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ANALYTICITY AND CHIRAL FERMIONS ON A RIEMANN SURFACE

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Abstract. The role of analyticity in a conformal field theory model consisting of a conjugate pair of chiral fermions on a compact Riemann surface is studied. It is shown that the physical analyticity properties coming from the *operator product expansion* and a *principle of maximal analyticity* determine rigorously all the correlation functions of the system.

1. Introduction

One of the areas to which Gérard Wanders has made important contributions is the programme of obtaining the strongest possible physical implications of the basic analyticity results of axiomatic field theory. Thus, while joining with the other contributors to this volume in offering him my most warm felicitations on his sixtieth birthday, it is a particular pleasure to contribute the present article in which I show how this philosophy can be used to analyse a model which is important in the currently fashionable string theory. In the context of string theory, it is amusing to note that in our paper on the Schwinger model [1] we had already come across what is now called an *infinite grassmannian* [2] and which plays an important role in string theory [3] as well as in soliton theory (see [4] for an exposition).

The system we consider is the *bc*-system, consisting of a pair b, c of conjugate chiral fermions with conformal spin J, $1 - J$ respectively, on a compact Riemann surface

M. The action S is

$$S \sim \int_M b\bar{\partial}c \tag{1.1}$$

where $\bar{\partial} = \partial/\partial\bar{z}$ in local coordinates. We study its correlation functions

$$C(m,n) \equiv < b(Q_1)\cdots b(Q_m)c(P_1)\cdots c(P_n) >, \tag{1.2}$$

where $Q_1, \cdots, P_n$ are arbitrary points on M. This system was studied over the complex plane $\mathbb{C}$ in a classic paper [5] and subsequently over a higher genus surface by a wide variety of techniques [6,3].

In this paper we shall determine $C(m,n)$ by postulating its physical analyticity properties following from the operator product expansion (OPE) for the bc-system over $\mathbb{C}$ obtained in [5], viz.

$$b(z)b(w) \sim O(z-w), \quad c(z)c(w) \sim O(z-w), \tag{1.3a}$$

$$b(z)c(w) \sim 1/(z-w), \tag{1.3b}$$

where the left-hand sides are assumed to be inside a correlation function $C(m,n)$. We also adopt a principle of maximal analyticity reminiscent of [7]. This extends and completes our earlier work on the spin 1/2 case [8,9].

2. An algebraic geometry formulation of the problem

From now on M denotes a compact Riemann surface of genus $g > 1$. We denote by $Pic(M)$ the *Picard group* [10] of holomorphic line bundles on M with tensor product $\otimes$ as group multiplication, the trivial line bundle as identity and the dual as inverse. $Pic^d(M)$ denotes the subset of line bundles of degree (or Chern class) $d \in \mathbb{Z}$ and K the holomorphic cotangent bundle of M.

To say that the field b has conformal spin J means it is an (operator-valued) section of K^J if J is an integer, or of $(\sqrt{K})^{2J}$, where $\sqrt{K}$ is a chosen square root bundle of K (a '*theta characteristic*'), if J is a half integer. In either case the line bundle has degree $2J(g-1)$. For generality we allow b to be a section of any $\zeta \in Pic^{2J(g-1)}(M)$. The restriction that the integrand of (1.1) be a volume form fixes c to be a section of $K \otimes \zeta^{-1}$. Thus $C(m,n)$, defined symbolically in (1.2), is a section of ζ in Q-variables and of $K \otimes \zeta^{-1}$ in P-variables.

The zero modes of the system (1.1) are holomorphic sections of ζ and $K \otimes \zeta^{-1}$. From the Riemann-Roch theorem [10] the b field contributes $N \equiv (2J-1)(g-1)$ zero modes while c has none if $J > 1$, which we suppose henceforth (the case $J = 1$ is easily handled separately). As is usual [6] the zero modes are eliminated by choosing an arbitrary set of N points $w_1, \cdots, w_N$ on M and requiring that $C(m,n)$ have a simple zero when a Q-variable takes w_i as its value and, since c is conjugate to b, a simple pole when a P-variable takes as value one of the w_i.

