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The Spherical Model and Bose-Einstein Condensation

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Abstract: In 1968 Gunton and Buckingham pointed out that there is a close relationship between the critical behaviour of the spherical model and that of the ideal Bose gas. In this paper we concentrate on this similarity between the two systems. We show that in the spherical model there is in fact Bose-Einstein condensation into the spin modes with low energy. As in the Bose gas we distinguish between generalized Bose-Einstein condensation and Bose-Einstein condensation. We find that the spherical model in certain cases has two critical temperatures: one temperature T_c corresponding to the onset of generalized Bose-Einstein condensation (and of spontaneous magnetization) and a lower temperature T_m at which generalized condensation becomes condensation into the spin mode with the lowest energy. We also study the fluctuations of the spin mode with lowest energy and investigate in detail some lattice interactions.

1. Introduction

The spherical model was introduced in 1952 by Berlin and Kac [1]; by using the *delta function technique* they evaluated the free energy per site and showed that the model exhibits spontaneous magnetization. Very soon after the publication of this paper Lewis and Wannier [2] pointed out that the calculations in [1] can be considerably simplified by the introduction of the grand canonical ensemble as in the ideal Bose gas. It was however very quickly realised that, although the thermodynamic functions can be calculated by using the grand canonical ensemble, the two ensembles are not equivalent below the critical temperature: the expectation values of some observables are not the same in the two ensembles in the thermodynamic limit. This means that as in the ideal Bose gas [3], [4], [5], [6] the probability measure connecting the two ensembles, the Kac density, is not degenerate below the critical temperature. This problem was studied in [7], [8], [9] and more recently in [10].

In 1968 Gunton and Buckingham [11] pointed out that there is a close relationship between the critical behaviour of the spherical model and that of the

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ideal Bose gas; they showed that the critical exponents are the same for the two systems. In this paper we concentrate on the similarity of the spherical model with the ideal Bose gas. We show that there is in fact Bose-Einstein condensation in the spherical model. The objects analogous to the occupation numbers in the Bose gas are the squares of the direction cosines of the spins with respect to the eigenvectors of the interaction matrix. We shall refer to these as the spin mode occupation numbers and we shall call the eigenvalues of the interaction the spin mode energies. We use the techniques developed for the Bose gas in particular in [12], [13] and [6]. As in the Bose gas we distinguish between generalized Bose-Einstein condensation and Bose-Einstein condensation. We shall say that there is generalized Bose-Einstein condensation when a set of the spin modes with low lying energies are occupied, while Bose-Einstein condensation requires that the spin mode with lowest energy is occupied. We find that the spherical model in certain cases has two critical temperatures: one temperature T_c corresponding to the onset of generalized Bose-Einstein condensation (and of spontaneous magnetization) and a lower temperature T_m at which generalized condensation becomes condensation into the spin mode with the lowest energy. This phenomenon occurs also in the ideal Bose gas [6].

The paper is set out as follows: in Theorem 1 we obtain the free-energy density for the spherical model for a very general class of interactions; by using the techniques of [13] we avoid the use of the saddle-point method which in many cases is difficult to make rigorous. In Theorem 2 we show that there is a critical temperature T_c below which there is generalized Bose-Einstein condensation and that the distribution of the generalized condensate is degenerate. In Theorem 3 we prove that there is a temperature $T_m \leq T_c$ such that for temperatures less than T_m there is Bose-Einstein condensation, again obtaining the distribution for the random variable representing the condensate; in many cases this distribution is degenerate. In Theorem 4 we study the fluctuations of this random variable. We find the relevant scale such that it attains asymptotically a non-degenerate finite distribution. These theorems are stated in Section 2 and proved in Section 3. In Section 4 we study examples of interactions on a lattice, some of which give rise to $T_c > T_m$. These examples are based on the work in [14] and [15].

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2. The model and the results

Let $\{n_l : l = 1, 2, 3 \ldots\}$ be a sequence of positive integers increasing to ∞ and let $\Omega_l = \mathbf{R}^{n_l}$. Let J_l be a linear operator on Ω_l and $e_l = (1, 1 \ldots, 1) \in \Omega_l$. For $h \in \mathbf{R}$ define $H_l^h : \Omega_l \to \mathbf{R}$ by

$$H_l^h(\omega) = -\frac{1}{2}\langle \omega, J_l \omega \rangle - h\langle e_l, \omega \rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on Ω_l. For $d \in \mathbf{N}$, $r > 0$ let $S(d, r)$

be the sphere in $\mathbf{R}^d$ with its centre at the origin and with radius r, and let

$$\Omega_l(r) = \{\omega : \omega \in \Omega_l, \|\omega\|^2 = rn_l\} = S(n_l, \sqrt{rn_l}).$$

The configuration space for the spherical model is $\Omega_l(1)$ and the canonical measure on $\Omega_l(1)$ at inverse temperature β is defined by:

$$\mathsf{P}_l^{\beta,h}(A) = (Z_l(\beta,h))^{-1} \int_A e^{-\beta H_l^h(\omega)} m_{n_l,\sqrt{n_l}}(d\omega) \tag{2.2}$$

where $m_{d,r}$ denotes Lebesgue measure on $S(d,r)$ and $Z_l(\beta,h)$ is the canonical partition function

$$Z_l(\beta,h) = \int_{\Omega_l} e^{-\beta H_l^h(\omega)} m_{n_l,\sqrt{n_l}}(d\omega). \tag{2.3}$$

In this paper we shall concentrate on the model with $h = 0$; we let $H_l = H_l^0$, $\mathsf{P}_l^\beta = \mathsf{P}_l^{\beta,0}$ and $Z_l(\beta) = Z_l(\beta,0)$. We shall assume that the operator J_l has an orthonormal set of n_l eigenvectors $\phi_l(j)$, $j = 1,\dots n_l$, with eigenvalues $\lambda_l(j)$, $j = 1,\dots n_l$. We order the eigenvalues so that

$$\lambda_l(1) \geq \lambda_l(2) \geq \lambda_l(3) \geq \dots \geq \lambda_l(n_l).$$

Let $\epsilon_l(j) = \frac{1}{2}(\lambda_l(1) - \lambda_l(j))$, so that

$$0 = \epsilon_l(1) \leq \epsilon_l(2) \leq \dots \leq \epsilon_l(n_l).$$

We shall call $\{\epsilon_l(j) : j = 1,\dots,n_l\}$ the spin mode energies and n_l the number of sites.

Let μ_l be the probability measure on $\mathbf{R}_+ \equiv [0,\infty)$ defined by

$$\mu_l(A) = \frac{1}{n_l}\sharp\{j : \epsilon_l(j) \in A\}, \tag{2.4}$$

that is

$$\int_{\mathbf{R}_+} f(t)\mu_l(dt) = \frac{1}{n_l}\sum_{j=1}^{n_l} f(\epsilon_l(j)).$$

We shall require that the eigenvalues of J_l have the following properties:
(A1) *the limit* $\lambda(1) = \lim_{l\to\infty}\lambda_l(1)$ *exists;*
(A2) *there is a measure* μ *on* $\mathbf{R}_+$ *such that for every* $f : \mathbf{R}_+ \to \mathbf{R}$ *which is continuous and bounded,* $\int_{\mathbf{R}_+} f(t)\ln(2+t)\mu_l(dt)$ *converges to* $\int_{\mathbf{R}_+} f(t)\ln(2+t)\mu(dt)$ *as* l *tends to infinity.*

We have above extracted the essential properties of J_l necessary for our results; We shall see below that these are satisfied in the case when J_l is given by a kernel on a lattice, see for example [16]:

Let $\{a_1, a_2, \ldots, a_\nu\}$ be a basis for $\mathbf{R}^\nu$ and for $l \in \mathbf{N}$ let Λ_l be the subset of Λ the Bravais lattice generated by this basis, given by

$$\Lambda_l = \{\sum_{i=1}^{\nu} m_i a_i \ : \ m \in \{-l, -l+1, \ldots, l-1, l\}^\nu\}.$$

Let n_l be the number of lattice points in Λ_l, that is $n_l = (2l+1)^\nu$. Let $u : \Lambda \to \mathbf{R}$ be a positive function such that $u(-x) = u(x)$ and $\sum_{x \in \Lambda} u(x) < \infty$. We then define $u_l : \Lambda \to \mathbf{R}$ in the following way:

$$u_l | \Lambda_l = u | \Lambda_l,$$

u_l is then extended to Λ in such a way that it is periodic with period Λ_l, that is, if $m \in \mathbf{Z}^\nu$, let $\tilde{m}$ have components

$$\tilde{m}_i = (m_i + l) \bmod (2l+1) - l,$$

then $\tilde{m} \in \{-l, -l+1, \ldots, l-1, l\}^\nu$ and

$$u_l(\sum_{i=1}^{\nu} m_i a_i) = u(\sum_{i=1}^{\nu} \tilde{m}_i a_i).$$

Let $\Lambda_l = \{x_1, x_2, \ldots, x_{n_l}\}$, that is choose a labelling of Λ_l and then define J_l to be the matrix with entries

$$J_l(i,j) = u_l(x_i - x_j). \tag{2.5}$$

Let $\{b_1, b_2, \ldots b_\nu\}$ be the basis of $\mathbf{R}^\nu$ satisfying $\langle a_i, b_j \rangle = 2\pi\delta_{ij}$ and let Λ_l^r be the lattice reciprocal to Λ_l:

$$\Lambda_l^r = \left\{ (2l+1)^{-1} \sum_{j=1}^{\nu} m_j b_j : m \in \{-l, -l+1, \ldots, l-1, l\}^\nu \right\}.$$

Choose Λ_l^{r+} and Λ_l^{r-} such that $\Lambda_l^{r+} \cup \Lambda_l^{r-} = \Lambda_l^r \setminus \{0\}$, $\Lambda_l^{r+} \cap \Lambda_l^{r-} = \emptyset$ and $T\Lambda_l^{r+} = \Lambda_l^{r-}$ under the mapping $T : k \mapsto -k$. For $k \in \Lambda_l^r$ let $\zeta_l(k) \in \mathbf{R}^{n_l}$ be defined as follows:

$$(\zeta_l(k))_i = \begin{cases} \left(\frac{2}{n_l}\right)^{\frac{1}{2}} \cos(\langle k, x_i \rangle) & \text{if } k \in \Lambda_l^{r+}, \\ \left(\frac{2}{n_l}\right)^{\frac{1}{2}} \sin(\langle k, x_i \rangle) & \text{if } k \in \Lambda_l^{r-}, \\ \left(\frac{1}{n_l}\right)^{\frac{1}{2}} & \text{if } k = 0. \end{cases} \tag{2.6}$$

$\{\zeta_l(k) : k \in \Lambda_l^r\}$ is an orthonormal set of eigenvectors of the matrix $J_l(i,j)$ with eigenvalues $\tilde{\lambda}_l(k)$ where

$$\tilde{\lambda}_l(k) = \sum_{x \in \Lambda_l} u(x) \cos(\langle k, x \rangle). \tag{2.7}$$

For $k \in \Lambda_l^r$ let

$$\tilde{\epsilon}_l(k) = \frac{1}{2}(\tilde{\lambda}_l(0) - \tilde{\lambda}_l(k)) = \sum_{x \in \Lambda_l} u(x) \sin^2 \frac{1}{2}\langle k, x \rangle. \tag{2.8}$$

(A1) is satisfied since $\lambda_l(1) = \tilde{\lambda}_l(0)$ converges to $\sum_{x\in\Lambda} u(x)$. Because $|\tilde{\lambda}_l(k)| \leq \tilde{\lambda}_l(0)$ for all $k \in \Lambda_l^r$, $\tilde{\epsilon}_l(k) \in [0, \tilde{\lambda}_l(0)]$ and therefore μ_l has support in $[0, \tilde{\lambda}_l(0)]$; thus to check (A2) it is sufficient to consider $\int_{\mathbf{R}_+} f(t)\mu_l(dt)$ where f is a bounded function on $\mathbf{R}_+$. Now

$$\int_{\mathbf{R}_+} f(t)\mu_l(dt) = \frac{1}{n_l} \sum_{k \in \Lambda_l^r} f(\tilde{\epsilon}_l(k))$$

which, if f is continuous, converges to $C^{-1} \int_{\Lambda^r} f(\tilde{\epsilon}(k))m(dk)$ where m is Lebesgue measure on $\mathbf{R}^\nu$, Λ^r is the parallelepiped

$$\{\sum_{i=1}^{n_l} k_i b_i : |k_i| \leq \frac{1}{2}, \quad i = 1, \ldots, n_l\},$$

C is the volume of Λ^r and $\tilde{\epsilon} : \Lambda^r \to \mathbf{R}$ is defined by

$$\tilde{\epsilon}(k) = \sum_{x \in \Lambda} u(x) \sin^2 \frac{1}{2}\langle k, x \rangle; \tag{2.9}$$

thus $\int_{\mathbf{R}_+} f(t)\mu_l(dt)$ converges to $\int_{\mathbf{R}_+} f(t)\mu(dt)$, where $\mu = C^{-1} m \circ \tilde{\epsilon}^{-1}$.
The sequence of operators $\{J_l\}$ converges strongly to the semi-infinite matrix J given by $J_{ij} = u(x_i - x_j)$ which defines a self-adjoint operator on $l^2(\Lambda)$; the measure μ is then the density of states of $\frac{1}{2}(\lambda(1) - J)$.

We now go back to the general case and state the results proved in this paper. In the first theorem we deal with the convergence of the free energy per site for the spherical model. Let $f_l(\beta)$ be the free energy per site for the finite system at inverse temperature β:

$$f_l(\beta) = -\frac{1}{\beta n_l} \ln Z_l(\beta).$$

Let

$$\beta_c = \int_{\mathbf{R}_+} \frac{1}{2t} \mu(dt).$$

Note that β_c can be equal to $+\infty$.

