Diffusive repair for the Ginzburg-Landau equation

Autor(en): **Collet, P. / Eckmann, J.-P. / Epstein, H.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **65 (1992)**

Heft 1

PDF erstellt am: **08.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-116387

Helv Phys Acta
Vol. 65 (1992)

0018-0238/92/010056-37$1.50+0.20/0

Diffusive repair for the Ginzburg-Landau equation

By P. Collet, J.-P. Eckmann and H. Epstein

Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau, France, Laboratoire UPR A14, CNRS
Département de Physique Théorique, Université de Genève, CH-1211 Geneva 4, Switzerland
IHES 35, Rte. de Chartres, F-91440 Bures-sur-Yvette, France

(15. VII. 1991)

Abstract. We consider the Ginzburg-Landau equation for a complex scalar field in one dimension and show that small phase and amplitude perturbations of a stationary solution repair diffusively to converge to a stationary solution. Our methods explain the range of validity of the phase equation, and the coupling between the "fast" amplitude equation and the "slow" phase equation.

1. Introduction

This study is motivated by a desire to gain a better understanding of the space-time dynamics occurring in hydrodynamic systems. Our global understanding of such problems is still very incomplete, and here we focus on a typical, simple example. However, it will turn out that even this simple example has a few unsuspected difficulties. They will shed some light on the relation between fast and slow modes, and their role in regularizing infrared singularities, which are typical for hydrodynamic systems in large containers, due to the translation invariance.

Another interesting aspect is the appearance of a sort of center manifold in space-time dynamics.

The equation we have in mind is the complex Ginzburg-Landau equation for the complex "field"

$$u : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{C} ,$$

satisfying

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t)\,(1 - |u(x,t)|^2) . \tag{1.1}$$

This equation has time-independent ("stationary") solutions of the form

$$u(x) = \sqrt{1-q^2} e^{iqx} e^{i\psi} , \tag{1.2}$$

for $q \in [-1,1]$, $\psi \in [0, 2\pi)$. We call these solutions "spirals," although for $q = 0$ their phase is constant.

We are interested in the initial value problem for initial data of the form

$$u(x,0) = r(x) e^{i\phi(x)} ,$$

with

$$\lim_{x\to\pm\infty} r(x) = \sqrt{1-q^2} ,$$
$$\lim_{x\to\pm\infty} \phi'(x) = q ,$$

but with a global phase shift

$$\delta = \int_{-\infty}^{\infty} dx\, (\phi'(x) - q) ,$$

which is not necessarily zero. This corresponds to "pulling" or "squeezing" the spirals of the corresponding stationary solution. The question is whether such a perturbation of the stationary solution will heal or whether it leads to a phase slip, or will even migrate to another stationary solution.

When the initial perturbation is sufficiently large, then one expects one of the latter two catastrophes to happen and such phenomena are still beyond control of rigorous mathematics. What we shall show here, and illustrate in a somewhat wider context, is the result that *small perturbations heal when q is in the Eckhaus stable domain* ($q^2 < 1/3$).

A very interesting problem is the global analysis of the initial value problem with different q-values at $+\infty$ and $-\infty$. One expects then the "invasion" of the less stable solution by the more stable one. This is again a problem which at this moment does not allow for precise mathematical statements.

An intuitively appealing argument of why repair happens goes as follows (but will turn out to miss some essential features of the problem). Consider Eq.(1.1) in polar coordinates, $u = r\, e^{i\phi}$. One gets

$$\partial_t r = \partial_x^2 r + r - r^3 - r\zeta^2 , \tag{1.3}$$

$$\partial_t \zeta = \partial_x^2 \zeta + 2\partial_x \left(\frac{\partial_x r}{r} \zeta \right) , \tag{1.4}$$

with $\zeta = \partial_x \phi$.

Note that since the r.h.s. of (1.4) is a derivative, $\int(\zeta - q)$ is a conserved quantity, so that the total phase shift δ is conserved. However, as a function of time, $\zeta(x,t) \to q$ for every fixed x when the initial data are suitably chosen. This is behavior similar to that which occurs, with $q = 0$, for the simple example of the pure diffusion equation $\partial_t \zeta = \partial_x^2 \zeta$.

Consider now the vicinity of a stationary solution,

$$r(x,t) = \sqrt{1-q^2} + s(x,t) , \quad \zeta(x,t) = q + \eta(x,t) .$$

The reduced equations for s, η are

$$\begin{aligned} \partial_t s &= \partial_x^2 s + s - 3(1-q^2)s - 3\sqrt{1-q^2}\, s^2 - s^3 \\ &\quad - \sqrt{1-q^2}\, \eta^2 - 2\sqrt{1-q^2}\, q\eta - sq^2 - 2sq\eta - s\eta^2 , \\ \partial_t \eta &= \partial_x^2 \eta + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2} + s} \right) q + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2} + s} \eta \right) . \end{aligned} \tag{1.5}$$

The corresponding linear system is (linearized around 0)

$$\begin{aligned} \partial_t s &= \partial_x^2 s - 2(1-q^2)s - 2\sqrt{1-q^2}\, q\eta , \\ \partial_t \eta &= \partial_x^2 \eta + 2\partial_x^2 s \frac{2q}{\sqrt{1-q^2}} . \end{aligned} \tag{1.6}$$

It is well-known, see e.g.[CE], that this system becomes unstable at small nonzero wavevectors when $q^2 > 1/3$, i.e., there are unstable eigenvectors which correspond to long-wavelength oscillations in space. Thus, one does not expect healing when $q^2 > 1/3$. When $q = 0$, the problem (1.5) reduces to

$$\begin{aligned} \partial_t s &= \partial_x^2 s - 2s - \eta^2 - 3s^2 - s^3 - s\eta^2 , \\ \partial_t \eta &= \partial_x^2 \eta + 2\partial_x \left(\frac{\partial_x s}{1+s} \eta \right) . \end{aligned} \tag{1.7}$$

In essence, the repair of the defect is now seen as follows: Due to the term $-2s$, the s variable relaxes quickly to its equilibrium value, and then the η variable diffusively tends to zero.

The problem with this argument comes from the η^2 term in the first equation of (1.7). In fact, s does *not* relax (fast) to zero but rather to something like η^2, or some other η dependent function $F(\eta)$. Substituting into the second equation, we get the slow equation

$$\partial_t \eta = \partial_x^2 \eta + 2\partial_x \left(\frac{DF(\eta)\partial_x \eta}{1+F(\eta)} \eta \right) . \tag{1.8}$$

Does this equation repair diffusively? We will see that this is indeed the case. However, at first sight, this is far from obvious, and it is useful to get an idea of the following, more general, class of problems. Let $N(u, u', u'')$ be a polynomial in the derivatives of u. Consider the equation

$$\partial_t u = u'' + N(u, u', u'') . \tag{1.9}$$

Then, essentially, the following is true. If N is a monomial, $N = (u)^{p_0} (u')^{p_1} (u'')^{p_2}$, denote by d its "degree," $d = p_0 + 2p_1 + 3p_2$, and assume $2p_1 + p_2 \leq 2$, and $p_0 + p_1 > 0$. If $d > 3$ then small initial conditions of the equation (1.9) tend to zero, and if $d < 3$ then there are arbitrarily small initial conditions which diverge in finite time. The case $d = 3$ is marginal, and depends on the details of N. For example, $N = u^3$ is *unstable* and $N = uu'$ is *stable*. The results mentioned above, and much more, are studied in many papers. For a review, see [L], and for an early reference, [W]. In Section 2 below, we give proofs for the cases which interest us in the sequel, and in order to familiarize the reader with the method. Note that the problem (1.8) is stable according to this power counting, but we will see several difficulties appear when $q \neq 0$. To illustrate our method, we will treat some of the cases for N in the next section. The general strategy of our proof for the full problem will then be more transparent.

The main result of this paper is the

Theorem 1.1. *Let $q^2 < 1/3$. There is an $\varepsilon = \varepsilon_q > 0$ such that the solutions (s, η) of the Eq.(1.7) tend to zero in L^∞ as $t \to \infty$ if the initial data satisfy*

$$\|k\tilde{s}_0\|_2 < \varepsilon , \quad \|\tilde{s}_0\|_\infty < \varepsilon ,$$

$$\|\frac{k}{|k|+1}\tilde{\eta}_0\|_2 < \varepsilon , \quad \|\tilde{\eta}_0\|_1 < \varepsilon , \quad \|\tilde{\eta}_0\|_\infty < \varepsilon .$$

Here, $\tilde{f}$ denotes the Fourier transform of f and $\|k\tilde{f}\|_2 = \left(\int dk\, k^2 |\tilde{f}(k)|^2\right)^{1/2}$.

2. Some Typical Cases

2.1. The Stable Side

In this section, we analyze the equation

$$\partial_t u = \partial_x^2 u + u^p , \tag{2.1}$$

in the case when $p > 3$. Then, sufficiently small initial data will converge to zero. We formulate this theorem with conditions on the Fourier transform of the initial data, since this will lead to a somewhat easier proof, although the direct space formulation would work as well.

Notation. Here, and in the sequel, we use $\tilde{f}$ to denote the Fourier transform of f, and we use u_t to denote $u(\cdot, t)$.

Proposition 2.1. *For every $p > 3$, $p \in \mathbf{N}$, there is a constant $\varepsilon_p > 0$ such that for*

$$\|\tilde{u}_0\|_\infty < \varepsilon_p , \quad \|\tilde{u}_0\|_1 < \varepsilon_p ,$$

the solution of the initial value problem (2.1) with $u(x,0) = u_0$ tends to zero. In addition, one has the bound

$$\|\tilde{u}_t\|_1 \le C_p (t+1)^{-1/2} ,$$

for some finite constant C_p.

Remark. The proof which we give here is based on methods in momentum space (conjugate to x). This will have the advantage of preparing the method of proof for Theorem 1.1. On the other hand, since fractional powers are awkward to bound in momentum space in the L^1 norm, we are forced to restrict the proof to integer values of $p > 3$. A proof for arbitrary $p > 3$ in x-space would look very similar to the one given here.

Proof. We shall work exclusively in the Fourier transformed space, and we call k the variable conjugate to x. We define $f^{*n} = f * f^{*(n-1)}$, $f^{*1} = f$. The problem (2.1) is equivalent to the integral equation

$$\partial_t \tilde{u}(k,t) = e^{-k^2 t}\tilde{u}_0(k) + \int_0^t d\tau e^{-k^2\tau}\tilde{u}^{*p}(k, t-\tau) . \tag{2.2}$$

We fix a maximal time $T > 0$ and work on the space $\mathcal{H}_T$ defined by

$$\mathcal{H}_T = \{h : \mathbf{R} \times [0,T] \to \mathbf{C}\} , \tag{2.3}$$

equipped with the norm

$$\|h\| = \max\bigl(\|h\|_{T,1}, \|h\|_{T,\infty}\bigr) ,$$

where

$$\|h\|_{T,q} = \sup_{0\le t\le T} \|h(\cdot,t)\|_q .$$

The Picard method consists now in viewing (2.2) as a fixed point problem on $\mathcal{H}_T$. The existence of a fixed point shows then the existence of solutions for times less than T, with bounds. Shifting the origin of time from 0 to T and repeating the procedure, we shall propagate these bounds and obtain the proof of Proposition 2.1.

In view of (2.2) we consider therefore the operator $\mathcal{T}$ defined by

$$\mathcal{T} : f \mapsto (\mathcal{T}f) ; \quad (\mathcal{T}f)(k,t) = e^{-k^2 t}\tilde{u}_0(k) + (\mathcal{M}f)(k,t) , \tag{2.4}$$

where $\mathcal{M}$ is defined by

$$\mathcal{M} : f \mapsto (\mathcal{M}f) ; \quad (\mathcal{M}f)(k,t) = \int_0^t d\tau e^{-k^2\tau} f^{*p}(k,t-\tau) .$$

Since

$$\|g^{*p}\|_1 \le \|g\|_1^p , \quad \|g^{*p}\|_\infty \le \|g\|_\infty \|g\|_1^{p-1} , \tag{2.5}$$

and

$$\int_0^t d\tau e^{-k^2\tau} \le t ,$$

we find

$$\|\mathcal{M}f\|_{T,1} \le T\|f\|_{T,1}^p , \tag{2.6}$$

and

$$\|\mathcal{M}f\|_{T,\infty} \le T\|f\|_{T,\infty} \|f\|_{T,1}^{p-1} . \tag{2.7}$$

Remark. It is the inequality (2.5) which does not generalize to non-integer p.

