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Classical theories, atomicity and nonclassical theories

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Abstract. The classical theories, represented by a special type of Boolean orthomodular lattices, are defined. By using the notion of classical theory, several reasons are given, permitting to assume that any orthomodular lattice which may be considered as a physical theory is "constructed" from a set of "classical components". This assumption leads to an interpretation of the atomicity of physical theories/orthomodular lattices. Arguments for the existence of nonclassical theories are examined by employing nonclassical observables, which are also defined in the paper.

1. Introduction

There are physical arguments to consider that any physical theory—if identified with a collection of "yes–no" experiments (tests)—has the mathematical structure of an ortholattice (a lattice with an orthocomplementation defined on it). Roughly speaking, a physical theory should be a mathematical object able to describe empirical states (modes of preparation) [1, 2]. Consequently, a physical theory must have a sufficiently rich mathematical structure, i.e. a structure which allows the correct description of the relations between states, time-evolutions of the states, symmetries, etc. It is clear that, from this point of view, ortholattices are too poor mathematical structures. Therefore, we have to find some other mathematical properties which confer to ortholattices the quality of being physical theories.

In a previous paper, using the fact that for any ortholattice which may be a physical theory there exists a compatibility relation defined on it, we proved that any physical theory is an orthomodular lattice [3]. This does not mean that any orthomodular lattice is a possible physical theory, so that it is necessary to complete the structure of orthomodular lattices with some new physically interpretable properties. It will be shown below that such a property is the atomicity (of orthomodular lattices).

One of the main purposes of this work is to make clear that any theory/orthomodular lattice may be completely embedded into an atomic theory and this is a physically interpretable statement. The argumentation of this fact is based on a careful analysis of the so-called classical theories. Since we consider that any theory may be thought of as a collection of observables, we will define first the observables as objects which are independent on any theory (Section 2). We will distinguish also between classical and nonclassical observables. Then it will be seen that there are theories which may be naturally considered as "constructed" from classical observables, so that they appear as a "superposition" of classical theories. These will be called **total** theories (Section 3). It will be also clear that, even if an arbitrarily given theory/orthomodular lattice is not total, it may be considered as a subtheory of a total theory. On the other hand, it will be proved that any total theory is atomic, and this fact offers an interpretation of the atomicity as a basic property of orthomodular lattices which are physical theories.

In Section 4 we will discuss the very interesting problem of the existence of nonclassical theories. It is well known that Heisenberg's uncertainty principle assures the existence of pairs of incompatible observables and this fact is naturally described in the language of Hilbert-space theory. The Hilbert-space theory is a non-Boolean one (classical theories are Boolean algebras) so that we might think that the existence of nonclassical theories is a direct consequence of the uncertainty principle. We will show that there is another interesting point of view. More precisely, the arguments used in Sections 2–3 do not imply necessarily the existence of non-Boolean physical theories, so that they appear as being independent on the uncertainty principle. In other words, the fact that any theory is an orthomodular atomic lattice does not imply directly the existence of incompatible pairs of tests/observables. It results that we have to look for a physical argument for the existence of non-Boolean theories and it seems to be interesting to find one which does not depend on the incompatibility of tests. It will be seen that it is sufficient to consider a total theory having a nonclassical observable described by an atomless Boolean algebra (such as the position of a microparticle). Such a theory is necessarily a non-Boolean orthomodular lattice. Taking into account this result, we may affirm that the existence—in a theory—of a nonclassical observable implies the existence of incompatible tests/observables and, consequently, the uncertainty principle. We could also say that there exists a close connection between atomicity and Heisenberg's uncertainty principle via nonclassical observables.

2. Observables

In this section it will be shown that any observable may be described, in principle, by an appropriate Boolean algebra. More precisely, it will be seen that the Boolean algebra associated with a given observable exists, but in order to construct it effectively we have either to consider some experimental facts, or/and to make specific physical hypotheses.

Let us consider an observable ω. From the empirical point of view, ω is a physical quantity and an experimental procedure permitting to measure it in any state. Intuitively, we understand by a state a mode of preparation [2]. Suppose that the result of any single measurement of ω is a real number (we restrict ouselves to this situation for the sake of simplicity, but it must be noted that there are observables whose "measured values" may be considered as elements of other sets, like $\mathbf{R}^3$ or appropriate spaces of functions—here $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}^3$ its third Cartesian power). The basic object of our discussion is the

σ-algebra $\mathscr{B}(\mathbf{R})$ of Borel subsets of the real line $\mathbf{R}$. The elements of $\mathscr{B}(\mathbf{R})$ may be interpreted as tests with respect to the observable ω : $B \in \mathscr{B}(\mathbf{R})$ represents the test which gives the answer "yes" when the measured value of the observable ω is a number from B. The set $\mathscr{B}(\mathbf{R})$ itself may be organized as a Boolean algebra if we put $B_1 \vee B_2 = B_1 \cup B_2$, $B_1 \wedge B_2 = B_1 \cap B_2$, $B^{\perp} = \mathbf{R} - B$ for all $B_1, B_2, B \in \mathscr{B}(\mathbf{R})$. Taking into account these facts, we may show easily—by a standard reasoning—that any arbitrarily fixed state σ defines an unique probability $p_\sigma : \mathscr{B}(\mathbf{R}) \to [0, 1]$.

