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The Geometric Schwinger Model on the Torus I [1]

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1 INTRODUCTION

The Schwinger Model [1] has a long history of illustrating various phenomena appearing in quantum field theory. Breakdown of global chiral symmetry by an axial anomaly [2], charge screening [3], vacuum structure and the realization of gauge transformations [4] are examples. In this note we analyze the Euclidean version of the geometric Schwinger Model on the torus.

1.1 The action of the **geometric Schwinger Model** (gSM) on the 2-dimensional torus $\mathcal{T}$ is

$$S = \frac{1}{2}\int_{\mathcal{T}}(F,F)_0 + \int_{\mathcal{T}}(\bar{\Phi},(d-\delta)_A\Phi)_0. \tag{1}$$

It leads to the Dirac-Kähler equation (DKE):

$$(d-\delta)_A\Phi = dx^\mu \vee (\partial_\mu - ieA_\mu)\Phi = (d-\delta)\Phi - ieA\vee\Phi = 0 \tag{2}$$

In this paper we use the well-known calculus of differential forms extended by a Clifford product as introduced by E.Kähler [5]. The relation of the Clifford product to the wedge product is described by: $dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}$, $g^{\mu\nu}$ is a metric tensor. We introduce on $\mathcal{T}$ the Euclidean standard metric $g^{\mu\nu} = \delta^{\mu\nu}$, and give the large and small circumference

[1]dedicated to my friend Gerard Wanders on the occasion of his 60 th birthday.

the length L_1 and L_2. $\Phi, \bar{\Phi}$ are inhomogeneous complex forms:
$\Phi = \phi(x,\emptyset) + \phi(x,\mu)dx^\mu + \phi(x,12)dx^{12} \equiv \sum_H \phi(x,H)dx^H$, etc. These Dirac Kähler forms (DK- forms) describe the fermions. For this reason we call the model 'geometric' [6]. $(\bar{\Phi},\Phi)_0$ denotes the 2-form of the scalar product: $(\bar{\Phi},\Phi)_0 = (\bar{\Phi},\Phi)dx^{12} = \sum_H \bar{\phi}(x,H)\phi(x,H)\cdot dx^{12}$. Exterior derivative and coderivative are denoted by d, δ. $F = dA$ is the 2-form of the field strength of the 2-dimensional abelian gauge potential $A = A_\mu(x)dx^\mu$.

The geometry of gauge fields on a manifold like a torus requires certain continuum conditions for the gauge fields. We consider A as a connection 1-form in an $U(1)$-bundle with base space $\mathcal{T}$. Its topological structure might be described by the periodicity conditions ($\hat{1} = (L_1, 0)$), etc.) :

$$A(x+\hat{\nu}) = A(x) - \frac{i}{e}\Lambda_\nu^{-1}(x)d\Lambda_\nu(x), \quad \Phi(x+\hat{\nu}) = \Lambda_\nu(x)\Phi(x), \tag{3}$$

where the transition functions $\Lambda_\nu(x) = \exp(iea_\nu(x))$ satisfy the cocycle condition $\Lambda_1(x)\Lambda_2(x+\hat{1}) = \Lambda_2(x)\Lambda_1(x+\hat{2})$. It is well-known [7], that under these requirements the set of gauge fields is distributed into classes $\mathcal{CH}^{(k)}$ ('topological sectors') which are characterized by the Chern index $k = \frac{e}{2\pi}\int F, \quad k = 0, \pm1, \pm2, \ldots$.

