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Non–Spreading Coherent States Riding on Kepler Orbits

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Abstract. We construct exact non–spreading coherent states for the Kepler problem and show that they must be two–dimensional lying on the plane of the orbit. The use of a new time variable in the associated auxiliary oscillator problem plays a crucial role.

Ever since the Schrödinger construction of non–spreading localized coherent states for the one–dimensional oscillator problem [1], many attempts have been made to construct the analogous non–spreading coherent states for the Kepler problem, a task Schrödinger himself found difficult to solve [2]. None have been found so far, although there are many ingenious constructions of the so–called "quasi–classical states" [3–12] which, however, eventually spread. There is now also considerable experimental interest in such states after the realization of localized wave packets in Rydberg orbits and their vanishing and revival in time [13–14].

We use the well–known transformation of the Coulomb Hamiltonian into an associated four–dimensional problem [15–16] whose evolution is oscillatory in a new time variable $dT =$

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dt/r. This turns out to play an important role. The four–dimensional motion is constraint and, transforming to the ordinary space property, we first show that the averaged motion is a Kepler ellipse and then that the coherent state around the average is a non–spreading two–dimensional gaussian wave function. We start from the well–known $SO(4)$–symmetry of the Schrödinger equation

$$H\psi \equiv \left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{r}\right)\psi = E\psi \tag{1}$$

for the Kepler problem, where μ is the reduced mass. The conserved generators of angular momentum $\mathbf{L}$ and the Lenz vector $\mathbf{A}$ satisfy

$$[\mathbf{L}, H] = [\mathbf{A}, H] = 0, \quad \mathbf{L}\times\mathbf{L} = ih\mathbf{L}, \quad \mathbf{L}\times\mathbf{A} = i\hbar\mathbf{A}, \quad \mathbf{A}\times\mathbf{A} = i\hbar\mathbf{L} \tag{2}$$

$$\mathbf{L}, \mathbf{A} = 0 \, . \tag{3}$$

In terms of two sets of boson operators $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, b' = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix}$, $[b_i, b_j^+] = \delta_{ij}$ we can represent

$$\begin{aligned} \mathbf{L} &= \frac{1}{2}(b^+\sigma b + b'^+\sigma b') \\ \mathbf{A} &= \frac{1}{2}(b^+\sigma b - b'^+\sigma b') \end{aligned} \tag{4}$$

so that the condition (3) gives

$$X = b_1^+ b_1 + b_2^+ b_2 - b_3^+ b_3 - b_4^+ b_4 = 0 \, . \tag{5}$$

Introducing new variables a by

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}(a_1 + ia_2), b_2 = \frac{1}{\sqrt{2}}(a_3 - ia_4) \\ b_3 &= \frac{1}{\sqrt{2}}(a_1 - ia_2), b_4 = \frac{1}{\sqrt{2}}(a_3 + ia_4) \end{aligned} \tag{6}$$

the condition (5) becomes

$$X = a_1^+ a_2 - a_2^+ a_1 - a_3^+ a_4 + a_4^+ a_3 = 0 \, . \tag{7}$$

From the set $\{a\}$ we pass to the canonical coordinates q_A, π_A of an auxiliary space

$$\begin{aligned} a_A &= (2m\hbar\omega)^{-1/2}(m\omega q_A + i\pi_A) \\ a_A^+ &= (2m\hbar\omega)^{-1/2}(m\omega q_A - i\pi_A), \; A = 1,2,3,4 \end{aligned} \tag{8}$$

so that

$$[q_A, \pi_B] = i\hbar\,\delta_{AB}, \quad \pi_A = -i\hbar\,\partial/\partial q_A \, . \tag{9}$$

The different operators b_i, a_i, q_A, π_A all have their distinct physical interpretations.

The condition (7) now becomes

$$X = q_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_4} - q_4 \frac{\partial}{\partial q_3} = 0 \ . \tag{10}$$

The four–dimensional q–space has a well–known Kustanheimo–Stiefel projection to the three–dimensional space [15–16]

$$x_1 = 2(q_1 q_3 - q_2 q_4), \ x_2 = 2(q_1 q_4 + q_2 q_3), \ x_3 = q_1^2 + q_2^2 - q_3^2 - q_4^2 \tag{11}$$

from which we obtain

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} = q_1^2 + q_2^2 + q_3^2 + q_4^2 = q^2 \tag{12}$$

and

$$\nabla^2 = \frac{1}{4r} \Box_q^2 - \frac{1}{4r^2} X^2 \tag{13}$$

where X is defined by (10). For the Kepler symmetry $X = 0$, and Eq.(1) becomes

$$\left(-\frac{\hbar^2}{2m} \Box_q + \frac{1}{2} m\omega^2 q^2 \right) \psi = \varepsilon\psi \tag{14}$$

$$m = 4\mu, \quad \omega = (-E/2\mu)^{1/2}, \quad \varepsilon = e^2 \ . \tag{15}$$