Taking $m+n$ copies M_i $(1 \leq i \leq m+n)$ of M, consider the product manifold $M^{m+n} \equiv \Pi_1^{m+n} M_i$. Let

$$\begin{aligned} p_i \ &: \ M^{m+n} \to M \\ (z_1, \cdots, z_i, \cdots, z_{m+n}) &\to z_i \end{aligned} \tag{2.1}$$

be the i-th projection. Then $C(m,n)$ is a section of the line bundle

$$\mathcal{F}_\zeta(m,n) \equiv p_1^\star(\zeta) \otimes \cdots \otimes p_m^\star(\zeta) \otimes p_{m+1}^\star(K \otimes \zeta^{-1}) \otimes \cdots \otimes p_{m+n}^\star(K \otimes \zeta^{-1}) \tag{2.2}$$

over M^{m+n}, where $\star$ denotes the pullback. We can now state our postulates for $C(m,n)$, which follow from the OPE (1.3), our procedure for elimination of zero modes and 'maximal analyticity':

($\mathcal{P}1$) $C(m,n)$ is a meromorphic section of $\mathcal{F}_\zeta(m,n)$.

($\mathcal{P}2$) $C(m,n)$ has a simple zero for $Q_i = Q_j$ $(1 \leq i < j \leq m)$, for $Q_i = w_k$ $(1 \leq i \leq m$, $1 \leq k \leq N)$, and for $P_i = P_j$ $(1 \leq i < j \leq n)$.

($\mathcal{P}3$) $C(m,n)$ has a simple pole for $Q_i = P_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ and for $P_i = w_k$ $(1 \leq i \leq n,\ 1 \leq k \leq N)$.

($\mathcal{P}4$) $C(m,n)$ is holomorphic apart from the poles required by ($\mathcal{P}3$).

Let pr_{ij} denote the projection

$$\begin{aligned} pr_{ij} \; : \; M^{m+n} &\to M_i \times M_j \\ (z_1, \cdots, z_i, \cdots, z_j, \cdots, z_{m+n}) &\to (z_i, z_j) \end{aligned} \tag{2.3}$$

and Δ_{ij} the diagonal of $M_i \times M_j$. The equations $z_i = z_j$ and $z_i = w_k$ define respectively the subvarieties $D_{ij} \equiv pr_{ij}^{-1}(\Delta_{ij})$ and $p_i^{-1}(w_k)$ of M^{m+n}. Recall that the *divisor* [10] of a meromorphic section is a formal combination of the subvarieties defined by the zeros and poles with integer coefficients whose magnitude gives the multiplicity and has a positive sign for zeros, negative for poles (see [11] for a pedagogic account). Thus ($\mathcal{P}2$), ($\mathcal{P}3$) define a divisor

$$D^J(m,n) = \Sigma' D_{ij} + \Sigma'' D_{ij} - \Sigma''' D_{ij} + \sum_1^m p_i^{-1}(W) - \sum_{m+1}^{m+n} p_i^{-1}(W), \tag{2.4}$$

where $\sum'$ runs over $1 \le i < j \le m$, $\sum''$ over $m+1 \le i < j \le m+n$, $\sum'''$ over $1 \le i \le m$, $m+1 \le j \le m+n$, and $W = \sum_{k=1}^{N} w_k$, $p_i^{-1}(W) = \sum_{k=1}^{N} p_i^{-1}(w_k)$. The divisor $D^J(m,n)$ defines a line bundle $\mathcal{O}(D^J(m,n))$ over M^{m+n} with a meromorphic section $S^J(m,n)$ (unique up to a multiplicative constant) whose divisor is $D^J(m,n)$. Thus our postulates can be rephrased to say that $C(m,n)/S^J(m,n)$ is a holomorphic section of

$$\mathcal{M}^J_\zeta(m,n) \equiv \mathcal{F}_\zeta(m,n) \otimes \mathcal{O}(-D^J(m,n)). \tag{2.5}$$

Thus $C(m,n)$ is given by $H^0(M^{m+n}, \mathcal{M}^J_\zeta(m,n))$ and our problem is well defined only if its dimension is 0 or 1; in the former case $C(m,n) = 0$.

3. Reduction to the spin 1/2 case

Comparing $D^J(m,n)$ with the divisor $D^{1/2}(m,n)$ that we encountered in the spin 1/2 case [8,9] we see that

$$D^J(m,n) = D^{1/2}(m,n) + \sum_1^m p_i^{-1}(W) - \sum_{m+1}^{m+n} p_i^{-1}(W). \tag{3.1}$$

Then (2.5) is easily seen to become

$$\mathcal{M}_\zeta^J(m,n) = \mathcal{M}_\alpha^{1/2}(m,n) \equiv \mathcal{F}_\alpha(m,n) \otimes \mathcal{O}(-D^{1/2}(m,n)) \tag{3.2a}$$

where

$$\alpha \equiv \zeta \otimes \mathcal{O}(-W) \in Pic^{g-1}(M). \tag{3.2b}$$

Moreover, for any generic choice of $w_1, \cdots, w_N$ we have

$$H^0(M,\alpha) = 0 \tag{3.3}$$

and this is the condition for the absence of zero modes in the spin 1/2 case [8,9]. Thus from (3.2), (3.3) and our earlier results [8,9] we get

$$\begin{aligned} dim\ H^0(M^{m+n}, \mathcal{M}_\zeta^J(m,n)) &= 0 \text{ if } m \neq n \\ &= 1 \text{ if } m = n. \end{aligned} \tag{3.4}$$