The functional derivative of $\mathcal{T}$ at f is $D\mathcal{M}_f$, since the inhomogeneous term is independent of f and hence we get from (2.6), (2.7), by polarization,

$$\|D\mathcal{T}_f g\|_{T,1} \le pT\|g\|_{T,1} \|f\|_{T,1}^{p-1} , \tag{2.8}$$

$$\|D\mathcal{T}_f g\|_{T,\infty} \le pT\|g\|_{T,\infty} \|f\|_{T,1}^{p-1} . \tag{2.9}$$

Thus, if we denote, for $\tilde{v} \in L^1$,

$$T_{\tilde{v}} = \frac{1}{2p(2\|\tilde{v}\|_1)^{p-1}} , \tag{2.10}$$

then, for $T \le T_{\tilde{u}_0}$ and $\|f\|_{T,1} \le 2\|\tilde{u}_0\|_1$, we find

$$\|D\mathcal{T}_f g\|_{T,1} \le \tfrac{1}{2}\|g\|_{T,1} , \quad \|D\mathcal{T}_f g\|_{T,\infty} \le \tfrac{1}{2}\|g\|_{T,\infty} .$$

We fix now $T = T_{\tilde{u}_0}$ and denote $U_0(k,t) = e^{-k^2 t}\tilde{u}_0(k)$. Then the above estimates imply that $\mathcal{T}$ maps the ball of radius $\|\tilde{u}_0\|_1$ around $U_0 \in \mathcal{H}_T$ into itself, since $\|U_0\|_{T,1} \le \|\tilde{u}_0\|_1$ and $\mathcal{T}$ is a contraction. Therefore the solution $\tilde{u}$ of (2.2) with initial data $\tilde{u}_0$ satisfies

$$\|\tilde{u}\|_{t,1} \le 2\|\tilde{u}_0\|_1 , \quad \text{for } t \le T_{\tilde{u}_0} . \tag{2.11}$$

The inequality

$$\|\tilde{u}\|_{t,\infty} \le \|\tilde{u}_0\|_\infty + pT\|\tilde{u}\|_{T,\infty}\|\tilde{u}\|_{T,1}^{p-1}$$

in turn implies

$$\|\tilde{u}\|_{t,\infty} \le 2\|\tilde{u}_0\|_\infty , \quad \text{for } t \le T_{\tilde{u}_0} . \tag{2.12}$$

We next show that $\|\tilde{u}(\cdot,t)\|_1$ converges to 0 when $t \to \infty$.

Remark. If $u_0(x) > 0$ for all x then the integral $\int dx\, u(x,t)$ is an *increasing* function of time. Therefore we cannot expect that $\|\tilde{u}_t\|_\infty$ tends to zero. In fact it must grow. However, we shall see that it stays bounded.

To show that $\|\tilde{u}_t\|_1$ tends to 0, we use the inequality $\|U_0(\cdot,t)\|_1 \le (\pi/t)^{1/2}$ $\|\tilde{u}_0\|_\infty$, so that

$$\begin{aligned} \|\tilde{u}_t\|_1 &\le \frac{\sqrt{\pi}\|\tilde{u}_0\|_\infty}{\sqrt{t}} + t \sup_{0\le\tau\le t} \|\tilde{u}_\tau\|_1^p \\ &\le \frac{\sqrt{\pi}\|\tilde{u}_0\|_\infty}{\sqrt{t}} + 2^p t\,\|\tilde{u}_0\|_1^p , \end{aligned} \tag{2.13}$$

by (2.8), (2.11). Similarly, using (2.9), (2.12), we get

$$\|\tilde{u}_t\|_\infty \le \|\tilde{u}_0\|_\infty + 2^p t\,\|\tilde{u}_0\|_\infty\|\tilde{u}_0\|_1^{p-1} . \tag{2.14}$$

We now denote

$$\begin{aligned} m_{t_1,t_2} &= \sup_{t_1\le t\le t_2} \|\tilde{u}_t\|_1 , \quad m_t = \|\tilde{u}_t\|_1 , \\ y_{t_1,t_2} &= \sup_{t_1\le t\le t_2} \|\tilde{u}_t\|_\infty , \quad y_t = \|\tilde{u}_t\|_\infty . \end{aligned}$$

Since the Eq.(2.1) does not depend explicitly on time, we can rewrite (2.13), (2.14) as

$$m_{t_1,t_2} \le \frac{\sqrt{\pi}y_{t_1}}{(t_2-t_1)^{1/2}} + 2^p(t_2-t_1)m_{t_1}^p \,, \tag{2.15}$$

$$y_{t_1,t_2} \le y_{t_1}\left(1+(t_2-t_1)p2^p m_{t_1}^{p-1}\right) , \tag{2.16}$$

and by (2.10), these inequalities hold as long as

$$0 \le t_2 - t_1 \le \frac{1}{p2^p m_{t_1}^{p-1}} \,. \tag{2.17}$$

Assume now that $m_{t_1} \le 1$ and $y_{t_1} \le \frac{1}{2}$. Henceforth, K denotes a constant which depends only on p and which can vary from equation to equation. Since $p > 3$ implies $p-1 > 2p/3$, there is a constant C^* such that if

$$t_2 = t_1 + C^* m_{t_1}^{-2p/3} ,$$

then t_2 satisfies the Eq.(2.17). Then, (2.15) leads to

$$m_{t_2} \le m_{t_1,t_2} \le K m_{t_1}^{p/3} , \tag{2.18}$$

and (2.16) leads to

$$y_{t_2} \le y_{t_1}\left(1+Km_{t_1}^{p-1-2p/3}\right) \le y_{t_1} e^{Km_{t_1}^q} , \tag{2.19}$$

with $q = p-1-2p/3 > 0$. We now define recursively $t_{n+1} = t_n + C^* m_{t_n}^{-2p/3}$, and iterate the process. Then (2.18) leads to

$$m_{t_{n+1}} \le C^{*\sum_{j=0}^{n-1}(p/3)^j} m_{t_1}^{(p/3)^n} \le \left(C^{*3/(p-3)} m_{t_1}\right)^{(p/3)^n} , \tag{2.20}$$

and hence (2.19) leads to

$$y_{t_{n+1}} \le y_{t_1} e^{K\sum_{j=1}^{n} m_{t_j}^q} \le 2y_{t_1} \le 1 \,.$$

The penultimate inequality follows from (2.20) and holds if m_{t_1} is sufficiently small. This essentially determines the constant ε_p of Proposition 2.1. Note that

$$\begin{aligned} t_{n+1} - t_1 &= \sum_{j=1}^{n}(t_{j+1}-t_j) = C^* \sum_{j=1}^{n}\left(C^{*3/(p-3)} m_{t_1}\right)^{-\frac{2p}{3}\left(\frac{p}{3}\right)^j} \\ &= \mathcal{O}\left(\left[C^{*3/(p-3)} m_{t_1}\right]^{-\frac{2p}{3}\left(\frac{p}{3}\right)^n}\right) . \end{aligned}$$

Thus, $t_{n+1} \to \infty$, and in fact the above estimate shows that if $t_1 = 0$, then $t_n = \mathcal{O}(m_{t_n}^{-2})$ so that $\|\tilde{u}_t\|_1 = \mathcal{O}((t+1)^{-1/2})$, as $t \to \infty$. The proof of Proposition 2.1 is complete.

2.2. The Unstable Side

In this case, we shall show that Gaussian initial conditions lead to solutions which diverge in finite time.

Proposition 2.2. *Every Gaussian initial condition for the problem (2.1) with $p < 3$ diverges in finite time.*

Proof. We shall use the Maximum Principle [F] to produce a diverging lower bound on the solution. Consider first the case $p < 2$. Then we use the two equations

$$\partial_t v = \partial_x^2 v \, , \tag{2.21}$$

$$\partial_t w = \partial_x^2 w + v^p \, , \tag{2.22}$$

and if both equations have the initial condition $u_0(x) = v(x,0) = w(x,0) = Ce^{-\alpha x^2}\alpha^{1/2}$, then $w(x,t)$ is a lower bound to u satisfying (2.1) with $u(x,0) = u_0(x)$.

Remark. The coefficient C can be eliminated by rescaling space and time and thus we consider only $C = 1$.

We shall show that $w(0,t)$ diverges as $t \to \infty$. The following observations will be useful. If $v(x,0) = u_0(x)$, then

$$v(x,t) = \left(\tfrac{\alpha}{\alpha+t}\right)^{1/2} e^{-\frac{\alpha}{1+\alpha t}x^2} \, , \tag{2.23}$$

$$\tilde{v}(k,t) = e^{-k^2(t+\frac{1}{\alpha})} \, , \tag{2.24}$$

$$\widetilde{v^p}(k,t) = \tfrac{1}{\sqrt{p}} \left(\tfrac{\alpha}{1+\alpha t}\right)^{\frac{p-1}{2}} e^{-k^2(\frac{t}{p}+\frac{1}{\alpha p})} \, , \tag{2.25}$$

as follows at once by Gaussian integration. We can write (2.22) as an integral equation

$$\tilde{w}(k,t) = e^{-k^2(t+\frac{1}{\alpha})} + \int_0^t d\tau \, e^{-k^2\tau} \widetilde{v^p}(k,t-\tau) \, . \tag{2.26}$$

Integrating over k, we find, using (2.25),

$$\begin{aligned} \int dk \, \tilde{w}(k,t) &\geq \int_0^t d\tau \frac{1}{\sqrt{p}} \left(\frac{\alpha}{1+\alpha\tau}\right)^{\frac{p-1}{2}} \frac{\sqrt{\pi}}{(t - \frac{p-1}{p}\tau + \frac{1}{\alpha p})^{1/2}} \\ &\geq \mathcal{O}(1) t^{1-\frac{1}{2}-\frac{p-1}{2}} \, , \end{aligned}$$

as $t \to \infty$. Thus, for $p < 2$, the quantity $u(0,t)$ diverges as $t \to \infty$.

If p satisfies $2 \leq p < 3$, we essentially iterate the above argument, using instead of (2.21), (2.22) the equations

$$\begin{aligned} \partial_t v &= \partial_x^2 v \,, \\ \partial_t w &= \partial_x^2 w + w v^{p-1} \,, \end{aligned}$$

and then iterating n times the integral equation

$$\tilde{w}(k,t) = e^{-k^2(t+\frac{1}{\alpha})} + \int d\ell \int_0^t d\tau \, e^{-k^2(t-\tau)} \tilde{w}(k-\ell,\tau)\, \widetilde{v^{p-1}}(\ell,\tau) \,. \quad (2.27)$$

Since all terms are positive, it suffices to consider one of them and to show its divergence. Setting $t = t_{n+1}$, this leads to an expression of the form

$$\begin{aligned} \tilde{w}^{(n+1)}(k_n, t_{n+1}) = & \int_0^{t_{n+1}} dt_n \cdots \int_0^{t_2} dt_1 \int dk_1 \cdots dk_{n-1} \\ & \times \prod_{j=1}^{n} e^{-k_j^2(t_{j+1}-t_j)} \\ & \times \widetilde{v^{p-1}}(k_1,t_1) \prod_{j=1}^{n-1} \widetilde{v^{p-1}}(k_{j+1}-k_j, t_j) \,. \end{aligned} \quad (2.28)$$

In the case of $p < 2$ we really used (2.28) with $n = 1$. If we integrate (2.28) over k_n, then dimensional analysis shows that the integral is bounded below by

$$\mathcal{O}(1) t^n \left(\frac{1}{t^{\frac{1}{2}}}\right) \left(\frac{1}{t^{\frac{p-1}{2}}}\right) \frac{1}{t^{\frac{p-1}{2}}} \,, \quad (2.29)$$

where we use, e.g.,

$$\int_0^t \frac{d\tau}{(1+\tau)^\gamma} = t \int_0^1 \frac{ds}{(1+ts)^\gamma} > t \int_{\frac{1}{2}}^1 \frac{ds}{(1+ts)^\gamma} \geq \mathcal{O}(t^{1-\gamma}) \,,$$

as $t \to \infty$. Combining the powers in (2.29), we find a lower bound of

$$\mathcal{O}(1) \left(\mathcal{O}(1) t\right)^{(\frac{3}{2}-\frac{p}{2})(n-1)} t^{1-\frac{p}{2}} \,.$$

If $p < 3$ and t is sufficiently large, this diverges as $n \to \infty$.