This is an auxiliary pseudo–Hamiltonian in a four–dimensional Euclidian space

$$H_q = -\frac{\hbar^2}{2m} \Box_q + \frac{1}{2} m\omega^2 q^2 = \sum_A (a_A^+ a_A + 2)\hbar\omega \tag{16}$$

with the eigenstates $|n_1 n_2 n_3 n_4)$ and eigenvalues

$$\in = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2) \equiv \hbar\omega 2n \ . \tag{17}$$

The condition (10) gives the restriction

$$n_2 + n_2 = n_3 + n_4 \equiv n - 1 \ . \tag{18}$$

It is easy to see that (17) together with (15) implies the Balmer formula with degeneracy n^2 so that the H–atom is equivalently described by two independent two–dimensional oscillators, or by 4 "H–atom quarks" [17].

From (16), (13) and (1) we obtain

$$H_q - \varepsilon = r(H_r - E) \ . \tag{19}$$

It has been known for some time [18] that the operator $\theta \equiv r(H_r - E)$ is actually an $SO(4,2)$–dynamical group operator for the Kepler problem and has a discrete spectrum – it is in fact essentially the principal quantum number operator.

According to a theorem in Hamiltonian systems, if for a Hamiltonian H with E being some value of energy, we define the new Hamiltonian $\bar{H} = G(p,q)[H - E]$, then the integral curves which belong to $H = E$ with time parameter t, now belong to new energy $\bar{H} = 0$ with time parameter T, where

$$T(t) = \int_{t_0}^{t} \frac{dt}{G(g(t))} \tag{20}$$

for each integral curve $g(t)$ of H. In our case it follows from (19) that for the q–system we have the new time variable

$$dT = dt/r \tag{21}$$

Comparing (1) and (14) we further see a new type of remarkable symmetry of the Kepler problem: states for fixed e^2 and variable energies $E_n = -\frac{1}{2}\frac{me^2}{n^2}$ on the one hand, and states of a family of Coulomb problems with fixed energy E, but variable coupling constants

$$e_n^2 = n\sqrt{-2E/m}\ , \tag{22}$$

form two isomorphic representations of the same dynamical algebra [19]. This is because the H–atom equation (19) after tilting is $[\sqrt{-2E/m}\ \Gamma_0 - e^2]\psi = 0$ where Γ_0 has the discrete spectrum $\Gamma_0|n> = n/n>$. Hence we get $(-2E/m)^{1/2}n = e^2$ so that we can either solve for $E_n = -me^4/2n^2$ for fixed e^2 (usual case), or we can fix E and solve for e^2: $e_n = n\sqrt{(-2E/m)}$.

In $q_A - \pi_A$ space we have the Heisenberg equations

$$\begin{gathered} \frac{m dq_A}{dT} = \pi_A, \quad \frac{d\pi_A}{dT} = -m\omega^2 q_A \\ \frac{d^2 q_A}{dT^2} + \omega^2 q_A = 0, \quad \frac{d^2\pi_A}{dT^2} + \omega^2\pi_A = 0\ . \end{gathered} \tag{23}$$

Transforming to the **r**–space with (20) we obtain the Heisenberg equations of the Kepler problem. In particular we have a relation between the momenta in two spaces

$$\pi_A = \sum_{k=1}^{3} \frac{\partial x^k}{\partial q_A}\, p_k\ . \tag{24}$$

The Schrödinger coherent states for the oscillators are well known [1], defined as the eigenvectors of the annihilation operators

$$a_A|\alpha> = \alpha_A|\alpha>\ . \tag{25}$$

For the ground state of the oscillator in the q–representation we have

$$\begin{aligned} < q_1 q_2 q_3 q_4|\alpha> &\equiv < q/\alpha > \\ &= \left(\frac{m\omega}{\pi h}\right) \exp\left[-\frac{1}{2}\left(\frac{m\omega}{\hbar}\right)\left\{\sum_A (q_A - \bar{q}_A(T))^2\right\}\right] \\ &\times \exp\left[\frac{i}{\hbar}\left(\sum_A\left\{q_A\bar{\pi}_A(T) - \frac{\hbar|\alpha_A|^2}{2}\sin 2(\phi_A + \omega T)\right\} - i\omega T\right)\right] \end{aligned} \tag{26}$$