Theorem 3.1. The correlation function $C(m,n)$ of the spin J, $1-J$ bc-system vanishes for $m \neq n$. The $2n$-point function $C(n,n)$ is uniquely determined (up to a multiplicative constant) by the postulates $(\mathcal{P}1)$-$(\mathcal{P}4)$.

Remark 3.2. Note that the vanishing of $C(m,n)$ for $m \neq n$ is obtained in earlier treatments [6,3] by appeal to charge conservation.

Remark 3.3. By the uniqueness of $C(n,n)$ its normalization can be fixed inductively by taking residues at poles and defining $C(0,0) = 1$.

Corollary 3.4. Wick's theorem holds for the spin J, $1-J$ bc-system.

Proof. Wick's theorem implies $C(m,n) = 0$ for $m \neq n$ because of the no zero modes condition $< b >= 0 =< c >$. Wick's theorem also says that

$$< b(Q_1)\cdots b(Q_n)c(P_1)\cdots c(P_n) >= det(< b(Q_i)c(P_j) >)|_{i,j=1}^{n} \tag{3.5}$$

It is clear that the right-hand side of (3.5) also satisfies $(\mathcal{P}1)$ - $(\mathcal{P}4)$ and the normalization condition of Remark 3.3. By the uniqueness of $C(n,n)$ the representation (3.5) holds.

4. Factors of automorphy and correlation functions

In section 3 we have already obtained all significant qualitative properties of the bc-system. We shall now show that we can also obtain an explicit expression for $C(n,n)$. We shall first briefly review function theory on a Riemann surface and show how to apply it to $C(n,n)$. We hope that this account will clarify many of the obscure features of the literature [6,3].

The principal idea is to do function theory on the universal cover of M, viz. the upper half plane H since $g > 1$. Thus we identify M with the fundamental domain of a discrete subgroup Γ of $PSL(2,R)$ acting by fractional linear transformations of H [12] and henceforth the arguments of $C(n,n)$ will refer to this fundamental domain in H. The pullback of a line bundle on M to H by the covering projection is trivial since H is simply connected and so the pullback of a meromorphic section is a meromorphic function on H with special transformation properties under Γ codified in the concept of an *automorphy factor.*

Definition 4.1. A *factor of automorphy* ϕ is a complex-valued, nowhere vanishing function on $\Gamma \times H$ such that (i) $\phi(\gamma, z)$ is holomorphic in $z \in H$ for each $\gamma \in \Gamma$, (ii) for γ_1, γ_2 in Γ and z in H,

$$\phi(\gamma_1\gamma_2, z) = \phi(\gamma_1, \gamma_2(z))\phi(\gamma_2, z). \tag{4.1}$$

A meromorphic function $f(z)$ on H is a *meromorphic section* of ϕ if

$$f(\gamma(z)) = \phi(\gamma, z)f(z), \quad \gamma \in \Gamma, \quad z \in H. \tag{4.2}$$

A holomorphic line bundle on M is associated to each automorphy factor [12]. Two automorphy factors ϕ_1, ϕ_2 are *equivalent,* i.e. define the same line bundle on M, if and only if there exists a nonvanishing holomorphic function $h(z)$ on H such that

$$\phi_2(\gamma, z)h(z) = \phi_1(\gamma, z)h(\gamma(z)). \tag{4.3}$$

Thus to the section $f_1(z)$ of ϕ_1 corresponds the section $f_2(z) \equiv f_1(z)h(z)$ of the equivalent automorphy factor ϕ_2. It is clearly sufficient to specify $\phi(\gamma, z)$ on a set of generators for Γ. We shall choose the canonical generators A_k, B_k $(1 \leq k \leq g)$ corresponding to a canonical basis of the fundamental group of M, homotopic to a given canonical basis of a-cycles and b-cycles on M.

For any $\delta \in Pic^0(M)$ we can choose its automorphy factor [12] to take the value

$$1 \text{ on } A_k,\ exp(2\pi i\delta_k) \text{ on } B_k \text{ (for some } \delta_k \in \mathbb{C}). \tag{4.4}$$

Since $Pic^0(M)$ acts transitively on $Pic^d(M)$ the factor of automorphy of an element of $Pic^d(M)$ is the product of the automorphy factor of some chosen fixed element of $Pic^d(M)$ and one of the form (4.4).