2.3. The Marginal Case

In this section, we consider the case $p = 3$. This case is unstable.

Proposition 2.3. *Consider the problem*

$$\partial_t u = \partial_x^2 u + u^3 , \tag{2.30}$$

with initial data $u(x,0) = u_0(x) = C\alpha^{1/2}e^{-\alpha x^2}$. *The solution diverges in finite time for any* $C > 0, \alpha > 0$.

Remark. The coefficient C can be eliminated by rescaling space and time and thus we consider only $C = 1$.

Proof. This proof is a variant of the case for p between 2 and 3, but the divergence can only be seen by tracking logarithmic corrections. We now use the induction

$$\begin{aligned} \partial_t v_0 &= \partial_x^2 v_0 , \\ \partial_t v_{n+1} &= \partial_x^2 v_{n+1} + v_n^3 , \end{aligned} \tag{2.31}$$

which leads to a lower bound for the solution of (2.30), if the v_n have u_0 as (positive) initial data. We will control inductively expressions of the form

$$\tilde{f}_n(k,t) = C_n e^{-k^2(t+\frac{1}{3^n\alpha})}\big(\log(1+\alpha t)\big)^{p_n} , \tag{2.32}$$

which are lower bounds on $\tilde{v}_n(k,t)$. The n^{th} integral leads then to an expression

$$\tilde{v}_{n+1}(k,t) \geq \int_0^t d\tau\, e^{-k^2(t-\tau)}\widetilde{f_n^3}(k,\tau) .$$

By (2.25), this leads to

$$\begin{aligned} \tilde{v}_{n+1}(k,t) &\geq \int_0^t d\tau\, e^{-k^2(t-\tau)-k^2(\frac{\tau}{3}+\frac{1}{3^{n+1}\alpha})}\big(\log(1+\alpha\tau)\big)^{3p_n} \\ &\quad \times C_n^3 \frac{1}{\sqrt{3}}\frac{3^n}{1+3^n\alpha\tau} \\ &\geq \frac{3^n\alpha C_n^3}{\sqrt{3}} e^{-k^2(t+\frac{1}{3^{n+1}\alpha})}\int_0^t d\tau\, \frac{\big(\log(1+\alpha\tau)\big)^{3p_n}}{3^n(1+\alpha\tau)} \\ &= \frac{C_n^3}{\sqrt{3}(3p_n+1)}\, e^{-k^2(t+\frac{1}{3^{n+1}\alpha})}\big(\log(1+\alpha\tau)\big)^{3p_n+1} \equiv \tilde{f}_{n+1}(k,t) . \end{aligned} \tag{2.33}$$

Note that for the pure Gaussian problem, $n = 0$, we have $C_0 = 1$, $p_0 = 0$. The relation (2.33) leads us to define a recursion $p_{n+1} = 3p_n + 1$, i.e., $p_n = (3^n - 1)/2$. Furthermore,

$$C_{n+1} = \frac{2C_n^3}{\sqrt{3}(3^{n+1} - 1)} \geq \frac{2C_n^3}{\sqrt{3}3^{n+1}} \geq \frac{C_n^3}{3^n} .$$

If we consider the recursion $D_{n+1} = D_n^3 \cdot 3^{-n}$, with $D_0 = 1$, we find for $E_n \equiv \log D_n$ the relation

$$E_{n+1} = 3E_n - n \log 3 ,$$

with the solution

$$E_n = -3^n \frac{\log 3}{4} + n\frac{\log 3}{2} + \frac{\log 3}{4} ,$$

since $E_0 = 0$. Using (2.32), (2.33), we see that

$$\tilde{v}_n(k,t) \geq \tilde{f}_n(k,t) \geq 3^{-3^n/4} 3^{n/2} 3^{1/4} e^{-k^2 t} e^{-k^2/(3^{n+1}\alpha)} \big(\log(1+\alpha t)\big)^{(3^n-1)/4} .$$

We see that if $\log(1 + \alpha t) > 3^{-1/2}$, then $\lim_{n\to\infty} \tilde{v}_n(k,t) = \infty$, for all k. Thus the solution diverges in finite time.

3. An Intermediate Equation

We consider here an equation which is obtained to lowest order from the problem of phase diffusion which is of main concern for this paper. Consider the Eq.(1.7). Since both equations are of diffusive type and in fact the s equation has a "mass," we expect that both s and η are smaller than they would have been in the case of pure diffusion. In fact, s should behave about as η^2. Therefore, the terms s^2, s^3, $s\eta^2$ are analogues of terms with $p > 3$ for the simplified problem and we have seen that such terms are not upsetting the diffusive repair of an initial perturbation. We therefore concentrate on the remaining terms which illustrate the precise mechanism of repair. Consider

$$\begin{aligned} \partial_t s &= \partial_x^2 s - 2s - \eta^2 , \\ \partial_t \eta &= \partial_x^2 \eta + 2\partial_x \big(\eta\, \partial_x s\big) . \end{aligned} \tag{3.1}$$

Assuming that s is a fast variable, and that η is slow, we find that the equilibrium value of s is

$$s = -(\partial_x^2 - 2)^{-1}\eta^2 . \tag{3.2}$$

This can be viewed as a center manifold for the motion of the (s,η) variable.

Substituting (3.2) into (3.1), we find

$$\partial_t\eta = \partial_x^2\eta - 2\partial_x\big(\eta\frac{\partial_x}{\partial_x^2 - 2}\eta^2\big) . \tag{3.3}$$

Sometimes, an expansion of the operator $\partial_x/(\partial_x^2 - 2)$ is performed in perturbation theory. However, as is easily seen in Fourier transform, the operator $-ik/(k^2+2)$ has much better properties at large k than its expansion in powers of k. Indeed, it is at once visible from (3.3) that the problem is *infrared and ultraviolet regular.* We will materialize this now in the proof of the following

Proposition 3.1. *There is an $\varepsilon > 0$ such that the solution of Eq.(3.3) with initial data η_0, satisfying*

$$\|\tilde\eta_0\|_\infty < \varepsilon , \quad \|\tilde\eta_0\|_1 < \varepsilon ,$$

tends to zero in L^∞ as $t \to \infty$.

Proof. In momentum space, the integral equation corresponding to (3.3) takes the form

$$\tilde\eta_t(k) = e^{-k^2t}\tilde\eta_0(k) - \int d\ell \int_0^t d\tau\, e^{-k^2(t-\tau)} k\, \tilde\eta_\tau(k-\ell)\cdot\frac{\ell}{\ell^2+m^2}\widetilde{\eta_\tau^2}(\ell) , \tag{3.4}$$

where we use the notation $2{=}m^2$ to make the analogy with quantum field theory more transparent.

Note that if $m \to 0$, we get an infrared singularity in (3.4) which would reflect that the time scale of s (given originally by $m^{-2} = \frac{1}{2}$) has become comparable to that of η.

We can now repeat the methods of Section 2.2. We use again the space $\mathcal{H}_T$, defined in (2.3), and we define maps

$$\mathcal{T} : f \mapsto (\mathcal{T}f) ; \quad (\mathcal{T}f)(k,t) = e^{-k^2t}\tilde\eta_0(k) + (\mathcal{M}f)(k,t) ,$$

where $\mathcal{M}$ is defined by $\mathcal{M} : f \mapsto (\mathcal{M}f)$,

$$(\mathcal{M}f)(k,t) = -\int d\ell \int_0^t d\tau\, e^{-k^2(t-\tau)} k\, \tilde f_\tau(k-\ell)\cdot\frac{\ell}{\ell^2+m^2}\widetilde{f_\tau^2}(\ell). \tag{3.5}$$

Since $|\ell/(\ell^2+m^2)| < 1/(2m)$, we see that the integral in (3.4) is bounded by

$$\int d\ell \int_0^t d\tau \, e^{-k^2(t-\tau)} |k| \cdot |\tilde{\eta}_\tau(k-\ell)| \cdot |\widetilde{\eta_\tau^2}(\ell)| \, .$$

Using

$$\int_0^t d\tau |k| e^{-k^2\tau} \le \mathcal{O}(t^{1/2}) \, ,$$

we find

$$\|\mathcal{M}f\|_{T,1} \le KT^{1/2} \|f\|_{T,1}^3 \, , \tag{3.6}$$

and

$$\|\mathcal{M}f\|_{T,\infty} \le KT^{1/2} \|f\|_{T,\infty} \|f\|_{T,1}^2 \, . \tag{3.7}$$

Therefore, we get for $D\mathcal{T}_f$,

$$\|D\mathcal{T}_f g\|_{T,1} \le K_0 T^{1/2} \|g\|_{T,1} \|f\|_{T,1}^2 \, , \tag{3.8}$$

$$\|D\mathcal{T}_f g\|_{T,\infty} \le K_0 T^{1/2} \|g\|_{T,\infty} \|f\|_{T,1}^2 \, . \tag{3.9}$$

For each $\tilde{v}$ we define a critical time

$$T_{\tilde{v}} = \frac{1}{4 \cdot 2^4 K_0^2 \|\tilde{v}\|_1^4} \, , \tag{3.10}$$

and for $T \le T_{\tilde{\eta}_0}$ and $\|f\|_{T,1} \le 2\|\tilde{\eta}_0\|_1$, we find

$$\|D\mathcal{T}_f g\|_{T,1} \le \tfrac{1}{2} \|g\|_{T,1} \, , \quad \|D\mathcal{T}_f g\|_{T,\infty} \le \tfrac{1}{2} \|g\|_{T,\infty} \, .$$

The above estimates imply that $\mathcal{T}$ contracts the ball of radius $\|\tilde{\eta}_0\|_1$ around $e^{-k^2 t}\eta_0(k) \in \mathcal{H}_T$ into itself. Therefore the solution $\tilde{\eta}$ of (3.4) with initial data $\tilde{\eta}_0$ satisfies

$$\|\tilde{\eta}\|_{t,1} \le 2\|\tilde{\eta}_0\|_1 \, , \quad \text{for } t \le T_{\tilde{\eta}_0} \, . \tag{3.11}$$

The same arguments as those leading to (2.12) imply then

$$\|\tilde{\eta}\|_{t,\infty} \le 2\|\tilde{\eta}_0\|_\infty \, , \quad \text{for } t \le T_{\tilde{\eta}_0} \, . \tag{3.12}$$

To show that $\|\tilde{\eta}_t\|_1$ tends to 0, we use the bound

$$\begin{aligned} \|\tilde{\eta}_t\|_1 &\le \frac{\sqrt{\pi}\|\tilde{\eta}_0\|_\infty}{t^{1/2}} + K_0 t^{1/2} \sup_{0\le\tau\le t} \|\tilde{\eta}_\tau\|_1^3 \\ &\le \frac{\sqrt{\pi}\|\tilde{\eta}_0\|_\infty}{t^{1/2}} + K_0 2^3 t^{1/2} \|\tilde{\eta}_0\|_1^3 \, , \end{aligned} \tag{3.13}$$

and

$$\|\tilde{\eta}_t\|_\infty \leq \|\tilde{\eta}_0\|_\infty + K_0 2^3 t^{1/2} \|\tilde{\eta}_0\|_\infty \|\tilde{\eta}_0\|_1^2 \,. \tag{3.14}$$

Introducing again

$$m_{t_1,t_2} = \sup_{t_1 \leq t \leq t_2} \|\tilde{\eta}_t\|_1 \,, \quad m_t = \|\tilde{\eta}_t\|_1 \,,$$

$$y_{t_1,t_2} = \sup_{t_1 \leq t \leq t_2} \|\tilde{\eta}_t\|_\infty \,, \quad y_t = \|\tilde{\eta}_t\|_\infty \,,$$

and defining

$$t_{n+1} = t_n + \frac{C^*}{m_{t_n}^3} \,,$$

we find a converging scheme as in Section 2.1. This completes the proof of Proposition 3.1.