Let us consider now a set of states denoted by $\mathscr{G}$. The set $I_{\mathscr{G}} = \{B \in \mathscr{B}(\mathbf{R}); p_\sigma(B) = 0 \text{ for all } \sigma \in \mathscr{G}\}$ is an ideal in $\mathscr{B}(\mathbf{R})$ [4]. If no state besides that contained in $\mathscr{G}$ is considered, any element of $I_{\mathscr{G}}$ corresponds to the "impossible test" (the test which gives always the answer "no"). This affirmation has an obvious statistical character since it does not exclude the possibility to obtain "sometimes" by single measurements of the test $B \in I_{\mathscr{G}}$ the answer "yes". Now it is almost evident that the quotient algebra $\mathscr{B}(\mathbf{R})/I_{\mathscr{G}}$ is the Boolean algebra describing the observable ω when a set of states $\mathscr{G}$ is fixed. This is because two Borel sets B_1, B_2 represent the same test if $B_1 - B_2, B_2 - B_1 \in I_{\mathscr{G}}$. All these facts lead to the conclusion that an observable ω is described by the quotient algebra $\mathscr{B}(\mathbf{R})I_{\mathscr{G}}$ when $\mathscr{G}$ is the set of all possible states. Unfortunately such a set cannot be defined, so that, for the moment, we know only that a Boolean algebra describing a given observable ω exists and is—in some sense—unique. It will be denoted by $\mathscr{B}_\omega$. For $\mathscr{B}_\omega$ we may propose different concrete forms, depending on the physical hypotheses which are made. A model of the Boolean algebra $\mathscr{B}_\omega$ may be obtained if there are physical reasons—related to the measurement of the observable ω—which permit to choose an ideal I whose elements may be considered as corresponding to the "impossible test". If such an ideal is given, we consider that $\mathscr{B}_\omega$ may be identified with $\mathscr{B}(\mathbf{R})/I$. This is the idea which will be used in what follows.

Definition 1. Let $\mathscr{B}_\omega = \mathscr{B}(\mathbf{R})/I$ be the Boolean algebra of the observable ω. We say that ω is a classical observable if there exists a set $S_\omega \in \mathscr{B}(\mathbf{R})$ such that $I = \{B \in \mathscr{B}(\mathbf{R}); B \cap S_\omega = \varnothing\}$.

It would be more correct to consider that $\mathscr{B}(\mathbf{R})/I$ is a classical model of the observable ω, but we will often prefer to say simply that ω is a classical observable.

The most important property—from the physical point of view—of classical observables is given in the following proposition, which may be proved without difficulty.

Proposition 1. *Let ω be a classical observable and $\mathscr{B}_\omega \equiv \mathscr{B}(\mathbf{R})/I$ its Boolean algebra. Then $\mathscr{B}_\omega$ is an atomic algebra and the set of its atoms is $\Omega(\mathscr{B}_\omega) = \{\{\hat{a}\}; a \in S_\omega\}$.*

Here $\Omega(L)$ denotes the set of all atoms of the orthomodular lattice L and $\{\hat{a}\}$ is the element of $\mathscr{B}(\mathbf{R})/I$ having as representant the one-element set $\{a\}$. The atomicity of the algebra $\mathscr{B}_\omega$ has a clear physical interpretation, which results directly from the physical meaning of the ideal I. Indeed, if $a \in \mathbf{R}$ has the property $\{a\} \notin I$, it results that we have accepted the existence of a state σ such that $p_\sigma(\{a\}) \neq 0$. Since $a \in S_\omega$ represents a possible result of a single measurement of the observable ω, we may say that there exists at least one mode of preparation (state) such that the measurement of the test corresponding to $\{a\}$, performed on identical copies of this state, gives the answer "yes" with a statistically significant frequency (we will say that $\{a\}$—or another Borel subset having this property—is statistically significant).

In conclusion, a classical model of an observable ω is obtained when a set S_ω of possible values of ω is chosen, all tests associated with the measurements of ω are described by subsets of S_ω and any one-element subset of S_ω is considered to be statistically significant. It follows that, when a Boolean algebra $\mathscr{A}$ is supposed to describe an observable ω and $\mathscr{A}$ is atomic, we may affirm that ω is a classical observable. Hence, the following definition appears as being quite natural.

Definition 2. An observable ω is said to be nonclassical if $\mathscr{B}_\omega$ is a nonatomic algebra.