where

$$\bar{q}_A(T) = <\alpha|q_A|\alpha> = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\alpha_A + \alpha_A^*)$$
$$\bar{\pi}_A(T) = \alpha|\pi_A|\alpha> = -i\left(\frac{m\omega\hbar}{2}\right)^{1/2} (\alpha_A - \alpha_A^*) \tag{27}$$

and satisfy, from (23), the classical equations of the motion

$$\frac{d^2}{dT^2}\,\bar{q}_A + \omega^2 \bar{q}_A = 0 \tag{28}$$

and the classical pseudo–Hamiltonian is

$$\bar{H}_q = <\alpha|H_q|\alpha> = \sum_A \frac{\overline{\pi_A^2}}{2m} + \frac{1}{2}\, m\omega^2\, \overline{q_A^2} = \varepsilon\ . \tag{29}$$

The condition $X = 0$, Eq.(10), applied to the coherent state (26) gives

$$\left(\frac{m\omega}{\hbar}\right)[(q_2\bar{q}_1 - q_1\bar{q}_2) + (q_3\bar{q}_4 - q_4\bar{q}_3)] + \frac{i}{\hbar}[q_2\bar{\pi}_1 - q_1\bar{\pi}_2 + q_3\bar{\pi}_4 - q_4\bar{\pi}_3] = 0\ .$$

Since our q's and π's are all real, each square bracket must vanish separately.

$$\begin{aligned} q_2\bar{\pi}_1 - q_1\bar{\pi}_2 + q_3\bar{\pi}_4 - q_4\bar{\pi}_3 &= 0 \\ q_2\bar{q}_1 - q_1\bar{q}_2 + q_3\bar{q}_4 - q_4\bar{q}_3 &= 0 \end{aligned}\ . \tag{30}$$

In the $\mathbf{r}$–space we have from (11) the following expectation values, or classical orbits

$$\begin{aligned} \bar{x}_1 = <x_1> &= 2[\bar{q}_1\bar{q}_3 - \bar{q}_2\bar{q}_4] \\ \bar{x}_2 &= 2[\bar{q}_1\bar{q}_4 + \bar{q}_2\bar{q}_3] \\ \bar{x}_3 &= <q_1^2> + <q_2^2> - <q_3^2> - <q_4^2> \\ \bar{r} &= <q_1^2> + <q_2^2> + <q_3^2> + <q_4^2>\ . \end{aligned} \tag{31}$$

It follows from (8) and (25) that in the coherent state $|\alpha>$

$$<q_A^2> = <q_A>^2 + \hbar/2m\omega\ . \tag{32}$$

We must distinguish between $\bar{r} = \overline{q^2}$ and $\tilde{r} = \bar{q}^2$

$$<r> \equiv \bar{r} = \tilde{r} + 2\hbar/m\omega \tag{33}$$

where

$$\tilde{r} \equiv (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)^{1/2},\quad \bar{r} = (\overline{x_1^2} + \overline{x_2^2} + \overline{x_3^2})^{1/2}\ . \tag{34}$$

Thus we can write for $\bar{x}_3$ in (31)

$$\bar{x}_3 = \bar{q}_1^2 + \bar{q}_2^2 - \bar{q}_3^2 - \bar{q}_4^2 \tag{35}$$

because the constant term in (33) drops out. We also have the relation between the two times, from (20), which we write as

$$dT = dt/\tilde{r} \ . \tag{36}$$

From (7), (30) and (36) we obtain

$$\mu \frac{d^2}{dt^2} \bar{\mathbf{r}} = -\frac{\varepsilon'}{\bar{\mathbf{r}}^3} \bar{\mathbf{r}} \tag{37}$$

where $\varepsilon' = e^2 - 2\hbar\omega$. This is the classical equation of motion (in the limit $\hbar \to 0$, the zero point energy drops out).

We shall now exhibit r–space motion in the coherent state and show that its centre describes an ellipse.