For $d = 2g-2$ the natural choice of the fixed element is K, whose automorphy factor $K(\gamma, z)$ is easily seen to be [12] $(d\gamma(z)/dz)^{-1}$. For $d = g-1$ we should choose an element the square of whose automorphy factor is $K(\gamma, z)$, for consistency. Thus the element must be a theta characteristic and the particular one we choose is the *Riemann constant* κ (see Clemens [10]), which is defined once the period matrix Ω of M is given. Recall that [10] once a symplectic basis a_i of a-cycles and b_i of b-cycles is chosen for M, we have a dual basis $\vec{v} = (v_1, \cdots, v_g)$ of $H^0(M, K)$ such that

$$\int_{a_j} v_k = \delta_{jk}, \quad \int_{b_j} v_k = \Omega_{jk}. \tag{4.5}$$

Thus Ω is specified, the Riemann theta function $\theta(\vec{z})$ ($\vec{z} \in \mathbb{C}^g$) is defined and κ is also fixed. We denote the automorphy factor of κ by $\kappa(\gamma, z)$.

For any $\xi \in Pic^{g-1}(M)$ we have

$$\xi = (\xi \otimes \kappa^{-1}) \otimes \kappa, \quad (\xi \otimes \kappa^{-1} \in Pic^0(M)) \tag{4.6}$$

and so its automorphy factor takes the value

$$\kappa(A_k, z) \text{ on } A_k, \quad \kappa(B_k, z) exp(2\pi i\xi_k) \text{ on } B_k \ (\xi_k \in \mathbb{C}). \tag{4.7}$$

Remark 4.2. We can now write down the automorphy factor for $\zeta \in Pic^{2J(g-1)}(M)$ (since $\zeta = (\zeta \otimes \kappa^{-2J}) \otimes \kappa^{2J}$) by replacing $\kappa(\gamma, z)$ in (4.7) by its $2J$-th power and ξ_k by ζ_k. Note that for $K \otimes \zeta^{-1}$ we replace $\kappa(\gamma, z)$ in (4.7) by its $2(1-J)$-th power and ξ_k by $-\zeta_k$.

For $\xi \in Pic^{g-1}(M)$ we denote by $\theta[\xi](Q-P)$ the theta function with characteristics $\kappa \otimes \xi^{-1}$ and argument the image under the Abel map of $(Q,P) \in M \times M$. It is a section of a degree g line bundle on M for fixed P and has automorphy factor (variable Q) [13,14],

$$1 \text{ on } A_k,\ exp[-\pi i\Omega_{kk} - 2\pi i(\int_P^Q v_k - \xi_k)] \text{ on } B_k \quad (\xi_k \text{ defined in (4.7)}). \tag{4.8}$$

The automorphy factor of the *prime form* $E(Q,P)$ [15] is easily seen to be

$$(\kappa(A_k,z))^{-1} \text{ on } A_k, \quad (\kappa(B_k,z))^{-1} exp[-\pi i\Omega_{kk} - 2\pi i \int_P^Q v_k] \text{ on } B_k. \tag{4.9}$$

Consider now the line bundle $\mathcal{O}(D)$ with $D = w_1 + \cdots + w_{g-1}$, where $w_i \in M$. As a product of line bundles of the form $\mathcal{O}(P)$ it has as factor of automorphy the product of (4.9) for $P = w_1, \cdots, w_{g-1}$ and canonical section $\Pi_{i=1}^{g-1} E(Q,w_i)$. However, as a line bundle of degree $g-1$ it should have an automorphy factor like (4.7). To determine the new equivalent automorphy factor, choose an odd theta characteristic η with automorphy factor (4.7) (η_k in place of ξ_k), with unique holomorphic section h_η vanishing linearly on $u_1, \cdots, u_{g-1}$. Thus the $h(Q)$ of (4.3) is

$$h(Q) = h_\eta(Q)/\Pi_{i=1}^{g-1}(E(Q,u_i)). \tag{4.10}$$

The new automorphy factor for $\mathcal{O}(D)$ is then

$$\kappa(A_k,Q) \text{ on } A_k, \quad \kappa(B_k,Q) exp[2\pi i(\eta_k + \sum_1^{g-1} \int_{u_i}^{w_i} v_k)] \text{ on } B_k. \tag{4.11}$$

The canonical section of $\mathcal{O}(D)$ w.r.t. the new automorphy factor is

$$\Pi_1^{g-1}(E(Q,w_i)/E(Q,u_i))h_\eta(Q). \tag{4.12}$$

It is easy to check that

$$h(Q)/h(P) = \sigma(Q)/\sigma(P) \tag{4.13}$$

where the function σ is defined in [13,15] and appears somewhat mysteriously in the literature [6,3]. It is now a simple exercise to write down the automorphy factor for $\mathcal{O}(W)$

and its canonical section.