Theorem 1. *The limit $f(\beta) = \lim_{l\to\infty} f_l(\beta)$ exists and is given by: if $\beta \leq \beta_c$,*

$$f(\beta) = \alpha(\beta) + \frac{1}{2\beta}\int_{\mathbf{R}_+} \ln(t-\alpha(\beta))\mu(dt) + \frac{1}{2\beta}\ln\frac{\beta}{\pi} - \frac{1}{2}\lambda(1) \tag{2.10}$$

where $\alpha(\beta) \leq 0$ is the unique solution of the equation

$$\beta = \frac{1}{2}\int_{\mathbf{R}_+} \frac{1}{t-\alpha}\mu(dt); \tag{2.11}$$

if $\beta > \beta_c$,

$$f(\beta) = \frac{1}{2\beta}\int_{\mathbf{R}_+} \ln t \; \mu(dt) + \frac{1}{2\beta}\ln\frac{\beta}{\pi} - \frac{1}{2}\lambda(1). \tag{2.12}$$

By analogy with the Bose gas we shall introduce the spin mode occupation numbers. For $j = 1, 2, \ldots n_l$ we let N_j be the random variable

$$N_j(\omega) = |\langle \phi_l(j), \omega\rangle|^2. \tag{2.13}$$

Note that for $\omega \in \Omega_l(1)$

$$N_j(\omega) \leq \|\phi_l(j)\|^2\|\omega\|^2 = n_l.$$

In the lattice model described above, if we take $\omega = (1, 1, \ldots, 1)$ then, since $\phi_l(1) = \frac{1}{\sqrt{n_l}}(1, 1, \ldots, 1)$, we have $N_1(\omega) = n_l$. Also the canonical partition function can be written in terms of the spin mode occupation numbers as follows:

$$Z_l(\beta) = \int_{\Omega_l(1)} \exp\{-\beta\sum_{j=1}^{n_l}\lambda_l(j)N_j(\omega)\} m_{n_l,\sqrt{n_l}}(d\omega).$$

We shall say that there is Bose-Einstein condensation into the spin mode with lowest energy if

$$\lim_{l\to\infty} \mathsf{E}_l^\beta\left(\frac{N_1}{n_l}\right) > 0, \tag{2.14}$$

where E_l^β is the expectation with respect to the probability measure P_l^β. It can happen that the limit in (2.14) is zero but there is still condensation into the spin modes with low lying energies $\epsilon_l(k)$; if this happens we shall say that the model exhibits generalized Bose-Einstein condensation. More exactly we shall say that there is generalized Bose-Einstein condensation if

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty} \mathsf{E}_l^\beta\left[\frac{1}{n_l}\sum_{\epsilon_l(j)<\delta} N_j\right] > 0. \tag{2.15}$$

Our next theorem shows that the spherical model exhibits generalized Bose- Einstein condensation if the temperature is sufficiently low:

Theorem 2. *In the spherical model there is no generalized condensation if* $\beta \leq \beta_c$*, while for* $\beta > \beta_c$ *the model exhibits generalized Bose-Einstein condensation: for* $s \geq 0$,

$$\lim_{\delta \downarrow 0} \lim_{l \to \infty} \mathsf{E}_l^\beta \left[\exp \left\{ -\frac{s}{n_l} \sum_{\epsilon_l(k) < \delta} N_k \right\} \right] = \exp -s \left\{ 1 - \frac{\beta_c}{\beta} \right\}. \tag{2.16}$$

To examine macroscopic occupation of the spin mode with lowest energy we need more information about the spin energy spectrum $\{\epsilon_l(j) : j = 1, 2, \ldots, n_l\}$. We introduce a second critical temperature β_m which is related to the maximum density of spin modes which have energies $\epsilon_l(k)$ tending to zero slower than ϵ / n_l as $l \to \infty$ for every $\epsilon > 0$. Let

$$\beta_m^+(\epsilon) = \limsup_{l \to \infty} \int_{[\frac{\epsilon}{n_l}, \infty)} \frac{1}{2t} \mu_l(dt), \tag{2.17}$$

and

$$\beta_m^-(\epsilon) = \liminf_{l \to \infty} \int_{[\frac{\epsilon}{n_l}, \infty)} \frac{1}{2t} \mu_l(dt).$$

We shall assume that $\lim_{\epsilon \to \infty} \beta_m^+(\epsilon)$ and $\lim_{\epsilon \to \infty} \beta_m^-(\epsilon)$ exist and are equal; we shall denote their common value by β_m. We introduce also a scaled density of states G_l; G_l is a measure on $\mathbf{R}_+$ defined by

$$G_l(A) = \sharp\{j : n_l \epsilon_l(j) \in A\} \tag{2.18}$$

and we denote its Laplace transform by γ_l:

$$\gamma_l(s) = \int_{[0,\infty)} e^{-st} G_l(dt). \tag{2.19}$$

We shall assume that $\gamma(s) = \lim_{l \to \infty} \gamma_l(s)$ exists for $s > 0$; this ensures that there is a measure G on $\mathbf{R}_+$ such that

$$\gamma(s) = \int_{[0,\infty)} e^{-st} G(dt) \tag{2.20}$$

for $s > 0$ and that $e^{-st} G_l(dt)$ converges weakly to $e^{-st} G(dt)$ for $s > 0$. In addition we shall assume that $\int_0^\infty e^{-as} \gamma(s) ds < \infty$ for all $a > 0$; since $s \mapsto \gamma(s)$ is monotonically decreasing this is equivalent to requiring that γ be locally integrable at zero.

In the case when $\mu(\{0\}) = 0$ clearly $\beta_c \leq \beta_m$. From the convergence of γ_l we can deduce that there exists a $K < \infty$ such that $\sharp\{j : \epsilon_l(j) < n_l^{-1}\} < K$ for all l; this implies that the inequality $\beta_c \leq \beta_m$ holds even when $\mu(\{0\}) > 0$.

In the following theorem we obtain the distribution of the random variable $\frac{N_1}{n_l}$ in the limit $l \to \infty$ by obtaining its Laplace transform.

Theorem 3. *If $\beta \leq \beta_m$*

$$\lim_{l\to\infty} \mathsf{E}_l^\beta \left[e^{-\lambda \frac{N_1}{n_l}} \right] = 1, \tag{2.21}$$

for all $\lambda \geq 0$.

Let $a > 0$; then the probability measure with Laplace transform

$$\exp -\{s\beta_m + \frac{1}{2}\int_0^\infty du \left(\frac{1-e^{-us}}{u}\right) e^{-au}\gamma(u)\} \tag{2.22}$$

has an absolutely continuous distribution F such that $F(\beta) = 0$ for all $\beta \leq \beta_m$ and $F(\beta)$ is strictly increasing for all $\beta > \beta_m$. If $\beta \geq \beta_m$ is such that F is differentiable at β then for all $\lambda \geq 0$,

$$\lim_{l\to\infty} \mathsf{E}_l^\beta \left[e^{-\lambda \frac{N_1}{n_l}} \right] = 1 - \frac{\lambda}{2\beta}(F'(\beta))^{-1} \int_{\beta_m}^{\beta} e^{-(a+\frac{\lambda}{2\beta})(\beta-\beta')} I\left(-\frac{\lambda}{2}(1-\frac{\beta'}{\beta})\right) dF(\beta'), \tag{2.23}$$

where I is the sum of the first two modified Bessel functions: $I(x) = I_0(x) + I_1(x)$.

Note that if F_1 and F_2 are defined through (2.22) with $a = a_1$ and $a = a_2$ respectively then $dF_2(\beta) = dF_1(\beta)e^{-(a_2-a_1)\beta} / \int_0^\infty e^{-(a_l-a_1)\beta'} dF_1(\beta')$ and therefore the righthand side of (2.23) is independent of a. If the interaction in the lattice model considered in Section 1 is between nearest neighbours then $\gamma(s) \equiv 1$ (see Section 4, Proposition 2); in that case, (2.22) yields $F'(\beta) = \sqrt{\frac{a}{\pi}} \frac{e^{-a(\beta-\beta_m)}}{\sqrt{\beta-\beta_m}}$ for $\beta > \beta_m$, and by using the identity:

$$1 - \kappa \int_0^1 (1-x)^{-\frac{1}{2}} e^{-\kappa x} I(-\kappa x) dx = e^{-\kappa},$$

(2.23) gives for $\beta > \beta_m$

$$\begin{aligned} &\lim_{l\to\infty} \mathsf{E}_l^\beta \left[e^{-\lambda \frac{N_1}{n_l}} \right] \\ &= 1 - \frac{\lambda}{2}\left(1 - \frac{\beta_m}{\beta}\right) \int_0^1 e^{-\frac{\lambda}{2}(1-\frac{\beta_m}{\beta})x} (1-x)^{-\frac{1}{2}} I\left(-\frac{1}{2}\lambda(1-\frac{\beta_m}{\beta})x\right) dx \\ &= e^{-\lambda(1-\frac{\beta_m}{\beta})}. \end{aligned} \tag{2.24}$$

Finally we examine the fluctuations for the random variable $\frac{N_1}{n_l}$ when $\gamma \equiv 1$. Let

$$\beta_l^m = \frac{1}{2}\int_{(0,\infty)} \frac{1}{t}\mu_l(dt) \tag{2.25}$$

and define, for $\sigma > 0$, the measure G_l^σ by

$$G_l^\sigma[A] = \sharp\{j : n_l^{1-\sigma}\epsilon_l(j) \in A \setminus \{0\}\}. \tag{2.26}$$

For $\sigma > 0$ for which there is a measure G^σ on $\mathbf{R}_+$ such that for every $f : \mathbf{R}_+ \to \mathbf{R}$ which is continuous and bounded, $\int_{\mathbf{R}_+} \frac{f(t)}{t(1+t)} G_l^\sigma(dt)$ converges to $\int_{\mathbf{R}_+} \frac{f(t)}{t(1+t)} G^\sigma(dt)$ as $l \to \infty$, we define the function $g^\sigma : \mathbf{R}_+ \setminus \{0\} \to \mathbf{R}$ by

$$g^\sigma(\zeta) = \int_{(0,\infty)} [\frac{\zeta}{t} - \ln(\frac{\zeta + t}{t})] G^\sigma(dt). \tag{2.27}$$

Theorem 4. *Suppose there exist a value σ such that the function g^σ is neither identically zero or plus infinity. Then this value of σ is unique and, whenever $\beta > \beta_m$, the following limit exists*

$$\lim_{l\to\infty} \mathsf{E}_l^\beta[\exp\{\zeta(\frac{N_1}{n_l} - (1 - \frac{\beta_l^m}{\beta}))\beta n_l^\sigma\}] = \exp\{g^\sigma(\zeta)\}. \tag{2.28}$$

3. Proofs of the Theorems

It is convenient to define a modified partition function $\tilde{Z}_l(\beta)$ by

$$\tilde{Z}_l(\beta) = \pi^{-\frac{1}{2}n_l}\sqrt{n_l}e^{-\frac{1}{2}\lambda_l(0)n_l\beta}\beta^{\frac{1}{2}(n_l-2)}Z_l(\beta). \tag{3.1}$$

If we let $\tilde{J}_l = \frac{1}{2}(\lambda_l(1) - J_l)$, then we can write

$$\tilde{Z}_l(\beta) = \pi^{-\frac{1}{2}n_l}\sqrt{\frac{n_l}{\beta}}\int_{\Omega_l(\beta)} e^{-\langle\omega,\tilde{J}_l\omega\rangle} m_{n_l,\sqrt{\beta n_l}}(d\omega). \tag{3.2}$$

If U is the orthogonal matrix which diagonalizes $\tilde{J}_l$, we have since the measure $m_{n_l,\sqrt{\beta n_l}}$ is invariant under U,

$$\begin{aligned}\tilde{Z}_l(\beta) &= \pi^{-\frac{1}{2}n_l}\sqrt{\frac{n_l}{\beta}}\int_{\Omega_l(\beta)} e^{-\langle\omega,U\tilde{J}_lU^*\omega\rangle} m_{n_l,\sqrt{\beta n_l}}(d\omega) \\ &= \pi^{-\frac{1}{2}n_l}\sqrt{\frac{n_l}{\beta}}\int_{\Omega_l(\beta)} e^{-\sum_{j=1}^{n_l}\epsilon_l(j)\omega_j^2} m_{n_l,\sqrt{\beta n_l}}(d\omega).\end{aligned} \tag{3.3}$$

We also introduce an auxiliary grand-canonical partition function $\Xi_l(\alpha)$ for $\alpha < 0$:

$$\Xi_l(\alpha) = \int_0^\infty e^{\beta n_l \alpha} \tilde{Z}_l(\beta) d\beta; \tag{3.4}$$

then

$$\Xi_l(\alpha) = \pi^{-\frac{1}{2}n_l} \int_{\mathbf{R}^{n_l}} e^{-\sum_{j=1}^{n_l}(\epsilon_l(j)-\alpha)\omega_j^2} d\omega = \Pi_{j=1}^{n_l}(\epsilon_l(j)-\alpha)^{-\frac{1}{2}}. \tag{3.5}$$

For $\alpha < 0$, let

$$p_l(\alpha) = \frac{1}{n_l} \ln \Xi_l(\alpha), \tag{3.6}$$

then

$$p_l(\alpha) = -\frac{1}{2} \int_{[0,\infty)} \ln(t-\alpha)\mu_l(dt), \tag{3.7}$$

and therefore because of the assumption (A2)

$$p(\alpha) = \lim_{l\to\infty} p_l(\alpha) = -\frac{1}{2} \int_{[0,\infty)} \ln(t-\alpha)\mu(dt). \tag{3.8}$$

Let $\tilde{f}_l(\beta) = -\frac{1}{n_l} \ln \tilde{Z}_l(\beta)$; we shall prove that $\tilde{f}(\beta) = \lim_{l\to\infty} \tilde{f}_l(\beta)$ exists and is given by:

$$\tilde{f}(\beta) = \alpha(\beta)\beta - p(\alpha(\beta)) \quad \text{for} \quad \beta < \beta_c \tag{3.9a}$$

where $\alpha(\beta) < 0$ is the unique solution of $\beta = p'(\alpha)$ and

$$\tilde{f}(\beta) = -p(0) \text{ for } \beta \geq \beta_c. \tag{3.9b}$$

Theorem 1 then follows immediately. We shall require the following two lemmas.