We can always choose the x–coordinates such that the classical orbit is in the x_1x_2–plane. Thus setting

$$< x_3 > = 0 \tag{38}$$

we obtain from (35)

$$\bar{q}_1 = \bar{q}_3 \quad \text{and} \quad \bar{q}_2 = \bar{q}_4 \tag{39}$$

(or equivalently $\bar{q}_1 = \bar{q}_4$ and $\bar{q}_2 = \bar{q}_3$. Then from (30) and (31)

$$\bar{x}_1 = 2(\bar{q}_1^2 - \bar{q}_2^2), \quad \bar{x}_2 = 4\bar{q}_1\bar{q}_2 \ . \tag{40}$$

Now from (27) and (28),

$$\bar{q}_i = \left(\frac{2\hbar}{m\omega}\right)^{1/2} |\alpha_i| \cos(\omega T + \phi_i), \quad i = 1, 2 \tag{41}$$

hence

$$\tilde{r} = 4\left(\frac{\hbar}{m\omega}\right) \left[|\alpha_1|^2 \cos^2(\omega T + \phi_1) + |\alpha_2|^2 \cos^2(\omega T + \phi_2)\right] \ . \tag{42}$$

Coming back to our coherent state (26), the conditions (30) now give

$$q_1 = q_3 \quad \text{and} \quad q_2 = q_4 \tag{43}$$

and this in turn implies

$$\bar{\pi}_1 = \bar{\pi}_3, \quad \bar{\pi}_2 = \bar{\pi}_4 \tag{44}$$

so that our coherent state simplifies to

$$\begin{aligned} < q|\alpha > &= \frac{m\omega}{\pi\hbar} \exp\left\{-\frac{m\omega}{\hbar}\left[(q_1 - \bar{q}_1)^2 + (q_2 - \bar{q}_2)^2\right] + \frac{2i}{\hbar}(q_1\bar{\pi}_1 + q_2\bar{\pi}_2) - \frac{i}{\hbar}\Phi\right\} \\ \Phi &= \hbar \sum_{A=1}^{2} |\alpha_A|^2 \sin 2(\phi_A + \omega T) - \hbar\,\omega T \ . \end{aligned} \tag{45}$$

Transforming (45) into the r–space we obtain

$$
\begin{aligned}
<\bar{r}|\alpha> \; = \left(\frac{m\omega}{\pi\hbar}\right) \exp\Bigg\{ &- \left[\sqrt{2}|\alpha_1|\cos(\omega T+\phi_1) \mp \frac{1}{\sqrt{2}}\left(\frac{m\omega}{2\hbar}\right)^{1/2}\sqrt{r(1+\cos\theta)}\right]^2 \\
&- \left[\sqrt{2}|\alpha_2|\cos(\omega T+\phi_2) \mp \frac{1}{\sqrt{2}}\left(\frac{m\omega}{2\hbar}\right)^{1/2}\sqrt{r(1-\cos\theta)}\right]^2 \\
&- i\left(\frac{2m\omega}{\hbar}\right)^{1/2}\Bigg[\sqrt{r(1+\cos\theta)}\;|\alpha_1|\sin(\omega T+\phi_1) \\
&+ \sqrt{r(1-\cos\theta)}\;|\alpha_2|\sin(\omega T+\phi_2)\Bigg] + \frac{i}{\hbar}\,\Phi\Bigg\} \,. \qquad (46)
\end{aligned}
$$

This is a non–spreading wave–form in time T. Its maximum is located at (r_m, θ_m):

$$
|\alpha_{\frac{1}{2}}|^2\cos^2(\omega T+\phi_{\frac{1}{2}}) = \frac{m\omega}{8\hbar}\, r_m(1\pm\cos\theta_m) \,. \qquad (47)
$$

Taking $\phi_1 = 0, \phi_2 = \pi/2$, we see that the orbit of the maximum is the standard Kepler–ellipse:

$$
\begin{aligned}
\frac{1}{r_m} &= \frac{1}{8}\,\frac{m\omega}{\hbar}\,\frac{|\alpha_1|^2+|\alpha_2|^2}{|\alpha_1|^2|\alpha_2|^2}\left(1+\left[1-\frac{4|\alpha_1|^2|\alpha_2|^2}{(|\alpha_1|^2+|\alpha_2|^2)^2}\right]^{1/2}\cos\theta_m\right) \\
&= \frac{\mu e^2}{L^2}\left(1+\left(1+\frac{2L^2E}{\mu e^4}\right)^{1/2}\cos\theta_m\right) \qquad (48)
\end{aligned}
$$

with

$$
\begin{aligned}
|\alpha_1|^2+|\alpha_2|^2 &= \frac{e^2}{2\hbar\omega} = \frac{e^2}{2\hbar}\left(-\frac{2\mu}{E}\right)^{1/2} \\
|\alpha_1|^2|\alpha_2|^2 &= \frac{1}{4\hbar^2}\,L^2 \,.
\end{aligned}
$$