The classification of observables into "classical" and "nonclassical", given in Definitions 1, 2, has an obvious intrinsic character since the observables are defined as objects which do not depend on any theory. A discussion concerning other possible classifications of observables into classical and nonclassical is given in Appendix A.

In what follows any orthomodular lattice will be denoted by a triple $(L, \leq, \perp)$, where L is a nonempty set, "$\leq$" an order relation on L—such that L is a lattice with respect to L having a lowest and a greatest element—and "$\perp$" an orthocomplementation on the lattice $(L, \leq)$.

Suppose now that an orthomodular lattice $(L, \leq, \perp)$—considered as a physical theory—and the Boolean algebra $\mathscr{B}_\omega$ of the observable ω are given. We want to obtain a mathematical form of the statement "ω is an observable of the theory L". Unfortunately, the whole "physical content" of this statement cannot be expressed in a purely mathematical form. Nevertheless, for our purposes it is sufficient to work with the following definition.

Definition 3. Let $(\mathscr{B}_\omega, \leq_\omega, \perp_\omega)$ be the Boolean algebra of the observable ω and $(L', \leq, \perp)$ a physical theory/orthomodular lattice. We say that ω is an observable of the theory L', if there exists a mapping $\varphi : \mathscr{B}_\omega \to L'$ such that:

(i) $x \leq_\omega y \Leftrightarrow \varphi(x) \leq \varphi(y)$;

(ii) $\varphi(x^{\perp_\omega}) = \varphi(x)^\perp$ for all $x \in L'$;

(iii) $x_i \in \mathscr{B}_\omega$, $i \in I$ and $\bigwedge_\omega i \in I\, x_i$ exists in $\mathscr{B}_\omega \Rightarrow \bigwedge_{i \in I} \varphi(x_i)$ exists in L' and $\varphi(\bigwedge_\omega i \in x_i) = \bigwedge_i \varphi(x_i)$.

We will say often that the pair (φ, ω) is an observable of the theory L'.

We want to notice also that a mapping $\varphi : L \to L'$, where L, L' are orthomodular lattices, having the properties (i)–(iii) from Definition 3—L instead of $\mathscr{B}_\omega$—will be called a complete embedding. Concerning this definition, we must take into account sometimes that it is not complete from the physical point of view. More precisely, the implication

$$\text{(I)}\quad \begin{pmatrix} \mathscr{B}_\omega \text{ is isomorphic to} \\ \text{an orthosublattice of } L \end{pmatrix} \Rightarrow \begin{pmatrix} \omega \text{ is an} \\ \text{observable of } L \end{pmatrix}$$

is true provided that nothing but the "logical structure" of the theory L is considered. If some other physical facts have to be taken into account, then φ must satisfy some additional conditions. Nevertheless, all results of this paper are physically correct since the implication (I) was not used for proving them (see also Appendix A).

3. Atomicity and classical theories

It has been seen in Section 2 that the Boolean algebras of classical observables are atomic and their atomicity may be interpreted in physical terms. In this section we will justify the following assertion: any theory is an atomic orthomodular lattice or may be embedded into such a lattice.

It is important to point out that classical observables exist. Indeed, any arbitrarily given test a generates a Boolean algebra $\mathscr{A}_a = \{0, a, a^\perp, 1\}$, where $a^\perp$ denotes the test which gives the answer "yes" if and only if a gives the answer "no". It is easy to understand that $\mathscr{A}_a$ may be considered as describing a physical quantity ω_a. It is also a trivial fact that ω_a is a classical observable.

Let $(L, \leq, \perp)$ be a theory/orthomodular lattice. Since $L = \bigcup_{a \in L} \mathscr{A}_a$, the theory L may be considered as a collection of classical observables interconnected by the lattice—operations "$\vee$" and "$\wedge$". In order to use efficiently this observation, we need some special technical results concerning orthomodular lattices.

Let $(L, \leq, \perp)$ be an orthomodular lattice and $A \subseteq L$. We will denote by $[A]$ the smallest orthomodular sublattice of L containing the set A and having the following property: $F \subseteq A$ and $\vee F$ exists $\Rightarrow \vee \mathrm{F} \in [\mathrm{A}]$. Given $\mathscr{F} = \{\mathscr{A}_i;\, 1 \leq i \leq n\}$ a finite family of atomic Boolean orthosublattices of L, the set

$$\Omega_{\mathscr{F}} = \{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k};\, \alpha_{i_k} \in \Omega(\mathscr{A}_k),\, 1 \leq k \leqslant n\}$$

will be called the nucleus of the family $\mathscr{F}$.

Definition 4. Let $(L, \leq, \perp)$ be an orthomodular lattice and $\mathscr{F} = \{(\varphi_i, \omega_i);\, 1 \leq i \leq n\}$ a finite family of classical observables of L.