Lemma 1. *Let X_1 and X_2 be independent non-negative random variables with means m_1 and m_2 respectively. Suppose that X_1 has density $\frac{1}{\sqrt{2\pi m_1 x}} \exp(-x/2m_1)$; if $x_0 > 0$ and $0 < \delta < x_0$ then*

$$\mathbf{P}\left[X_1 + X_2 \in [x_0, x_0+\delta]\right] \geq \frac{e^{\frac{-(x_0+\delta)}{2m_1}}}{\sqrt{2\pi m_1(x_0+\delta)}} \delta(1 - \frac{m_2}{x_0}).$$

Proof: The random variable X has density ρ given by [17]:

$$\rho(x) = \int_{[0,x]} \frac{1}{\sqrt{2\pi m_1(x-y)}} e^{-\frac{(x-y)}{2m_1}} F_2(dy).$$

where F_2 is the distribution of X_2. Thus $\rho(x) \geq \frac{e^{-\frac{x}{2m_1}}}{\sqrt{2\pi m_1 x}} \int_{[0,x]} F_2(dy)$; therefore

$$\begin{aligned}
\mathsf{P}[X_1 + X_2 \in [x_0, x_0+\delta]] &= \int_{x_0}^{x_0+\delta} \rho(x)dx \\
&\geq \int_{x_0}^{x_0+\delta} dx \frac{e^{-\frac{x}{2m_1}}}{\sqrt{2\pi m_1 x}} \int_{[0,x]} F_2(dy) \\
&\geq \frac{e^{-\frac{(x_0+\delta)}{2m_1}}}{\sqrt{2\pi m_1 (x_0+\delta)}} \int_{x_0}^{x_0+\delta} dx \int_{[0,x]} F_2(dy) \\
&\geq \frac{e^{-\frac{(x_0+\delta)}{2m_1}}}{\sqrt{2\pi m_1 (x_0+\delta)}} \int_{x_0}^{x_0+\delta} \int_{[0,\infty)} (1-\frac{y}{x}) F_2(dy) \\
&= \frac{e^{-\frac{(x_0+\delta)}{2m_1}}}{\sqrt{2\pi m_1 (x_0+\delta)}} (\delta - m_2 \ln\left(\frac{x_0+\delta}{x_0}\right)) \\
&\geq \frac{e^{-\frac{(x_0+\delta)}{2m_1}}}{\sqrt{2\pi m_1 (x_0+\delta)}} \delta (1 - \frac{m_2}{x_0}).
\end{aligned}$$

□

The mapping $\alpha \mapsto p_l'(\alpha)$ is strictly increasing for $\alpha < 0$; $p_l'(\alpha) \to 0$ as $\alpha \to -\infty$ and $p_l'(\alpha) \to \infty$ as $\alpha \to 0$. Therefore for each $\beta > 0$ and each $l \in \mathbb{N}$ there is a unique value of $\alpha < 0$ such that $\beta = p_l'(\alpha)$; denote this value of α by $\alpha_l(\beta)$).

Lemma 2. *$\lim_{l\to\infty} \alpha_l(\beta)$ exists and is given by*

$$\lim_{l\to\infty} \alpha_l(\beta) = \begin{cases} \alpha(\beta); & \beta < \beta_c, \\ 0; & \beta \geq \beta_c. \end{cases} \tag{3.10}$$

Furthermore

$$\lim_{l\to\infty} p(\alpha_l(\beta)) = \begin{cases} p(\alpha(\beta)); & \beta < \beta_c, \\ p(0); & \beta \geq \beta_c. \end{cases}$$

Proof: We first observe that $p_l'(\alpha) \leq -\frac{1}{2\alpha}$ and therfore $\alpha_l(\beta)$ lies in the interval $[-\frac{1}{2\beta}, 0)$ for each l . The sequence $(\alpha_l(\beta))$ thus has an accumulation point in the closure of this interval. This accumulation point is readily shown to be unique and given by (3.10); the proof of this is identical with that of Lemma 3 in [6] and we omit it.

The function $\alpha \mapsto p_l(\alpha)$ is convex and therefore the convergence of $p_l(\alpha)$ to $p(\alpha)$ is uniform on compact subsets of $(-\infty, 0)$; the convergence of $p_l(\alpha_l(\beta))$ to $p(\alpha(\beta))$ is then immediate in the case $\beta < \beta_c$. Next we suppose $\beta_c < \infty$; this implies that μ does not have an atom at the origin. To deal with the case $\beta \geq \beta_c$ we define, for $\delta > 0$,

$$p_l(\alpha;\delta) = -\frac{1}{2}\int_{[\delta,\infty)} \ln(t-\alpha)\mu_l(dt) \tag{3.11a}$$

and

$$p(\alpha;\delta) = -\frac{1}{2}\int_{[\delta,\infty)} \ln(t-\alpha)\mu(dt). \tag{3.11b}$$

If μ does not have an atom at δ, then $p_l(\alpha;\delta)$ converges to $p(\alpha;\delta)$ and again the convergence is uniform in α on compact subsets of $(-\infty,\delta)$. Now

$$p_l(\alpha_l(\beta)) = -\frac{1}{2}\int_{[0,\delta]} \ln(t-\alpha_l(\beta))\mu_l(dt) + p_l(\alpha_l(\beta));\delta). \tag{3.12}$$

Using the fact that $-\ln x < \frac{1}{x}$ for all $x > 0$ and that $t - \alpha_l(\beta) \leq \delta + \frac{1}{2\delta}$ for $t \in [0,\delta]$ we have

$$-\frac{1}{2}\ln(\delta+\frac{1}{2\beta}) \leq -\frac{1}{2}\ln(t-\alpha_l(\beta)) \leq \frac{1}{2}\frac{1}{t-\alpha_l(\beta)} < \frac{1}{2t}$$

for $t \in [0,\delta]$. Integrating over $[0,\delta]$ with respect to μ_l we obtain

$$-\frac{1}{2}\ln(\delta+\frac{1}{2\beta})\mu_l[\delta] \leq -\frac{1}{2}\int_{[0,\delta]} \ln(t-\alpha_l(\beta))\mu_l(dt) \leq \int_{[0,\delta]} \frac{1}{2t}\mu_l(dt). \tag{3.12}$$

Let (δ_n) be a sequence of positive numbers converging to zero such that for each n, $\mu(\{\delta_n\}) = 0$. From (3.13) we get

$$\begin{aligned}-\frac{1}{2}\ln(\delta+\frac{1}{2\beta})\mu[\delta_n] &\leq \liminf_{l\to\infty} -\frac{1}{2}\int_{[0,\delta_n]} \ln(t-\alpha_l(\beta))\mu_l(dt)\\ &\leq \limsup_{l\to\infty} -\frac{1}{2}\int_{[0,\delta_n]} \ln(t-\alpha_l(\beta))\mu_l(dt) \leq \int_{[0,\delta_n]} \frac{1}{2t}\mu_l(dt).\end{aligned} \tag{3.14}$$

Since μ does not have an atom at zero this yields

$$\lim_{n\to\infty}\limsup_{l\to\infty}\int_{[0,\delta_n]} \ln(t-\alpha_l(\beta))\mu_l(dt) = \lim_{n\to\infty}\liminf_{l\to\infty}\int_{[0,\delta_n]} \ln(t-\alpha_l(\beta))\mu_l(dt) = 0,$$

which combined with (3.12) gives

$$\lim_{l\to\infty} p_l(\alpha_l(\beta)) = \lim_{n\to\infty} p(0;\delta_n) = p(0).$$

□

Proof of Theorem 1:
We first note that the function

$$\beta \mapsto \pi^{-\frac{1}{2}n_l} \frac{\sqrt{n_l}}{\beta} \int_{\Omega_l(1)} e^{-\beta<x,\tilde{J}_l x>} m_{n_l,\sqrt{n_l}}(dx)$$

is decreasing and therefore $\beta \mapsto \tilde{f}_l(\beta) + \frac{1}{2}\ln\beta$ is increasing. Now for $\alpha < 0$ and $\delta > 0$

$$\begin{aligned}\Xi_l(\alpha) &\geq \int_{\beta-\delta}^{\beta} e^{\beta' n_l \alpha} \tilde{Z}_l(\beta')d\beta' \\ &= \int_{\beta-\delta}^{\beta} e^{-n_l\{\tilde{f}_l(\beta')+\frac{1}{2}\ln\beta'-\alpha\beta'\}} e^{\frac{1}{2}n_l \ln\beta'} d\beta' \\ &\geq e^{-n_l\{\tilde{f}_l(\beta)+\frac{1}{2}\ln\beta-\alpha\beta\}} e^{\frac{1}{2}n_l\ln(\beta-\delta)}\delta.\end{aligned}$$

Therefore

$$p_l(\alpha) \geq \frac{1}{n_l}\ln\delta - \tilde{f}_l(\beta) + \alpha\beta + \frac{1}{2}\ln(\frac{\beta-\delta}{\beta});$$

and so

$$\liminf_{l\to\infty} \tilde{f}_l(\beta) \geq \alpha\beta - p(\alpha) + \frac{1}{2}\ln(\frac{\beta-\delta}{\beta});$$

since δ is arbitrary we have, letting $\delta \to 0$

$$\liminf_{l\to\infty} \tilde{f}_l(\beta) \geq \alpha\beta - p(\alpha).$$

Because this is true for all $\alpha < 0$ it follows that

$$\liminf_{l\to\infty} \tilde{f}_l(\beta) \geq \sup_{\alpha<0}\{\alpha\beta - p(\alpha)\}.$$

To prove the upper bound we introduce the Kac probability measure K_l^α on R_+; K_l^α is absolutely continuous and is defined by

$$\mathsf{K}_l^\alpha(d\beta) = e^{-n_l p_l(\alpha)} e^{\beta n_l \alpha} \tilde{Z}_l(\beta)d\beta. \tag{3.15}$$

We first consider the case $\beta < \beta_c$; in this case we can find a unique value of $\alpha < 0$ such that $p'(\alpha) = \beta$. From (3.4) we get

$$\int_0^\infty e^{-\delta\beta'} \mathsf{K}_l^\alpha(d\beta') = \exp - s\frac{\{p_l(\alpha) - p_l(\alpha-\delta_l)\}}{\delta_l} \tag{3.16}$$

where $\delta_l = s/n_l$. For $\alpha < 0$ we have that

$$\lim_{l\to\infty} \frac{p_l(\alpha) - p_l(\alpha-\delta_l)}{\delta_l} = p'(\alpha); \tag{3.17}$$

therefore

$$\lim_{l\to\infty}\int_0^\infty e^{-s\beta'}\mathbf{K}_l^\alpha(d\beta') = e^{-sp'(\alpha)},$$

and hence $\mathbf{K}_l^\alpha$ converges weakly to $\delta_{p'(\alpha)}$. Thus for $0<\delta<-\frac{1}{2}\alpha(\beta)$,

$$\lim_{l\to\infty}\mathbf{K}_l^{\alpha(\beta)+\delta}\left[\beta, p'(\alpha(\beta)+2\delta)\right] = 1.$$

Now

$$\begin{aligned}
&\frac{1}{n_l}\ln\mathbf{K}_l^{\alpha(\beta)+\delta}\left[\beta, p'(\alpha(\beta)+2\delta)\right]\\
&= \frac{1}{n_l}\ln\int_\beta^{p'(\alpha(\beta)+2\delta)} e^{-n_l\{\tilde{f}_l(\beta')+\frac{1}{2}\ln\beta'-(\alpha(\beta)+\delta)\beta'\}}e^{\frac{1}{2}n_l\ln\beta'}d\beta - p_l(\alpha(\beta)+\delta)\\
&\le -\tilde{f}_l(\beta)+\alpha(\beta)\beta+\delta\beta+\frac{1}{2}\ln\left(\frac{p'(\alpha(\beta)+2\delta)}{\beta}\right) - p_l(\alpha(\beta)+\delta);
\end{aligned}$$

therefore

$$\limsup_{l\to\infty}\tilde{f}_l(\beta)\le\alpha(\beta)\beta - p(\alpha(\beta)+\delta)+\delta\beta+\frac{1}{2}\ln\left(\frac{p'(\alpha(\beta)+2\delta)}{\beta}\right)$$

since p and p' are continuous, letting $\delta\to 0$ we get

$$\limsup_{l\to\infty}\tilde{f}_l(\beta)\le\alpha(\beta)\beta-p(\alpha(\beta)) = \sup_{\alpha<0}[\alpha\beta-p(\alpha)].$$

We now take $\beta\ge\beta_c$; let $Y_1, Y_2\ldots, Y_{n_l}$ be independent random variables with density $(2\pi m_j x)^{-\frac{1}{2}}\exp(-x/2m_j), j=1,\ldots,n_l$ where the mean $m_j = (2n_l(\epsilon_l(j)-\alpha))^{-1}$. Then from (3.16) we see that $X=\sum_{j=1}^{n_l}Y_j$ is distributed with probability measure $\mathbf{K}_l^\alpha$ and has mean $p_l'(\alpha)$. Putting $X_1=Y_1$ and $X_2=\sum_{j=2}^{n_l}Y_j$ in Lemma 1 we get

$$\mathbf{K}_l^\alpha[\beta,\beta+\delta]\ge\left(\frac{n_l(-\alpha)}{\pi(\beta+\delta)}\right)^{\frac{1}{2}} e^{\frac{n_l\alpha}{(\beta+\delta)}}\delta\left(1-(p_l'(\alpha)+\frac{1}{2n_l\alpha})\beta^{-1}\right).$$