In terms of the (r_m, θ_m), the absolute value squared coherent state wave function can be simply expressed

$$
\begin{aligned}
|<\bar{r}|\alpha>|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^2 \exp\Bigg[-\frac{m\omega}{2\hbar}\Bigg\{&\left(\sqrt{r(1+\cos\theta)}-\sqrt{r_m(1+\cos\theta_m)}\right)^2 \\
&+\left(\sqrt{r(1-\cos\theta)}-\sqrt{r_m(1-\cos\theta_m)}\right)^2\Bigg\}\Bigg] \,. \qquad (49)
\end{aligned}
$$

Finally, we construct the (nonstationary) coherent states of the time–dependent Schrödinger equation

$$
i\hbar\,\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2\mu}\,\nabla^2-\frac{e^2}{r}\right)\psi \,. \qquad (50)
$$

Set $\psi = e^{-i\frac{E}{\hbar}t}\chi$. Then χ satisfies

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-\frac{\hbar^2}{2\mu} \nabla^2 - \frac{e^2}{r} - E \right) \chi \, . \tag{51}$$

We now pass to the new time T, Eq.(20), and set

$$\chi = e^{i\epsilon T/\hbar}\varphi$$

so that φ satisfies

$$i\hbar \frac{\partial \varphi}{\partial T} = \left(-\frac{\hbar^2}{2m} \Box_q + \frac{1}{2} m\omega^2 q^2 \right) \varphi, m = 4\mu \tag{52}$$

whose coherent states are precisely $< q|\alpha >$ given in Eq.(26), or $< \mathbf{r}|\alpha >$ given in Eq.(46). Consequently our final solution for ψ is

$$\psi = e^{-\frac{E}{\hbar}t} \, e^{i\epsilon T/\hbar} \, < \mathbf{r}|\alpha > \, . \tag{53}$$

Hence

$$|\psi|^2 = | < \mathbf{r}|\alpha > |^2 \tag{54}$$

as given by Eq.(49), is a nonspreading Gaussian around the average orbit. The phase of ψ is, however, rather complicated and involves both times t and T.

In Reference 6 the transformation to the 4–dimensional oscillator was also considered, but the authors did not take into account that the time has to be changed as well.

It is instructive to verify directly that the form (53) satisfies the time dependent Schrödinger Eq.(50) using (52) and (51).

From (53)

$$i\hbar \frac{\partial \psi}{\partial t} = \left(E - e^2 \frac{dT}{dt} \right) \psi + e^{-i/\hbar E t} \, e^{ie^2/\hbar T} \, i\hbar \frac{\partial \varphi}{\partial T} \frac{dT}{dt} \, ;$$

insert this into (50) using $\frac{dT}{dt} = \frac{1}{r}$ from (20), we get

$$i\hbar \frac{\partial \varphi}{\partial T} = \left(-\frac{\hbar^2}{2\mu} r\nabla^2 - Er \right) \varphi \, .$$

Using now the KS–transformation, Eqs.(13), (15), this is Eq.(52) and our $< r|\alpha >$, Eq.(46), are precisely the nonstationary coherent states of this oscillator problem in time T. The time $T(t)$ can be calculated from (47) in terms of the maximum of the localized state around the orbit

$$T = \left(-\frac{2\mu}{E} \right)^{1/2} \tan^{-1} \left[\frac{|\alpha_1|}{|\alpha_2|} \tan \frac{\theta_m(t)}{2} \right] \, .$$

We remark finally that it is not unusual to express a solution in terms of an auxilliary function $T(t)$. The Kepler problem (even classical) has a conformal symmetry in *velocity* *space* (not in coordinate space) which can be best expressed in T as Levi–Civita, Hadamard and Beltrametti have shown a long time ago [20]. Many properties are expressed in simplest way in time T, hence so our coherent states, as we see. However using $r_{\max}$ and $\theta_{\max}$ we can express $| < r|\alpha > |^2$ independent of T, as in Eq.(49), and we see that $|\psi|^2$ is a Gaussian relative to its maximum.

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