The calculation of the quantum mechanical vacuum expectation values (VEV) of observables $\Omega(\bar{\Phi},\Phi,A)$ is performed with help of the path integral formula

$$\langle\Omega\rangle = \frac{1}{Z}\int \mathcal{D}[A]\mathcal{D}[\Phi,\overline{\Phi}]\Omega[\mathbf{A},\overline{\Phi},\Phi]e^{-S(\mathbf{A},\overline{\Phi},\Phi)}. \tag{4}$$

This expression is purely formal:

-it needs regularization,

- for the gauge field integration we have to give meaning to the 'measure' $\mathcal{D}[A]$ ('gauge fixing'), and describe the space of gauge field configurations ('topological sectors'),

- the 'fermion integration' must take care of the appearence of 'zero modes' related to the topologically non-trivial gauge field configurations. Thus we get after fermion integration [8]

$$\langle\bar{\Phi}(x_1)\Phi(y_1)\ldots\bar{\Phi}(x_n)\Phi(y_n)\rangle =$$
$$\frac{1}{Z}\sum_k\int \mathcal{D}^{(k)}[A]e^{-S_F[A]}\det{}'[D_A]\times\sum_{perm}\left\{\begin{array}{c}(\pm1)\chi_1(x_{i_1})\ldots\chi_l(x_{i_l})\bar{\chi}_1(y_{j_1})\ldots\bar{\chi}_l(y_{j_l})\\ \mathcal{G}^{(k)}(x_{i_{l+1}},y_{j_{l+1}};A)\ldots\mathcal{G}^{(k)}(x_{i_n},y_{j_n};A).\end{array}\right\} \tag{5}$$

$\sum_k$ denotes summation over the different topological sectors.

$\chi_i(x)$, $\bar{\chi}_j(y)$, $i,j = 1,\ldots,l$ is an orthogonal set of zero mode wave functions. We have $l = 2k$. (Consider the 'Index Theorem' for DK fermions [7]).

$\det'[D_A]$, and $\mathcal{G}^{(k)}(x,y;A)$ denote the determinant and the Green's function of the DK operator $D_A = (d-\delta)_A$ with the zero modes omitted.

In the following we want to treat all these problems with appropriate exactness.

1.2. After this short description of the scope of our investigation, we want to add some remarks on our **motivations**. The discussions on the Schwinger model are so numerous, that we can not refer to them all. Our treatment is special in two aspects: We consider the geometric version, and we treat the gSM on a torus with Euclidean metric. Both choices originate in the interest in a systematic analysis of the lattice approximation of the model in the future.

There is a systematic lattice approximation for DK-forms provided geometrically by DeRham mapping [9]. This procedure avoids additional spectrum doubling. Therefore a comparison of the lattice approximation of geometric models with the continuum theory is not hindered by this lattice phenomen. DK-forms can be expressed by Dirac fields $\phi_a^{(b)}(x)$.

$$(\phi_a^b(x)) = \sum_H \phi(x,H)(\gamma^H)_a^b, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^{12} = i\gamma^5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad (6)$$

$a = 1,2$ Dirac index, $b = 1,2$ isospin index : It follows from this description of DK forms by Dirac components that the gSM contains formally an isospin doublet of fermions. Such kind of models were treated before [10].

Most of the mathematical problems of quantum field theory mentioned above can be treated more exactly in a compactified version of Euclidean space time. Under these circumstances the spectrum of $(d-\delta)_A$ becomes discrete. This allows e.g. a precise calculation of the regularized determinant of $(d-\delta)_A$ appearing as a result of the fermion integration. Furthermore, compactification leads to a precise definition of topological sectors together with its related zero modes. A particular transparent treatment of the Schwinger model on a compact space was given recently by C.Jayewardena: 'Schwinger Model on S^2' [11]. Our investigations were inspired by this paper.

However, compactification on $\mathcal{T}$ is much better suited for lattice approximation than compactification on the sphere. It is a torus which is most naturally approximated by the finite cubical lattices on which the numerical calculations are preformed [12]. The symmetry groups of the gSM on the torus and on the finite lattice are closely related [13]. Furthermore, there is a first study of the lattice approximation of the topological zero modes for the Schwinger model [14].

Here we do not treat the chiral Schwinger model which recently did rise so must interest [15].The study of a geometric version of this model along the lines of this note will be of great interest for the important problem of the lattice approximation of chiral gauge theories like the standard model [17] [16].