Choose $\alpha_l<0$ such that $p_l'(\alpha_l)=\beta$ and let $\delta=1/n_l$, then

$$\begin{aligned}
\mathbf{K}_l^{\alpha_l}\left[\beta,\beta+\frac{1}{n_l}\right] &\ge \left(\frac{-n_l\alpha_l}{\pi(\beta+\frac{1}{n_l})}\right)^{\frac{1}{2}} e^{\frac{n_l\alpha_l}{\beta+\frac{1}{n_l}}}\frac{1}{2n_l^2(-\alpha_l)}\\
&\ge\frac{1}{(\pi(\beta+\frac{1}{n_l}))^{\frac{1}{2}}}\frac{1}{2n_l^{3/2}}e^{\frac{n_l\alpha_l}{\beta+\frac{1}{n_l}}},
\end{aligned}$$

for large l, since $\alpha_l \to 0$ as $l \to \infty$. Thus $\liminf_{l\to\infty} \frac{1}{n_l} \ln \mathsf{K}_l^{\alpha_l}\left[\beta, \beta + \frac{1}{n_l}\right] \geq 0$ and by the same argument as for $\beta < \beta_c$ this gives

$$\limsup_{l\to\infty} \tilde{f}_l(\beta) \leq -\limsup_{l\to\infty} p_l(\alpha_l);$$

but $p_l(\alpha_l)$ converges to $p(0)$ by Lemma 2 and therefore

$$\limsup_{l\to\infty} \tilde{f}_l(\beta) \leq -p(0) \leq \sup_{\alpha<0}(\alpha\beta - p(\alpha)).$$

Combining the upper and lower bounds we then get:

$$\tilde{f}(\beta) = \lim_{l\to\infty} \tilde{f}_l(\beta) = \sup_{\alpha<0}(\alpha\beta - p(\alpha));$$

the Legendre transform of p can be readily calculated to give (3.9 a,b).

□

Proof of Theorem 2:
Let $X_l^\delta(\omega) = \sum_{\epsilon_l(j)\geq\delta} N_j(\omega)$; let

$$Z_l^\delta(s,\beta) = \int_{\Omega_l(1)} e^{-\beta\{H_l(\omega) - sX_l^\delta(\omega)\}} m_{n_l,\sqrt{n_l}}(d\omega) \tag{3.18}$$

and let $f_l^\delta(s,\beta) = -(\beta n_l)^{-1} \ln Z_l^{s,\delta}(\beta)$. We shall prove that $f^\delta(s,\beta) = \lim_{l\to\infty} f_l^\delta(s,\beta)$ exists and for $\delta > 0, s \mapsto f^\delta(s,\beta)$ is differentiable in s at $s = 0$. But

$$\mathsf{E}_l^\beta\left[e^{\beta s \frac{X_l^\delta}{n_l}}\right] = \exp\left[-\beta s \frac{(f_l^\delta(\sigma_l,\beta) - f_l^\delta(0,\beta))}{\sigma_l}\right] \tag{3.19}$$

where $\sigma_l = s/n_l$ and therefore since $s \mapsto f^\delta(s,\beta)$ is convex,

$$\lim_{l\to\infty} \mathsf{E}_l^\beta\left[e^{\beta s \frac{X_l^\delta}{n_l}}\right] = \exp\left[-\beta s \frac{\partial f^\delta}{\partial s}(0,\beta)\right]. \tag{3.20}$$

Now $N_j(\omega) = |\langle\phi_j,\omega\rangle|^2 = \langle\omega, P_j\omega\rangle$, where P_j is the orthogonal projection onto ϕ_j, and therefore $X_l^\delta(\omega) = \langle\omega, \sum_{\epsilon_l(j)\geq\delta} P_j\omega\rangle$; thus

$$Z_l^\delta(s,\beta) = \int_{\Omega_l(1)} e^{\frac{1}{2}\beta\langle\omega, J_l^{s,\delta}\omega\rangle} m_{n_l,\sqrt{n_l}}(d\omega) \tag{3.21}$$

where $J_l^{s,\delta} = J_l + 2s\sum_{\epsilon_l(j)\geq\delta} P_j$. The eigenvalues of this operator are $\lambda_l^{s,\delta}(j)$ where

$$\lambda_l^{s,\delta}(j) = \lambda_l(j) \text{ if } \epsilon_l(j) < \delta$$

and

$$\lambda_l^{s,\delta}(j) = \lambda_l(j) + 2s \text{ if } \epsilon_l(j) \geq \delta.$$

Thus we can apply the result of Theorem 1 with μ_l and μ replaced by $\mu_l^{s,\delta}$ and $\mu^{s,\delta}$ where $\mu_l^{s,\delta}(A) = \frac{1}{n_l}\sharp\{j : \frac{1}{2}(\lambda_l(0) - \lambda_l^{s,\delta}(j)) \in A\}$ and $\mu^{s,\delta}$ is the limit of $\mu_l^{s,\delta}$ in the sense of (A2). If $g : \mathbf{R}_+ \to \mathbf{R}$ is a continuous function such that $|g(t)| < C\ln(2+t)$ for some constant C then

$$\int_{[0,\infty)} g(t)\mu^{s,\delta}(dt) = \int_{[0,\delta)} g(t)\mu(dt) + \int_{[\delta,\infty)} g(t-s)\mu(dt). \tag{3.22}$$

Let $p^{s,\delta}(\alpha) = -\frac{1}{2}\int_{[0,\infty)} \ln(t-\alpha)\mu^{s,\delta}(dt)$ for $\alpha < 0$, let $\beta_c^{s,\delta} = \int_{[0,\infty)} \frac{1}{2t}\mu^{s,\delta}(dt)$ and for $\beta < \beta_c^{s,\delta}$ let $\alpha^{t,\delta}(\beta) < 0$ be the unique solution of $\beta = \frac{\partial}{\partial\alpha}p^{t,\delta}(\alpha)$. Then by Theorem 1 we have that $f_l^\delta(s,\beta)$ converges as $l \to \infty$ to $f^\delta(s,\beta)$ where

$$f^\delta(s,\beta) = \alpha^{s,\delta}(\beta) - p^{s,\delta}\frac{(\alpha^{s,\delta}(\beta))}{\beta} - \frac{1}{2\beta}\ln\frac{\beta}{\pi} - \frac{1}{2}\lambda(0),$$

if $\beta < \beta_c^{s,\delta}$, and

$$f^\delta(s,\beta) = -\frac{p^{s,\delta}}{\beta}(0) - \frac{1}{2\beta}\ln\frac{\beta}{\pi} - \frac{1}{2}\lambda(0),$$

if $\beta \geq \beta_c^{s,\delta}$.

We can calculate the derivative of $f^\delta(s,\beta)$ with respect to s at $s = 0$ to get

$$\frac{\partial f^\delta}{\partial s}(0,\beta) = \begin{cases} 1 - \frac{1}{2\beta}\int_{(0,\delta)} \frac{1}{t-\alpha(\beta)}\mu(dt) & \text{if } \beta \leq \beta_c \\ \frac{\beta_c}{\beta} - \frac{1}{2\beta}\int_{(0,\delta)} \frac{1}{t}\mu(dt) & \text{if } \beta > \beta_c. \end{cases}$$

Therefore

$$\lim_{\delta\to 0}\frac{\partial f^\delta}{\partial s}(0,\beta) = \begin{cases} 1 & \text{if } \beta \leq \beta_c, \\ \frac{\beta_c}{\beta} & \text{if } \beta > \beta_c. \end{cases}$$

Thus using (3.20) and $\sum_{j\geq 1} N_j(\omega) = n_l$ we get the required result.

□

We shall say that $F : \mathbf{R}_+^n \to \mathbf{R}$ is monotone if

$$F(x_1,\ldots,x_n) \geq F(y_1,\ldots,y_n)$$

whenever $x_i \geq y_i$ for $i = 1, \dots, n$.

Lemma 3: Suppose $F : \mathbf{R}^{n_l - 1} \to \mathbf{R}$ and $G : \mathbf{R}^{n_l - 1} \to \mathbf{R}$ are monotone; define $f : \mathbf{R}^{n_l} \to \mathbf{R}$ and $g : \mathbf{R}^{n_l} \to \mathbf{R}$ by $f(\omega) = F(N_1, N_2, \dots, N_{n_l - 1})$ and $g(\omega) = G(N_1, N_2, \dots, N_{n_l - 1})$.
Then

$$\mathsf{E}_l^\beta[fg] \geq \mathsf{E}_l^\beta[f]\mathsf{E}_l^\beta[g].$$

Proof:

$$\mathsf{E}_l^\beta[f] = \frac{\int_{\Omega_l(1)} \exp\{\beta \sum_{j=1}^{n_l - 1}(\epsilon_l(n_l) - \epsilon_l(j))\omega_j^2\} F(\omega_1^2, \omega_2^2, \dots, \omega_{n_l-1}^2) m_{n_l, \sqrt{n_l}}(d\omega)}{\int_{\Omega_l(1)} \exp\{-\beta \sum_{j=1}^{n_l - 1}(\epsilon_l(n_l) - \epsilon_l(j))\omega_j^2\} m_{n_l, \sqrt{n_l}}(d\omega)}$$

$$= \frac{\int_0^1 dr_{n_l-1} \int_0^{r_{n_l-1}} dr_{n_l-2} \dots \int_0^{r_2} dr_1 F(r_1, r_2, \dots r_{n_l-1}) D(r_1, r_2, \dots r_{n_l-1})}{\int_0^1 dr_{n_l-1} \int_0^{r_{n_l-1}} dr_{n_l-2} \dots \int_0^{r_2} dr_1 D(r_1, r_2, \dots r_{n_l-1})}$$

where

$$D(r_1, r_2, \dots r_{n_l-1}) = \frac{\exp\{\beta \sum_{j=1}^{n_l-1}(\epsilon_l(n_l) - \epsilon_l(j)) r_j\}}{\{(1 - r_{n_l-1})(r_{n_l-1} - r_{n_l-2}) \dots (r_2 - r_1) r_1\}}.$$

For $h : \mathbf{R}_+^k \to \mathbf{R}$ and $r > 0$ let

$$\langle h \rangle_r^k = (Z_r^k)^{-1} \int_0^r dr_k \int_0^{r_k} dr_{k-1} \dots \int_0^{r_2} dr_1 \frac{\exp\{\mathcal{H}(r_1, \dots, r_k) h(r_1, \dots, r_k)\}}{\mathcal{P}_k(r_1, \dots, r_k, r)}$$

where

$$Z_r^k = \int_0^r dr_k \int_0^{r_k} dr_{k-1} \dots \int_0^{r_2} dr_1 \frac{\exp\{\mathcal{H}(r_1, \dots, r_k)\}}{\mathcal{P}_k(r_1, \dots, r_k, r)},$$

$$\mathcal{H}(r_1, \dots, r_k) = a_1 r_1 + \dots + a_k r_k,$$

$a_1 \geq 0, \quad a_2 \geq 0, \quad \dots \quad a_k \geq 0$, and

$$\mathcal{P}_k(r_1, \dots, r_k, r) = \{(r - r_k)(r_k - r_{k-1}) \dots (r_2 - r_1) r_1\}^{\frac{1}{2}}.$$

We shall prove by induction on k that if h_1 and h_2 are monotone then

$$\langle h_1 h_2 \rangle_r^k \geq \langle h_1 \rangle_r^k \langle h_2 \rangle_r^k. \tag{3.23}$$

If $h_1 : \mathbf{R}_+ \to \mathbf{R}$ and $h_2 : \mathbf{R}_+ \to \mathbf{R}$ are monotone then

$$(h_1(r) - h_1(r'))(h_2(r) - h_2(r')) \geq 0$$

for all $r, r' \in \mathbf{R}_+$. Therefore

$$(Z_r^1)^{-2}\int_0^r dr_1 \frac{e^{a_1 r_1}}{\{(r-r_1)r_1\}^{\frac{1}{2}}}\int_0^r dr_2 \frac{e^{a_1 r}}{\{(r-r_1')r_1'\}^{\frac{1}{2}}}(h_1(r_1)-h_1(r_1'))(h_2(r_1)-h_2(r_1')) \geq 0\,,$$

or

$$\langle h_1 h_2\rangle_r^1 \geq \langle h_1\rangle_r^1 \langle h_2\rangle_r^1.$$

Suppose that the inequality (3.23) is true for $k = 1, \ldots, m-1$.

$$\int_0^r dr_m \cdots \int_0^{r_2} dr_1 \frac{\exp\{\mathcal{H}(r_1,\ldots,r_m)\}}{\mathcal{P}(r_1,\ldots,r_m,r)}\int_0^r dr_m' \cdots \int_0^{r_2'} dr_1' \frac{\exp\{\mathcal{H}(r_1',\ldots,r_m')\}}{\mathcal{P}(r_1',\ldots,r_m',r)}$$

$$\times\{(h_1(r_1,\ldots,r_m)-h_1(r_1',\ldots,r_m')\}\{h_2(r_1,\ldots,r_m)-h_2(r_1',\ldots,r_m')\} \geq 0. \quad (3.24)$$

