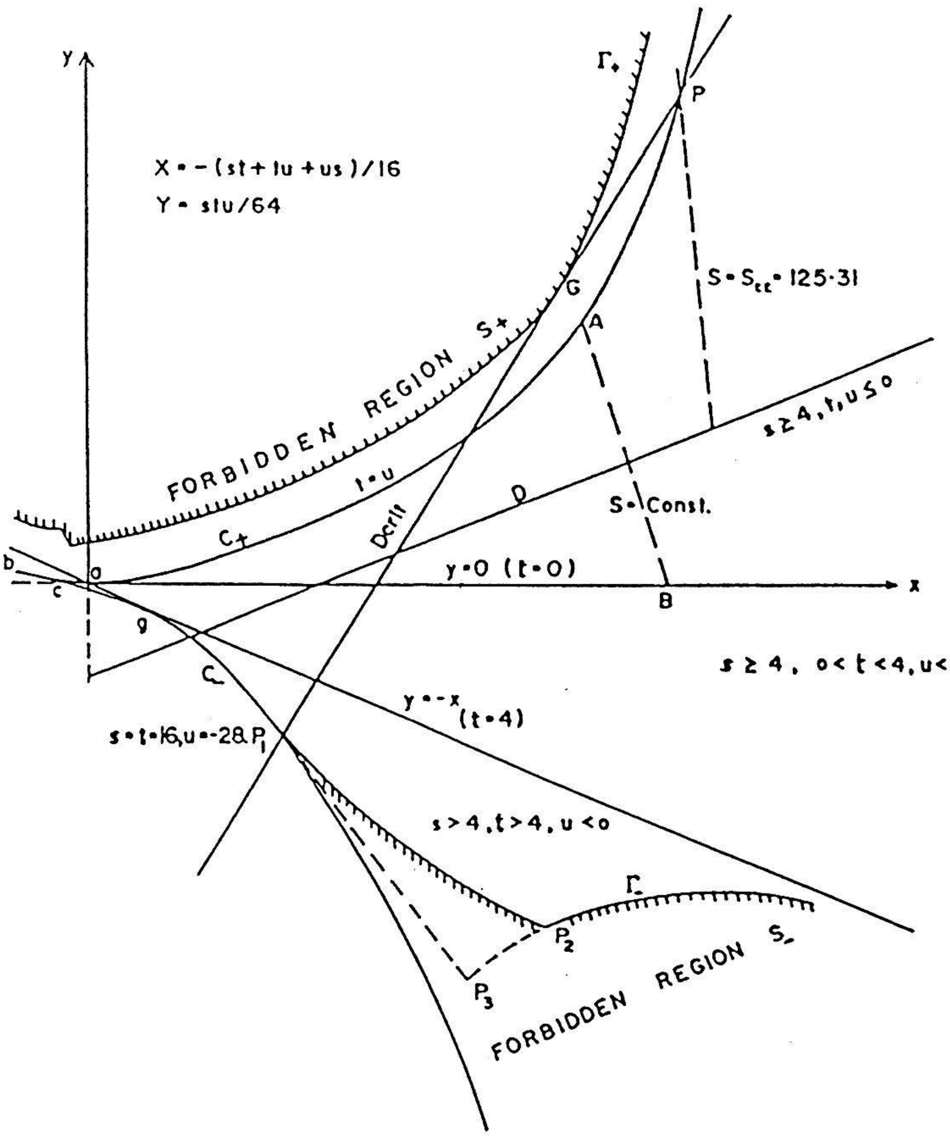


**Fig.3.** **The Wanders $x-y$ plot. The real $s,t,u$ plane is mapped to the right of the curve built up of $C_+$ and $C_-$ which constitute the image of the $t=u$ line. The triangle $0<s,t,u<4$ is mapped inside the region $abc$. The lines $y=0$ and $y=-x$ are images of $t=0$ and $t=4$ respectively. The forbidden regions $S_+$ and $S_-$ are bounded by the curves $\Gamma_+$ and $\Gamma_-$ constituting the images of the end-points of the minor-axis and the major-axis respectively of the large Lehmann-Martin ellipse $E(s)$. Partial wave-equations can be written for a given physical $s$, if through every point of the part of the straight line $AB$ with $t,u\leq 0$ and $s=$ const., a straight line $D$ not intersecting $S_+$ and $S_-$ can be drawn. The maximum such value of $s$ ($s_{max}=125.31$) is obtained when for the point $P$ with $t=u$, the straight line $D_{crit.}$ through $P$ touches $S_+$ at $G$ and $S_-$ at $P_1$.**