1.3. We add some short remarks on the **organization** of the two papers: 'The gMS on the Torus' I + II. As seen from Eq. (5) the zero mode wave functions associated with the different toplogical sectors play an important role in the calculation of VEV of fermions. Section 2. in Part I is devoted to the calculation of these wave functions. It turns out that for constant field strength F these wave functions can be expressed by θ-functions defined on $\mathcal{T}$. One may say without exaggeration that our problem is a physical illustration of the algebraic theory of θ-functions [18] [19]. In Section 3. Part II, we determine the regularized effective action Γ_{reg}, $\exp \frac{1}{2}\Gamma \sim \det' D_A$ by standard methods [20], and we discuss the propagator related to it. Here the theory of θ-functions appears in the form of 'Kronecker's Double Series' as discussed in the book by A.Weil [21]. Because of limited space, we have to use without further comments the formulas on θ-functions as given in the different references. In the final Section 4. we study applications to the standard questions like the particle spectrum, the screening of the static potential, and the appearence of the anomaly. The treatment of the SM on the torus allows a systematic study of finite size effects in the limit of a large torus. In this limit we only study the Euclidean version of the geometric model. It would be particularly interesting to analyze the relation to the geometric model in Minkowsky space, e.g. the effective action calculated along the line of G.Wanders [22].

2 SPECTRUM AND EIGENFUNCTIONS OF THE DK OPERATOR

The solubility of the Schwinger model follows essentially from the fact that for gauge potentials of the form $A_\mu(x) = -\epsilon_\mu^\nu \partial_\nu b(x)$, the spinor $\psi(x) = \exp(-e\gamma_5 b(x))\psi_0(x)$ is a solution of the 2-dimensional Dirac equation $\gamma^\mu(\partial_\mu - ieA_\mu(x))\psi(x) = 0$ iff $\psi_0(x)$ is a solution of the free equation. Our first task is now to find the solutions of the massless DKE for the gauge fields in the different topological sectors which satisfy the appropriate boundary conditions on the torus.

2.1. The U(1)-gauge fields on $\mathcal{T}$ decompose in **Chern classes**. We may choose as a representative of each class a gauge potential $C^{(k)}$ with constant field strength:

$$C^{(k)} = \frac{B}{2}(x^1dx^2 - x^2dx^1), \quad F^{(k)} = dC^{(k)} = Bdx^1 \wedge dx^2$$

$$k = \frac{e}{2\pi}\int F^{(k)}, \quad i.e.\ B = \frac{2\pi k}{eL_1L_2}. \tag{7}$$

The transition functions $\Lambda_\nu(x)$, Eq. (3), are in this case the gauge transformations:

$$\Lambda_1(x) = \exp(\frac{i}{2}eBL_1x^2), \quad \Lambda_2(x) = \exp(-\frac{i}{2}eBL_2x^1) \tag{8}$$

These describe the continuation of $C^{(k)}$ in the non-trivial principle U(1) bundles along a cycle in $\mathcal{T}$.

A general gauge potential of a given Chern class has the form

$$A^{(k)} = A + C^{(k)} = da + t - \delta b + C^{(k)} \equiv (d-\delta)\alpha + t + C^{(k)}. \tag{9}$$

Here A is single valued 'continuous' on $\mathcal{T}$. Thus we may apply the Hodge decomposition theorem [7], and represent A by a 'pure gauge': $da = \partial_\mu a(x)dx^\mu$, a 'toron field': $t = t_\mu dx^\mu$, t_μ constant, restricted to $0 \le t_\mu < 2\pi/eL_\mu$. and a coderivative of a 2-form: $b = b(x)dx^1 \wedge dx^2$, $\delta b = -\epsilon_\mu^\nu \partial_\nu b(x)dx^\mu$.