The integral in (3.24) is equal to

$$\int_0^r dr_m \frac{e^{a_m r_m}}{\sqrt{r-r_m}}\int_0^r dr_m' \frac{e^{a_m r_m'}}{\sqrt{r-r_m'}} Z_{r_m}^{m-1} Z_{r_m'}^{m-1}$$

$$\{\langle h_1(\ldots,r_m)h_2(\ldots,r_m)\rangle_{r_m}^{m-1} - \langle h_1(\ldots,r_m)\rangle_{r_m}^{m-1}\langle h_2(\ldots,r_m')\rangle_{r_m'}^{m-1}$$
$$- \langle h_1(\ldots,r_m')\rangle_{r_m'}^{m-1}\langle h_2(\ldots,r_m)\rangle_{r_m}^{m-1} + \langle h_1(\ldots,r_m')h_2(\ldots,r_n')\rangle_{r_m'}^{m-1}\}.$$

By the induction hypothesis the quantity in $\{\ldots\}$ is greater than or equal to

$$(\langle h_1(\ldots,r_m)\rangle_{r_m}^{m-1} - \langle h_1(\ldots,r_m')\rangle_{r_m'}^{m-1})(\langle h_2(\ldots,r_m)\rangle_{r_m}^{m-1} - \langle h_2(\ldots,r_m')\rangle_{r_m'}^{m-1});$$

therefore it is sufficient to prove that if h is monotone $r \to \langle h(\ldots,r)\rangle_r^{m-1}$ is monotone. Now if $r > r'$

$$\begin{aligned}\langle h(\ldots,r)\rangle_r^{m-1} - \langle h(\ldots,r'\rangle_{r'}^{m-1} &= \langle h(\ldots,r)\rangle_r^{m-1} - \langle h(\ldots,r')\rangle_r^{m-1}\\ &\quad + \langle h(\ldots,r')\rangle_r^{m-1} - \langle h(\ldots,r')\rangle_{r'}^{m-1}\\ &\geq \langle h(\ldots,r')\rangle_r^{m-1} - \langle h(\ldots,r')\rangle_{r'}^{m-1}.\end{aligned}$$

Therefore it is enough to prove that for fixed $r', r \mapsto \langle h(\ldots,r')\rangle_r^{m-1}$ is monotone. We can write

$$\langle h(\ldots,r')\rangle_r^{m-1} = \frac{\int_0^1 dr_{m-1}\cdots\int_0^{r_2} dr_1 h(rr_1,\ldots,rr_{m-1},r')W(r_1,\ldots,r_{m-1})}{\int_0^1 dr_{m-1}\cdots\int_0^{r_2} dr_1 W(r_1,\ldots,r^{m-1})};$$

where

$$W(r_1,\ldots,r_{m-1}) = \frac{\exp\{r\mathcal{H}(r_1,r_2,\ldots,r_{m-1})\}}{\mathcal{P}(r_1,\ldots,r_{m-1},1)}.$$

Since if $r > s$, $h(rr_1, \ldots, rr_{m-1}, r') \geq h(sr_1, \ldots, sr_{m-1}, r')$, it is sufficient to prove that

$$r \mapsto \frac{\int_0^1 dr_{m-1} \ldots \int_0^{r_2} dr_1 W(r_1, \ldots, r_{m-1}) h(r_1, \ldots, r_{m-1}, r')}{\int_0^1 dr_{m-1} \ldots \int_0^{r_2} dr_1 W(r_1, \ldots, r_{m-1})},$$

is monotone; but this quantity can be differentiated with respect to r to give

$$\langle \mathcal{H}\, h(\ldots, r')\rangle_1^{m-1} - \langle \mathcal{H}\rangle_1^{m-1} \langle h(\ldots, r')\rangle_1^{m-1} \tag{3.25}$$

with $a_1, a_2, \ldots, a_{m-1}$ replaced by $ra_1, ra_2, \ldots, ra_{m-1}$. But by the induction hypothesis (3.25) is non-negative and thus the lemma is proved.

□

For the next theorem we require also the following information about the sequence $\alpha_l(\beta)n_l$.

Lemma 4. *For $\beta > 0$ let $b_l = -\alpha_l(\beta)n_l$. If $\beta \leq \beta_m$, b_l diverges to $+\infty$ and if $\beta > \beta_m$, b_l converges to $b(\beta) \geq 0$ where $b(\beta)$ is the unique root of*

$$\frac{1}{2} \int_{[0,\infty)} e^{-sb(\beta)} \gamma(s) ds = \beta - \beta_m. \tag{3.26}$$

We do not give the proof of this lemma since it is almost identical with that of Theorem 3 in [6].

Proof of Theorem 3:
For $\zeta \geq 0$ let

$$g_l(\zeta, \beta) = \mathsf{E}_l^\beta \left[e^{-\zeta\beta \frac{N_1}{n_l}} \right];$$

then

$$g_l(n_l\zeta, \beta) = (Z_l(\beta))^{-1} \int_{\Omega_l(1)} e^{-\beta(H_l(\omega) + \zeta N_1(\omega))} m_{n_l, \sqrt{n_l}}(d\omega)$$

$$= \pi^{-\frac{1}{2}n_l} \sqrt{\frac{n_l}{\beta}} \left(\tilde{Z}_l(\beta)\right)^{-1} \int_{\Omega_l(\beta)} e^{-\langle\omega, (\tilde{J}_l + \zeta P_1)\omega\rangle} m_{n_l, \sqrt{\beta n_l}}(d\omega). \tag{3.27}$$

Let

$$\Xi_l(\zeta, \alpha) = \pi^{-\frac{1}{2}n_l} \int_{\mathbf{R}^{n_l}} e^{-\langle\omega, (\tilde{J}_l + \zeta P_1 - \alpha)\omega\rangle} d\omega; \tag{3.28}$$

then on the one hand we have

$$\Xi_l(\zeta, \alpha) = \left(\frac{-\alpha}{\zeta - \alpha}\right)^{\frac{1}{2}} \Xi_l(\alpha), \tag{3.29}$$

and on the other hand by (3.27)

$$\Xi_l(\zeta,\alpha) = \int_0^\infty e^{\beta n_l \alpha} \tilde{Z}_l(\beta) g_l(n_l\zeta,\beta) d\beta. \tag{3.30}$$

Therefore

$$\int_0^\infty e^{\beta n_l \alpha} \tilde{Z}_l(\beta) g_l(n_l\zeta,\beta) d\beta = \left(\frac{-\alpha}{\zeta-\alpha}\right)^{\frac{1}{2}} \int_0^\infty e^{\beta n_l \alpha} \tilde{Z}_l(\beta) d\beta \tag{3.31}$$

which we can rewrite as

$$\int_0^\infty e^{-sx} \tilde{Z}_l(x/n_l) g_l(n_l\zeta, x/n_l) dx = \left(\frac{s}{\zeta+s}\right)^{\frac{1}{2}} \int_0^\infty e^{-sx} \tilde{Z}(x/n_l) dx \tag{3.32}$$

for $s > 0$. Using the identity

$$\left(\frac{s}{\zeta+s}\right)^{\frac{1}{2}} = 1 - \int_0^\infty e^{-sx} e^{-\frac{1}{2}\zeta x} \frac{1}{2}\zeta I(-\frac{1}{2}\zeta x) dx \tag{3.33}$$

where I is the sum of the first two modified Bessel functions: $I(x) = I_0(x) + I_1(x)$, we can invert the Laplace transforms in (3.32) to get

$$g_l(n_l\zeta, x/n_l) = 1 - \frac{1}{2}\zeta(\tilde{Z}_l(x/n_l))^{-1} \int_0^x \tilde{Z}_l(y/n_l) e^{-\frac{1}{2}\zeta(x-y)} I(-\frac{1}{2}\zeta(x-y)) dy$$

or

$$g_l(\zeta,\beta) = 1 - \frac{1}{2}\zeta(\tilde{Z}_l(\beta))^{-1} \int_0^\beta \tilde{Z}_l(\beta') e^{-\frac{1}{2}\zeta(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta')) d\beta'. \tag{3.34}$$

Let $a > 0$, then by multiplying (3.34) by $\tilde{Z}_l(\beta)e^{-a\beta}$ and integrating we get

$$\begin{aligned}\int_{\beta_1}^{\beta_2} g_l(\zeta,\beta) \mathbf{K}_l^{-a/n_l}(d\beta) &= \mathbf{K}_l^{-a/n_l}(\beta_1,\beta_2) \\ &- \frac{1}{2}\zeta \int_{\beta_1}^{\beta_2} d\beta \int_0^\beta e^{-(a+\frac{1}{2}\zeta)(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta')) \mathbf{K}_l^{-a/n_l}(d\beta').\end{aligned} \tag{3.35}$$

Now $\int_0^\infty e^{-s\beta} \mathbf{K}_l^{-a/n_l}(d\beta) = \exp -\psi_l(s)$ where

$$\begin{aligned}\psi_l(s) &= -n_l \left\{ p_l\left(-\frac{(s+a)}{n_l}\right) - p_l\left(-\frac{a}{n_l}\right) \right\} \\ &= \frac{1}{2} \int_{[0,b)} \{\ln(t+s+a) - \ln(t+a)\} G_l(dt) \\ &+ \frac{1}{2} n_l \int_{[\frac{b}{n_l},\infty)} \ln\left(1 + \frac{s}{n_l(t+\frac{a}{n_l})}\right) \mu_l(dt).\end{aligned}$$

Using the inequality

$$(1+y)^{-1}y < \ln(1+y) < y \tag{3.36}$$

for $y > 0$ we get that

$$\lim_{b\to\infty}\lim_{l\to\infty}\frac{1}{2}n_l\int_{[b/n_l,\infty)}\ln\left(1+\frac{s}{n_l(t+\frac{a}{n_l})}\right)\mu_l(dt) = s\beta_m\ .$$

If $b > 0$ is chosen so that G does not have an atom at b then

$$\lim_{l\to\infty}\frac{1}{2}\int_{[0,b)}\{\ln(t+s+a)-\ln(t+a)\}G_l(dt)$$

$$= \lim_{l\to\infty}\frac{1}{2}\int_0^\infty du\left(\frac{1-e^{-us}}{u}\right)e^{-au}\int_{[0,b)}e^{-ut}G_l(dt)$$

$$= \frac{1}{2}\int_0^\infty du\left(\frac{1-e^{-us}}{u}\right)e^{-au}\int_{[0,b)}e^{-ut}G(dt);$$

we have used here the dominated convergence theorem: $\int_{[0,b)}e^{-ut}G_l(dt) < G_l[0,b)$, which is bounded since it converges to $G[0,b)$. If a is chosen such that $\int_0^\infty due^{-au}\gamma(u) < \infty$ then by letting $b\to\infty$ along the points of continuity of G we get

$$\psi(s) \equiv \lim_{l\to\infty}\psi_l(s) = s\beta_m + \frac{1}{2}\int_0^\infty du\left(\frac{1-e^{-us}}{u}\right)e^{-au}\gamma(u).$$

Therefore the sequence of probability measures $\{\mathbf{K}_l^{-a/n_l}\}$ converges weakly to an infinitely divisible measure $\mathbf{K}$ whose Laplace transform is $\exp -\psi(s)$; since $G_l(\{0\}) \geq 1$, $\gamma(u) \geq 1$ and therefore we can write

$$e^{-\psi(s)} = \left(\frac{a}{s+a}\right)^{\frac{1}{2}} e^{-\tilde{\psi}(s)}\ ,$$

where

$$\tilde{\psi}(s) = s\beta_m + \frac{1}{2}\int_0^\infty du\left(\frac{1-e^{-us}}{u}\right)e^{-au}(\gamma(u)-1).$$

Since $(a/(s+a))^{\frac{1}{2}}$ is the Laplace transform of $\rho_0(x) = (\frac{a}{\pi})^{\frac{1}{2}}e^{-ax}x^{-\frac{1}{2}}$, then $\mathbf{K}$ is the convolution of the measure with density ρ_0 with another measure $\tilde{\mathbf{K}}$ whose Laplace transform is $\exp -\tilde{\psi}(s)$. Therefore $\mathbf{K}$ is absolutely continuous; let F be its distribution. Clearly $F(\beta) = 0$ for $\beta \leq \beta_m$. Also $\tilde{\mathbf{K}}[0,\beta_m) = 0$. Suppose there is $\delta > 0$ such that $\tilde{\mathbf{K}}[\beta_m,\beta_m+\delta] = 0$; if this is true for one value of a

then it is true for all possible values of a. Then $\tilde{\psi}(s)/s \geq \beta_m + \delta$ and therefore $\liminf_{s\to 0} \frac{\tilde{\psi}(s)}{s} \geq \beta_m + \delta$. But

$$\lim_{s\to 0} \frac{\tilde{\psi}(s)}{s} = \beta_m + \frac{1}{2}\int_0^\infty (\gamma(u) - 1)e^{-au} < \beta_m + \delta$$

for a sufficiently large; therefore $\tilde{\mathsf{K}}[\beta_m, \beta_m + \delta] > 0$ for all $\delta > 0$. Now for $\beta > \beta_m$

$$F'(\beta) = \sqrt{\frac{a}{\pi}} \int_{[\beta_m,\beta]} \frac{e^{-a(\beta-\beta')}}{\sqrt{\beta-\beta'}} \tilde{\mathsf{K}}(d\beta') \geq \frac{\sqrt{a}}{\sqrt{\pi(\beta-\beta_m)}} e^{-a(\beta-\beta_m)} \tilde{\mathsf{K}}[\beta_m, \beta] > 0.$$

The right-hand side of equation (3.35) converges to

$$\int_{\beta_1}^{\beta_2} dF(\beta) - \frac{1}{2}\zeta \int_{\beta_1}^{\beta_2} d\beta \int_0^\beta e^{-(a+\frac{1}{2}\zeta)(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta'))dF(\beta'). \qquad (3.37)$$

Now

$$\frac{\partial g_l}{\partial \beta}(\zeta, \beta) = -\zeta \mathsf{E}_l^\beta \left[\frac{N_1}{n_l} e^{-\zeta\beta\frac{N_1}{n_l}} \right] - \left\{ \mathsf{E}_l^\beta \left[e^{-\zeta\beta\frac{N_1}{n_l}} H \right] - \mathsf{E}_l^\beta \left[e^{-\zeta\beta\frac{n_1}{n_l}} \right] \mathsf{E}_l^\beta[H] \right\}.$$