4. The Full Equation

In this section, we provide the proof of Theorem 1.1. Since the proof is somewhat lengthy, we explain first the main steps. If we denote by s, η the two components of the problem, then the *linear* part of the problem is, in k-space, equal to a matrix operator $-L$, where L is of the form

$$L = \begin{pmatrix} k^2 + 2(1-q^2) & 2q\sqrt{1-q^2} \\ \frac{2q}{\sqrt{1-q^2}} k^2 & k^2 \end{pmatrix} .$$

We shall treat this operator differently for $|k| \leq 1$ and for $|k| \geq 1$. The large momenta are easy to handle since then L has spectrum in the right half-plane, which is bounded away from the imaginary axis. To handle the case $|k| \leq 1$, we diagonalize L by a transformation R,

$$R = \begin{pmatrix} 1 & \hat{B} \\ \hat{A} & 1 \end{pmatrix} , \tag{4.1}$$

where we define $\hat{A}$ by

$$(\hat{A}f)(k) = \begin{cases} 0 \,, & \text{if } |k| \geq 1 \,, \\ (Af)(k) \,, & \text{if } |k| < 1 \,, \end{cases} \tag{4.2}$$

and

$$A = \frac{\sqrt{1-q^2}(\Delta - 1)}{2q} , \tag{4.3}$$

with

$$\Delta = \sqrt{1 + 4q^2k^2/(1-q^2)^2} .$$

Similarly,

$$(\hat{B}f)(k) = \begin{cases} 0 , & \text{if } |k| \geq 1 , \\ (Bf)(k) , & \text{if } |k| < 1 , \end{cases} \tag{4.4}$$

where

$$B = \frac{-2q}{\sqrt{1-q^2}(1+\Delta)} . \tag{4.5}$$

Note that R commutes with multiplication by k and is the identity when $|k| \geq 1$.

We consider, for small momenta, $L_0 = RLR^{-1}$, and we work in the basis $(v, w) = R(s, \eta)$. In the new variables, (v, w), there is a "fast" coordinate, v, and a "slow" one, w, and therefore the ideas of the simplified model of Section 3 apply. We do not work directly with the operator $k/(k^2 + 2)$ we found there, but the choice of the norms which we shall use will allow us to transfer some regularity from the v variable onto the w variable, and this process mimicks the explicit regularization of $k/(k^2 + 2)$. In fact, $(k^2 + 2)^{-1/2}$ would have sufficed in the simplified model. Of course, the "positivity of the mass," i.e., the observation that for $k = 0$ one of the two eigenvalues of L is strictly positive, will also be used in all estimates of e^{-Lt} and $e^{-L_0 t}$.

4.1. The Operator L

We consider first the operator L. By the change of variables (4.1), we find, for $|k| \leq 1$,

$$RLR^{-1} = L_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} , \tag{4.6}$$

with

$$\begin{aligned} \lambda &= k^2 + (1-q^2)(1+\Delta) , \\ \mu &= k^2 + (1-q^2)(1-\Delta) , \\ \Delta &= \sqrt{1 + 4q^2k^2/(1-q^2)^2} . \end{aligned} \tag{4.7}$$

The Eckhaus instability occurs when μ behaves, for small k, like $-\text{const.}\,k^2$, with a positive constant. This occurs when $q^2 \geq 1/3$ and is the reason for restricting the analysis to $q^2 < 1/3$. One has

$$e^{-L_0 t} = \begin{pmatrix} e^{-\lambda t} & 0 \\ 0 & e^{-\mu t} \end{pmatrix} . \tag{4.8}$$

When $|k| > 1$ then we consider L itself and get

$$e^{-Lt} = \frac{1}{1 - AB} \begin{pmatrix} e^{-\lambda t} - ABe^{-\mu t} & B\big(e^{-\mu t} - e^{-\lambda t}\big) \\ A\big(e^{-\lambda t} - e^{-\mu t}\big) & e^{-\mu t} - ABe^{-\lambda t} \end{pmatrix} , \tag{4.9}$$

where A has been defined in (4.3) and B has been defined in (4.5). Note that

$$\begin{aligned} A &= \mathcal{O}(k^2) \text{ when } k \to 0 , \\ A &= \mathcal{O}(|k|) \text{ when } k \to \infty , \\ B &= \mathcal{O}(1) \text{ when } k \to 0 , \\ B &= \mathcal{O}(|k|^{-1}) \text{ when } k \to \infty . \end{aligned} \tag{4.10}$$

Finally, observe that $(1 - AB)^{-1}$ is uniformly bounded for $k \in \mathbf{R}$.

4.2. The Spaces

We begin by defining the spaces in which we are going to work. All norms are in momentum space, e.g., $\|kf\|_2$ is a short-hand notation for

$$\|kf\|_2 = \left(\int dk\, k^2 |f(k)|^2\right)^{1/2} .$$

Also, we omit the ˜ which denotes Fourier transform and work exclusively with functions in k-space. As in the proofs of the simplified model or in Section 3 we have spaces at "fixed" time, denoted by $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$, and spaces for intervals of time, denoted by $\mathcal{H}_T$, $\mathcal{K}_T$, and $\mathcal{L}_T$. We denote by $P_<$ and $P_>$ the operators of multiplication by the characteristic functions of $\{|k| \leq 1\}$ and $\{|k| \geq 1\}$. We also define $Q(k) = \min(|k|, 1)$.

Definition.

$$\mathcal{H} = \{X_0 = (s,\eta) \,:\, ks \in L^2,\ s \in L^1 \cap L^\infty,\ Q(k)\eta \in L^2,\ \eta \in L^1 \cap L^\infty\}\,,$$
$$\mathcal{K} = \{Y_0 = (v,w) \,:\, kv \in L^2,\ v \in L^1 \cap L^\infty,\ Q(k)w \in L^2,\ w \in L^1 \cap L^\infty\}\,,$$
$$\mathcal{L} = \{Z_0 = (f,g) \,:\, Q(k)f \in L^2,\ f \in L^1 \cap L^\infty,\ g \in L^2 \cap L^\infty\}\,.$$

Remark. The spaces $\mathcal{H}$ and $\mathcal{K}$ are equal, but we distinguish them and will view R as a map from $\mathcal{H}$ to $\mathcal{K}$. Also, the inclusion of the condition $s \in L^1$ is redundant, but convenient.

Since we shall use the strategy of Section 2.1 to show that the solution tends to zero, we again need to define the "integral" and the "sup" part of the norms, so we define

$$\begin{aligned} m_{\mathcal{H}}(X_0) &= \max(\|ks\|_2,\ \|s\|_1,\ \|Q(k)\eta\|_2,\ \|\eta\|_1)\,, \\ y_{\mathcal{H}}(X_0) &= \max(\|s\|_\infty,\ \|\eta\|_\infty)\,, \end{aligned} \tag{4.11}$$

and similarly for $\mathcal{K}$. For the space $\mathcal{L}$, we have by analogy

$$\begin{aligned} m_{\mathcal{L}}(Z_0) &= \max(\|Q(k)f\|_2,\ \|f\|_1,\ \|g\|_2)\,, \\ y_{\mathcal{L}}(Z_0) &= \max(\|f\|_\infty,\ \|g\|_\infty)\,. \end{aligned}$$

We consider on $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ the corresponding norms

$$\begin{aligned} \|X_0\|_{\mathcal{H}} &= \max(m_{\mathcal{H}}(X_0), y_{\mathcal{H}}(X_0))\,, \\ \|Y_0\|_{\mathcal{K}} &= \max(m_{\mathcal{K}}(Y_0), y_{\mathcal{K}}(Y_0))\,, \\ \|Z_0\|_{\mathcal{L}} &= \max(m_{\mathcal{L}}(Z_0), y_{\mathcal{L}}(Z_0))\,. \end{aligned}$$

We also define the spaces

$$\begin{aligned} \mathcal{H}_T &= \left\{X = \{X_t\}_{t\in[0,T]}\,,\ X_t \in \mathcal{H} \text{ for } t \in [0,T]\right\}\,, \\ \mathcal{K}_T &= \left\{Y = \{Y_t\}_{t\in[0,T]}\,,\ Y_t \in \mathcal{K} \text{ for } t \in [0,T]\right\}\,, \\ \mathcal{L}_T &= \left\{Z = \{Z_t\}_{t\in[0,T]}\,,\ Z_t \in \mathcal{L} \text{ for } t \in [0,T]\right\}\,, \end{aligned}$$

with the corresponding norms

$$\begin{aligned} \|X\|_{\mathcal{H}_T} &= \sup_{t\in[0,T]} \|X_t\|_{\mathcal{H}}\,, \\ \|Y\|_{\mathcal{K}_T} &= \sup_{t\in[0,T]} \|Y_t\|_{\mathcal{K}}\,, \\ \|Z\|_{\mathcal{L}_T} &= \sup_{t\in[0,T]} \|Z_t\|_{\mathcal{L}}\,. \end{aligned}$$

Finally, we extend the other definitions to the full spaces:

$$\begin{aligned} m_{\mathcal{H}_T}(X) &= \sup_{t\in[0,T]} m_{\mathcal{H}}(X_t) , \\ y_{\mathcal{H}_T}(X) &= \sup_{t\in[0,T]} y_{\mathcal{H}}(X_t) , \end{aligned} \tag{4.12}$$

and similarly for $\mathcal{K}$, $\mathcal{L}$.

4.3. The Integral Equation

Before we state the estimates, we reformulate the problem as an integral equation. We recall the main equation, written in x-space,

$$\begin{aligned} \partial_t s &= \partial_x^2 s + s - 3(1-q^2)s - 3\sqrt{1-q^2}\, s^2 - s^3 \\ &\quad - \sqrt{1-q^2}\, \eta^2 - 2\sqrt{1-q^2}\, q\eta - sq^2 - 2qs\eta - s\eta^2 , \\ \partial_t \eta &= \partial_x^2 \eta + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2}+s} \right) q + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2}+s} \eta \right) . \end{aligned} \tag{4.13}$$

We define the nonlinearities, written in x-space, by

$$\mathcal{N}_0(s,\eta) = -3\sqrt{1-q^2}\, s^2 - s^3 - \sqrt{1-q^2}\, \eta^2 - 2qs\eta - s\eta^2 , \tag{4.14}$$

$$\mathcal{N}_1(s,\eta) = 2q \left(\frac{\partial_x s}{\sqrt{1-q^2}+s} - \frac{\partial_x s}{\sqrt{1-q^2}} \right) + 2\frac{\partial_x s}{\sqrt{1-q^2}+s}\eta .$$

Note that the overall derivative in the second component has been omitted. It will be taken care of below. This term will in fact show that the derivative in the nonlinearity in the second equation of (4.13) is *compensated* by the semigroup. Define $\mathcal{M}$ by

$$\mathcal{M} : Z \to \mathcal{M}Z ; \quad (\mathcal{M}Z)_t = \int_0^t d\tau\, e^{-\tau L} Z_{t-\tau} . \tag{4.15}$$

The problem we consider is of the form: Given $X_0 \in \mathcal{H}$, and $T > 0$, try to find an $X \in \mathcal{H}_T$ such that

$$X = \mathcal{T} X , \tag{4.16}$$

where

$$(\mathcal{T}X)_t = e^{-Lt}X_0 + (\mathcal{M}\mathcal{D}\mathcal{N}(X))_t \,, \tag{4.17}$$

with $X \in \mathcal{H}_T$, $\mathcal{N} = (\mathcal{N}_0, \mathcal{N}_1)$ and $\mathcal{D}$ defined by

$$\mathcal{D}\begin{pmatrix} v \\ w \end{pmatrix}(k) = \begin{pmatrix} v(k) \\ ikw(k) \end{pmatrix} ;$$

in other words, $\mathcal{D}$ is the operator we omitted in the operator $\mathcal{N}_1$.