2.2. There is a **local solution of the DKE**, Eq. (2), with this external potential:

$$\Phi = e^{iet_\mu x^\mu} e_\vee^{ie\alpha} \vee \Phi_0. \tag{10}$$

Here $e_\vee^\psi$ denotes the formal Clifford power series $e_\vee^\psi = 1 + \psi + \frac{1}{2!}\psi \vee \psi + \frac{1}{3!}\psi \vee \psi \vee \psi + \ldots$, Φ_0 is a solution of the free DKE. The statement Eq. (10) follows immediately from the product differentiation formula:

$$(d-\delta)[(\chi(x,\emptyset)+\chi(x,12)dx^{12})\vee\Phi]$$

$$=[(d-\delta)(\chi(x,\emptyset)+\chi(x,12)dx^{12})]\vee\Phi+[(\chi(x,\emptyset)-\chi(x,12)dx^{12})]\vee(d-\delta)\Phi$$

Of course, the Dirac components of Φ are the solutions of the ordinary Dirac equation in which we mentioned at the beginning of this Section. Further we shall see that the expression Eq. (10) provides also a guideline for the calculation of global solutions on the torus which describe the zero modes of the DK operator.

2.3 First we calculate **the spectrum and the eigenfunctions** of the DK operator with a pure toron field as representative **of the trivial gauge sector**. The eigenvalue equation of the anti-Hermitean Euclidean DK operator with toron field

$$dx^{\mu}\vee(\partial_{\mu}-iet_{\mu})\Phi=E\Phi. \tag{11}$$

is invariant under Clifford right multiplication by a constant form ('flavour transformation'[9]). As a special case, right multiplication by dx^{12} induces Hodge duality: $\star\Phi=\Phi\vee dx^{12}$ with a certain phase convention: $\star dx^{\mu}=\epsilon^{\mu}_{\nu}dx^{\nu}$, $\star dx^{12}=-1$, $\star\star=-1$. The invariance of the DKE under this duality allows a separation of the eigenfunctions into dual: $\star{}^{d}\Phi=i{}^{d}\Phi$ and anti-dual $\star{}^{a}\Phi=-i{}^{a}\Phi$ forms. With such an ansatz it is a straightforward calculation of the standard type to find the eigenfunctions:

$$\begin{aligned}{}^{d}\Phi_{\pm} &= \frac{1}{\sqrt{4|\bar{n}_-|L_1L_2}}\{\sqrt{\bar{n}_-}(dx^1-idx^2)\pm\sqrt{\bar{n}_+}(1-idx^{12})\}e^{2\pi i(n_1\frac{x_1}{L_1}+n_2\frac{x_2}{L_2})}\\ {}^{a}\Phi_{\pm} &= \frac{1}{\sqrt{4|\bar{n}_-|L_1L_2}}\{\sqrt{\bar{n}_+}(dx^1+idx^2)\pm\sqrt{\bar{n}_-}(1+idx^{12})\}e^{2\pi i(n_1\frac{x_1}{L_1}+n_2\frac{x_2}{L_2})}\end{aligned} \tag{12}$$

The eigenfunctions are normalized according to $\int(\bar{\Phi}\Phi)_0=1$. The square root must be taken in the complex plane cut along the negative axis such that $(\sqrt{a})^*=\sqrt{a^*})$. These belong to the eigenvalues with multiplicity 2:

$$E_n=\pm i\sqrt{\bar{n}_+\bar{n}_-},\quad \bar{n}_{\pm}=2\pi(\frac{n_1}{L_1}\pm i\frac{n_2}{L_2})-e(t_1\pm it_2),\quad n_i=0,\pm1,\pm2,\dots. \tag{13}$$

In the trivial sector the transition functions $\Lambda_{\nu}(x)=1$ lead to simple periodic boundary condtions, Eq. (3). This results in the usual discretization of the momenta $p_i=((2\pi)/(L_i)\cdot n_i$. It is important to remark that there are no zero eigenvalues for non-exceptional toron fields: $t_\mu \neq (2\pi)/(eL_\mu)\cdot n'_\mu$, n'_μ is an integer. Thus there is no necessity to avoid such states by partly anti-periodic boundary conditions. These have the same effect on the eigenvalues as toron fields with half integer n'_μ.