parts provided $a(s,t,u,x_0) \in \bar{V}(x_0)$. Next (III.24) is substituted into the unsubtracted relation (III.21) for $G_2(4,0,0)$ to obtain a sum-rule (the Wanders-sum rule (W2)) expressing the $P$-wave scattering length $a_1^{(1)}$ in terms of $(2a_0^{(0)} - 5a_0^{(2)})$ and the physical absorptive parts. Finally, we are able to express $G_k(s,t,u)$ for $a(s,t,u,x_0) \in \bar{V}(x_0)$ in terms of $a_0^{(0)}$, $a_0^{(2)}$ and the physical absorptive parts. Their partial wave projections constitute the partial wave equations; the condition $a(s,t,u,x_0) \in \bar{V}(x_0)$ leads to the region of validity $-28 < s < 125.31$. Fig. 3 illustrates the method of calculating $s_{max} = 125.31$ graphically.

For a given $x_0$, a representation for $f_\ell^I(s)$ valid in an interval $s_1(x_0) \le s \le s_2(x_0)$ is obtained; two values of $x_0$ are enough to cover the region (4,125.31). We have

$$s_1(0) = 4, \quad s_2(0) = 90.20, \quad s_1(50.41) = 39.78, \quad s_2(50.41) = 125.31. \tag{III.25}$$

### III.3. Extension of axiomatic analyticity domain of the pion-pion amplitudes.

An important application of the crossing symmetric dispersion relations (III.21) not yet fully investigated is that they yield new points of axiomatic analyticity of the scattering amplitude, in particular of the fixed angle amplitudes $F^I(s, \cos\theta)$ and the partial wave amplitudes. Martin (M10) showed that the analyticity domain in $s$ of $F^I(s, \cos\theta = 0)$ plays a crucial role in determining the analyticity domain of $f_\ell^I(s)$. Roy and Wanders (R7) have obtained an extension of Martin's analyticity domain of $F^I(s, \cos\theta = 0)$ using the crossing symmetric dispersion relations (III.21). The new domain obtained and the previous Martin-domain are shown in Fig. 4, for $Re\ s > 75$ and $Im\ s > 0$. If $s$ is a point inside the analyticity domain of $F^I(s, \cos\theta = 0)$ so is $s^*$; hence the part $Im\ s < 0$ is not shown on the figure. A cut along the real $s$-axis is understood. The new domain extends upto $Re\ s = 125.31$ where $(Im\ s)_{max} = 0$; at $Re\ s = 120$ where the Martin domain has $(Im\ s)_{max} \sim 0$, the new domain has $(Im\ s)_{max} = 22$; at $Re\ s = 78$, $(Im\ s)_{max} \approx 63$ for both the domains; for $Re\ s < 78$ the Martin-domain is larger than that obtained from the new equations.

We indicate the method briefly. For any complex $s$, $\cos\theta = 0$, let

$$\begin{aligned} &disc\ G_k(s', t'(s', a, x_0)) \equiv disc\ G_k(s', t'(s', x_0, s)), \\ &\cos\theta'(s') \equiv 1 + \frac{2t'(s', x_0, s)}{(s'-4)}, \end{aligned} \tag{III.26}$$

where $a$ is eliminated in favour of $s$. Let $A(s')$ denote the semi-major axis of the large

