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GAUGE-INVARIANT O(N) NON-LINEAR SIGMA MODEL(S) AND GAUGE-INVARIANT KLEIN-GORDON THEORY: WESS-ZUMINO TERMS AND HAMILTONIAN AND BRST FORMULATIONS

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Abstract

The Wess–Zumino terms for the gauge–non–invariant O(N) non–linear sigma model and for the gauge–non–invariant Klein–Gordon theory, both in one–space one–time dimension are calculated and the Hamiltonian and BRST formulations of the resulting gauge–invariant theories (obtained by the inclusion of the corresponding Wess–Zumino terms) are investigated.

1. Introduction

The O(N) nonlinear sigma models in one–space one–time dimension have some striking qualitative similarities with quantum chromodynamics and have attracted wide interest in the recent years [1–8]. Some of the common features of both the field theories are e.g., renormalizability and asymptotic freedom [1–8]. The quantum sigma model in (1+1)–dimension is a non–trivially solvable [5] quantum field theory. Sigma models provide a laboratory for various nonperturbative techniques e.g., the 1/N expansion, operator–product expansion and low–energy theorems [7]. They are also of importance in the context of string theories [6] where they appear in the classical limit. The model exhibits a nonperturbative particle spectrum, has no intrinsic scale parameter and possesses topological charges.

The Hamiltonian formulation and Dirac quantization of the gauge–non–invariant O(N) nonlinear sigma model in (1+1)–dimension has been studied, in particular, by Maharana [1]. The model is seen to possess a set of four second–class constraints, reflecting a lack of gauge symmetry. The gauge symmetry, however, when present in a theory has many beneficial consequences. It is rather well known that the addition of an appropriate Wess–Zumino kind of term [9–11,2] to the action of a gauge–non–invariant theory possessing a set of second–class constraints converts it into a gauge–invariant theory possessing a set of first–class constraints.

Several procedures exist in the literature [9–11,2] for the calculation of the so–called Wess–Zumino term. One of the simplest and perhaps the oldest procedures (sometimes called the theta–trick) was introduced originally by Stuckelberg about five decades ago [10] in the context of a study of the renormalization properties of massive gauge theories [10]. Another method for reformulating a gauge–non–invariant theory possessing a set of second–class constraints into a gauge–invariant theory possessing a set of first–class constraints is due to Mitra and Rajaraman [17,2]. In the present work we employ both of the above methods, namely, the Stuckelberg method and the Mitra–Rajaraman method, for constructing two different gauge–invariant versions of the gauge–non–invariant O(N) non–linear sigma model in (1+1) dimension possessing a set of four second–class constraints [1–2]. One of the gauge–invariant models so constructed (called model A in our text) is obtained by calculating the Wess–Zumino term [9,10] (that transforms the second–class constaints of the theory into the first–class ones) by enlarging the Hilbert space of the corresponding gauge–non–invariant theory [1–3] using the Stuckelberg method [10]. This gauge–invariant model (model A) is seen to possess a set of three first–class constraints. The other gauge–invariant model (called model B in our text) is obtained by using the Mitra–Rajaraman method [17,2]. The Hamiltonian [12] and Becchi–Rouet–Stora–Tyutin (BRST) [13,14,15] fomulations of these gauge–invariant models (model A and model B) are then investigated. Before coming to the sigma model, however, we first consider a rather trivial and well known example, namely, that of the Klein–Gordon theory in one– space one–time dimension in light–cone coordinates (redefined as space and time coordinates) which is seen to possess one (primary) second–class constraint. We first construct a gauge–invariant model corresponding to this gauge–non–invariant Klein–Gordon theory by calculating the Wess–Zumino term [5] using the Stuckelberg method [10] and then study the Hamiltonian and BRST formulations of the gauge–invariant Klein–Gordon theory so obtained.

In the context of the nonlinear sigma model it is important to mention (as is also pointed out in Ref. [1], and as we would see in Secs. 3 and 4), that the constraints of the theory involve the products of canonical variables in the classical description of the theory as well as in the calculation of the Dirac brackets. These variables are, however, envisaged as noncommuting operators in the quantized theory and therefore one encounters the problem of operator ordering [1,16,17]. This problem can, however, be resolved if one demands that all the relevant brackets be consistent with the hermiticity of the operators (i.e. the fields and the momenta canonically conjugate to the fields [1,18]).

The work on the Klein–Gordon theory is presented in the next section (Sec. 2), and the Hamiltonian and BRST formulations of the gauge–invariant O(N) non–linear sigma models, namely, models A and B are presented in Sections 3 and 4 respectively. Finally, the summary and discussion is given in Sec. 5.

