Jones Polynomial and the Potts Model

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Abstract. It is shown that the Jones polynomial occurring in the theory of links as a knot invariant is expressible directly as the partition function of a q^2-state Potts model.

An exciting current development in the mathematical theory of knots and links is the discovery of new and direct ways of constructing knot invariants using the method of statistical mechanics. The connection between the two seemingly unrelated fields of knot theory and statistical mechanics was first noted by Jones [1]. The precise connection was made explicit soon thereafter by Kauffman [2, 3], who showed that the consideration of a state model leads directly to the Jones polynomial. It turns out that in these and other considerations [2, 3, 4] it is necessary to first consider invariants for unoriented knots. The Jones polynomial is subsequently obtained through the introduction of the writhe of the knot. Here, we report on a new derivation which gives rise to the Jones polynomial for oriented links directly as the partition function of a Potts model, without the use of the writhe. Details will be given elsewhere [5].

The projection of an oriented knot on a plane consists of lines intersecting with two different types of crossings shown in Fig. 1, indicating the relative spatial arrangement of the two intersecting lines.

Fig. 1. Two types of line crossings.

The crux of matter in the statistical mechanical approach is to construct for each knot a statistical mechanical model whose partition function gives rise to the desired knot invariant. For a given knot the lines define a lattice with the two aforementioned kinds of lattice points. We now consider a q-state vertex model considered by Perk and Wu [6, 7]. Color the lattice edges with q different colors such that only $q(2q-1)$ vertex configuration forming nonintersecting polygons are permitted. These allowed vertices are assigned vertex weights

$$\omega_i(a, b|x, y) = A_i\delta(a, x)\delta(b, y) + B_i\delta(a, y)\delta(b, x), \qquad i = \pm \tag{1}$$

where a, b, x, y are colors of the four edges incident at a vertex shown in Fig. 1. Here, in (1), we have followed the notations of Perk and Wu [7] and Jones [8]. The partition function of the vertex model is

$$Z(q, A_+, A_-, B_+, B_-) = \sum \prod \omega_i(a, b|c, d), \tag{2}$$

where the summation is taken over all lattice edge colorings, now independently, the product over all lattice sites, and ω_i is either ω_+ or ω_- depending on the knot crossing configuration. Furthermore, Perk and Wu [6] have established that $Z(q, A_+, A_-, B_+, B_-)$ is also the partition function of a q^2-state Potts model.

We have established [5] that with the choice of

$$\begin{aligned} &B_+ \equiv B, \qquad &B_- = B^{-1} \\ &A_+ = -B^2, \qquad &A_- = -B^{-2} \\ &q = B + B^{-1} \end{aligned} \tag{3}$$

where B is arbitrary, the function

$$f(B) \equiv q^{-1} Z(q, A_+, A_-, B_+, B_-) \tag{4}$$

obeys the requirements of all Reidemeister moves of a knot invariant. Furthermore, we have $f(B)_{\mathrm{unknot}} = 1$, and $V(t) \equiv f(-\sqrt{t})$ satisfies the Skein relation of the Jones polynomial. It follows that $V(t)$ is precisely the Jones polynomial and is given directly as the partition function of a q^2-state Potts model.

This work is supported in part by the National Science Foundation grant DMR-9015489. After the completion of this work we received a preprint [9] from Professor L. H. Kauffman in which an oriented state model is also considered and used to derive the Jones polynomial.

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