2.4. Next we consider the **solutions** of the DKE, Eq. (2), **with the gauge field** $C^{(k)} + t$, the representatives **of the non-trivial gauge sectors**. The massless DKE is invariant under Hodge duality, and does not mix even and odd forms. Therefore we can make an ansatz of the type Eq. (10):

$$\Phi = e^{iet_\mu x^\mu} e^{\mp \frac{k\pi}{2}\frac{x^2}{L_1 L_2}} F(x)\omega \tag{14}$$

with $\omega = dx^1 - idx^2,\ 1 - idx^{12};\ dx^1 + idx^2,\ 1 + idx^{12}$ denoting odd and even, dual and anti-dual forms. Since $C^{(k)}(x) = -\frac{B}{4}\delta x^2$, *i.e.* $\exp(ie\alpha(x))\Phi_0 = \exp(\mp\pi k/2)(x^2/L_1L_2)\Phi_0$, and it follows that Φ is a solution of the DKE if $\Phi_0 = F(x)\omega$ is a free solution. This means

$$\begin{aligned} (I)\ (\partial_1 + i\partial_2)F(x) &= 0 \text{ for } \omega = dx^1 + idx^2,\ 1 - idx^{12}, \\ (II)\ (\partial_1 - i\partial_2)F(x) &= 0 \text{ for } \omega = dx^1 - idx^2,\ 1 + idx^{12}. \end{aligned} \tag{15}$$

The ω of 'type I' are odd anti-dual and even dual, those of 'type II' are odd dual and even anti-dual. The signs in Eq. (14) refer to the cases (I) and (II). These equations have the form of the Cauchy Riemann differential equations. Therefore it is natural to introduce complex coordinates:

$$z = \frac{1}{L_1}(x^1 + ix^2),\ \ \bar z = \frac{1}{L_1}(x^1 - ix^2),\ \ \frac{\partial}{\partial z} = \frac{L_1}{2}(\partial_1 - i\partial_2)\ \ \frac{\partial}{\partial \bar z} = \frac{L_1}{2}(\partial_1 + i\partial_2) \tag{16}$$

$F(x)$ is an analytic function of z in case (I), and conjugate analytic in case (II).

The main problem is now to determine $F(x)$ in such a way that Φ satisfies the periodicity conditions Eqs. (3), (8). A short calculation shows that this means for the two cases:

$$\begin{aligned} (I)\ F(z+1) &= e^{\frac{\pi k z}{|\tau|} + \frac{\pi k}{2|\tau|} - ieL_1t_1} F(z) \quad F(z+\tau) = e^{-i\pi k z + \frac{\pi}{2}k|\tau| - ieL_1|\tau|t_2} F(z) \\ (II)\ F(\bar z+1) &= e^{\frac{-\pi k \bar z}{|\tau|} - \frac{\pi k}{2|\tau|} - ieL_1t_1} F(\bar z) \quad F(\bar z+\bar\tau) = e^{-i\pi k \bar z - \frac{\pi}{2}k|\tau| - ieL_1|\tau|t_2} F(\bar z) \end{aligned} \tag{17}$$

$\tau = iL_2/L_1$.

Analytic functions which are double periodic up to an exponentional factor can be expressed by θ-functions [18] [19]:

$$\theta_{ab}(z|\tau) = \sum_{n=-\infty}^{+\infty} \exp[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)] \tag{18}$$

($a, b = 0, \frac{1}{2}$; in the notation of [23]: $\theta_1 = -\theta_{\frac{1}{2}\frac{1}{2}}$ $\theta_2 = \theta_{\frac{1}{2}0}$, $\theta_3 = \theta_{00}$, $\theta_4 = \theta_{0\frac{1}{2}}$.)