2. The Gauge–Invariant Klein–Gordon Theory

2A. The Gauge–Non–Invariant Theory

We start with the gauge–non–invariant Klein–Gordon theory in one–space one–time dimension in light–cone coordinates (redefined as space and time coordinates) described and defined by the Lagrangian density [2]:

$$\mathcal{L}^{N} = \dot{\phi}\,\phi' - \tfrac{1}{2}\,m^{2}\phi^{2} \tag{2.1}$$

Where overdot and prime denote time and space derivatives respectively (i.e., $\partial_0\phi = \dot{\phi}$ and $\partial_1\phi = \phi'$). Throughout this work we would work with the Lorentz metric: $g^{\mu\nu} := \text{diag}(+1,-1)$. The momentum canonically conjugate to ϕ is

$$\pi := \frac{\partial \mathcal{L}^N}{\partial \dot{\phi}} = \phi' \tag{2.2}$$

implying that $\mathcal{L}^N$ possesses a primary constraint

$$\Omega := (\pi - \phi') \approx 0 \tag{2.3}$$

where the symbol ($\approx$) represents a weak equality in the sense of Dirac [12]. The canonical Hamiltonian density corresponding to $\mathcal{L}^N$ (2.1) is

$$\mathcal{H}_c^{\ N} = \pi\dot{\phi} - \mathcal{L}^N = \tfrac{1}{2} m^2 \phi^2 \tag{2.4}$$

After including the primary constraint Ω in the canonical Hamiltonian density $\mathcal{H}_c^{\ N}$ with the help of the Lagrange multiplier field w, one can write the total Hamiltonian density $\mathcal{H}_T^{\ N}$ as:

$$\mathcal{H}_T^{\ N} = \tfrac{1}{2} m^2 \phi^2 + (\pi - \phi')w \tag{2.5}$$

For the Poisson bracket $\{,\}_p$ of two functions A and B, we choose the convention:

$$\{A(x),B(y)\}_p := \int dz \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \tag{2.6}$$

Demanding that the primary constraint Ω be maintained in the course of time one does not obtain any secondary constraint but instead gets a condition on the Lagrange multiplier field w namely,

$$2w' - m^2\phi \approx 0 \tag{2.7}$$

and therefore the theory is seen to possess only one constraint Ω. Also, the Poisson bracket of Ω with itself is

$$\{\Omega(z),\Omega(z')\}_p = -2\delta'(z-z') \tag{2.8}$$

The matrix of the Poisson brackets of the constraints of the theory is therefore

$$J_{\alpha\beta}(z,z') := \{\Omega_\alpha(z), \Omega_\beta(z')\}_p = [-2\delta'(z-z')]\ 1_{1\times 1} \tag{2.9}$$

with the inverse

$$J^{-1}_{\alpha\beta}(z,z') = [-\tfrac{1}{4}\,\epsilon(z-z')]\ 1_{1\times 1} \tag{2.10}$$

and

$$\int dz\ J(x,z)\ J^{-1}(z,y) = 1_{1\times 1}\ \delta(x-y) \tag{2.11}$$

Here $\epsilon(z-z')$ is a step function defined as

$$\epsilon(z-z') := \begin{cases} +1, & (z-z') > 0 \\ -1, & (z-z') < 0 \end{cases} \tag{2.12}$$

The nonsingular nature of the matrix $J_{\alpha\beta}$ implies that the constraint Ω is a second–class constraint. The Dirac bracket $\{,\}_D$ of two functions A and B is defined as [12]:

$$\{A,B\}_D := \{A,B\}_p - \iint dz\ dz' \sum_{\alpha,\beta} \Big[\{A,\Gamma_\alpha(z)\}_p \left[\Delta^{-1}_{\alpha\beta}(z,z')\right] \{\Gamma_\beta(z'),B\}_p\Big] \tag{2.13}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(z,z')$ $[:= \{\Gamma_\alpha(z), \Gamma_\beta(z')\}_p]$ is the matrix of the Poisson brackets of the constraints Γ_i.