The last expression in $\{\ldots\}$ is positive; this can be seen by putting $F(r_1, \ldots, r_{n_l-1}) = -e^{-\zeta\beta\frac{r_1}{n_l}}$ and $G(r_1, \ldots, r_{n_l-1}) = \sum_{j=1}^{n_l-1}(\epsilon_l(n_l) - \epsilon_l(j))r_j$ in Lemma 3, therefore $\beta \mapsto g_l(\zeta, \beta)$ is decreasing. Thus

$$\limsup_{l\to\infty} g_l(\zeta, \beta_2) \int_{\beta_1}^{\beta_l} dF(\beta) \leq \lim_{l\to\infty} \int_{\beta_1}^{\beta_2} g_l(\zeta, \beta) \mathsf{K}_l^{\frac{a}{n_l}}(d\beta) \leq \liminf_{l\to\infty} g_l(\zeta, \beta_1) \int_{\beta_1}^{\beta_2} dF(\beta). \qquad (3.38)$$

Let F be differentiable at β; from (3.37) and (3.38) we get

$$\liminf_{l\to\infty} g_l(\zeta, \beta) \frac{1}{\delta} \int_\beta^{\beta+\delta} dF(\beta') \geq \frac{1}{\delta} \int_\beta^{\beta+\delta} dF(\beta') - \frac{1}{2}\frac{\zeta}{\delta} \int_\beta^{\beta+\delta} d\beta'' \int_0^{\beta''} e^{-(a+\frac{1}{2}\zeta)(\beta''-\beta')} I(-\frac{1}{2}\zeta(\beta''-\beta'))dF(\beta'). \qquad (3.39)$$

Letting $\delta \to 0$ we get

$$\liminf_{l\to\infty} g_l(\zeta, \beta) F'(\beta) \geq F'(\beta) - \frac{1}{2}\zeta \int_0^\beta e^{-(a+\frac{1}{2}\zeta)(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta'))dF(\beta'). \qquad (3.40)$$

Integrating from $\beta-\delta$ to β and using the other side of the inequality in (3.38) we get

$$\limsup_{l\to\infty} g_l(\zeta,\beta)F'(\beta) \le F'(\beta) - \frac{1}{2}\zeta\int_0^{\beta} e^{-(a+\frac{1}{2}\zeta)(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta'))dF(\beta'). \tag{3.41}$$

If $\beta \ge \beta_m, F'(\beta) > 0$,then (3.40) and (3.41) then give

$$\lim_{l\to\infty} g_l(\zeta,\beta) = 1 + \frac{1}{2}(F'(\beta))^{-1}\zeta\int_0^{\beta} e^{-(a+\frac{1}{2}\zeta)(\beta-\beta')} I(-\frac{1}{2}\zeta(\beta-\beta'))dF(\beta').$$

We shall now study the case $\beta < \beta_m$. We start from (3.31); this can be written in the form

$$\int_0^\infty g_l(\zeta,\beta')\mathbf{K}_l^\alpha(d\beta') = \left(\frac{-\alpha}{\zeta/n_l - \alpha}\right)^{\frac{1}{2}}. \tag{3.42}$$

Let $\beta_1 < \beta_m$; from the inequality (3.36) we get

$$\frac{1}{2}\int_{[0,\infty)} \frac{s}{t-\alpha_l(\beta_1)+\frac{s}{n_l}}\mu_l(dt) \le -\ln\int_{[0,\infty)} e^{-s\beta}\mathbf{K}_l^{\alpha_l(\beta_1)}(d\beta) \le s\beta_1.$$

Therefore

$$\lim_{l\to\infty} -\ln\int_{[0,\infty)} e^{-s\beta}\mathbf{K}_l^{\alpha_l(\beta_1)}(d\beta) = s\beta_1,$$

and thus $\mathbf{K}_l^{\alpha_l(\beta_1)}$ converges weakly to δ_{β_1}. Also $-\alpha_l(\beta_1)n_l \to \infty$ by Lemma 4 and so (3.42) gives

$$\lim_{l\to\infty}\int_0^\infty g_l(\zeta,\beta')\mathbf{K}_l^{\alpha_l(\beta_1)}(d\beta') = 1. \tag{3.43}$$

Suppose $\beta < \beta_m$ and choose $\beta_1 < \beta$, then since $\beta \mapsto g_l(\zeta_1\beta)$ is decreasing and $\mathbf{K}_l^{\alpha_l(\beta_1)} \to \delta_{\beta_1}$ from (3.43) we get

$$\limsup_{l\to\infty} g_l(\zeta,\beta) \le \lim_{l\to\infty}\int_0^{\beta} g_l(\zeta,\beta')\mathbf{K}_l^{\alpha_l(\beta_1)}(d\beta') = 1.$$

Similarly by choosing $\beta_1 > \beta$ we get

$$\liminf_{l\to\infty} g_l(\zeta,\beta) \ge \lim_{l\to\infty}\int_{\beta_1}^{\infty} g_l(\zeta,\beta')\mathbf{K}_l^{\alpha_l(\beta_1)}(d\beta') = 1.$$

Therefore $\lim_{l\to\infty} g_l(\zeta,\beta)$ exists and is equal to 1.

□

Proof of Theorem 4: The uniqueness of σ follows from the scaling of the measures G_l^σ:

$$\int f(t)G_l^\sigma(dt) = \int f(n^{\sigma'-\sigma}t)G_l^{\sigma'}(dt).$$

We have also

$$\lim_{l\to\infty}\beta_l^m = \lim_{b\to\infty}\lim_{l\to\infty}\frac{1}{2}\int_{[b/n_l,\infty)}\frac{1}{t}\mu_l(dt) + \lim_{b\to\infty}\lim_{l\to\infty}\frac{1}{2}n_l^{-\sigma}\int_{(0,b/n_l^\sigma)}\frac{1}{t}G_l^\sigma(dt).$$

The first term is equal to β_m and the second term is zero; thus

$$\lim_{l\to\infty}\beta_l^m = \beta_m.$$

For $\zeta > 0$ and $\sigma \geq 0$, let

$$h_l(\zeta,\beta) = \mathsf{E}_l^\beta\left[e^{-\zeta\beta n_l^\sigma(1-\frac{\beta_l^m}{\beta}-\frac{N_1}{n_l})}\right];$$

then $h_l(\zeta,\beta) = g_l(-n_l^\sigma\zeta,\beta)e^{-\zeta n_l^\sigma(\beta-\beta_l^m)}$ and from (3.34) we get

$$h_l(\zeta,\beta) = e^{-\zeta n_l^\sigma(\beta-\beta_l^m)} + (\frac{1}{2}\zeta)^{\frac{1}{2}}n_l^{\frac{1}{2}\sigma}\left(\tilde{Z}_l(\beta)\right)^{-1}\int_0^\beta \tilde{Z}_l(\beta')e^{-\zeta n_l^\sigma(\beta'-\beta_l^m)}\frac{\tilde{I}(\frac{1}{2}\zeta n_l^\sigma(\beta-\beta'))}{\sqrt{\beta-\beta'}}d\beta' \tag{3.44}$$

where $\tilde{I}(x) = \sqrt{x}e^{-x}I(x)$. We let $\tilde{p}_l(\alpha) = -\frac{1}{2}\int_{(0,\infty)}\ln(t-\alpha)\mu_l(dt)$ for $\alpha < 0$ and define a measure m_l on $\mathbf{R}_+$ in the following way:

$$m_l(A) = n_l^{-\frac{\sigma+1}{2}}e^{-n_l\tilde{p}_l(0)}\int_{A\cap(-n_l^\sigma\beta_l^m,\infty)}\tilde{Z}_l(\beta_l^m + y/n_l^\sigma)dy. \tag{3.45}$$

We can rewrite (3.44) in the following form with $a < 0$

$$\int_{\beta_1}^{\beta_2} h_l(\zeta,\beta)\mathsf{K}_l^{-\frac{a}{n_l}}(d\beta) = \int_{\beta_1}^{\beta_2} e^{-\zeta n_l^\sigma(\beta-\beta_l^m)}\mathsf{K}_l^{-\frac{a}{n_l}}(d\beta) + (\frac{1}{2}\zeta)^{\frac{1}{2}}n_l^{\frac{1}{2}}e^{-n_l(p_l(-\frac{a}{n_l})-\tilde{p}_l(0))}$$

$$\times\int_{\beta_1}^{\beta_2} d\beta e^{-a\beta}\int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)}\frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\beta-\beta_l^m-yn_l^{-\sigma}}}e^{-\zeta y}m_l(dy). \tag{3.46}$$

Let $\beta_m < \beta_1 < \beta_2$; the first term in right-hand side of the equation is bounded above by $e^{-\zeta n_l^\sigma(\beta_1-\beta_l^m)}$ and since it is positive it vanishes in the limit $l \to \infty$. We shall prove that the second term converges to $\frac{a^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}e^{g(\zeta)}e^{a\beta_m}\int_{\beta_1}^{\beta_2}d\beta\frac{e^{-a\beta}}{\sqrt{\beta-\beta_m}}$. Now by

Lemma 3, $\beta \mapsto h_l(\zeta,\beta)$ is increasing and in the case studied here K_l^{-a/n_l} converges to the measure with density $\rho_0(\beta) = \frac{a^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\frac{e^{-a(\beta-\beta_m)}}{\sqrt{\beta-\beta_m}}$; therefore using the argument employed in the proof of Theorem 3 we obtain

$$\lim_{l\to\infty} h_l(\zeta,\beta) = e^{g^\sigma(\zeta)}. \tag{3.47}$$

We now study the last term in (3.46); we want to show that

$$\begin{aligned}|\zeta^{\frac{1}{2}} &\int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)} \frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\beta-\beta_l^m-yn_l^{-\sigma}}} e^{-\zeta y} m_l(dy)\\ &-\frac{\zeta^{\frac{1}{2}}}{\sqrt{\beta-\beta_l^m}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}e^{-\zeta y}m_l(dy)| \to 0 \text{ as } l\to\infty.\end{aligned} \tag{3.48}$$

We first note that

$$\begin{aligned}\lim_{l\to\infty}\zeta^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\zeta y} m_l(dy) &= \lim_{l\to\infty}\zeta^{\frac{1}{2}} n_l^{\frac{1}{2}(\sigma-1)} e^{-n_l\tilde{p}_l(0)}\int_0^\infty \tilde{Z}_l(\beta')e^{-\zeta n_l^\sigma(\beta'-\beta_l^m)}d\beta'\\ &= \lim_{l\to 0}\exp\left[\zeta n_l^\sigma\beta_l^m + n_l\{\tilde{p}_l(-\zeta/n_l^{1-\sigma})-\tilde{p}_l(0)\}\right]\\ &= \lim_{l\to\infty}\exp\frac{1}{2}\int_{(0,\infty)}\left\{\frac{\zeta}{t}-\ln(\frac{\zeta+t}{t})\right\}G_l^\sigma(dt)\\ &= \exp g^\sigma(\zeta).\end{aligned} \tag{3.49}$$

We also know that $\lim_{x\to\infty}\tilde{I}(x) = \sqrt{\frac{2}{\pi}}$ and that $\tilde{I}(x)$ and $\frac{\tilde{I}(x)}{\sqrt{x}}$ are bounded; let

$$A_1 = \sup_{x\in[0,\infty)}\tilde{I}(x)$$

and

$$A_2 = \sup_{x\in[0,\infty)}\frac{\tilde{I}(x)}{\sqrt{x}}.$$

Then

$$\begin{aligned}&\zeta^{\frac{1}{2}}\int_{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}}^{n_l^\sigma(\beta-\beta_l^m)}\frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\beta-\beta_l^m-yn_l^{-\sigma}}}e^{-\zeta y}m_l(dy)\\ &=\frac{1}{\sqrt{2}}\zeta n_l^{\sigma/2}\int_{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}}^{n_l^\sigma(\beta-\beta_l^m)}\frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y)}}e^{-\zeta y}m_l(dy)\end{aligned}$$

$$\leq \frac{A_2}{\sqrt{2}}\zeta n_l^{\sigma/2} e^{-\frac{\zeta}{2}(n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3})}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\zeta y} m_l(dy)$$

$$\to 0 \text{ as } l \to \infty.$$

Since $x \to x^{-\frac{1}{2}}$ is convex

$$0 < (\beta - \beta_l^m - y n_l^{-\sigma})^{-\frac{1}{2}} - (\beta - \beta_l^m)^{-\frac{1}{2}} < \frac{y n_l^{-\sigma}}{2(\beta - \beta_l^m - y n_l^{-\sigma})^{3/2}}$$

and therefore for $0 < y < n_l^\sigma(\beta - \beta_l^m) - n_l^{2\sigma/3}$

$$\left|(\beta - \beta_l^m - y n_l^{-\sigma})^{-\frac{1}{2}} - (\beta - \beta_l^m)^{-\frac{1}{2}}\right| < \frac{y n_l^{-\sigma/2}}{2},$$

and for $y < 0$

$$\left|(\beta - \beta_l^m - y n_l^{-\sigma})^{-\frac{1}{2}} - (\beta - \beta_l^m)^{-\frac{1}{2}}\right| < \frac{|y| n_l^{-\sigma}}{2(\beta - \beta_l^m)^{3/2}}.$$