(i) $\mathscr{F}$ is said to be reproducible if $\bigcup_{i=1}^n \varphi_i(\mathscr{B}_\omega) \subseteq [\Omega_{\mathscr{F}}]$

(ii) $\mathscr{F}$ is said to be independent if $0 \notin \Omega_{\mathscr{F}}$.

(iii) $\mathscr{F}$ is said to be maximal if $[\bigcup_{i=1}^n \varphi_i(\mathscr{B}_{\omega_i})]$ is a maximal Boolean sublattice of L.

The following proposition characterizes the reproducible families of classical observables. It gives also a characterization of independent families of classical observables when these families are reproducible.

Proposition 2. *Let $(L, \leq, \perp)$ be an orthomodular lattice and $\mathscr{F} = \{(\varphi_i, \omega_i); 1 \leq i \leq n\}$ a finite family of classical observables of L. Then the following statements are equivalent*:

(i) *$\mathscr{F}$ is reproducible*;

(ii) $\vee \Omega_{\mathscr{F}} = 1$;

(iii) *$[\bigcup_{i=1}^{n} \varphi_i(\mathscr{B}_{\omega_i})]$ is a Boolean sublattice of L.*

Proof. Let us denote $\varphi_i(\mathscr{B}_{\omega_i})$ by $\mathscr{A}_i$. We will prove first the implication (ii) $\Rightarrow$ (i). Since $(\alpha, \gamma)K$ (the relation K is defined by the equivalence $(a, b)K \Leftrightarrow a = (a \wedge b) \vee (a \wedge b^{\perp})$ for all $\gamma \in \Omega_{\mathscr{F}}, \alpha \in \Omega(\mathscr{A}_i)$ $(1 \leq i \leq n)$, we have $\alpha = \bigvee_{\gamma \in \Omega_{\mathscr{F}}} (\alpha \wedge \gamma)$.

The sublattices $\mathscr{A}_i$ are atomic, so that we may write $\bigcup_{i=1}^{n} \mathscr{A}_i \subseteq [\Omega_{\mathscr{F}}]$. In order to prove that (i) $\Rightarrow$ (iii), let us notice first that the elements of $\Omega_{\mathscr{F}}$ are mutually orthogonal. Indeed, if $\alpha_1 \wedge \cdots \wedge \alpha_n, \alpha_1' \wedge \cdots \wedge \alpha_n' \in \Omega_{\mathscr{F}}$ are two different elements and $\alpha_k \neq \alpha_k'$, we may write

$$\alpha_1 \wedge \cdots \wedge \alpha_n \leq \alpha_k \leq \alpha_k'^{\perp} \leq \alpha_1'^{\perp} \wedge \cdots \wedge \alpha_n'^{\perp} = (\alpha_1' \wedge \cdots \wedge \alpha_n')^{\perp}.$$

From this fact we get immediately that $[\Omega_{\mathscr{F}}]$ is a Boolean sublattice. Since $\mathscr{A}_i \subseteq [\Omega_{\mathscr{F}}]$ for all i, $1 \leq i \leq n$, it results that $[\bigcup_{i=1}^{n} \mathscr{A}_i] \subseteq [\Omega_{\mathscr{F}}]$. Therefore, $[\bigcup_{i=1}^{n} \mathscr{A}_i]$ is a Boolean sublattice.

It remains to prove the implication (iii) $\Rightarrow$ (ii). It is sufficient to examine the case $n = 2$, since then the proof may be easily obtained by induction. Let $\mathscr{F} = \{\mathscr{A}_1, \mathscr{A}_2\}$ be a family of classical observables such that $[\mathscr{A}_1 \cup \mathscr{A}_2]$ is a Boolean sublattice of L. Since $\vee \Omega(\mathscr{A}_2) = 1$, we may write for all $\alpha \in \Omega(\mathscr{A}_1)$ the equality $\alpha = \bigvee_{\beta \in \Omega(\mathscr{A}_2)} (\alpha \wedge \beta)$. Hence,

$$1 = \vee \Omega(\mathscr{A}_1) = \bigvee_{\substack{\alpha \in \Omega(\mathscr{A}_1) \\ \beta \in \Omega(\mathscr{A}_2)}} (\alpha \wedge \beta) = \vee \Omega_{\mathscr{F}}, \qquad \text{Q.E.D.}$$

It may be proved by examples that the reproducibility and the independence of a family of observables do not imply each other (see Appendix B). The result expressed in Proposition 1 is important when we have to decide whether a subtheory of a given nonclassical theory is or not a classical one (an example is given in [5]).

Definition 5. Let $(L, \leq, \perp)$ be a theory.

(i) A set $\mathscr{F}$ of classical observables of the theory L is said to be complete if it is finite, reproducible, independent and maximal.

(ii) L is said to be a total theory if, given (φ, ω) any classical observable of L, there exists $\mathscr{F}$ a complete set of classical observables, such that the family $\mathscr{F} \cup \{(\varphi, \omega)\}$ is reproducible.