These functions are analytic and satisfy the 'periodicity conditions'

$$\theta_{a,b}(z + m\tau + n|\tau) = \exp[-2\pi i(mz + \frac{m^2\tau}{2} + bm - an)]\theta_{ab}(z|\tau) \tag{19}$$

They are related by

$$\theta_{ab}(z|\tau) = \exp[2\pi i(a^2\tau/2 + a(z+b)]\theta_{00}(z+b+a\tau|\tau) \tag{20}$$

2.5. We first construct by this procedure the **solutions for** $k = 1$ without torons: $t = 0$. As a special case of Eq. (19 we have

$$\theta_3(z+1|\tau) = \theta_3(z|\tau), \quad \theta_3(z+\tau|\tau) = e^{-i\pi(2z+\tau)}\theta_3(z|\tau). \tag{21}$$

Then one sees that $F(z) = \exp \pi z^2/2|\tau| \cdot \theta_3(z)$ satisfies the conditions Eq. (17) for $k = 1$. Thus we get the two orthonormal solutions of the DKE

$$\Phi = \frac{|2\tau|^{-1/4}}{L_1} e^{\frac{\pi}{2|\tau|}(z^2 - |z|^2)}\theta_3(z)\omega \quad \omega = dx^1 + idx^2,\ 1 - idx^{12}. \tag{22}$$

We have normalized this solution as $\int(\Phi, \Phi)_0 = 1$. In Fig.1. we show phase and absolute value of $e^{\frac{\pi}{2}(z^2-|z|^2)}\theta_3(z)$ for $L_1 = L_2 = 1$ The characteristic feature is the zero together with the singularity of the section in the associated $U(1)$-bundle of the phase. This feature follows from the extended Hopf index theorem [24].

2.6. The set of 'periodicity conditions' Eq. (17) describes an interesting algebraic structure which allows to find **the general solutions**. We put $t = 0$, consider $k \geq 1$, and transform Eq. (17) by $F(z) = \exp(\pi k z^2/2|\tau|)T_k(z)$ into :

$$T_k(z+1) = T_k(z), \quad T_k(z+\tau) = e^{-ik\pi(2z+\tau)}T_k(z). \tag{23}$$

One sees immediately that if $T_k(z)$, $T_{k'}(z)$ satisfy Eq. (23), then $\tilde{T}_{k+k'}(z) = T_k(z)T_{k'}(z)$ is a solution for $k + k'$. Since the conditions (23) are linear, one may say that these periodicity conditions define a k-graded ring of analytic functions over $\mathbf{C}$. It is a simple consequence of Eq. (19) that