The nonvanishing equal–time Dirac brackets obtained for the theory $\mathcal{L}^N$ (2.1) are:

$$\{\phi(x),\pi(y)\}_D = \tfrac{3}{2}\,\delta(x-y) \tag{2.14a}$$

$$\{\pi(x),\pi(y)\}_D = -\tfrac{1}{2}\,\delta'(x-y) \tag{2.14b}$$

$$\{\phi(x),\phi(y)\}_D = -\tfrac{1}{4}\,\epsilon(x-y) \tag{2.14c}$$

2B. The Wess–Zumino Term

In constructing a gauge–invariant Klein–Gordon theory from the gauge–non–invariant one described by $\mathcal{L}^N$ [2], we calculate the Wess–Zumino term for $\mathcal{L}^N$. For this following the Stuckelberg method [10] we enlarge the Hilbert space of the quantum theory described by $\mathcal{L}^N$ [9,10], and introduce a new field θ, called the Wess–Zumino field, through the following redefinition of field ϕ in the original Lagrangian density $\mathcal{L}^{\mathrm{N}}$ [9,10] (the motivation for which comes from the gauge–transformations (2.26) under which the proposed gauge–invariant theory $\mathcal{L}^{\mathrm{I}}$ (2.16) is expected to be invariant):

$$\phi \longrightarrow \Phi = \phi - \theta \tag{2.15}$$

The Wess–Zumino field θ is a full quantum field [9,10]. Performing the changes (2.15) in $\mathcal{L}^{\mathrm{N}}$ (2.1), we obtain the modified Lagrangian density as

$$\mathcal{L}^{\mathrm{I}} = (\dot{\phi}-\dot{\theta})\,(\phi'-\theta') - \tfrac{1}{2}\,\mathrm{m}^2(\phi-\theta)^2 = \mathcal{L}^{\mathrm{N}} + \mathcal{L}^{\mathrm{WZ}} \tag{2.16}$$

$$\mathcal{L}^{\mathrm{WZ}} = \dot{\theta}\theta' - \dot{\theta}\,\phi' - \dot{\phi}\theta' + \mathrm{m}^2\phi\theta - \tfrac{1}{2}\,\mathrm{m}^2\theta^2 \tag{2.17}$$

where $\mathcal{L}^{\mathrm{WZ}}$ is the appropriate Wess–Zumino term corresponding to $\mathcal{L}^{\mathrm{N}}$. We shall see later that $\mathcal{L}^{\mathrm{I}}$ describes a gauge–invariant theory. In fact, we will be able to recover the physical content of the gauge–non–invariant theory described by $\mathcal{L}^{\mathrm{N}}$ under some special choice of gauge. The Euler–Lagrange equations obtained from $\mathcal{L}^{\mathrm{I}}$ are:

$$2(\dot{\theta}' - \dot{\phi}') + \mathrm{m}^2\,(\theta-\phi) = 0 \tag{2.18a}$$

$$2(\dot{\phi}' - \dot{\theta}') + \mathrm{m}^2\,(\phi-\theta) = 0 \tag{2.18b}$$

2C. Hamiltonian Formulation of the Gauge–Invariant Theory

The canonical momenta for the gauge–invariant theory described by $\mathcal{L}^{\mathrm{I}}$ are:

$$\pi := \frac{\partial\mathcal{L}^{\mathrm{I}}}{\partial\dot{\phi}} = \phi' - \theta'; \tag{2.19a}$$

$$\pi_\theta := \frac{\partial\mathcal{L}^{\mathrm{I}}}{\partial\dot{\theta}} = \theta' - \phi' \tag{2.19b}$$

The equations (2.19) imply that the theory $\mathcal{L}^I$ possesses two primary constraints

$$\Omega_1 := (\pi - \phi' + \theta') \approx 0 \quad \text{and} \quad \Omega_2 := (\pi_\theta - \theta' + \phi') \approx 0 \tag{2.20}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}^I$ is

$$\mathcal{H}_c^I = \frac{1}{2} m^2 (\phi - \theta)^2 \tag{2.21}$$

After including the primary constraints Ω_1 and $\Omega_2$of the theory in the canonical Hamiltonian density $\mathcal{H}_c^I$ with the help of the Lagrange multipliers u and v, one can write the total Hamiltonian density $\mathcal{H}_T{}^I$ as

$$\mathcal{H}_T{}^I = \frac{1}{2} m^2(\phi - \theta)^2 + (\pi - \phi' + \theta')u + (\pi_\theta - \theta' + \phi')v \tag{2.22}$$

The Hamilton's equations obtained from the total Hamiltonian $H_T^I = \int dx\ \mathcal{H}_T{}^I$ are:

$$\dot{\phi} = \frac{\partial H_T{}^I}{\partial \pi} = u; \qquad -\dot{\pi} = \frac{\partial H_T{}^I}{\partial \phi} = m^2(\phi - \theta) + u' - v' \tag{2.23a}$$

$$\dot{\theta} = \frac{\partial H_T{}^I}{\partial \pi_\theta} = v; \qquad -\dot{\pi}_\theta = \frac{\partial H_T{}^I}{\partial \theta} = -m^2(\phi - \theta) - u' + v' \tag{2.23b}$$

$$\dot{u} = \frac{\partial H_T{}^I}{\partial p_u} = 0; \quad -\dot{p}_u = \frac{\partial H_T{}^I}{\partial u} = (\pi - \phi' + \theta') \tag{2.23c}$$

$$\dot{v} = \frac{\partial H_T{}^I}{\partial p_v} = 0; \quad -\dot{p}_v = \frac{\partial H_T{}^I}{\partial v} = (\pi_\theta - \theta' + \phi') \tag{2.23d}$$

These are the equations of motion of the theory that preserve the constraints of the theory Ω_i in the course of time.