(iii) L is said to be a classical theory if there exists $\mathscr{F}$, a complete set of classical observables, such that for any (φ, ω), classical observable of L, the set $\mathscr{F} \cup \{(\varphi, \omega)\}$ is reproducible.

Let us discuss first the classical theories. They are characterized by the following proposition, which has some common points with a category of mathematical results concerning the representation of lattice morphisms by measurable functions [6].

Proposition 3. *Let $(L, \leq, \perp)$ be a classical theory. Then the following statements are true*:

(i) *L is an atomic Boolean algebra*;

(ii) *there exist S a set and $(L_s, \subseteq, \perp)$ (here $A^{\perp} = S - A$ for any $A \in L_s$) a Boolean algebra of subsets of S, having the property $x \in S \Rightarrow \{x\} \in L_s$, such that L is isomorphic to L_s and for any (φ, ω) a classical observable of L there exists a unique function $f_\omega : S \to \Omega(\mathscr{B}_\omega)$ satisfying the property $f_\omega^{-1}(B) = \varphi(B)$ for all $B \in \mathscr{B}_\omega$.*

Proof. (i) Let $\mathscr{F}$ be a complete set of classical observables having the properties required by Definition 5(iii) and $a \in L$. Since $\mathscr{A}_a \subseteq L$ represents a classical observable and $\mathscr{F}$ is maximal, we get $a \in [\bigcup_{(\varphi, \omega) \in \mathscr{F}} \varphi(\mathscr{B}_\omega)] = [\Omega_{\mathscr{F}}]$. It results that $L = [\Omega_{\mathscr{F}}]$. Therefore, L is a Boolean algebra and $\Omega(L) = \Omega_{\mathscr{F}}$.

(ii) Let us denote by (φ_i, ω_i), $1 \leq i \leq n$, the observables of the set $\mathscr{F}$. Since $\mathscr{F}$ is independent and $L = [\Omega_{\mathscr{F}}]$, it results that L is a Boolean product of Boolean algebras $\mathscr{B}_{\omega_i}$, $1 \leq i \leq n$. Moreover, if we consider the set $S = \Omega(\mathscr{B}_{\omega_1}) \times \cdots \times \Omega(\mathscr{B}_{\omega_n})$, then L is isomorphic to a Boolean algebra L_s of subsets of S, having the property $\Omega(L_s) = \{\{x\}; x \in S\}$. Therefore, any observable $(\tilde{\varphi}, \omega)$ of the theory L is completely described by a complete embedding $\varphi : \mathscr{B}_\omega \to L_s$. It remains to prove that there exists a mapping $f_\omega : S \to \Omega(\mathscr{B}_\omega)$ such that $f_\omega^{-1}(B) = \varphi(B)$ for all $B \in \mathscr{B}_\omega$. It is easy to see that $S = \bigcup_{\alpha \in \Omega(\mathscr{B}_\omega)} \varphi(\alpha)$ and $\alpha \neq \alpha' \Rightarrow \varphi(\alpha) \cap \varphi(\alpha') \neq \varnothing$. Hence, for any $x \in S$ there exists a unique $\alpha_x \in \Omega(\mathscr{B}_\omega)$ such that $x \in \varphi(\alpha_x)$. It results that we may define a mapping $f_\omega : S \to \Omega(\mathscr{B}_\omega)$ by the equality $f_\omega(x) = \alpha_x$ for all $x \in S$. If $B \in \mathscr{B}_\omega$, then $B = \vee \{\alpha; \alpha \in \Omega(\mathscr{B}_\omega), \alpha \leq B\}$ and, since φ is a complete embedding, we may write $\varphi(B) = \bigcup \{\varphi(\alpha); \alpha \in \Omega(\mathscr{B}_\omega), \alpha \leq B\}$. On the other hand, $f_\omega^{-1}(B) = \{x \in S$; there exists $\alpha \leq B, \alpha \in \Omega(\mathscr{B}_\omega), x \in \varphi(\alpha)\}$ and it is easy to verify that $\varphi(B) = f_\omega^{-1}(B)$, Q.E.D.

Proposition 3 may be considered as justifying our definition of classical theories. Indeed, if we take L a theory, having a complete set of n classical real observables, then $S \subseteq \mathbf{R}^n$ and all real classical observables of L are described by Borel—measurable real functions.

Since there are experimental facts which confirm that any classical observable may be described in the framework of a classical theory, we will accept the following axiom.

A. For any arbitrarily given theory $(L, \leq, \perp)$, there exists a total theory $(L', \leq', \perp')$ and a complete embedding $\varphi : L \to L'$.

It is obvious that for any arbitrarily given classical observable there exist many classical theories "containing" it. Axiom A affirms that it is possible to choose, for any classical observable of a theory L, a classical theory such that the union of these theories may be organized as a total theory having a subtheory isomorphic to L.