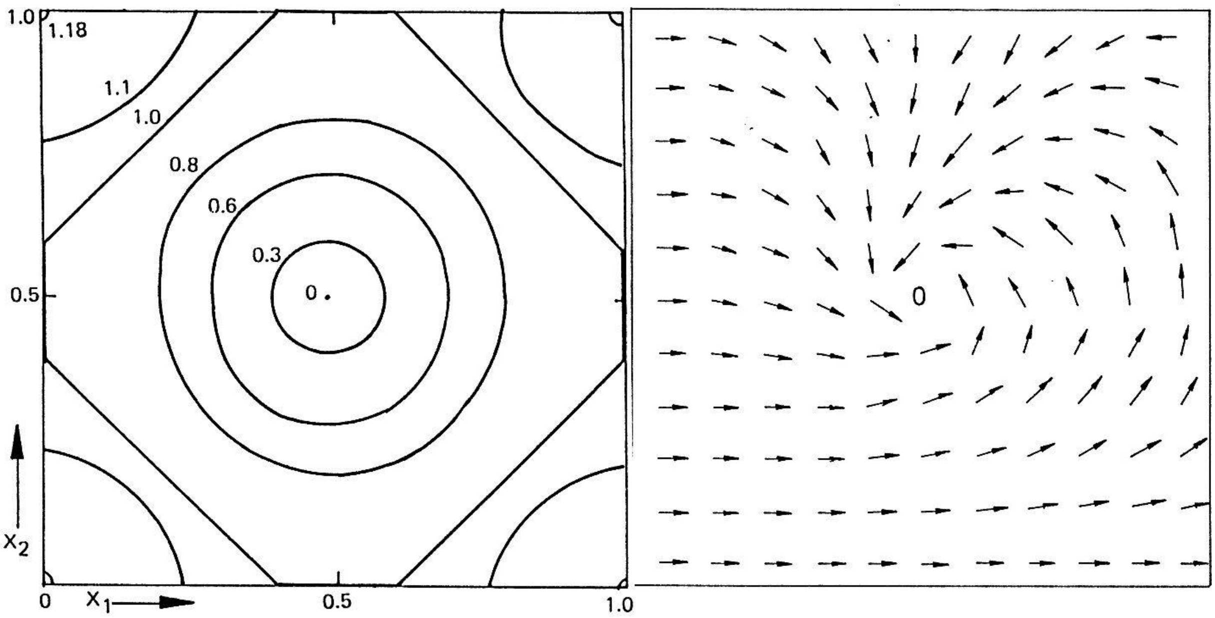

Figure 1: Phase and absolute value of the zero mode wave function $F(z)$.

$$(\theta_1(z|\tau))^{2n}(\theta_3(z|\tau))^{2p-2n},\ (\theta_1(z|\tau))^{2m+1}(\theta_3(z|\tau))^{2p-2m-3}(\theta_2(z|\tau))(\theta_4(z|\tau)),$$

$$n = 0,\ldots,p,\ m = 0,\ldots,p-2 \text{ for } k = 2p;$$

and

$$(\theta_1(z|\tau))^{2n}(\theta_3(z|\tau))^{2p-2n+1},\ (\theta_1(z|\tau))^{2m+1}(\theta_3(z|\tau))^{2p-2m-2}(\theta_2(z|\tau))(\theta_4(z|\tau)),$$

$$n = 0,\ldots,p,\ m = 0,\ldots,p-2 \text{ for } k = 2p+1; \tag{24}$$

satisfy Eq. (23). It is a result of the theory of θ- functions [19] that these k different monomials of θ's form a linear independent base of analytic functions satisfying condition Eq. (23). Thus we have finally the complete set of $2k$ linear independent solutions of the DKE with $C^{(k)}$, $k \geq 1$ as backgrond field:

$$\Phi = e^{\frac{\pi k}{2|\tau|}(z^2-|z|^2)} H^{(k,i)}(z)\omega \quad \omega = dx^1 + idx^2,\ 1 - idx^{12}. \tag{25}$$

where $H^{(k,i}(z)$, $i = 1,\ldots,k$ form a possibly orthonormalized base of the functions spanned by the θ-monomials defined in Eq.(24).

The index theorem for the DKE states [25] that the number of solutions with ω of type I minus the number of solutions with ω of type II is equal to 2k. There are no solutions of type II. The $2k$ solutions of type I we have explicitly constructed above. It can be seen easily that one gets the solutions for $k \le -1$ by complex conjugation. This transforms solutions of type I into such of type II.

For the solutions with arbitrary $A \in \mathcal{CH}^{(k)}$ it remains to include the toron field and the periodic part $(d-\delta)\alpha$, Eq. (9) of the gauge potential. A direct calculation shows that we can include the toron field by a translation $z' = z + \frac{eL_1|\tau|}{2\pi k}(t_2 - it_1)$. The modification required by the periodic part follows from the local solution, Eq. (10). Thus we get as the main result of this Section the solutions of the DKE on the torus with general potential, Eq. (9), for $k \ge 1$:

$$\Phi = e^{ie(a(x)+ib(x)}e^{\frac{ie}{2}t_\mu x^\mu}e^{\frac{\pi k}{2|\tau|}(z'^2-\bar{z}'z')}H^{(k,i)}(z)\omega, \quad \omega = dx^1 + idx^2,\ 1 - idx^{12}. \tag{26}$$

The solutions for $k \le 1$ follow essentially from complex conjugation.