Recall that $P_<$ and $P_>$ denote the operators of multiplication by the characteristic functions of $\{|k| \le 1\}$ and $\{|k| \ge 1\}$. Then we can rewrite the operator $\mathcal{T}$ in the form

$$\mathcal{T}X = \mathcal{T}_<X + \mathcal{T}_>X \,,$$

where

$$(\mathcal{T}_>X)_t = e^{-Lt}P_>X_0 + (\mathcal{M}\mathcal{D}P_>\mathcal{N}(X))_t \,.$$

We really work on $\mathcal{K} = R\mathcal{H}$ (see below) and therefore, we consider instead of $\mathcal{T}$ the operator $\mathcal{U} = R\mathcal{T}R^{-1}$. We next rewrite $\mathcal{U}_<$ as

$$(\mathcal{U}_<Y)_t = e^{-L_0t}P_<Y_0 + \int_0^t d\tau\, e^{-L_0\tau}\mathcal{D}P_<R(\mathcal{N}(R^{-1}Y))_{t-\tau} \,, \tag{4.18}$$

where $L_0 = RLR^{-1}$. Recall the definition of R, Eq.(4.1). Since the operator $\hat{A}$ has an explicit factor of k^2 by Eq.(4.10), we can write $R\mathcal{D}P_< = \mathcal{D}SP_<$, where

$$S = i\begin{pmatrix} 1 & k\hat{B} \\ \frac{1}{k}\hat{A} & 1 \end{pmatrix} , \tag{4.19}$$

and hence we shall consider

$$\begin{aligned}(\mathcal{U}_<Y)_t &= e^{-L_0t}P_<Y_0 + \int_0^t d\tau\, e^{-L_0\tau}\mathcal{D}SP_<(\mathcal{N}(R^{-1}Y))_{t-\tau} \,, \\ (\mathcal{U}_>Y)_t &= e^{-Lt}P_>Y_0 + (\mathcal{M}\mathcal{D}P_>\mathcal{N}(R^{-1}Y))_t \,.\end{aligned} \tag{4.20}$$

Some factors of R and R^{-1} have been omitted in the last equation since $R^{\pm 1}P_> = P_>$ (strictly speaking, they should have been replaced by the natural isomorphisms between $\mathcal{H}$ and $\mathcal{K}$). Note that the operators R, S, L, L_0 and $\mathcal{D}$, but not $\mathcal{N}$, commute with multiplication by k. We will control the operator $\mathcal{U}$ by studying the operators as maps between the following spaces:

$$R : \mathcal{H} \to \mathcal{K} \,, \quad R^{-1} : \mathcal{K} \to \mathcal{H} \,, \quad \mathcal{N} : \mathcal{H} \to \mathcal{L} \,, \quad \mathcal{M} : \mathcal{L} \to \mathcal{K} \,, \quad S : \mathcal{L} \to \mathcal{L} \,.$$

4.4. The Operators R and S

Remark. Here, and in the sequel, K denotes a constant which may vary from equation to equation, and which is independent of the ingredients of the equation, except of q, but which is bounded uniformly in q for $|q| < q_0 < 3^{-1/2}$.

In this subsection, we bound R, R^{-1}, and S. Since R, and S are nontrivial only for $|k| \leq 1$ and since R and R^{-1} have the same form up to the factor $1/(1-AB)$, and signs, it suffices to bound the operators $\hat{A}$, $k^{-1}\hat{A}$, $\hat{B}$ and $k\hat{B}$. In the next two lemmas we bound R, resp. S. Recall that $\hat{A}P_> = \hat{B}P_> = 0$.

Lemma 4.1. *One has the following bounds*

$$\begin{aligned} \|k\hat{A}P_< s\|_2 &\leq K\|ks\|_2 , & (4.21)\\ \|\hat{A}P_< s\|_1 &\leq K\|s\|_1 , & (4.22)\\ \|\hat{A}P_< s\|_\infty &\leq K\|s\|_\infty , & (4.23)\\ \|k\hat{B}P_<\eta\|_2 &\leq K\|Q(k)\eta\|_2 , & (4.24)\\ \|\hat{B}P_<\eta\|_1 &\leq K\|\eta\|_1 , & (4.25)\\ \|\hat{B}P_<\eta\|_\infty &\leq K\|\eta\|_\infty . & (4.26) \end{aligned}$$

Lemma 4.2. *One has the following bounds*

$$\begin{aligned} \|k^{-1}\hat{A}P_< f\|_2 &\leq K\|P_< kf\|_2 , & (4.27)\\ \|k^{-2}\hat{A}P_< f\|_\infty &\leq K\|f\|_\infty , & (4.28)\\ \|k(k\hat{B}P_< g)\|_2 &\leq K\|g\|_2 , & (4.29)\\ \|k\hat{B}P_< g\|_1 &\leq K\|g\|_2 , & (4.30)\\ \|k\hat{B}P_< g\|_\infty &\leq K\|g\|_\infty . & (4.31) \end{aligned}$$

Proof. Since, by Eq.(4.10), we have $|k^{-2}\hat{A}| < K$, and $\hat{B} = \mathcal{O}(1)$, the proof is obvious.

Corollary 4.3. *The following linear operators are bounded:*

$$R : \mathcal{H} \to \mathcal{K} , \quad R^{-1} : \mathcal{K} \to \mathcal{H} , \quad SP_< : \mathcal{L} \to \mathcal{L} .$$

Furthermore,

$$m_{\mathcal{K}_T}(RX) \le K m_{\mathcal{H}_T}(X)\,, \quad y_{\mathcal{K}_T}(RX) \le K y_{\mathcal{H}_T}(X)\,,$$
$$m_{\mathcal{H}_T}(R^{-1}Y) \le K m_{\mathcal{K}_T}(Y)\,, \quad y_{\mathcal{H}_T}(R^{-1}Y) \le K y_{\mathcal{K}_T}(Y)\,. \tag{4.32}$$

Finally,

$$m_{\mathcal{L}_T}(SP_< Z) \le K m_{\mathcal{L}_T}(Z)\,, \quad y_{\mathcal{L}_T}(SP_< Z) \le K y_{\mathcal{L}_T}(Z)\,. \tag{4.33}$$

Proof. The proof is obvious from the definitions of the spaces, and from (4.21)–(4.31).

4.5. The Linear Semigroup for $|k| \le 1$

Here, we consider the semigroup $e^{-L_0 t}P_<$ acting on $\mathcal{K}$. We denote the matrix elements of $e^{-L_0 t}P_<$ by $G_{ij}(t)$, $i, j = 0, 1$. Note that $G_{10} = G_{01} = 0$. From the definition (4.7) of λ, μ, we have the bounds, valid for $q^2 < 1/3$: $\lambda \ge k^2 + 1$ and $\mu \ge k^2(1 - 3q^2)$. Therefore,

$$|G_{00}(t)| \le e^{-(k^2+1)t}\,,$$
$$|G_{11}(t)| \le e^{-(1-3q^2)k^2 t}\,. \tag{4.34}$$

This implies that

$$\|G_{00}(t)\|_p \le K e^{-t}\,, \text{ for } p = 1, 2\,,$$
$$\|G_{00}(t)\|_\infty \le 1\,,$$
$$\|G_{11}(t)\|_\infty \le 1\,, \tag{4.35}$$
$$\|kG_{11}(t)\|_2 \le K(1+t)^{-3/4}\,,$$
$$\|G_{11}(t)\|_1 \le K(1+t)^{-1/2}\,.$$

The last two inequalities are obtained by bounding the integral over k either by the sup of the integrand (which leads to an $\mathcal{O}(1)$ bound since $|k| \le 1$), or by integrating over all of k which leads to the inverse powers of t. Using (4.34), we find

$$\|G_{00}(t)v\|_{\infty} \leq \|v\|_{\infty}, \quad \|G_{11}(t)w\|_{\infty} \leq \|w\|_{\infty}. \tag{4.36}$$

Furthermore, we have

$$\begin{aligned} \|kG_{00}(t)v\|_2 &\leq e^{-t}\|kv\|_2, \quad \|G_{00}(t)v\|_1 \leq e^{-t}\|v\|_1, \\ \|kG_{11}(t)w\|_2 &\leq 4\|P_{<}kw\|_2, \quad \|G_{11}(t)w\|_1 \leq 4\|w\|_1. \end{aligned} \tag{4.37}$$

Finally, as in Eq.(2.13), we shall need the more interesting bounds where the integral norms are bounded in terms of the sup norm, which are readily obtained from (4.35):

$$\begin{aligned} \|kG_{00}(t)v\|_2 &\leq Ke^{-t}\|v\|_{\infty}, \\ \|G_{00}(t)v\|_1 &\leq Ke^{-t}\|v\|_{\infty}, \\ \|kG_{11}(t)w\|_2 &\leq K(t+1)^{-3/4}\|w\|_{\infty}, \\ \|G_{11}(t)w\|_1 &\leq K(t+1)^{-1/2}\|w\|_{\infty}. \end{aligned} \tag{4.38}$$

4.6. The Linear Semigroup for $|k| \geq 1$

We next produce the analogous bounds for the case when $|k| \geq 1$, i.e., we work on the space $P_{>}\mathcal{H}$. Using now the form (4.9) for $e^{-Lt}P_{>}$, we call its matrix elements $H_{ij}(t)$. Recall that $\lambda > k^2$ and $\mu \geq k^2(1-3q^2) \equiv k^2\rho$. Note that ρ depends on q, but for fixed $q^2 < 1/3$, ρ is a positive (q-dependent) constant. Since K is also defined to be q-dependent, the estimates in this section are valid for every fixed $q^2 < 1/3$. We get, using always that $|(1-AB)^{\pm 1}| \leq K$, and $|k| \geq 1$,

$$\begin{aligned} |H_{00}(t)| &\leq Ke^{-\rho t - k^2\rho t}, \\ |H_{01}(t)| &\leq K\frac{1}{|k|}e^{-\rho t - k^2\rho t}, \\ |H_{11}(t)| &\leq Ke^{-\rho t - k^2\rho t}. \end{aligned} \tag{4.39}$$

In order to bound H_{10}, we need to take into account the cancellation of the two terms. Recall that $H_{10}(t) = A(1-AB)^{-1}(e^{-\lambda t} - e^{-\mu t})P_{>}$, and that

$\lambda - \mu = 2\Delta(1 - q^2)$. Thus, we can write

$$H_{10}(t) = \frac{-A}{1-AB} e^{-\mu t}(1 - e^{-2\Delta(1-q^2)t}) P_> \, .$$

We also have $\Delta = \mathcal{O}(|k|)$ and $A = \mathcal{O}(|k|)$ as $k \to \infty$, so that we find a bound of the form

$$|H_{10}(t)| \le K|k| e^{-\rho t - k^2 \rho t}(1 - e^{-\mathcal{O}(1)|k|t}) \, .$$

When $|k|t > 1$, we can bound $(1 - e^{-\mathcal{O}(1)|k|t})$ by 2 and get

$$|H_{10}(t)| \le Kk^2 t e^{-\rho t - k^2 \rho t} \le K e^{-\rho t - k^2 \rho t/2} \, ,$$

and when $|k|t < 1$, we bound $(1 - e^{-\mathcal{O}(1)|k|t}) \le \mathcal{O}(|k|t)$ and hence we find in all cases,

$$|H_{10}(t)| \le K e^{-\rho t - k^2 \rho t/2} \, . \tag{4.40}$$

Clearly, for all $p \ge 0$, we have bounds of the form

$$|k^p H_{ij}(t)| \le K_p t^{-p/2} e^{-\rho t - k^2 \rho t/2} \, .$$

We can now proceed to the analogues of the bounds on G_{ij}. We have

$$\|H_{ij}(t)h\|_\infty \le K e^{-\rho t} \|h\|_\infty \, . \tag{4.41}$$

Furthermore, we get

$$\begin{aligned}
\|kH_{00}(t)s\|_2 &\le Ke^{-\rho t}\|ks\|_2 \, , & \|H_{00}(t)s\|_1 &\le Ke^{-\rho t}\|s\|_1 \, , \\
\|kH_{01}(t)\eta\|_2 &\le Ke^{-\rho t}\|P_>\eta\|_2 \, , & \|H_{01}(t)\eta\|_1 &\le Ke^{-\rho t}\|\eta\|_1 \, , \\
\|H_{10}(t)s\|_2 &\le Ke^{-\rho t}\|ks\|_2 \, , & \|H_{10}(t)s\|_1 &\le Ke^{-\rho t}\|s\|_1 \, , \\
\|H_{11}(t)\eta\|_2 &\le Ke^{-\rho t}\|P_>\eta\|_2 \, , & \|H_{11}(t)\eta\|_1 &\le Ke^{-\rho t}\|\eta\|_1 \, .
\end{aligned} \tag{4.42}$$

Finally, one can bound the integral norms in terms of the sup norms and one gets

$$\begin{aligned}
\|kH_{00}(t)s\|_2 &\le Kt^{-3/4}e^{-\rho t}\|s\|_\infty \, , & \|H_{00}(t)s\|_1 &\le Kt^{-1/2}e^{-\rho t}\|s\|_\infty \, , \\
\|kH_{01}(t)\eta\|_2 &\le Kt^{-1/4}e^{-\rho t}\|\eta\|_\infty \, , & \|H_{01}(t)\eta\|_1 &\le Kt^{-1/2}e^{-\rho t}\|\eta\|_\infty \, , \\
\|H_{10}(t)s\|_2 &\le Kt^{-1/4}e^{-\rho t}\|s\|_\infty \, , & \|H_{10}(t)s\|_1 &\le Kt^{-1/2}e^{-\rho t}\|s\|_\infty \, , \\
\|H_{11}(t)\eta\|_2 &\le Kt^{-1/4}e^{-\rho t}\|\eta\|_\infty \, , & \|H_{11}(t)\eta\|_1 &\le Kt^{-1/2}e^{-\rho t}\|\eta\|_\infty \, .
\end{aligned} \tag{4.43}$$

4.7. The Inhomogeneous Term

We combine here the estimates of the last two subsections to formulate the bounds on the inhomogeneous term $e^{-Lt}X_0$ for later use.