The most important result of this section is:

Proposition 4. *Any total theory is an atomic orthomodular lattice.*

Proof. Let $(L, \leq, \perp)$ be a total theory and $\mathscr{U}$ the set of all its classical maximal subtheories. It may be proved easily that $L = \cup \mathscr{U}$. Let us consider $U \in \mathscr{U}$. If $\alpha \in \Omega(U)$, then $\alpha \in \Omega(L)$. Indeed, let us suppose that $\alpha \notin \Omega(L)$. Then we may find $\beta \in L, 0 < \beta < \alpha$. Obviously, $(\beta, \alpha')K$ for all $\alpha' \in \Omega(U)$ and $\beta \notin U$. Then $[\{\beta\} \cup U]$ is a Boolean sublattice including strictly the sublattice U, which is impossible since U is maximal. It results that $\alpha \in \Omega(L)$. If $a \in L, A > 0$, then there exists $U \in \mathscr{U}$ such that $a \in U$ and we can find $\alpha \in \Omega(U) \subseteq \Omega(L)$ such that $\alpha \leq a$. Therefore, $\Omega(L) = \bigcup_{U \in \mathscr{U}} \Omega(U)$, Q.E.D.

The physically significant result of this section is expressed in the following statement.

Corollary. *Any theory may be completely embedded into an atomic theory.*

This result represents—in our opinion—a sufficiently good interpretation of the atomicity axiom.

4. Nonclassical theories

In previous sections we saw that there are natural arguments to consider classical models for physical observables. Henceforth, taking into account only such a kind of arguments, we have no serious motivation to consider nonclassical theories. In this section we will show that nonclassical theories must be considered if there are physical motivations for describing some observables by nonclassical models. This statement is based on the following proposition.

Proposition 5. *Let L be a total theory having a nonclassical observable. Then L is a non-Boolean orthomodular lattice.*

Proof. Let $\{\mathscr{K}_i; i \in I\}$ be the family of all classical maximal theories of L. We know that any $\mathscr{K}_i$ is an atomic maximal Boolean orthosublattice of L and $L = \bigcup_{i \in I} \mathscr{K}_i$. Let (φ, ω) be a nonclassical observable of L. For any $b \in \varphi(\mathscr{B}_\omega)$ there exists $i \in I$ such that $b \in \mathscr{K}_i$. Suppose that $b \in \mathscr{K}_i$ for all $b \in \varphi(\mathscr{B}_\omega)$. Then it results that φ is a complete embedding of a nonatomic Boolean algebra into an atomic Boolean algebra, which is impossible. Indeed, it is known that if a complete Boolean algebra B may be completely embedded into a complete Boolean algebra A, then B is also atomic [7]. Therefore, let us consider the minimal completions $\mathscr{B}^*_\omega, \mathscr{K}^*_i$ of the Boolean algebras $\mathscr{B}_\omega, \mathscr{K}_i$ respectively [8]. Since there exists a complete embedding $\xi : \mathscr{B}^*_\omega \to \mathscr{K}^*_i$ and $\mathscr{K}^*_i$ is obviously atomic, we find that $\mathscr{B}^*_\omega$ is atomic. It remains to observe that, in this case, $\mathscr{B}_\omega$ must be also atomic [8]. But $\mathscr{B}_\omega$ is nonatomic, so that we obtained a contradiction. It follows that $\{\mathscr{K}_i; i \in I\}$ has at least two elements, so that L is a non-Boolean orthomodular lattice, Q.E.D.

We will consider now—as an example of nonclassical observable—the case of the observable Q corresponding to the measurement of the position of a microparticle in the "physical space" $\mathbf{R}^3$. It is clear that the possible values of this observable are points in $\mathbf{R}^3$. The Boolean algebra $\mathscr{B}_Q$ may be constructed by considering the Lebesgue—measurable subsets of $\mathbf{R}^3$ as tests which determine the position of the considered microparticle. Let us denote by $\mathscr{L}$ and μ the class of Lebesgue—measurable subsets of $\mathbf{R}^3$ and the Lebesgue measure, respectively. It is well known that—when microparticles are considered—there are physical reasons to accept that $M \in \mathscr{L}$ is statistically significant if and only if $\mu(M) > 0$. Taking account of this hypothesis, we may affirm that $\mathscr{B}_Q = \mathscr{L}/\mathscr{N}$, where $\mathscr{N} = \{N \in \mathscr{L}; \mu(N) = 0\}$. The set $\mathscr{N}$ is obviously an ideal of the Boolean algebra $\mathscr{L}$. The Boolean algebra $\mathscr{B}_Q$ has no atoms. Indeed, let us consider $M \in \mathscr{L}, \mu(M) > 0$. Then there exists $M' \in \mathscr{L}, M' \subset M$, such that $0 < \mu(M') < \mu(M)$. It follows that $\hat{M}$ is not an atom and, consequently, $\mathscr{B}_Q$ is a nonclasical observable. Therefore, any theory describing systems of microparticles is non-Boolean since it must "contain" the position observables of the microparticles. It remains to show that there exists at least one theory having Q as one of its observables. But this is a well known fact: Q is an observable of the theory whose elements are orthogonal projectors in $L^2(\mathbf{R}^3)$ (the space of all square—integrable complex functions defined on $\mathbf{R}^3$).