Demanding that the primary constraints Ω_1 and Ω_2 be maintained in the course of time leads (in both cases) to the condition

$$m^2(\phi - \theta) - 2u' + 2v' = 0 \tag{2.24}$$

which involves Lagrange multiplier fields u and v. Thus there are no further secondary constraints and therefore the theory is seen to possess only two constraints Ω_1 and Ω_2. The matrix of the Poisson brackets of the constraints Ω_i is

$$K_{\alpha\beta}(z,z') := \{\Omega_\alpha(z),\Omega_\beta(z')\}_p = \begin{bmatrix} -2\delta'(z-z') & 2\delta'(z-z') \\ 2\delta'(z-z') & -2\delta'(z-z') \end{bmatrix} \tag{2.25}$$

the above matrix $K_{\alpha\beta}(z,z')$ is clearly singular implying that the constraints Ω_1 and Ω_2 are first class and that the theory described by $\mathcal{L}^I$ is a gauge–invariant theory. In fact, the Lagrangian density $\mathcal{L}^I$ is seen to be invariant under the time–dependent gauge–transformations:

$$\delta\phi = \dot{\mu}(x,t), \quad \delta\theta = \dot{\mu}(x,t), \quad \delta\pi = 0, \quad \delta\pi_\theta = 0, \tag{2.26}$$

where $\mu(x,t)$ is an arbitrary function of the coordinates. In quantizing the theory with Dirac's procedure [12], we have to convert the first–class constraints of the theory into second–class ones. This we achieve by imposing, arbitrarily, some additional constraints on the system, in the form of gauge–fixing conditions. Following the work of Ref. [15], we go to a special gauge given by $\theta = 0$ (or equivalently, $\partial_1\theta = \theta' = 0$), and accordingly choose the gauge fixing condition of the theory as [15]:

$$\mathcal{G} = \theta' \approx 0; \tag{2.27}$$

With the gauge–fixing condition (2.27) the total set of constraints of the theory becomes

$$\tau_1 = \Omega_1 = (\pi - \phi' + \theta') \approx 0; \tag{2.28a}$$
$$\tau_2 = \Omega_2 = (\pi_\theta - \theta' + \phi') \approx 0 \tag{2.28b}$$
$$\tau_3 = \mathcal{G} = \theta' \approx 0; \tag{2.28c}$$

The matrix of the Poisson brackets of the constraints τ_i is obtained as

$$M_{\alpha\beta}(z,z') := \{\tau_\alpha(z),\tau_\beta(z')\}_p = \begin{bmatrix} -2\delta'(z-z') & 2\,\delta'(z-z') & 0 \\ 2\delta'(z-z') & -2\,\delta'(z-z') & +\delta'(z-z') \\ 0 & +\delta'(z-z') & 0 \end{bmatrix} \tag{2.29}$$

with the inverse

$$M^{-1}_{\alpha\beta}(z,z') = \begin{bmatrix} -\frac{1}{4}\epsilon(z-z') & 0 & \frac{1}{2}\epsilon(z-z') \\ 0 & 0 & +\frac{1}{2}\epsilon(z-z') \\ \frac{1}{2}\epsilon(z-z') & +\frac{1}{2}\epsilon(z-z') & 0 \end{bmatrix} \tag{2.30}$$

and

$$\int dz\; M(x,z)\; M^{-1}(z,y) = 1_{4\times 4}\; \delta(x-y) \tag{2.31}$$

Finally, the nonvanishing equal–time Dirac brackets of the gauge–invariant theory described by $\mathcal{L}^I$ under the gauge (2.27) are finally obtained as:

$$\{\phi(x),\pi(y)\}_D = \frac{3}{2}\,\delta(x-y) \tag{2.32}$$
$$\{\pi(x),\pi(y)\}_D = -\frac{1}{2}\,\delta'(x-y) \tag{2.33}$$
$$\{\phi(x),\phi(y)\}_D = -\frac{1}{4}\,\epsilon(x-y) \tag{2.34}$$
$$\{\phi(x),\pi_\theta(y)\}_D = \frac{1}{2}\,\delta(x-y); \quad \{\theta(x),\pi_\theta(y)\}_D = 2\delta(x-y) \tag{2.35}$$
$$\{\pi(x),\pi_\theta(y)\}_D = \frac{1}{2}\,\delta'(x-y); \quad \{\pi_\theta(x),\pi_\theta(y)\}_D = -\frac{1}{2}\delta'(x-y) \tag{2.36}$$