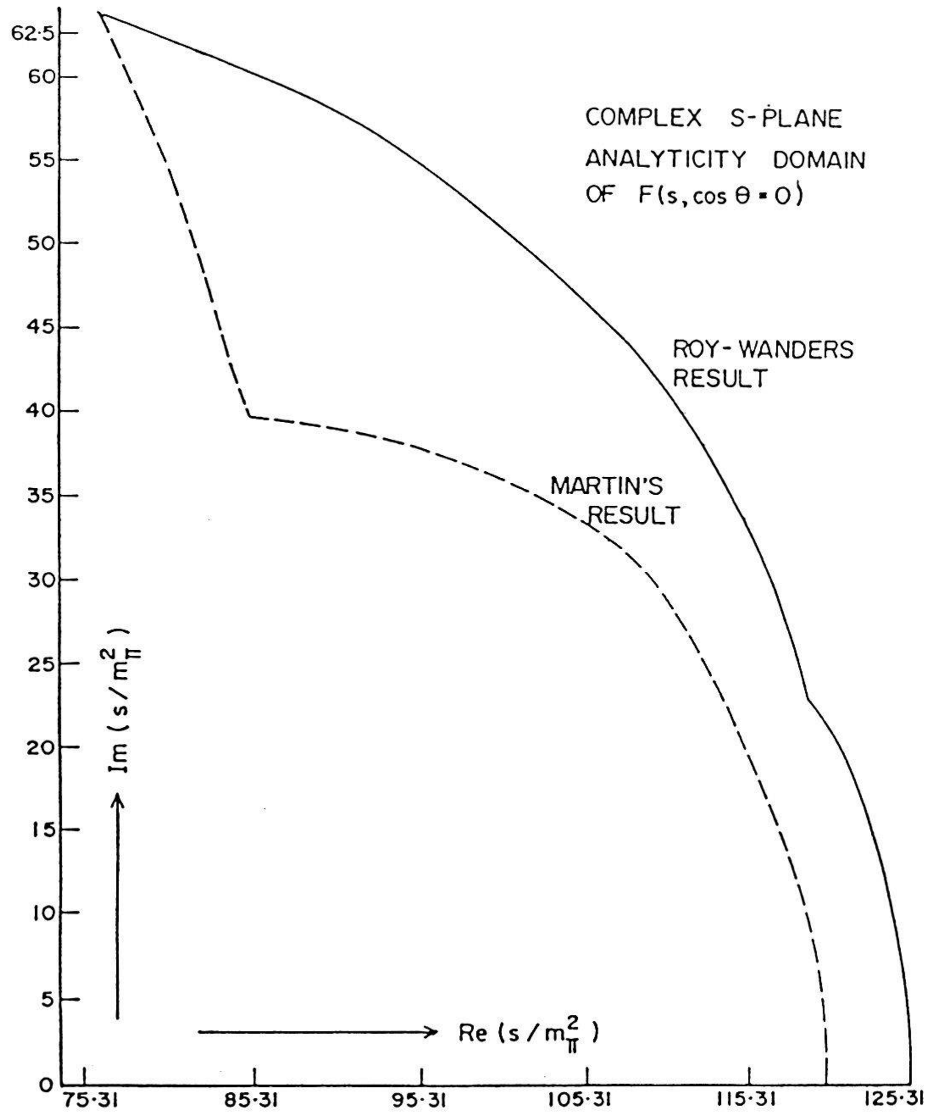


**Fig.4. The improvement of Martin's complex $s$-plane analyticity domain (M10) of $F(s, \cos\theta = 0)$ obtained by Roy and Wanders (R7) using the Mahoux-Roy-Wanders dispersion relation (M1) is shown for $Re\ s > 75$ and $Im\ s > 0$. A cut along the real $s$-axis is understood. A reflection of the boundary curve across the real $s$-axis yields the analyticity points for $Im\ s < 0$.**

Lehmann-Martin ellipse $E(s')$ in the $\cos\theta'$-plane. Let

$$X(s') \equiv \left[\frac{Re\ \cos^2\theta'(s') - \frac{1}{2}}{A^2(s') - \frac{1}{2}}\right]^2 + \left[\frac{Im\ \cos^2\theta'(s')}{A(s')\sqrt{A^2(s') - 1}}\right]^2 - 1. \qquad (III.27)$$

Then, $\cos\theta'(s') \in E(s')$ if and only if $X(s') < 0$. For a given complex $s$, if there exists any $x_0$ for which $X(s') < 0$ for all $s' \geq 4$, then $F^I(s, \cos\theta = 0)$ is analytic in $s$ at this point. The domain in Fig. 4 is obtained by searching over all $x_0$; we note that $x_0 \sim 50$ for the part displayed in Fig. 4.