It is important to remark that, in our formalism, the existence of non-Boolean theories results directly from the existence of a nonclassical observable. In other words, the existence of a nonclassical observable implies the existence of pairs of tests/observables which are not compatible. It is also interesting to note that a total theory having Q as an observable has an infinite family of maximal atomic Boolean sublattices (classical subtheories or classical components).

Proposition 6. *Let* $(L, \leq, \perp)$ *be a total theory such that* (φ, Q) *is an observable of* L *and* $\mathscr{U}$ *the set of all classical components of* L. *Then* $\mathscr{U}$ *is an infinite set.*

Proof. Let $U \in \mathscr{U}$ be an arbitrarily fixed classical component and $B_Q = \varphi(\mathscr{B}_Q)$. The Boolean sublattice $B_Q \cap U$ is complete. Indeed, let $\{b_i; i \in I\}$ be a family of elements of $B_Q \cap U$ and $a \in U$. Since B_Q is complete and $(b_i, a)K$ for all $i \in I$, we get $(\bigvee_{i \in I} b_i, a)K$. By applying Definition 4(iii) we get $\bigvee_{i \in I} b_i \in U$. Now we will prove that $B_Q \cap U$ is an atomic sublattice of B_Q. Let us denote $B_Q \cap U$ by B and consider $a \in B, a > 0$. Since U is atomic, there exists $\alpha \in \Omega(U), \alpha \leq a$. Let $p : U \to [0, 1]$ be the probability defined by the equality

$$p(a') = \begin{cases} 1, & \alpha \leq a' \\ 0, & \alpha \nleq a' \end{cases}$$

for all $a' \in U$ and consider the set $B_a = \{b \in B, p(b) = 1\}$. Since B is complete, $\beta = \wedge B_a \in B$ and β is an atom of B (which is obviously contained in a). Since $\alpha \le b$ for all $b \in B_a$, we get $\alpha \le \beta$, so that $\beta \in B_a$. Obviously, β is the smallest element of B_a. If β is not an atom of B, then there exists $\beta' \in B$ such that $0 < \beta' < \beta$. We have also $\beta - \beta' > 0$ and $\beta = \beta' \vee (\beta - \beta')$ so that $p(\beta) = p(\beta') + p(\beta - \beta') = 1$. Consequently, $p(\beta') = 1$ or $p(\beta - \beta') = 1$. It follows that we may find $\gamma < \beta$, $\gamma \in B_a$, which is absurd; the atomicity of $B_Q \cap U$ is proved. Suppose now that $\mathscr{U}$ is finite and let $U_1, U_2, \ldots, U_n$ be its elements. The family $\mathscr{F} = \{B_i; B_i = B_Q \cap U_i, 1 \le i \le n\}$ may be considered as a family of classical observables of L. Since $B_Q = \bigcup_{i=1}^n B_i$, we get from Proposition 2 that $\mathscr{F}$ is a reproducible family and, therefore, $B_Q = [\Omega_{\mathscr{F}}]$. This result is absurd since $[\Omega_{\mathscr{F}}]$ is an atomic sublattice of L and it follows that $\mathscr{U}$ must be an infinite set, Q.E.D.

Appendix A

We will present here another possibility of classifying observables into classical and nonclassical. Let G be a classical theory and ω an observable. Any triple (H, ξ, φ), where H is a total theory and ξ, φ are complete embeddings of G respectively $\mathscr{B}_\omega$ into H, is called an extension of G containing ω. Usually we are interested in such extensions which satisfy certain physical properties, so that we will consider the so-called $\mathscr{P}$-extensions, where $\mathscr{P}$ is a set of physical properties. If H is a classical (nonclassical) theory we will say that (H, ξ, φ) is a $\mathscr{P}$-classical ($\mathscr{P}$-nonclassical) extension.

By using these notions we may define $\mathscr{P}$-classical and $\mathscr{P}$-nonclassical observables with respect to a given classical theory G. We say that ω is a $\mathscr{P}$-classical observable with respect to G if there exists a $\mathscr{P}$-classical extension of G containing ω. The observable ω is said to be $\mathscr{P}$-nonclassical if any $\mathscr{P}$-extension of G containing ω is nonclassical.