2.7. We shall complete our consideration of the DK-operator with the gauge potential $C^{(k)} + t$ by the calculation of its **spectrum**. Like in paragraph 2.3 we make the ansatz

$$\Phi = F_1(x)(dx^1 \mp idx^2) + F_2(x)(1 \mp idx^{12}) \tag{27}$$

for the solution of the equation

$$D\Phi \equiv [(d-\delta) - ie(C^{(k)} + t)\vee]\Phi = E\Phi \tag{28}$$

A short calculation shows that with help of the differential operators

$$D^+ = \sqrt{\frac{|\tau|}{\pi|k|}}\left(\frac{\partial}{\partial z} - \frac{\pi k}{2|\tau|}\bar{z}'\right), \quad D^- = -\sqrt{\frac{|\tau|}{\pi|k|}}\left(\frac{\partial}{\partial \bar{z}} + \frac{\pi k}{2|\tau|}z'\right), \tag{29}$$

Eq. (28) gets the form

$$\begin{aligned} \pm D^\pm F_2(x)(dx^1 \pm idx^2) \mp D^\mp F_1(x)(1 \pm idx^{12}) &= \\ E\sqrt{\frac{L_1L_2}{4\pi|k|}}\{F_1(x)(dx^1 \pm idx^2) + F_2(x)(1 \pm idx^{12})\} \end{aligned} \tag{30}$$

For the solution of this equation it is essential that the operators $D^\pm$ satisfy the the commutation relations:

$$[D^-, D^+] = \frac{k}{|k|} \tag{31}$$

As differential operators they are adjoint operators: $(D^+)^\dagger = D^-$. Thus we may consider for $k \geq 1$, $(k \leq 1)$ D^+, (D^-) as creation (annihilation) operator, and D^-, (D^+) as annihilation (creation) operator. The functions $\tilde{H}^{(k,i)}(x) = e^{\frac{\pi k}{2|\tau|}(z'^2 - \bar{z}'z')} H^{(k,i)}(z')$ of the $2k$ zero modes Eq. (26) describe the corresponding 'vacuum states'.

Now we solve first the eigenvalue problem for the 'iterated' DKE: $D_C D_C^\dagger \Phi = |E|^2 \Phi$. It follows from Eq. (28) that $D_C D_C^\dagger$ has the form

$$\begin{aligned} D_C D_C^\dagger F(x)\omega &= \frac{4\pi|k|}{L_1 L_2} D^+ D^- F(x)\omega \;\; \text{for } \omega \text{ of type } I \\ D_C D_C^\dagger F(x)\omega &= \frac{4\pi|k|}{L_1 L_2} D^- D^+ F(x)\omega \;\; \text{for } \omega \text{ of type } II \end{aligned} \tag{32}$$

Therefore the spectrum of $D_C D_C^\dagger$ follows from the standard calculations with creation and annihilation operators:

$$|E_n|^2 = \frac{4\pi|k|}{L_1 L_2} n, \;\; n = 0, 1, \ldots, \;\; \text{with multiplicity } 2|k| \text{ for } n = 0, \; 4|k| \text{ for } n > 0,. \tag{33}$$

with eigendifferentials for $k > 0$ of the form

$$\frac{1}{N}(D^+)^n \tilde{H}^{(k,i)}\omega, \;\; \omega \text{ of type } I, \; n \geq 0, \quad \frac{1}{N}(D^+)^{n-1} \tilde{H}^{(k,i)}\omega, \;\; \omega \text{ of type } II, \; n \geq 1. \tag{34}$$

In the case $k < 0$, the types of the ω's according to Eq. (15) have to be exchanged, and D^+ must be substituted by D^-.

Since we have $D_C^\dagger = -D_C$, the eigen differentials of D_C can be easily constructed from those of $D_C D_C^\dagger$, Eq. (34). The non-zero eigenvalues of D are

$$E = \pm i \sqrt{\frac{4\pi|k|}{L_1 L_2} n}, \;\; n = 1, 2, \ldots \text{ with multiplicity } 2|k|. \tag{35}$$

with the dual eigendifferentials for $k > 0$:

$$^d\Phi = \frac{1}{N}\{(D^+)^n \tilde{H}^{(k,i)}(x)(dx^1 - idx^2) \pm i\sqrt{n}\{(D^+)^{n-1} \tilde{H}^{(k,i)}(x)(1 - idx^{12})\}. \tag{36}$$

The anti-dual solution has the form $^a\Phi =^d \Phi \vee dx^1$. For $k \leq 1$ one has to make similar substitutions as discussed above.

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