Following the sequence of reasoning offered in Ref. [15], where the quantization of a gauge–invariant theory of chiral bosons (obtained by the inclusion of an appropriate Wess–Zumino term) has been treated along similar lines, it is easy to see that the above relations (2.32) – (2.36), together with $\mathcal{H}_c^{\,I}$ (2.21) under the gauge (2.27), reproduce precisely the quantum system described by $\mathcal{L}^N$ (2.1). It is easy to see that (2.27) when inserted in (2.20), yields the constraint (2.3) of $\mathcal{L}^N$, thus implying that under (2.27), $\tau_i \approx \Omega$. Also, when (2.27) is inserted in (2.21), one recovers exactly the Hamiltonian density ($\mathcal{H}_c^{\,N}$) corresponding to $\mathcal{L}^N$, and thus implying that under (2.27), $\mathcal{H}_c^{\,I} \approx \mathcal{H}_c^{\,N}$.

Thus under the gauge–fixing condition (2.27), one is able to recover the physical content of the theory described by $\mathcal{L}^N$. The difference of the descriptions of the theories given by $\mathcal{L}^I$ and $\mathcal{L}^N$ appears in terms of the additional constraint $\mathcal{G}$ which serves the purpose of eliminating θ and π_θ. In fact, in view of the above, we see that we have succeeded in finding a gauge (namely (2.27)) which translates the gauge–invariant version of the theory described by $\mathcal{L}^I$ into the gauge–non– invariant one described by $\mathcal{L}^N$. A comparison of (2.32)–(2.36) and (2.14) reveals that (2.32)–(2.34) coincide completely with (2.14) as they should. The additional commutators (2.35), (2.36) express merely the dependence on θ and π_θ. It is important to observe here that the gauge–non–invariant theory described by $\mathcal{L}^N$ is equivalent to working in a specific gauge of the corresponding gauge invariant formulation of the theory [15]. In fact, the physical Hilbert spaces of the two theories ($\mathcal{L}^I$ and $\mathcal{L}^N$) are the same. The addition of the Wess–Zumino term $\mathcal{L}^{WZ}$ to the theory (i.e. to $\mathcal{L}^N$) enlarges only the unphysical part of the full Hilbert space of the theory $\mathcal{L}^N$, without modifying the physical content of the theory. The Wess–Zumino field θ itself, infact, represents only an unphysical degree of freedom and consequently the physics of the theories with and without the Wess–Zumino term remains the same.

2D BRST Formulation of the Gauge–Invariant Theory

In considering the BRST formulation of the gauge–invariant theory described by $\mathcal{L}^I$, we first convert the total Hamiltonian density $\mathcal{H}_T^{\ I}$ into the first–order Lagrangian density:

$$\mathcal{L}^I_{IO} = \pi\dot{\phi} + \pi_\theta\dot{\theta} + p_u\dot{u} + p_v\dot{v} - \mathcal{H}_T^{\ I}$$

$$= p_u\dot{u} + p_v\dot{v} + (\dot{\phi} - \dot{\theta})\,(\phi' - \theta') - \tfrac{1}{2}m^2\,(\phi - \theta)^2 \qquad (2.37)$$

In (2.37), the terms $\pi(\dot{\phi}-u)$, and $\pi_\theta(\dot{\theta}-v)$ drop out in view of the Hamilton's equations (2.23a) and (2.23b).

2D1. The BRST Invariance

We rewrite the gauge–invariant theory described by $\mathcal{L}^{I}$ as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of the quantum theory described by $\mathcal{L}^{I}$ and replace the notion of gauge transformation which shifts operators by c–number functions by a BRST transformation which mixes operators having different statistics. We then introduce new anti–commuting variables c and $\bar{c}$ (called the Faddeev–Popov ghost and anti–ghost fields which are Grassmann numbers on the classical level, and operators in the quantized theory) and a commuting variable b called the Nakanishi–Lautrup field (with the property $\hat{\delta}^2 = 0$) such that

$$\hat{\delta}\phi = \dot{c}, \quad \hat{\delta}\theta = \dot{c}, \quad \hat{\delta}\pi = 0, \quad \hat{\delta}\pi_{\theta} = 0; \tag{2.38a}$$

$$\hat{\delta}c = 0, \qquad \hat{\delta}\,\bar{c} = b, \qquad \hat{\delta}b = 0 \tag{2.38b}$$

The transformations for the Lagrange multiplier fields and their canonical momenta need not be specified as they are not needed. We then define a BRST–invariant function of the dynamical variables to be a function $f(\pi,\pi_{\theta},p_b,\pi_c,\pi_{\bar{c}},\phi,\theta,b,c,\bar{c})$ such that $\hat{\delta}f = 0$.