From Proposition 5 we get immediately that, given G a classical theory and ω a nonclassical observable, any $\mathscr{P}$-extension of G containing ω is nonclassical. In other words any nonclassical observable is $\mathscr{P}$-nonclassical with respect to any classical theory, irrespective of the set $\mathscr{P}$.

The problem of classifying classical observables is completely solved by Proposition 2. Indeed, from this proposition we get that a classical observable ω is $\mathscr{P}$-nonclassical with respect to G if $\mathscr{P}$ contains a condition which implies that $\vee\Omega \neq 1$, where Ω is the nucleus of the family $\{\xi(G), \varphi(\mathscr{B}_\omega)\}$. Such a condition may be, for example, the existence of a state σ whose correspondent $p_\sigma : H \to [0, 1]$ has the property $\sum_{a \in \Omega} p_\sigma(a) < 1$ (here Ω is assumed to be denumerable). An important case is that when $\mathscr{P}$ contains only the condition of independence of G and ω. Then ω is $\mathscr{P}$-classical with respect to G since the Boolean product of the family $\{G, \mathscr{B}_\omega\}$ is obviously a $\mathscr{P}$-classical extension of G containing ω. It is also interesting to notice that, even if G, ω are independent, ω may be $\mathscr{P}$-nonclassical with respect to G since $\mathscr{P}$ may contain a condition which entails $\vee\Omega \neq 1$. In Appendix B an example is given which proves that such a situation is possible. Finally, if there exists a complete embedding $\varphi : \mathscr{B}_\omega \to G$ and we take $\mathscr{P} = \varnothing$, then we see that G is a classical extension of G containing ω, situation which corresponds to Definition 3.

In conclusion, besides the "intrinsic" classification of observables into classical and nonclassical adopted in Paragraph 2, there are many other classifications, depending on the "reference classical theory" G and the set $\mathscr{P}$ of physical properties. For any concrete problem which requires such a classification of observables, we have to choose an appropriate classical theory and a set of relevant physical properties.

Appendix B

Given $\rho \subset M \times M$ a relation on M, we will write $(a, b)\rho$ if $(a, b) \in \rho$ and $(a, b)\not\rho$ if $(a, b) \notin \rho$. We will denote by L_n the lattice of all orthogonal projections in the real Hilbert space $\mathbf{R}^n$ and by "$\perp$" the orthogonality relation on L_n.

Let us construct now an example of a family of independent observables, which is not reproducible. The Hilbert space $\mathbf{R}^6$ is ismorphic to the direct sum $\mathbf{R}^4 \oplus \mathbf{R}^2$. We consider the mutually orthogonal one-dimensional projectors $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in L_4$ and the orthogonal pairs of one-dimensional projectors $\beta_1, \beta_2 \in L_2, \beta_3, \beta_4 \in L_2$ such that $(\beta_1, \beta_j)\not\perp, (\beta_2, \beta_j)\not\perp$ for $j = 3, 4$. By using the projectors $\alpha_1' = \alpha_{11} \vee \alpha_{12}$, $\alpha_2' = \alpha_{21} \vee \alpha_{22}$, $b_1' = \alpha_{11} \vee \alpha_{21}$, $b_2' = \alpha_{12} \vee \alpha_{22}$ we may construct the following projectors from L_6: $a_1 = a_1' \vee \beta_1$, $a_2 = a_2' \vee \beta_2$, $b_1 = b_1' \vee \beta_3$, $b_2 = b_2' \vee \beta_4$. Since $a_1^\perp = a_2, b_1^\perp = b_2, a_1 \vee a_2 = 1, b_1 \vee b_2 = 1$, it is easy to see that $\mathscr{A} = \{0, 1, a_1, a_2\}$, $\mathscr{B} = \{0, 1, b_1, b_2\}$ are atomic Boolean sublattices of L_6. The family $\{\mathscr{A}, \mathscr{B}\}$ is obviously independent. We will show now that $(a_1, b_1)\not K$. Indeed, if $(a_1, b_1)K$, then, since $(a_1', b_1)K, (b_1', a_1)K, (a_1', b_1')K$, it results that a_1, b_1, a_1', b_1' are all elements of a maximal Boolean orthosublattice $\mathscr{K}$. Therefore, $\beta_1, \beta_3 \in \mathscr{K}$ since $\beta_1 = a_1 \wedge a_1'^\perp$, $\beta_3 = b_1 \wedge b_1'^\perp$. It follows that $(\beta_1, \beta_3)K$, which is absurd, so that $[\mathscr{A} \cup \mathscr{B}]$ is not Boolean and $\{\mathscr{A}, \mathscr{B}\}$ is not reproducible. If we denote by β the projector $\beta_1 \vee \beta_2$, we see that the family $\{\mathscr{A}', \mathscr{B}'\}$, where $\mathscr{A}' = [\{a_1', a_2', \beta\}]$, $\mathscr{B}' = [\{b_1', b_2', \beta\}]$, is reproducible but not independent.

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