Therefore

$$\begin{aligned}
&|\zeta^{\frac{1}{2}} \int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}} \frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\beta-\beta_l^m - y n_l^{-\frac{1}{2}}}} e^{-\zeta y} m_l(dy) \\
&- \frac{\zeta^{\frac{1}{2}}}{\sqrt{\beta-\beta_l^m}} \int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}} \tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y)) e^{-\zeta y} m_l(dy)| \\
&\leq \frac{\zeta^{\frac{1}{2}}}{2} n_l^{-\sigma/2} A_1 \int_0^\infty y e^{-\zeta y} m_l(dy) \\
&+ \frac{\zeta^{\frac{1}{2}} n_l^{-\sigma}}{2(\beta-\beta_l^m)^{3/2}} A_1 \int_{-\infty}^0 |y| e^{-\zeta y} m_l(dy) \to 0 \text{ as } l \to \infty.
\end{aligned}$$

Given $\epsilon > 0$ there is x_0 such that $|\tilde{I}(x) - \sqrt{\frac{2}{\pi}}| < \epsilon$ for $x > x_0$. Then for l such that $\frac{1}{2}\zeta n_l^{2\sigma/3} > x_0$ we have

$$\begin{aligned}
&|\int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}} \tilde{I}(\frac{1}{2}\zeta(n^\sigma(\beta-\beta_l^m)-y)) e^{-\zeta y} m_l(dy) \\
&- \int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)-n_l^{2\sigma/3}} \frac{1}{\sqrt{2\pi}} e^{-\zeta y} m_l(dy)| < \epsilon \int_{-\infty}^{\infty} e^{-\zeta y} m_l(dy).
\end{aligned}$$

We note that $\gamma = 1$ implies that $\epsilon_l(1) < \epsilon_l(2)$; using this fact and an argument similar to that used at the beginning of the proof we obtain

$$\lim_{l\to\infty} n_l^{\frac{1}{2}} e^{-n_l(p_l(-\frac{a}{n_l})-\tilde{p}_l(0))} = a^{\frac{1}{2}} e^{a\beta_m}. \tag{3.50}$$

Combining these results we see that (3.48) is satisfied and thus by (3.49) we have

$$\lim_{l\to\infty} \zeta^{\frac{1}{2}} \int_{-\infty}^{n_l^\sigma(\beta-\beta_l^m)} \frac{\tilde{I}(\frac{1}{2}\zeta(n_l^\sigma(\beta-\beta_l^m)-y))}{\sqrt{\beta-\beta_l^m-yn_l^{-\sigma}}} e^{-\zeta y} m_l(dy) = \frac{1}{\sqrt{\beta-\beta_m}} e^{g^\sigma(\zeta)}.$$

From the above inequalities we see also that the integral is uniformly bounded for β in compact subsets of (β_m, ∞); therefore the Lebesgue dominated convergence theorem together with (3.50) yields the required result.

□

4. Some Examples on the Lattice

The finite lattice Λ_l introduced in section 2, can be replaced by a more general parallelepiped $\tilde{\Lambda}_l$ whose sides do not all scale proportionally;

$$\tilde{\Lambda}_l = \{r = \sum_{i=1}^{\nu} m_i a_i \ : \ m_i = 0, \pm 1, \ldots, \pm l_i\} .$$

$\tilde{\Lambda}_l$ consists of $n_l = L_1 L_2 \ldots L_\nu$ sites, where $L_1 = 2l_i + 1$. We shall assume that the basis is labelled so that

$$L_1 \geq L_2 \geq \ldots L_\nu . \tag{4.1}$$

The lattice $\tilde{\Lambda}_l^r$, which is reciprocal to $\tilde{\Lambda}_l$, is given by

$$\tilde{\Lambda}_l^r = \{k = \sum_{i=1}^{\nu} k_i b_i \ : \ L_i k_i = 0, \pm 1, \ldots, \pm l_i\} .$$

In the bulk thermodynamic limit, which we shall denote simply by "$l \to \infty$ ", we consider the limit

$$L_1, L_2 \ldots, L_\nu \to \infty . \tag{4.2}$$

We shall consider the problem in dimensions $\nu \geq 3$. As an interaction we take the isotropic simple-cubic nearest-neighbour interaction which has the following kernel;

$$u_l(x) = \begin{cases} \frac{1}{2}, & x = \pm a_i, i = 1, \ldots, \nu; \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

The energy eigenvalue corresponding to $k = \sum_{i=1}^{\nu} k_i b_i \in \tilde{\Lambda}_l^r$ is

$$\tilde{\epsilon}_l(k) = \tilde{\epsilon}(k) = \sum_{i=1}^{\nu} \sin^2(\pi k_i) \ . \tag{4.4}$$

The smallest non-zero energy eigenvalue $\epsilon_l(2)$ is therefore given by

$$\epsilon_l(2) = \tilde{\epsilon}(\pm \frac{a_1}{L_1}) = \sin^2(\frac{\pi}{L_1}) \ .$$

Proposition 1. *Suppose*

$$\lim_{l\to\infty} \frac{n_l}{L_1^2} = A \in (0, \infty], \tag{4.5}$$

and

$$\lim_{l\to\infty} \frac{\ln L_2}{L_3 \dots L_\nu} = B \in (0, \infty), \tag{4.6}$$

then the second critical temperature β_m exists and is given by

$$\beta_m = \beta_c + \frac{B}{\pi}. \tag{4.7}$$

Proof : Choose $\delta > 0$; we define the non-negative number $m(\delta)$ by

$$m(\delta) = \begin{cases} 0, & A = \infty, \\ \frac{1}{\pi\delta\sqrt{A}}, & 0 < A < \infty \ . \end{cases} \tag{4.8}$$

Then for all l sufficiently large we have

$$\{k \in \tilde{\Lambda}_l^r \ : \tilde{\epsilon}(k) < \frac{1}{n_l \delta^2}\} = \{k = \frac{s_1}{L_1} a_1 \ : \ s_1 \in \mathbf{Z}, |s_1| \leq m(\delta)\} \ .$$

For $\epsilon > 0$, we define

$$\beta_c(\epsilon; l) = \int_{(\epsilon,\infty)} \frac{1}{2t} \mu_l(dt) \ ; \tag{4.9}$$

one notes that $\beta_c(\epsilon) = \lim_{l\to\infty} \beta_c(\epsilon; l)$ exists and converges to β_c as $\epsilon \to 0^+$. Finally, we define

$$\beta_m(\epsilon; l) = \frac{1}{2n_l} \sum_{\{k\in\tilde{\Lambda}_l^r : \tilde{\epsilon}(k) > \frac{1}{n_l}\epsilon\}} \frac{1}{\tilde{\epsilon}(k)} \ .$$

The functions β_m^- and β_m^+, introduced in section 2, are defined by

$$\beta_m^-(\epsilon) = \liminf_{l\to\infty} \beta_m(\epsilon; l)$$

and

$$\beta_m^+(\epsilon) = \limsup_{l\to\infty} \beta_m(\epsilon; l) \ .$$

We have the following relation, for $\delta > 0$;

$$\beta_m(\frac{1}{\delta^2}; l) = \beta_c(\delta^2; l) + \frac{1}{2n_l}\sum_K \frac{1}{\tilde{\epsilon}(k)} \ , \tag{4.10}$$

where K is the set $\{k \in \tilde{\Lambda}_l^r : \frac{1}{n_l\delta^2} < \tilde{\epsilon}(k) < \delta^2\}$. Now, for $0 < k_0 \le \frac{1}{2}$ we have

$$(\frac{\sin \pi k_0}{k_0})k \le \sin \pi k \le \pi k \ , \quad \text{for} \quad |k| < k_0 \ . \tag{4.11}$$

Hence we have it that, on setting $k_0 = \frac{1}{2}$;

$$E_l \ \subseteq \ \{k \in \tilde{\Lambda}_l^r \ : \ \frac{1}{\delta^2 n_l} < \tilde{\epsilon}(k) < \delta^2\} \ \subseteq \ F_l$$

where

$$E_l = \{k \in \tilde{\Lambda}_l^r \ : \ \frac{1}{4\delta^2 n_l} < k_1^2 + k_2^2 + \ldots + k_\nu^2 < \frac{\delta^2}{\pi^2}\} \tag{4.12a}$$

and

$$F_l = \{k \in \tilde{\Lambda}_l^r \ : \ \frac{1}{\pi^2\delta^2 n_l} < k_1^2 + k_2^2 + \ldots + k_\nu^2 < \frac{\delta^2}{4}\} \ . \tag{4.12b}$$

For $k \in E_l$ we have

$$\frac{1}{\tilde{\epsilon}(k)} \ge \frac{1}{\pi^2}\frac{1}{k_1^2 + \ldots + k_\nu^2} \ ,$$

while for $k \in F_l$ we have, setting $k_0 = \frac{\delta}{2}$;

$$\frac{1}{\tilde{\epsilon}(k)} \le \frac{\delta^2}{4\sin^2(\frac{1}{2}\pi\delta)}\frac{1}{k_1^2 + \ldots + k_\nu^2} \ .$$

This provides us with the following bounds ;

$$\frac{1}{2\pi^2 n_l}\sum_{E_l}\frac{1}{k_1^2 + \ldots + k_\nu^2} \le \beta_m(\frac{1}{\delta^2}; l) - \beta_c(\delta^2; l) \le \frac{\delta^2}{4\sin^2(\frac{1}{2}\pi\delta)}\frac{1}{2n_l}\sum_{F_l}\frac{1}{k_1^2 + \ldots + k_\nu^2} \ . \tag{4.13}$$

The lower bound yields, for l sufficient large ;

$$\beta_m(\frac{1}{\delta^2};l)-\beta_c(\delta^2;l)\geq$$

$$\frac{1}{2n_l\pi^2}\sum{}'(\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2}+\ldots+\frac{s_\nu^2}{L_\nu^2})^{-1}$$

$$\geq\frac{1}{2n_l\pi^2}\sum{}''(\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2})^{-1}$$

$$\geq\frac{1}{2n_l\pi^2}L_1L_2\int_{r_1}^{\delta/\pi}[R^{-2}]2\pi R dR-\frac{1}{2n_l\pi^2}[2\int_{m(\delta)+1}^{L_1\delta/\pi}ds_1\frac{L_1^2}{s_1^2}+2\int_1^{L_2\delta/\pi}ds_2\frac{L_2^2}{s_2^2}] \tag{4.14}$$

where the primed and double primed summations are to be carried out over the sets $\{s\in\mathbf{Z}^\nu:|s_1|>m(\delta)\text{ if }s_2,\ldots,s_\nu=0,\frac{s_1^2}{L_1^2}+\ldots+\frac{{s_\nu}^2}{L_\nu^2}<\frac{\delta^2}{\pi^2}\}$ and $\{s\in\mathbf{Z}^2:|s_1|>m(\delta)\text{ if }s_2=0,\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2}<\frac{\delta^2}{\pi^2}\}$ respectively and where $r_1=\max\{\frac{m(\delta)+1}{L_1},\frac{1}{L_2}\}$.

For δ fixed, if we take l sufficiently large then we have $r_1=\frac{1}{L_2}$. We observe that the first term in inequality (4.14) is convergent;

$$\frac{L_1L_2}{n_l\pi}\ln(\frac{\delta L_2}{\pi})\to\frac{B}{\pi},$$

as $l\to\infty$. The second term in (4.14) depends on the limiting value A of $\frac{n_l}{L_1^2}$ in (4.5);

case 1 : $A=\infty$, and therefore $m(\delta)=0$;

$$\frac{L_1^2}{n_l}\int_1^{L_1\delta/\pi}ds_1\frac{1}{s_1^2}=\frac{L_1^2}{n_l}[1-\frac{\pi}{L_1\delta}]\to 0,\ \text{as } l\to\infty$$

the other integral being similarly bounded.

case 2 : $0<A<\infty$, and therefore $m(\delta)>0$;

$$\frac{L_1^2}{n_l}\int_{m(\delta)}^{L_1\delta/\pi}ds_1\frac{1}{s_1^2}=\frac{L_1^2}{n_l}[\frac{1}{1+m(\delta)}-\frac{\pi}{L_1\delta}]\to\frac{1}{A(1+m(\delta))}$$

as $l\to\infty$. However $m(\delta)\to\infty$ as $\delta\to 0^+$ in this case. Therefore, it follows that

$$\beta_m^-(\frac{1}{\delta^2})=\liminf_{l\to\infty}\beta_m(\frac{1}{\delta^2};l)\geq\beta_c(\delta^2)+\frac{B}{\pi}+\frac{1}{A(1+m(\delta))}, \tag{4.15}$$

where the last term in (4.15) is interpreted as zero for the case $A=\infty$. On taking the limit $\delta\to 0^+$ we obtain

$$\lim_{\delta\to 0^+}\beta_m^-(\frac{1}{\delta^2})\geq\beta_c+\frac{B}{\pi}. \tag{4.16}$$

The upper bound, for l sufficient large, becomes

$$\beta_m(\frac{1}{\delta^2};l) - \beta_c(\delta^2;l) \leq \frac{\delta^2}{4\sin^2(\frac{1}{2}\delta\pi)}\{\mathrm{I}_l(\delta) + \mathrm{II}_l(\delta)\} , \tag{4.17}$$

where

$$\mathrm{I}_l(\delta) = \frac{1}{2n_l}\sum_{F(\mathrm{I})}(\frac{s_2^1}{L_1^2} + \frac{s_2^2}{L_2^2})^{-1}$$

with $F(\mathrm{I}) = \{s \in \mathbb{Z}^2 \ : |s_1| > m(\delta) \ \text{if} \ s_2 \neq 0, \ \frac{s_1^2}{L_2^2} + \frac{s_2^2}{L_2^2} < \frac{\delta^2}{4}\}$ and

$$\mathrm{II}_l(\delta) = \frac{1}{2n_l}\sum_{F(\mathrm{II})}(\frac{s_1^2}{L_1^2} + \frac{s_2^2}{L_2^2} + \ldots + \frac{s_\nu^2}{L_\nu^2})^{-1}$$

with $F(\mathrm{II}) = \{s \in \mathbb{Z}^\nu \ : s_3, \ldots, s_\nu \neq 0, \frac{s_1^2}{L_1^2} + \ldots + \frac{s_\nu^2}{L_\nu^2} < \frac{\delta^2}{4}\}$.