2D2. Gauge–Fixing in the BRST Formalism

Performing gauge–fixing in the BRST formalism implies adding to the first–order Lagrangian density (2.37) a trivial BRST–invariant function [14,15]. We could thus write the quantum Lagrangian density (taking e.g., a trivial BRST–invariant function as follows) [14,15]:

$$\begin{aligned}\mathcal{L}_{BRST} &= \mathcal{L}^{I}_{IO} + \hat{\delta}[\,\bar{c}(2\dot{\phi} - \dot{\theta} + \tfrac{1}{2}b)]\\ &= p_u\dot{u} + p_v\dot{v} + (\dot{\phi} - \dot{\theta})(\phi' - \theta') - \tfrac{1}{2}m^2(\phi - \theta)^2 + \hat{\delta}[\,\bar{c}(2\dot{\phi} - \dot{\theta} + \tfrac{1}{2}b)]\end{aligned} \tag{2.39}$$

The last term in the above equation (Eq. (2.39)) is the extra BRST–invariant gauge–fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathcal{L}_{BRST}$ (with one integration by parts) as:

$$\mathcal{L}_{BRST} = p_u \dot{u} + p_v \dot{v} + (\dot{\phi} - \dot{\theta})(\phi' - \theta') - \frac{1}{2} m^2 (\phi - \theta)^2 + \frac{1}{2} b^2 + b(2\dot{\phi} - \dot{\theta}) + \dot{\bar{c}}\dot{c} \tag{2.40}$$

Proceeding classically, the Euler–Lagrange equation for b reads:

$$-b = (2\dot{\phi} - \dot{\theta}) \tag{2.41}$$

Also, the requirement $\hat{\delta}b = 0$ (cf. Eq. (2.38b)) implies:

$$-\hat{\delta}b = (2\hat{\delta}\dot{\phi} - \hat{\delta}\dot{\theta}) = 0 \tag{2.42}$$

which in turn implies

$$\ddot{c} = 0 \tag{2.43}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{BRST}$ with respect to $\bar{c}$. In introducing momenta we have to be careful in defining those for fermionic variables. Thus we define the bosonic momenta in the usual way so that

$$\pi = \frac{\partial}{\partial\dot{\phi}} \mathcal{L}_{BRST} = \phi' - \theta' + 2b \tag{2.44a}$$

$$\pi_\theta = \frac{\partial}{\partial\dot{\theta}} \mathcal{L}_{BRST} = \theta' - \phi' - b \tag{2.44b}$$

implying that $b = (\pi + \pi_\theta)$. For the fermionic momenta with directional derivatives, we set

$$\pi_c := \mathcal{L}_{BRST} \frac{\overleftarrow{\partial}}{\partial\dot{c}} = \dot{\bar{c}}\,; \quad \pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial\dot{\bar{c}}} \mathcal{L}_{BRST} = \dot{c} \tag{2.45}$$

implying that the variable canonically conjugate to c is $\dot{\bar{c}}$ and the variable conjugate to $\bar{c}$ is $\dot{c}$. In forming the Hamiltonian density $\mathscr{H}_{BRST}$ from the Lagrangian density in the usual way we remember that the former has to be Hermitian. Then

$$\mathscr{H}_{BRST} = \pi\dot{\phi} + \pi_\theta\dot{\theta} + p_u\dot{u} + p_v\dot{v} + \pi_c\dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - \mathcal{L}_{BRST}$$

$$= \dot{\phi}\,(\pi-\phi'+\theta'-2b) + \dot{\theta}(\pi_\theta-\theta'+\phi'+b) + \tfrac{1}{2}m^2(\phi-\theta)^2 - \frac{b^2}{2} + \pi_c\pi_{\bar{c}}$$

$$= \tfrac{1}{2}m^2(\phi-\theta)^2 - \tfrac{1}{2}(\pi+\pi_\theta)^2 + \pi_c\pi_{\bar{c}} \qquad (2.46)$$

We can check the consistency of (2.45) and (2.46) by looking at Hamilton's equations for the fermionic variables, i.e.