### III.4. Phenomenological applications

The phenomenological understanding of pion-pion scattering became particularly interesting after the discovery of the $K\bar{K}$-threshold effect, the resolution of the up-down ambiguity in the $I = 0$ $s$-wave in the energy region above 750 MeV in favour of the 'down' branch, (see the review of Morgan (M12)), and the appearance of phase-shift analysis upto 1400 MeV (E1,H2). The physical region partial wave equations based on fixed-$t$ dispersion relations (R6) are a convenient tool to impose unitarity, analyticity and crossing on the phase-shift data. For example, if the absorptive parts for $s' \geq 60$ are considered given, the $Im\ f_l^I(s')$ for $4 < s' < 60$ from a phase-shift analysis can be inserted into the partial wave equations to calculate $Re\ f_l^I(s')$ which can then be tested against the input phase-shifts; or with more courage one can attempt a solution of the integral equation. In practice, there are many problems such as the non-existence of phase-shift data at very low energies, the difficulty of obtaining the high energy absorptive parts, the estimation of the error involved in taking only a few low partial waves into account, and the difficulty of ensuring the validity of the subsidiary crossing conditions on the absorptive part.

The phenomenological use of the partial wave equations to the testing of phase-shift data and the determination of the low energy parameters is one of the most impressive applications of axiomatic analyticity properties. It was initiated by Basdevant, Le Guillou and Navelet (B5) and carried out to perfection by other authors (B6, B10, B11, P3). For example, Basdevant, Frogatt and Petersen (B6) choose the $Im\ f_l^I(s)$ to be the imaginary part of an amplitude $g_l^I(s)$ parametrized to satisfy unitarity with the experimentally observed inelasticity, and to reproduce the experimental mass and width of the $\rho$-meson;

considering the absorptive parts for $s > 60$ to be given by phenomenological considerations (such as Regge behaviour), they vary the parameters in $g_\ell(s)$ to obtain a numerical fit to the partial wave equations. Thus they obtained $S$- and $P$-wave amplitudes fitting the partial wave equations and the data, for a range of scattering lengths given in Fig. 5, the range includes in particular Weinberg's current algebra values (W4). The $P$-wave scattering length so deduced (B6) ($0.040 \pm .004$) is however in strong disagreement with the CERN-MUNICH experimental result ($\approx 0.1$) (M11). We refer to (B6) and (S2) for different views on this problem. The $S$- and $P$-wave scattering lengths from the physical region eqns. are in good agreement with those deduced from QCD ideas by Gasser and Leutwyler (G5).

It is hoped that similar use of the crossing symmetric equations of Mahaox, Roy and Wanders (M1) valid upto $s = 125.31$ would yield further interesting information, particularly for the higher partial waves. For the special case of forward scattering, dispersion relations for $G_k$ and positivity yield rigorous sum rules and bounds for scattering lengths and amplitudes (R8).