Proposition 4.4. *Let $X_0 \in \mathcal{H}$, $Y_0 \in \mathcal{K}$. There is a constant C_1 for which the operators e^{-Lt}, $e^{-L_0 t}$ satisfy*

$$\begin{aligned} m_{\mathcal{H}_T}(\{e^{-Lt}X_0\}_{t\in[0,T]}) &\leq C_1 m_{\mathcal{H}}(X_0)\,, && (4.44)\\ y_{\mathcal{H}_T}(\{e^{-Lt}X_0\}_{t\in[0,T]}) &\leq C_1 y_{\mathcal{H}}(X_0)\,, && (4.45)\\ y_{\mathcal{K}}(e^{-L_0 t}P_< Y_0) &\leq y_{\mathcal{K}}(P_< Y_0)\,, && (4.46)\\ y_{\mathcal{H}}(e^{-Lt}P_> X_0) &\leq C_1 e^{-t/2} y_{\mathcal{H}}(P_> X_0)\,. && (4.47) \end{aligned}$$

Finally, there is a $t_0 > 0$ such that for $t > t_0$,

$$\begin{aligned} m_{\mathcal{H}}(e^{-Lt}X_0) &\leq C_1(t+1)^{-1/2} y_{\mathcal{H}}(X_0)\,, && (4.48)\\ y_{\mathcal{K}}(e^{-L_0 t}P_< Y_0 + Re^{-Lt}R^{-1}P_> Y_0) &\leq y_{\mathcal{K}}(Y_0)\,. && (4.49) \end{aligned}$$

Proof. The inequality (4.44) follows from (4.37) and (4.42). The inequality (4.45) follows from (4.36) and (4.41). Eq.(4.46) follows from (4.36) and Eq.(4.47) follows from (4.41). Eq.(4.48) follows from (4.38) and (4.43). Finally, choosing $t_0 \geq 2\log(C_1)$, we see that (4.46), (4.47) and $R^{\pm 1}P_> = P_>$ imply (4.49).

4.8. The Nonlinearity

We consider the nonlinearity in (4.14) as a map from $\mathcal{H}_T$ to $\mathcal{L}_T$. Note that the norms have been chosen to make these estimates easy. We need the following version of the Sobolev inequalities:

Lemma 4.5. *Let $kf \in L^2$ and $f \in L^\infty$. Then we have $f \in L^2$ and*

$$\|f\|_2 \leq 3\|f\|_\infty^{2/3}\|kf\|_2^{1/3}\,. \qquad (4.50)$$

Proof. Assume for simplicity that $f \geq 0$. To prove (4.50), we use

$$\|f\|_2^2 = \int_{|k|\leq\alpha} dk f^2(k) + \int_{|k|\geq\alpha} dk\, k^2 f^2(k)\frac{1}{k^2}$$
$$= 2\alpha\|f\|_\infty^2 + \frac{1}{\alpha^2}\|kf\|_2^2 .$$

Setting $\alpha = \frac{1}{2}\|kf\|_2^{2/3}\|f\|_\infty^{-2/3}$, the bound (4.50) follows.

Lemma 4.6. *There are constants $\beta_0 > 0$, $C_2 > 1$, such that the following is true if $y \leq \beta_0$, $m \leq \beta_0$: Assume that*

$$m_{\mathcal{H}}(X_0) \leq m\,, \quad y_{\mathcal{H}}(X_0) \leq y\,.$$

Then one has, in k-space

$$\|Q(k)\mathcal{N}_0(X_0)\|_2 \leq C_2(m^2 + m^{3/2}y^{1/2})\,, \tag{4.51}$$
$$\|\mathcal{N}_0(X_0)\|_1 \leq C_2 m^2\,, \tag{4.52}$$
$$\|\mathcal{N}_1(X_0)\|_2 \leq C_2 m^2\,, \tag{4.53}$$
$$\|\mathcal{N}_0(X_0)\|_\infty \leq C_2 my\,, \tag{4.54}$$
$$\|\mathcal{N}_1(X_0)\|_\infty \leq C_2(m^{4/3}y^{2/3} + m^{3/2}y^{1/2})\,. \tag{4.55}$$

Proof. The proof follows by a multiple application of the Young inequality. We denote convolution by $*$, and assume $X_0 = (s, \eta)$. For example, in order to prove (4.52), we observe that

$$\|s * s\|_1 \leq \|s\|_1^2 \leq m^2\,, \quad \|\eta * \eta\|_1 \leq \|\eta\|_1^2 \leq m^2\,,$$

and all other terms in $\mathcal{N}_0$ lead to even better bounds. In order to bound the difference term in $\mathcal{N}_1$, we write it as a geometric series, and bound each term individually. The most dangerous term is, written in x-space,

$$\frac{2q}{\sqrt{1-q^2}}\frac{-s}{\sqrt{1-q^2}}\partial_x s\,, \tag{4.56}$$

and this is bounded by $K\|s\|_1\|ks\|_2 \leq Km^2$. All other terms are smaller and the geometric series converges. This proves (4.53). The bounds for the norms in L^∞ are obtained similarly. For example, we have

$$\|s * s\|_\infty \leq \|s\|_1\|s\|_\infty \leq my\,,$$
$$\|\eta * \eta\|_\infty \leq \|\eta\|_1\|\eta\|_\infty \leq my\,.$$

Again, for $\mathcal{N}_1$, we use the geometric series, and get, for example for the term (4.56), a bound of the form

$$K\|(ks)*s\|_\infty \le K\|ks\|_2\|s\|_2 \le K\|ks\|_2^{4/3}\|s\|_\infty^{2/3} \le Km^{4/3}y^{2/3}, \quad (4.57)$$

by Eq.(4.50). Similarly,

$$K\|(ks)*\eta\|_\infty \le K\|ks\|_2\|\eta\|_2 \le K\|ks\|_2\|\eta\|_1^{1/2}\|\eta\|_\infty^{1/2} \le Km^{3/2}y^{1/2}. \quad (4.58)$$

These bounds prove (4.53)–(4.55).

Finally, we prove Eq.(4.51). We have

$$\|kP_<(s*s)\|_2 = 2\|P_<\big((ks)*s\big)\|_2 \le 2\|(ks)*s\|_2 \le 2\|ks\|_2\|s\|_1 \le m^2 .$$

Similarly,

$$\|kP_<(\eta*\eta)\|_2 \le \|P_<(\eta*\eta)\|_2 \le \|\eta\|_2\|\eta\|_1 \le \|\eta\|_1^{3/2}\|\eta\|_\infty^{1/2} \le m^{3/2}y^{1/2} .$$

For $|k| > 1$, we have

$$\|P_>(s*s))\|_2 \le \|P_>k(s*s)\|_2 \le 2\|(ks)*s\|_2 \le 2\,\|ks\|_2\|s\|_1 \le m^2 ,$$

and

$$\|P_>(\eta*\eta)\|_2 \le \|\eta\|_2\|\eta\|_1 \le \|\eta\|_1^{3/2}\|\eta\|_\infty^{1/2} \le m^{3/2}y^{1/2} .$$

All other terms are smaller, and hence (4.51) follows.

We finally bound the tangent map.

Corollary 4.7. *There are constants $\beta_0 > 0$, $C_2 > 1$, such that the following holds when $y < \beta_0$ and $m < \beta_0$: Assume that $X_0 \in \mathcal{H}$ and that*

$$m_{\mathcal{H}}(X_0) < m , \quad \|X_0\|_{\mathcal{H}} < z .$$

Assume $V_0 \in \mathcal{H}$, and let $m' = m_{\mathcal{H}}(V_0)$, $\|V_0\|_{\mathcal{H}} = z'$. Then the tangent map $D\mathcal{N}$ satisfies

$$\|Q(k)D\mathcal{N}_{0,X_0}V_0\|_2 \le C_2(m + m^{1/2}z^{1/2})m' , \quad (4.59)$$

$$\|D\mathcal{N}_{0,X_0}V_0\|_1 \le C_2mm' , \quad (4.60)$$

$$\|D\mathcal{N}_{1,X_0}V_0\|_2 \le C_2mm' , \quad (4.61)$$

$$\|D\mathcal{N}_{0,X_0}V_0\|_\infty \le C_2mz' , \quad (4.62)$$

$$\|D\mathcal{N}_{1,X_0}V_0\|_\infty \le C_2(mz'^{2/3}m'^{1/3} + m^{1/3}z^{2/3}m') . \quad (4.63)$$

Proof. Note that $\|X_0\|_{\mathcal{H}} = \max(y_{\mathcal{H}}(X_0), m_{\mathcal{H}}(X_0))$. The proof follows by polarization from the proof of Lemma 4.6. The critical case in (4.63) is handled as follows: Let $X_0 = (s,\eta)$, $V_0 = (s',\eta')$. To prove (4.63), we have to bound a term of the form $\|(ks) * s'\|_\infty$. Using (4.57), (4.58), we see that

$$\|(ks) * s'\|_\infty \le K\|ks\|_2\|ks'\|_2^{1/3}\|s'\|_\infty^{2/3} ,$$
$$\|(ks) * \eta'\|_\infty \le K\|ks\|_2\|\eta'\|_1^{1/2}\|\eta'\|_\infty^{1/2} .$$

Similarly,

$$\|(ks') * s\|_\infty \le K\|ks\|_2^{1/3}\|s\|_\infty^{2/3}\|ks'\|_2 ,$$
$$\|(ks') * \eta\|_\infty \le K\|\eta\|_1^{1/2}\|\eta\|_\infty^{1/2}\|ks'\|_2 .$$

Other such terms are handled analogously and, using $m' \le z'$, these bounds prove (4.63).

Without loss of generality, we assume the constants in Lemma 4.6 and Corollary 4.7 are the same.

4.9. The Operator $\mathcal{M}$

In this subsection, we bound the operator $\mathcal{M}$ as a map from $\mathcal{L}$ to $\mathcal{H}$ (or $\mathcal{K}$, which is the same).

The bounds are divided into two groups, one for $P_<$ and one for $P_>$. We begin by estimating the matrix elements of $\int d\tau e^{L_0\tau}$. These estimates are all based on the following type of inequality:

$$\|k\int_0^t d\tau\, G_{00}(\tau)f_{t-\tau}\|_2 \le \int_0^t d\tau\, \|G_{00}(\tau)\|_\infty \sup_{0\le\tau\le t} \|kf_\tau\|_2 .$$

We shall henceforth write $\sup_\tau$ instead of $\sup_{0\le\tau\le t}$. We will use various norms and powers of k, but the principle will always remain the same.