We shall, first of all, estimate the contribution made to $\mathrm{I}_l(\delta)$ by the elements of $F(\mathrm{I})$ for which $s_2 = 0$.

case 1 : $A = \infty$ and therefore $m(\delta) = 0$;

$$\frac{1}{2n_l}\sum_{F(\mathrm{I})\cap\{s_2=0\}}(\frac{s_1^2}{L_1^2} + \frac{s_2^2}{L_2^2})^{-1} = \frac{L_1^2}{2n_l}\sum_{\{s_1\in\mathbb{Z}:m(\delta)<|s_1|<\frac{1}{2}\delta L_1\}}\frac{1}{s_1^2}$$

$$\leq \frac{L_1^2}{n_l}[1 + \int_1^{\frac{1}{2}\delta L_1}\frac{1}{s^2}ds]$$

$$= \frac{L_1^2}{n_l}[2 - \frac{2}{\delta L_1}] ,$$

which tends to zero as $l \to \infty$.

case 2 : $0 < A < \infty$;

$$\frac{L_1^2}{2n_l}\sum_{\{s_1\in\mathbb{Z}:m(\delta)<|s_1|<\frac{1}{2}\delta L_1\}}\frac{1}{s_1^2} \leq \frac{L_1^2}{n_l}\int_{m(\delta)}^{\frac{1}{2}\delta L_1}\frac{1}{s^2}ds$$

$$= \frac{L_1^2}{n_l}[\frac{1}{m(\delta)} - \frac{2}{\delta L_1}] ,$$

which converges to $\frac{1}{Am(\delta)}$ as $l \to \infty$.

Similarly, one obtains the result that the contributions made to $\mathrm{I}_l(\delta)$ from the elements of $F(\mathrm{I})$, for which we have respectively $s_1 = 1, s_1 = -1, s_2 = 0, s_2 = 1$ and $s_2 = -1$, are likewise bounded.

One notes that, in the case $0 < A < \infty$, the number $m(\delta) \to \infty$ as $\delta \to 0^+$. Hence these particular low energy bands of states can be ignored as they make no overall contribution to β_m.
The remainder of $\mathrm{I}_l(\delta)$ is

$$\frac{1}{2n_l}\sum_{K'}(\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2})^{-1}$$
$$\leq \frac{1}{2n_l}\int_{A_1} ds_1 ds_2 [\frac{(|s_1|-1)^2}{L_1^2}+\frac{(|s_2|-1)^2}{L_2^2}]^2$$
$$\leq \frac{1}{2n_l}L_1L_2\int_{A_2} dt_1 dt_2 (t_1^2+t_2^2)^{-1}$$
$$= \frac{1}{2n_l}L_1L_2\int_{\sqrt{3/2}\sqrt{1/L_1^2+1/L_2^2}}^{\delta/2}[\frac{1}{R^2}]2\pi R dR$$
$$= \frac{\pi}{n_l}L_1L_2[\ln\sqrt{\frac{2}{3}}\frac{\delta}{2}-\frac{1}{2}\ln(\frac{1}{L_1^2}+\frac{1}{L_2^2})] = \frac{\pi}{L_3\ldots L_\nu}[\ln\sqrt{\frac{2}{3}}\frac{\delta}{2}-\frac{1}{2}\ln(1+\frac{L_2^2}{L_1^2})+\ln L_2],$$

where K' is the set $\{s \in \mathbf{Z}^2 : |s_1|,|s_2| > 1, \frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2} < \frac{\delta^2}{4}\}$, A_1 and A_2 are the regions in $\mathbf{R}^2$ defined by $A_1 = \{(s_1,s_2) : \frac{3}{2} < s_1^2+s_2^2, \frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2} < \frac{\delta^2}{4}\}$ and $A_2 = \{(t_1,t_2) : \frac{3}{2}(\frac{1}{L_1^2}+\frac{1}{L_2^2}) < t_1^2+t_2^2 < \delta^2\}$.

As the limit of $\frac{L_2}{L_1}$ always exists and is in $[0,1]$, so we have that

$$\lim_{l\to\infty}\frac{\pi}{L_3\ldots L_\nu}[\ln\frac{2}{3}\delta-\frac{1}{2}\ln(1+\frac{L_2^2}{L_1^2})+\ln L_2] = \lim_{l\to\infty}\pi\frac{\ln L_2}{L_3\ldots L_\nu} = \pi B.$$

Therefore, for $0 < A < \infty$ we have

$$\lim_{\delta\to 0^+}\lim_{l\to\infty}\mathrm{I}_l(\delta) \leq \pi B. \tag{4.18}$$

Finally we examine the term $\mathrm{II}_l(\delta)$. As before, we shall remove the contributions from the bands $s_1 = -1,0,1$ and $s_2 = -1,0,1$ occuring in $F(\mathrm{II})$ as they do not effect the value of β_m. We therefore concentrate on

$$\frac{1}{2n_l}\sum{}'(\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2}+\ldots+\frac{s_\nu^2}{L_\nu^2})^{-1},$$

where the primed summation is over the set $\{s \in \mathbf{Z} : |s_1|,|s_2|,|s_3| > 1$ if$s_4,\ldots,s_\nu \neq 0.\frac{s_1^2}{L_1^2}+\ldots+\frac{s_\nu^2}{L_\nu^2} < \delta^2/4\}$; this sum is bounded above by

$$\frac{1}{2n_l}\sum{}''(\frac{s_1^2}{L_1^2}+\frac{s_2^2}{L_2^2}+\frac{s_3^2}{L_3})^{-1},$$

where the double primed summation is over the set $\{s \in \mathbf{Z} : |s_1|, |s_2|, |s_3| > 1, \frac{s_1^2}{L_1^2} + \frac{s_2^2}{L_2^2} + \frac{s_3^2}{L_3^2} < \delta^2/4\}$; this set is in turn bounded above by the integral

$$\leq \frac{1}{2} \int_{A_3} dt_1 dt_2 dt_3 (t_1^2 + t_2^2 + t_3^2)^{-1}$$

$$= \frac{1}{2} \int_{\sqrt{\frac{3}{2}}\sqrt{\frac{1}{L_1^2}+\frac{1}{L_2^2}+\frac{1}{L_3^2}}}^{\delta/2} [\frac{1}{R^2}] 4\pi R^2 dR$$

$$= 2\pi [\frac{1}{4}\delta - \sqrt{\frac{3}{2}}\sqrt{\frac{1}{L_1^2} + \frac{1}{L_2^2} + \frac{1}{L_3^2}}],$$

where $A_3 = \{(t_1, t_2, t_3) \in \mathbf{R}^3 : \frac{3}{2}\frac{1}{L_1^2} + \frac{1}{L_2^2} + \frac{1}{L_3^2} < t_1^2 + t_2^2 + t_3^2 < \delta^2/4\}$. The above bound tends to $\pi\delta/2$ as $l \to \infty$.
This gives

$$\lim_{\delta \to 0+} \lim_{l \to \infty} \mathrm{II}_l(\delta) = 0. \tag{4.19}$$

Hence combining the inequalities (4.17) and (4.18) with (4.19), we obtain, for $A = \infty$,

$$\beta_m^+(\frac{1}{\delta^2}) \leq \beta_c(\delta^2) + \frac{\delta^2}{4\sin^2(\frac{1}{2}\delta\pi)}[\pi B + \frac{\pi\delta}{2}] \tag{4.20}$$

and by taking the limit $\delta \to 0^+$ we obtain

$$\lim_{\delta \to 0+} \beta_m^+(\frac{1}{\delta^2}) = \beta_c + \frac{B}{\pi}. \tag{4.21}$$

The same result is obtained for the case $A < \infty$.

□

Proposition 2. *Suppose that conditions (4.5) and (4.6) of Propostion 2 are satisified then the function γ given by (2.20), exists and is given by*

$$\gamma(t) = \begin{cases} 1, & A = \infty, \\ \sum_{z \in \mathbf{Z}} \exp\{-A\pi^2 z^2 t\}, & 0 < A < \infty. \end{cases} \tag{4.22}$$

Proof : Now $\gamma(t) = \lim_{l \to \infty} \gamma_l(t)$, where

$$\gamma_l(t) = \sum_{\{|s_i| \leq l_i; i=1,\ldots,\nu\}} \prod_{j=1}^{\nu} \exp\{-n_l \sin^2(\pi \frac{s_j}{L_j})t\}.$$

Now we introduce the function $g_j(t;l)$ defined by

$$g_j(t;l) = \sum_{s_j=1}^{l_j} \exp\{-n_l \sin^2(\pi \frac{s_j}{L_j})t\}.$$

It is evident that

$$|\gamma_l(t) - \sum_{z\in\mathbf{Z}} \exp\{-A\pi^2 z^2 t\}| \leq$$

$$|\sum_{z\in\mathbf{Z}} \exp\{-A\pi^2 z^2 t\} - \sum_{|s_1|\leq l_1} \exp\{-n_l \sin^2(\pi\frac{s_1}{L_1})t\}| + |2\sum_{i_1=2}^{\nu} g_{i_1}(t;l)|$$

$$+|2^2 \sum_{i_1\neq i_2} g_{i_1}(t;l)g_{i_2}(t;l) + ... + 2^{\nu-1} \sum_{\{i_1,i_2,...,i_\nu:\text{distinct}\}} g_{i_1}(t;l)g_{i_2}(t;l)...g_{i_{\nu-1}}(t;l)|$$

$$+|2^\nu \sum_{s_1=1}^{l_1} ... \sum_{s_\nu=1}^{l_\nu} \prod_{j=1}^{\nu} \exp\{-n_l \sin^2(\pi\frac{s_j}{L_j})t\}|. \tag{4.23}$$

The first term on the right-hand side of (4.23) tends to zero because

$$1 + 2g_1(t;l) \to \sum_{z\in\mathbf{Z}} \exp\{-A\pi^2 z^2 t\ \}, \quad \text{as} \quad l\to\infty.$$

Next of all, we consider $i = 2, ..., \nu$:

$$g_i(t;l) \leq g_2(t;l) \leq l_2 \exp\{n_l \sin^2(\pi\frac{1}{L_2)}t\} \leq L_2 \exp\{-4t\frac{n_l}{L_2^2}\}$$

now we have that $\lim_{l\to\infty} \frac{1}{L_3...L_\nu}\frac{n_l}{L_2^2} = \lim_{l\to\infty}\frac{L_1^2}{n_l} = \frac{1}{A}$. It follows that given $0 < \epsilon < \frac{1}{A}$, we have, for l sufficiently large;

$$0 < g_i(t;l) \leq L_2 \exp\{-(\frac{1}{A} - \epsilon)(L_3...L_\nu)^2\}$$

$$\leq \exp\{(B+\epsilon)(L_3...L_\nu) - (\frac{1}{A} - \epsilon)(L_3...L_\nu)^2\},$$

where $0 \leq B < \infty$, is the parameter introduced in proposition 1.

It follows that $g_i(t;l) \to 0$ as $l\to\infty$ for $i > 1$. Similarly one shows that the other summations in the second term on the right-hand side of (4.23) vanish as $l\to\infty$.

Finally, the last term is bounded above by

$$2^\nu \int_0^{l_1-1} \dots \int_0^{l_\nu-1} \prod_{j=1}^{\nu} \exp\{-n_l \sin^2(\pi \frac{s_j}{L_j})t\}$$

$$\leq \prod_{j=1}^{\nu} \int_{-\infty}^{+\infty} \exp\{-n_l 4 \frac{s_j^2}{L_j^2} t\} \leq (\frac{\pi}{4t})^{\frac{1}{2}\nu} n_l^{1-\frac{1}{2}\nu},$$

which tends to zero as $l \to \infty$.

□

Let σ be the critical exponent describing the fluctuations of $\frac{N_1}{n_l}$ as in the statement of Theorem 4.

Proposition 3. *In the case of nearest neighbour interactions on the original lattice Λ_l, the critical exponent σ describing the fluctuations of $\frac{N_1}{n_l}$ is well-defined and given by $1 - \frac{2}{\nu}$.*

Proof: Now we have $n_l = L^\nu = (2l+1)^\nu$, so by choosing $\sigma = 1 - \frac{2}{\nu}$ we have

$$G_l^{1-\frac{2}{\nu}}[A] = \sharp\{j : n_l^{\frac{2}{\nu}} \epsilon_l(j)\} = \sharp\{j : L^2 \epsilon_l(j) \in A\}$$

thereby anticipating the $\frac{1}{L^2}$ scaling of the low level energy values in the nearest neighbour interaction. It is sufficient for our purposes to calculate the Laplace transform ω_l of the measure $G_l^{1-\frac{2}{\nu}}$;

$$\omega_l(s) = \int_{(0,\infty)} e^{-st} G_l^{1-\frac{2}{\nu}}(dt) = \sum_{k \in \Lambda_l^r} \exp\{-s n_l^{\frac{2}{\nu}} \tilde{\epsilon}_l(k)\}$$

$$= \sum_{|m_i| \leq l} \exp\{-sL^2[\sin^2(\frac{\pi m_1}{L}) + \dots + \sin^2(\frac{\pi m_\nu}{L})]\}$$

$$\to \sum_{m \in \mathbb{Z}^\nu} \exp\{-s\pi^2 |m|^2\}.$$

where $|m|^2 = m_1^2 + \dots + m_\nu^2$. This gives

$$g^{1-\frac{2}{\nu}}(\zeta) = \frac{1}{2} \sum_{m \in \mathbb{Z}^\nu/\{0\}} \{\frac{\zeta}{\pi^2|m|^2} - \ln(1 + \frac{\zeta}{\pi^2|m|^2})\}. \tag{4.24}$$

□

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