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial\pi_c}\mathscr{H}_{BRST}, \qquad \dot{\bar{c}} = \mathscr{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial\pi_{\bar{c}}} \qquad (2.47)$$

Thus

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial\pi_c}\mathscr{H}_{BRST} = \pi_{\bar{c}}\,; \qquad \dot{\bar{c}} = \mathscr{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial\pi_{\bar{c}}} = \pi_c \qquad (2.48)$$

in agreement with (2.45).

For the operators $c,\bar{c},\dot{c}$ and $\dot{\bar{c}}$ one needs to specify the anti–commutation relations of $\dot{c}$ with $\bar{c}$ or of $\dot{\bar{c}}$ with c, but not of c with $\bar{c}$. c and $\bar{c}$ are, in general, independent canonical variables and one assumes that [14,15]:

$$\{\pi_c,\pi_{\bar{c}}\} = \{\bar{c},c\} = 0\,; \qquad \frac{d}{dt}\{\bar{c},c\} = 0; \qquad (2.49a)$$

$$\{\dot{\bar{c}},c\} = -\{\dot{c},\bar{c}\} \qquad (2.49b)$$

where {,} means anticommutator. We thus see that the anti–commutators in (2.49b) are non–trivial and need to be fixed. In order to fix these we demand that c satisfy the Heisenberg equation of motion [14,15]:

$$[c, \mathscr{H}_{BRST}] = i\dot{c} \tag{2.50}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathscr{H}_{BRST}] = \{\dot{\bar{c}}, c\}\, \dot{c}. \tag{2.51}$$

Eqs. (2.49) – (2.51) then imply:

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} = i\ . \tag{2.52}$$

Here the minus sign in the above equation is non–trivial and implies the existence of states with negative norm in the space of state vectors of the theory [14,15].

2D3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformation (2.38). It is nilpotent and therefore satisfies $Q^2 = 0$. It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anti–commutators with Fermi operators, in the present case, satisfy:

$$[\phi, Q] = \dot{c}\ ; \quad [\theta, Q] = \dot{c}; \quad \{\bar{c}, Q\} = (\pi + \pi_\theta) \tag{2.53}$$

All other commutators and anti–commutators involving Q vanish. In view of (2.53), the BRST charge operator of the present theory can be written as

$$\begin{aligned} Q &= \int dx\, [-i\dot{c}\, (\pi - \phi' + \theta' + \pi_\theta + \phi' - \theta')] \\ &= \int dx\, [-i\dot{c}\, (\pi + \pi_\theta)] \end{aligned} \tag{2.54}$$

This equation implies that the set of states satisfying the condition $(\pi - \phi' + \theta')|\psi\rangle = 0$ and $(\pi_\theta - \theta' + \phi')|\psi\rangle = 0$ belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST–invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Eq. (2.43) (namely, $\ddot{c} = 0$). The solution of this equation gives the Heisenberg operator c(t) (and correspondingly $\bar{c}(t)$) as:

$$c(t) = Gt + F; \quad \bar{c}(t) = G^{\dagger}t + F^{\dagger}; \tag{2.55}$$

which at time t = 0 imply

$$c \equiv c(0) = F, \quad \bar{c} \equiv \bar{c}(0) = F^{\dagger} \tag{2.56a}$$

$$\dot{c} \equiv \dot{c}(0) = G, \quad \dot{\bar{c}} \equiv \dot{\bar{c}}(0) = G^{\dagger} \tag{2.56b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c},c\} = \{\dot{\bar{c}},\dot{c}\} = 0; \quad \{\dot{\bar{c}},c\} = i = -\{\dot{c},\bar{c}\} \tag{2.57}$$

one then obtains

$$F^2 = F^{\dagger 2} = \{F^{\dagger},F\} = \{G^{\dagger},G\} = 0 \tag{2.58a}$$

$$\{G^{\dagger},F\} = -\{G,F^{\dagger}\} = i \tag{2.58b}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$G|0\rangle = F|0\rangle = 0; \tag{2.59}$$