We have the bounds, using (4.34), (4.35),

$$\int_0^t d\tau \, \|G_{00}(\tau)\|_\infty \le K ,$$
$$\int_0^t d\tau \, \|kG_{11}(\tau)\|_2 \le Kt^{1/4} ,$$
$$\int_0^t d\tau \, \|k^2 G_{11}(\tau)\|_\infty \le K \log(1+t) ,$$
$$\int_0^t d\tau \, \|kG_{11}(\tau)\|_\infty \le Kt^{1/2} .$$

The bounds on G_{ij} which we need below are:

$$\begin{aligned} &\|k \int_0^t d\tau \, G_{00}(\tau) f_{t-\tau}\|_2 \\ &\quad \le \int_0^t d\tau \, \|G_{00}(\tau)\|_\infty \sup_\tau \|P_< k f_\tau\|_2 \le K \sup_\tau \|Q(k) f_\tau\|_2 , \\ &\| \int_0^t d\tau \, G_{00}(\tau) f_{t-\tau}\|_1 \le \int_0^t d\tau \, \|G_{00}(\tau)\|_\infty \sup_\tau \|f_\tau\|_1 \le K \sup_\tau \|f_\tau\|_1 . \end{aligned} \tag{4.64}$$

Similarly,

$$\begin{aligned} &\|k \int_0^t d\tau \, G_{11}(\tau) k g_{t-\tau}\|_2 \\ &\qquad \le \int_0^t d\tau \, \|k^2 G_{11}(\tau)\|_\infty \sup_\tau \|g_\tau\|_2 \le Kt^{1/4} \sup_\tau \|g_\tau\|_2, \\ &\| \int_0^t d\tau \, G_{11}(\tau) k g_{t-\tau}\|_1 \le \int_0^t d\tau \, \|kG_{11}(\tau)\|_2 \sup_\tau \|g_\tau\|_2 \le Kt^{1/4} \sup_\tau \|g_\tau\|_2 . \end{aligned} \tag{4.65}$$

We obtain similar bounds for the L^∞ norms:

$$\begin{aligned} &\| \int_0^t d\tau \, G_{00}(\tau) f_{t-\tau}\|_\infty \le K \sup_\tau \|f_\tau\|_\infty , \\ &\| \int_0^t d\tau \, G_{11}(\tau) k g_{t-\tau}\|_\infty \le Kt^{1/2} \sup_\tau \|g_\tau\|_\infty , \\ &\| \int_0^t d\tau \, G_{11}(\tau) k^2 g_{t-\tau}\|_\infty \le K \log(1+t) \sup_\tau \|g_\tau\|_\infty . \end{aligned} \tag{4.66}$$

We next handle the momenta $|k| \geq 1$. Then, by Eq.(4.39), (4.40), we have

$$\int_0^t d\tau \, \|kH_{00}(\tau)\|_\infty \leq K \,, \quad \int_0^t d\tau \, \|k^2 H_{01}(\tau)\|_\infty \leq K \,,$$
$$\int_0^t d\tau \, \|kH_{01}(\tau)\|_2 \leq K \,, \quad \int_0^t d\tau \, \|kH_{10}(\tau)\|_\infty \leq K \,,$$
$$\int_0^t d\tau \, \|kH_{11}(\tau)\|_2 \leq K \,, \quad \int_0^t d\tau \, \|kH_{11}(\tau)\|_\infty \leq K \,.$$

All these inequalities follow by straightforward integration. Using these inequalities, we find

$$\begin{aligned}\|k \int_0^t d\tau \, H_{00}(\tau) f_{t-\tau}\|_2 &\leq K \int_0^t d\tau \, \|kH_{00}(\tau)\|_\infty \sup_\tau \|P_> f_\tau\|_2 \\ &\leq K \sup_\tau \|P_> f_\tau\|_2 \,, \\ \|\int_0^t d\tau \, H_{00}(\tau) f_{t-\tau}\|_1 &\leq K \int_0^t d\tau \, \|H_{00}(\tau)\|_\infty \sup_\tau \|f_\tau\|_1 \leq K \sup_\tau \|f_\tau\|_1 \,.\end{aligned} \tag{4.67}$$

Next,

$$\begin{aligned}\|k \int_0^t d\tau \, H_{01}(\tau) k g_{t-\tau}\|_2 &\leq \int_0^t d\tau \, \|k^2 H_{01}(\tau)\|_\infty \sup_\tau \|g_\tau\|_2 \leq K \sup_\tau \|g_\tau\|_2, \\ \|\int_0^t d\tau \, H_{01}(\tau) k g_{t-\tau}\|_1 &\leq \int_0^t d\tau \, \|kH_{01}(\tau)\|_2 \sup_\tau \|g_\tau\|_2 \leq K \sup_\tau \|g_\tau\|_2.\end{aligned} \tag{4.68}$$

For H_{10}, we get

$$\begin{aligned}\|\int_0^t d\tau \, H_{10}(\tau) f_{t-\tau}\|_2 &\leq \int_0^t d\tau \, \|H_{10}(\tau)\|_\infty \sup_\tau \|P_> f_\tau\|_2 \leq K \sup_\tau \|P_> f_\tau\|_2, \\ \|\int_0^t d\tau \, H_{10}(\tau) f_{t-\tau}\|_1 &\leq \int_0^t d\tau \, \|H_{10}(\tau)\|_\infty \sup_\tau \|f_\tau\|_1 \leq K \sup_\tau \|f_\tau\|_1.\end{aligned} \tag{4.69}$$

Finally,

$$\begin{aligned}\|\int_0^t d\tau \, H_{11}(\tau) k g_{t-\tau}\|_1 &\leq \int_0^t d\tau \, \|kH_{11}(\tau)\|_2 \sup_\tau \|g_\tau\|_2 \leq K \sup_\tau \|g_\tau\|_2 \,, \\ \|\int_0^t d\tau \, H_{11}(\tau) k g_{t-\tau}\|_2 &\leq \int_0^t d\tau \, \|kH_{11}(\tau)\|_\infty \sup_\tau \|g_\tau\|_2 \leq K \sup_\tau \|g_\tau\|_2 \,.\end{aligned} \tag{4.70}$$

The L^∞ bounds are easy, and we get, for $i,j \in \{0,1\}$,

$$\| \int_0^t d\tau\, H_{ij}(\tau) k h_{t-\tau} \|_\infty \le \sup_\tau \|h_\tau\|_\infty \,. \tag{4.71}$$

4.10. The Completion of the Proof of Theorem 1.1

In this section, we combine the various terms to show, first, that the map $\mathcal{U}$, defined in Eq.(4.20) is a contraction of a small ball in $\mathcal{K}_T$. Then, we shall show that the estimates on $m_{\mathcal{K}_T}$ can be iterated and lead to the convergence of the solution to zero.

Let $T > 0$ be given. This is the time interval during which we control the solution of Eq.(4.16). We shall fix it below. Let $Y \in \mathcal{K}_T$. We start by giving bounds on the nonlinear contribution to $\mathcal{U}$, which, at fixed time t, is given by

$$\begin{aligned} &\big(R\mathcal{M}\mathcal{D}\mathcal{N}(R^{-1}Y)\big)_t \\ &= \big(\mathcal{M}\mathcal{D}P_>\mathcal{N}(R^{-1}Y)\big)_t + \int_0^t d\tau\, e^{-L_0\tau}\mathcal{D}SP_<(\mathcal{N}(R^{-1}Y))_{t-\tau} \\ &\equiv \big(\mathcal{M}\mathcal{D}P_>\mathcal{N}(R^{-1}Y)\big)_t + \big(\mathcal{M}_0\mathcal{D}SP_<\mathcal{N}(R^{-1}Y)\big)_t \,. \end{aligned}$$

By Corollary 4.3, the operator R^{-1} is bounded. Therefore, there is a constant $\beta_1 > 0$ such that if

$$\|Y\|_{\mathcal{K}_T} \le \beta_1 \,, \tag{4.72}$$

then $\|R^{-1}Y\|_{\mathcal{H}_T} < \beta_0$, so that Lemma 4.6 applies. Henceforth, we assume (4.72) holds. In fact, we shall make increasingly stronger assumptions throughout the proof, which guarantee that at every step of the proof all quantities are sufficiently small for the various bounds to apply. Using (4.51)–(4.55), we find that

$$\|\mathcal{N}(R^{-1}Y)\|_{\mathcal{L}_T} \le K\|Y\|_{\mathcal{K}_T}^2 \,. \tag{4.73}$$

In fact, while this estimate is very suggestive, we will use directly the bounds obtained from Corollary 4.7 to control the tangent map, because (4.73) is not a good enough bound to prove convergence in time. We start by estimating the tangent map of the nonlinearity.

Proposition 4.8. *Let β_1 be the constant defined above. There is a constant D_1, with $1 < D_1 < \infty$, such that if $\|Y\|_{\mathcal{K}_T} < \beta_1$, then the derivative of the nonlinear term is bounded by*

$$\begin{aligned} m_{\mathcal{K}_T}\big(DR\mathcal{M}\mathcal{D}\mathcal{N}(R^{-1}Y)Y'\big) &\le D_1(1+T)^{1/4} \\ &\quad \times\big(m_{\mathcal{K}_T}(Y)+m_{\mathcal{K}_T}(Y)^{1/2}\|Y\|_{\mathcal{K}_T}^{1/2}\big)m_{\mathcal{K}_T}(Y') , \qquad (4.74) \\ y_{\mathcal{K}_T}\big(DR\mathcal{M}\mathcal{D}\mathcal{N}(R^{-1}Y)Y'\big) &\le D_1(1+T)^{1/4}m_{\mathcal{K}_T}(Y)\,\|Y'\|_{\mathcal{K}_T} \\ &\quad +D_1(1+T)^{1/2}m_{\mathcal{K}_T}(Y)^{1/3}\,\|Y\|_{\mathcal{K}_T}^{2/3}\,m_{\mathcal{K}_T}(Y') \qquad (4.75) \\ &\quad +D_1(1+T)^{1/2}m_{\mathcal{K}_T}(Y)\,\|Y'\|_{\mathcal{K}_T}^{2/3}\,m_{\mathcal{K}_T}(Y')^{1/3} . \end{aligned}$$

Proof. We start by considering $\mathcal{D}\mathcal{N}$. We have from (4.62), (4.63), the bounds

$$\begin{aligned} y_{\mathcal{L}_T}(D\mathcal{N}_{0,X}V_0) &\le C_2 m_{\mathcal{H}_T}(X)\cdot\|V_0\|_{\mathcal{H}_T} , \\ y_{\mathcal{L}_T}(D\mathcal{N}_{1,X}V_0) &\le C_2 m_{\mathcal{H}_T}(X)\cdot m_{\mathcal{H}_T}(V_0)^{1/3}\cdot\|V_0\|_{\mathcal{H}_T}^{2/3} \qquad (4.76) \\ &\quad +C_2 m_{\mathcal{H}_T}(X)^{1/3}\cdot\|X\|_{\mathcal{H}_T}^{2/3}\cdot m_{\mathcal{H}_T}(V_0) . \end{aligned}$$

Similarly, taking the worst bound among (4.59)–(4.61), we get

$$m_{\mathcal{L}_T}(D\mathcal{N}_X V_0) \le C_2\big(m_{\mathcal{H}_T}(X)+m_{\mathcal{H}_T}(X)^{1/2}\cdot\|X\|_{\mathcal{H}_T}^{1/2}\big)\cdot m_{\mathcal{H}_T}(V_0) . \qquad (4.77)$$

We next combine these bounds with those on $\mathcal{M}$. By (4.67)–(4.71), we see that

$$m_{\mathcal{K}_T}(\mathcal{M}\mathcal{D}P_{>}Z) \le K m_{\mathcal{L}_T}(Z) , \quad y_{\mathcal{K}_T}(\mathcal{M}\mathcal{D}P_{>}Z) \le K y_{\mathcal{L}_T}(Z) .$$

By (4.64)–(4.65), we see that

$$m_{\mathcal{K}_T}(\mathcal{M}_0\mathcal{D}SP_{<}Z) \le K(1+T)^{1/4}m_{\mathcal{L}_T}(Z) .$$

The case of the bounds on y for $|k| < 1$ is more complicated, because the two components have different growth behavior in time, but also different powers of the norms. We handle this in the following

Lemma 4.9. *Let $Z = (f, g) \in \mathcal{L}$ and let*

$$W_t = (\mathcal{M}_0\mathcal{D}SP_{<}Z)_t .$$

There is a constant K such that one has the bound

$$y_{\mathcal{K}}(W_t) \le K(t+1)^{1/4} \sup_{0\le\tau\le t} \|f_\tau\|_\infty + K(t+1)^{1/2} \sup_{0\le\tau\le t} \|g_\tau\|_\infty \,.$$

Proof. For the first component of W, these bounds are obvious, and in fact better, as is easily seen from (4.66) and from the bounds (4.33) on S in Corollary 4.3. For the second component, we note that S maps (f, g) to

$$i(k^{-1}\hat{A}f + g) = i(k\,k^{-2}\hat{A}f + g) \,.$$

By the last two inequalities of (4.66) and since $k^{-2}\hat{A}$ is bounded, this leads to a bound of the form

$$\| \int_0^t d\tau\, G_{11}(\tau) k(k\,k^{-2}\hat{A}f + g)\|_\infty$$
$$\le K \int_0^t d\tau\, \|G_{11}(\tau)k^2 f\|_\infty + K \int_0^t d\tau\, \|G_{11}(\tau)kg\|_\infty \,,$$

from which the assertion follows.