The line bundle α defined in (3.2b) has an automorphy factor of the form (4.7) with α_k instead of ξ_k. Eqn. (3.2b) and the preceding discussion gives the following relation between α_k, ζ_k and η_k:

$$\alpha_k = \zeta_k - (2J-1)\eta_k - \sum_{r=0}^{2J-2}\sum_{i=1}^{g-1}\int_{u_i}^{w_{i+r(g-1)}} v_k. \tag{4.14}$$

One difficulty we have apparently overlooked is that $C(n,n)$ is a section of a line bundle over M^{2n}. The notion of automorphy factor indeed extends to compact complex manifolds [16], but is simplified in our case by the fact that $M^{2n} = H^{2n}/\Gamma^{2n}$. Moreover $\mathcal{F}_\zeta(n,n)$ is simply the cross product of line bundles on $2n$ copies of M. Hence its automorphy factor corresponds to that of ζ in each Q-variable and of $K \otimes \zeta^{-1}$ in each P-variable as given in Remark 4.2.

5. The $2n$-point function and Fay's identity

We saw in section 3 that $C(n,n)$ is the product of the unique meromorphic section $S^J(n,n)$ of $\mathcal{O}(D^J(n,n))$ with divisor $D^J(n,n)$ and the unique holomorphic section of $\mathcal{M}^J_\zeta(n,n)$, which by (3.2) can be identified with the result of our earlier calculations [8,9], viz. $\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)$. From (3.1) we see that $S^J(n,n)$ can be obtained from the canonical section of $\mathcal{O}(D^{1/2}(n,n))$ written earlier [8,9] and that of $\mathcal{O}(W)$, which follows from the analysis of section 4. We thus get:

Theorem 5.1. The unique normalised $2n$-point function of the spin J, $1-J$ bc-system $(J > 1)$ is given by

$$< b(Q_1)\cdots b(Q_n)c(P_1)\cdots c(P_n) >=$$
$$\frac{\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)}{\theta[\alpha](0)}\frac{\Pi_{i<j}E(Q_i,Q_j)E(P_j,P_i)}{\Pi_{i,j}E(Q_i,P_j)}\Pi_{i=1}^n(\Pi_{j=1}^N\frac{E(Q_i,w_j)}{E(P_i,w_j)})\left(\frac{h(Q_i)}{h(P_i)}\right)^{2J-1} \tag{5.1}$$

where $\theta[\alpha] \neq 0$ if and only if $H^0(M,\alpha) = 0$, ensured by (3.3).

Proof. It is sufficient to check the factor of automorphy of the two sides of (5.1) following the discussion of section 4.

Recalling eqn. (3.5) of Corollary 3.4 we see that $C(n,n)$ is also a determinant of the 2-point function

$$< b(Q)c(P) >= \frac{\theta[\alpha](Q-P)}{\theta[\alpha](0)} \frac{1}{E(Q,P)} \Pi_{j=1}^{N} \frac{E(Q,w_j)}{E(P,w_j)} \left(\frac{h(Q)}{h(P)}\right)^{2J-1} . \tag{5.2}$$

Substituting (5.1), (5.2) in (3.5) we obtain, after cancelling common factors:

Theorem 5.2 (Fay's identity).

$$\frac{\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)}{\theta[\alpha](0)} \frac{\Pi_{i<j} E(Q_i,Q_j)E(P_j,P_i)}{\Pi_{i,j} E(Q_i,P_j)} = \det\left(\frac{\theta[\alpha](Q_i - P_j)}{\theta[\alpha](0)} \frac{1}{E(Q_i,P_j)}\right)\Bigg|_{i,j=1}^{n} . \tag{5.3}$$

The identity (5.3), first obtained by Fay [13], is of fundamental importance in algebraic geometry in the case $n = 2$, known as the *trisecant identity* [14]. The identities for $n > 2$ are consequences of this one [13], in perfect agreement with the physical intuition that if the 4-point function of a quantum field theory is free, then so are the higher point functions.

6. Conclusions and outlook

We have thus shown that the physical singularity structure of the correlation functions of the bc-system rigorously determine them. This illustrates the power of analyticity. Our methods can be extended to situations involving branch point singularities, as occur in the presence of spin fields or for orbifolds [17]. Further extensions are under investigation.

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