## IV. Absolute bounds on the pion-pion amplitudes

Martin discovered the remarkable result (M3) that the pion-pion amplitudes obey absolute bounds involving the pion-mass alone. Improving these first results Lukaszuk and Martin (L2) obtained,

$$-50 \leq F^{\pi^0\pi^0\to\pi^0\pi^0}\left(\frac{4}{3},\frac{4}{3},\frac{4}{3}\right) \leq 8, \qquad (IV.1)$$

The existence of bounds on the coupling constant is explained by Martin by the argument that if the coupling were too strong it would produce bound states of the pion-pion system. Martin (M3), and Lukaszuk and Martin (L2) also derived from these absolute bounds upper limits on integrals of total cross section in terms of pion-mass alone. These bounds were considerably improved by Common (C4) and Yndurain (Y3). Another important result extracted by Martin from the absolute bounds is a lower bound on the $\pi^0\pi^0 \to \pi^0\pi^0$ $S$-wave scattering length. Bonnier and Vinh Mau (B7) in an ingenious application of these

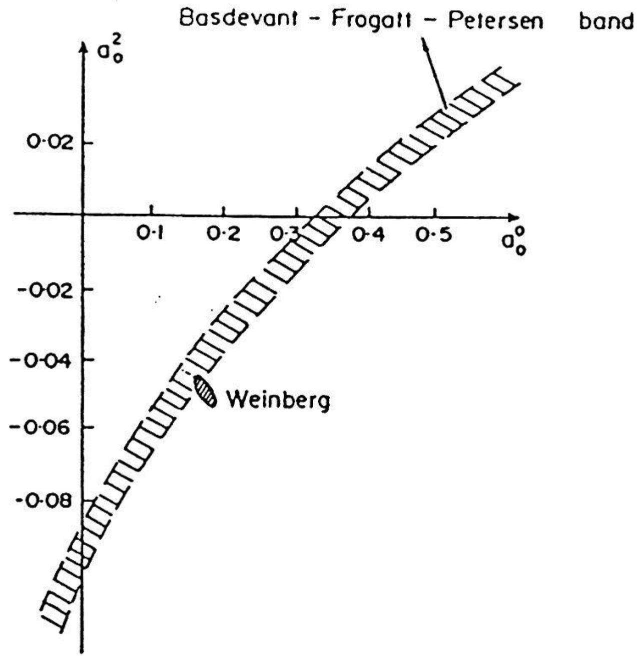


**Fig.5. The band of values of the $S$-wave scattering lengths $a_0^0$, $a_0^2$ for which Basdevant, Frogatt and Petersen (B6) have obtained fits to the physical region partial wave equations (R6) and to the data upto 1100 MeV.**

Table 1

Absolute bounds on the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitudes given by Lopez and Mennessier (L4) improving the original results of Martin (M3,L2).

| $\pi^0\pi^0 \to \pi^0\pi^0$ Amplitude | Absolute Lower Bound (units $m_\pi^{-1}$) | Absolute Upper Bound (Units $m_\pi^{-1}$) |
|---|---|---|
| $F(4,0,0) = a_0^{\pi^0\pi^0 \to \pi^0\pi^0}$ | -1.75 | NONE |
| $F(3,0,1)$ | -3.30 | 3.20 |
| $F(2,0,2)$ | -3.50 | 2.85 |
| $F(\frac{4}{3},0,\frac{8}{3})$ | -3.40 | 3.00 |
| $F(2,1,1)$ | -7.25 | 2.75 |
| $F(\frac{4}{3},\frac{4}{3},\frac{4}{3})$ | -8.10 | 2.70 |
| $F(3,2,-1)$ | -1.30 | 14.5 |

ideas, considerably improved Martin's original result and obtined,

$$a_0^{\pi^0\pi^0\to\pi^0\pi^0} > -3m_\pi^{-1}. \qquad (IV.2)$$

For the other iso-spin combinations Common (C2) obtains,

$$a_0^0 > -7.5, \quad a_0^2 > -7.1 \qquad (IV.3)$$

One also obtains bounds on energy averages of physical real parts e.g. (G6)

$$-7.9 \leq \frac{1}{2}\int_4^6 ds\ Re\ F^{\pi^0\pi^0\to\pi^0\pi^0}(s,0) \leq 9.6 \qquad (IV.4)$$

After much painstaking effort [A8,L3,L4,B7] the present bounds on the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitudes given by Lopez and Mennessier [L4] are as in Table 1. The existence of these bounds is a testimony of the power of axiomatic analyticity and unitarity.

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