Returning to the proof of Proposition 4.8 we observe that by Corollary 4.3, we have that R^{-1} and R are bounded linear operators in the various norms. We can now combine the various estimates, in particular, by setting $Z = D\mathcal{N}_{R^{-1}Y}Y'$ in Lemma 4.9, and the assertion of Proposition 4.8 follows at once.

We can now proceed to the proof of existence of solutions for a time span T. We need to define the subspace of $\mathcal{K}_T$ of functions with the same initial condition. Let $U_0 \in \mathcal{K}$. Then we define

$$\mathcal{K}_T(U_0) = \{Y \in \mathcal{K}_T : Y_0 = U_0\} \,. \tag{4.78}$$

All norms below will be on the space $\mathcal{K}_T$ and we shall omit the corresponding index from the norms, i.e., we write

$$m(Y) = m_{\mathcal{K}_T}(Y)\,, \quad y(Y) = y_{\mathcal{K}_T}(Y) \,.$$

We consider initial data $U_0 \in \mathcal{K}$. By Proposition 4.4, and Corollary 4.3, we conclude that there is a constant D_2, such that the inhomogeneous term

$$U = \{R\,e^{-Lt}R^{-1}U_0\}_{t\in[0,T]} \tag{4.79}$$

satisfies

$$y_{\mathcal{K}_T}(U) \leq y_{\mathcal{K}_T}(e^{-L_0 t}P_< U_0) + y_{\mathcal{K}_T}(e^{-Lt}P_> U_0) \leq D_2 y_{\mathcal{K}_T}(U_0) . \quad (4.80)$$

Similarly,

$$m_{\mathcal{K}_T}(U) \leq D_2 m_{\mathcal{K}}(U_0) . \quad (4.81)$$

In principle, we would like to apply the contraction mapping theorem to show that the equation $\mathcal{U}(Y) = Y$ has a solution, and to bound it for a time T of order $(1/m_{\mathcal{K}}(U_0))^2$. However, the nonlinear nature of Eq.(4.75) does not allow this, and we need a direct control of convergence to provide the necessary bounds. Let U_0 and T be given, and define U by (4.79). Then we construct a solution Y in $\mathcal{K}_T(U_0)$ by iteration, setting

$$Y^{(0)} = U , \quad Y^{(n+1)} = \mathcal{U}(Y^{(n)}) .$$

We shall show that the $Y^{(n)}$ form a Cauchy sequence by using the identity

$$\begin{aligned} Y^{(n+1)} - Y^{(n)} &= \mathcal{U}(Y^{(n)}) - \mathcal{U}(Y^{(n-1)}) \\ &= \int_0^1 d\sigma D\mathcal{U}_{Y^{(n-1)}+\sigma(Y^{(n)}-Y^{(n-1)})}\big(Y^{(n)} - Y^{(n-1)}\big) . \end{aligned} \quad (4.82)$$

The difference between (4.82) and the general contraction principle is the availability of some relation between the tangent vector and the point at which the derivative is evaluated. This will allow to overcome the nonlinear bound in Proposition 4.8.

We shall apply the inequalities of Proposition 4.8 not directly to the norms, but instead to upper bounds on these norms. We assume $Y^{(0)}$ is given and we set $Y^{(-1)} = 0$. We assume

$$m(Y^{(0)}) \leq m_0 , \quad y(Y^{(0)}) \leq y_0 , \quad m_0^{1/3} \leq y_0 .$$

(It is the last inequality which may only hold for the bounds, but not for the norms themselves, and this is the reason for introducing the bounds.) We define $\Delta m_0 = m_0$ and $\Delta y_0 = y_0$, and

$$m_n = \sum_{j=0}^{n} \Delta m_j , \quad y_n = \sum_{j=0}^{n} \Delta y_j .$$

We shall show by induction that

$$\Delta m_n \leq 8^{-n}\Delta m_0 , \quad \Delta y_n \leq 2^{-n}\Delta y_0 , \quad (4.83)$$

and

$$m(Y^{(n)} - Y^{(n-1)}) \leq \Delta m_n\,, \quad y(Y^{(n)} - Y^{(n-1)}) \leq \Delta y_n\,. \tag{4.84}$$

By (4.82), this implies

$$m(Y^{(n)}) \leq m_n\,, \quad y(Y^{(n)}) \leq y_n\,. \tag{4.85}$$

Note also that the bounds (4.83) imply that

$$m_n \leq 2m_0\,, \quad y_n \leq 2y_0\,.$$

Throughout, we assume that m_0 and y_0 are sufficiently small; this will give conditions for the applicability range of Theorem 1.1.

We now prove (4.83) and (4.84) by using Proposition 4.8 and (4.82). Note that $(1+T)^{1/4} \leq (1+T)^{1/2}$. Using the inductive assumption for n, we get the bounds

$$\begin{aligned}
m(Y^{(n+1)} - Y^{(n)}) &\leq \Delta m_n (1+T)^{1/4} D_1 (m_n + m_n^{1/2} y_n^{1/2}) \\
&\leq \Delta m_n (1+T)^{1/4} 2D_1 (m_0 + m_0^{1/2} y_0^{1/2})\,, \\
y(Y^{(n+1)} - Y^{(n)}) &\leq \Delta y_n (1+T)^{1/4} D_1 m_n \\
&\quad + (1+T)^{1/2} D_1 \left(\Delta m_n m_n^{1/3} y_n^{2/3} + (\Delta m_n)^{1/3} (\Delta y_n)^{2/3} m_n\right) \\
&\leq \Delta y_n (1+T)^{1/4} 2D_1 m_0 \\
&\quad + (1+T)^{1/2} 2D_1 \left(\Delta m_n m_0^{1/3} y_0^{2/3} + (\Delta m_n)^{1/3} (\Delta y_n)^{2/3} m_0\right) \\
&\leq \Delta y_n (1+T)^{1/4} 2D_1 m_0 + (\Delta m_n)^{1/3} (1+T)^{1/2} 4D_1 \left(m_0 y_0^{2/3} + y_0^{2/3} m_0\right)\,.
\end{aligned}$$

Therefore, if $(1+T)^{1/4} m_0^{1/2}$ is sufficiently small, we see that the inductive hypothesis follows for $(n+1)$ if it was valid for $0, \ldots, n$. We have thus shown

Proposition 4.10. *There are constants $D^* > 0$, $\beta > 0$ such that if the initial data satisfy*

$$m_{\mathcal{K}}(U_0) \leq m_0 \leq \beta\,, \quad y_{\mathcal{K}}(U_0) \leq y_0 \leq \beta\,,$$

with

$$m_0^{1/3} \leq y_0\,,$$

the problem $\mathcal{U}(Y) = Y$ has a unique solution Y with $Y \in \mathcal{K}_T(U_0)$ for T satisfying $(1+T)^{1/4} \leq D^* m_0^{-1/2}$. Furthermore, this solution satisfies

$$m_{\mathcal{K}_T}(Y) \leq 2m_0\,, \quad y_{\mathcal{K}_T}(Y) \leq 2y_0\,.$$

Remark. The above statement says that solutions exist for time T and bounds on their norms double at most during this time.

We finally want to show the convergence to 0 of the solution as $t \to \infty$. As in the earlier examples, we do not expect convergence in the norm $y_{\mathcal{K}}(\cdot)$, but only of the integral norms $m_{\mathcal{K}}(\cdot)$. In fact, the bounds on the y-norms will be seen to *grow* to a finite multiple of their initial value. We therefore will assume that the initial value has been chosen small enough to offset this growth. We divide the time into segments whose length T is of the order of the value of m^{-2} at the beginning of the considered time span.

Assume U_0 and T are given, and let $Y \in \mathcal{K}_T(U_0)$. Assume furthermore that $m(U_0) \le m$, $y(U_0) \le y$, and assume $y \le \varepsilon$, $m^{1/3} \le y$. If $\varepsilon > 0$ is sufficiently small, then the constant t_0 of Proposition 4.4 will satisfy $(1+2t_0)^{1/4} m^{1/2} \le D^*$. We define T by $(1+T)^{1/4} m^{1/2} = D^*$. By Proposition 4.10, the solution $Y \in \mathcal{K}_T(U_0)$ of $\mathcal{U}(Y) = Y$ satisfies

$$m_{\mathcal{K}_T}(Y) \le 2m\,, \quad y_{\mathcal{K}_T}(Y) \le 2y\,.$$

Since $T > t_0$, we have by Eqs.(4.48), (4.49), the bounds, valid for $t_0 \le t \le T$,

$$\begin{aligned} m_{\mathcal{K}}(Y_t) &\le C_1 y(1+t)^{-1/2} + 8D_1(1+t)^{1/4} m^{3/2} y^{1/2}\,, \\ y_{\mathcal{K}}(Y_t) &\le y + 8D_1(1+t)^{1/4} m\, y + 8D_1(1+t)^{1/2} m^{4/3} y^{2/3}\,. \end{aligned}$$

This implies

$$\begin{aligned} m_{\mathcal{K}}(Y_t) &\le \left(C_1 \frac{1}{D^*} y + 8D_1 {D^*}^{1/4} y^{1/2}\right) m \le \delta(y + y^{1/2}) m\,, \\ y_{\mathcal{K}}(Y_t) &\le y + 8D_1 {D^*}^{1/4} y\, m^{1/2} + 4D_1 y^{2/3} {D^*}^{1/2} m^{1/3} \\ &\le y + \delta\,(y + y^{2/3}) m^{1/3}\,, \end{aligned} \tag{4.86}$$

with δ as small as we like if ε is sufficiently small. (We have also used $m < 1$.) Taking now Y_T as the new initial data, we find from the above bounds that for

$$m' = \delta(y + y^1/2) m\,, \quad y' = y + \delta(y + y^{2/3}) m^{1/3}\,,$$

we have

$$m_{\mathcal{K}}(Y_T) \le m'\,, \quad y_{\mathcal{K}}(Y_T) \le y'\,.$$

Since $m^{1/3} \le y$ we see that

$$m'^{1/3} \le \delta^{1/3}(y + y^{1/2})^{1/3} m^{1/3} \le y \le y'\,,$$

provided δ is sufficiently small. So Y_T satisfies the same bounds as U_0 but with m and y replaced by m' and y'. Therefore we can iterate the argument. If we call y_k the k^{th} iterate of the bound above, and determine always T from the preceding bound on m, then, if ε is sufficiently small we can achieve

$$\delta(y_k + y_k^{1/2} + y_k^{2/3}) \leq \tfrac{1}{8}\,, \quad m_k \leq \tfrac{1}{8} m_{k-1}\,,$$

and therefore, by induction,

$$y_k \leq y_0 e^{8D_1 \sum_{j=1}^{k} (1/2)^j} \leq y_0 e^{8D_1}\,.$$

Therefore y_k *is bounded for all* k *if* ε *is sufficiently small, and* m_k *tends to 0.* Furthermore, $T \to \infty$ as $k \to \infty$ and hence the proof of Theorem 1.1 is complete.

In fact, the construction of $(1+t_k)^{1/4} = D^* m_k^{-1/2}$ shows that $m_{\mathcal{K}}(Y_t) \leq \mathcal{O}(t^{-1/2})$, in other words, the convergence to zero is of diffusive type.

Acknowledgements. We thank J. Stubbe for a critical reading of the manuscript and H. Brézis, Th. Cazenave, H. Weinberger, and H. Weissler for useful information concerning the results of Section 2. Most of this work was done while JPE was enjoying the usual warm hospitality at the IHES. This work was partially supported by the Fonds National Suisse and DRET.

References

[CE] Collet, P. and J.-P. Eckmann. *Instabilities and Fronts in Extended Systems*, Princeton University Press (1990).

[F] A. Friedman. *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs (1964).

[L] H.A. Levine. The role of critical exponents in blow up theorems. (To appear). SIAM Review

[W] F.B. Weissler. Existence and non-existence of global solutions for a semilinear heat equation. Israel J. Math. **38**, 29–40 (1981).