Defining $|0\rangle$ to have norm one, (2.58b) implies

$$\langle 0|FG^{\dagger}|0\rangle = i; \quad \langle 0|GF^{\dagger}|0\rangle = -i \tag{2.60}$$

so that

$$G^{\dagger}|0\rangle \neq 0; \qquad F^{\dagger}|0\rangle \neq 0 \tag{2.61}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{BRST}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators the Hamiltonian density is

$$\mathscr{H}_{BRST} = \frac{1}{2}m^2(\phi - \theta)^2 - \frac{1}{2}(\pi + \pi_\theta)^2 + G^\dagger G \tag{2.62}$$

and the BRST charge operator Q is

$$Q = \int dx \, \{-i[G(\pi + \pi_\theta)]\} \tag{2.63}$$

Now, because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (2.20) holds but also additional states for which $G|\psi\rangle = F|\psi\rangle = 0$ but (2.20) does not hold. However, the Hamiltonian is also invariant under the anti–BRST transformation (in which the role of c and $-\bar{c}$ is interchanged) given by

$$\hat{\bar{\delta}}\phi = -\dot{\bar{c}}, \quad \hat{\bar{\delta}}\theta = -\dot{\bar{c}}, \quad \hat{\bar{\delta}}\pi = 0, \quad \hat{\bar{\delta}}\pi_\theta = 0; \tag{2.64a}$$

$$\hat{\bar{\delta}}\bar{c} = 0, \quad \hat{\bar{\delta}}c = -b, \quad \hat{\bar{\delta}}b = 0 \tag{2.64b}$$

with generator or anti–BRST charge

$$\bar{Q} = \int dx \, [i\dot{\bar{c}}(\pi + \pi_\theta)] = \int dx \, \{i[G^\dagger(\pi+\pi_\theta)]\} \tag{2.65}$$

We now have $[Q,H] = 0$ and $[\bar{Q},H] = 0$, and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states implying that

$$Q|\psi\rangle = 0 \text{ and } \bar{Q}|\psi\rangle = 0 \tag{2.66}$$

The states for which (2.20) holds strongly satisfy both of these conditions and, in fact, are the <u>only</u> states satisfying both the conditions since, although with (2.58)

$$G^\dagger G = -GG^\dagger \tag{2.67}$$

there are no states of this operator with $G^\dagger|0\rangle = 0$ and $F^\dagger|0\rangle = 0$ (cf. (2.61)), and hence no free eigenstates of the fermionic part of $\mathscr{H}_{BRST}$ which are annihilated by each of $G,G^{\dagger},F,F^{\dagger}$. Thus the only states satisfying (2.66) are those satisfying the constraints (2.20).

The states for which $(\pi-\phi'+\theta')|\psi\rangle = 0$ and $(\pi_\theta-\theta'+\phi')|\psi\rangle = 0$ satisfy both of these conditions (2.66) and, in fact, are the only states satisfying both of these conditions (2.66) because in view of (2.57), one cannot have simultaneously $c,\dot{c}$ and $\bar{c},\dot{\bar{c}}$ applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (2.66) are those that satisfy the constraints of the theory (2.20), and they belong to the set of BRST–invariant and anti–BRST–invariant states.

One can understand the above point in terms of fermionic annihilation and creation operators as follows: the condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only states for which $(\pi-\phi'+\theta')|\psi\rangle = 0$ and $(\pi_\theta-\theta'+\phi')|\psi\rangle = 0$, but also additional states for which $G|\psi\rangle = F|\psi\rangle = 0$, but $(\pi-\phi'+\theta')|\psi\rangle \neq 0$ and $(\pi_\theta-\theta'+\phi')|\psi\rangle \neq 0$. However $\overline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which $(\pi-\phi'+\theta')|\psi\rangle = 0$ and $(\pi_\theta-\theta'+\phi')|\psi\rangle = 0$ simply because $G^{\dagger}|\psi\rangle \neq 0$ and $F^{\dagger}|\psi\rangle \neq 0$. Thus in this alternative way also we see that the states satisfying $Q|\psi\rangle = \overline{Q}|\psi\rangle = 0$ (i.e. (2.66)) are only those that satisfy the constraints (2.20) and also that these states belong to the set of BRST–invariant and anti–BRST– invariant states.

3. The Gauge–Invariant O(N) Non–Linear Sigma Model (Model A)

3A. The Gauge–Non–Invariant Model

We start with the gauge–non–invariant O(N) non–linear–sigma model in one–space one–time dimension described by the Lagrangian density [1,2]:

$$\mathcal{L}^N = \frac{1}{2}\,\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda(\sigma_k^2-1); \quad k = 1,2,\ldots,N \tag{3.1a}$$

$$= \frac{1}{2}\,\dot{\sigma}_k^2 - \frac{1}{2}\,\sigma_k'^2 + \lambda(\sigma_k^2-1); \; k = 1,2,\ldots,N \tag{3.1b}$$

Here $\vec{\sigma} \equiv \{\sigma_k(x,t);\ k = 1,2,\ldots,N\}$ is a multiplet of N real scalar fields in one–space one–time dimension and $\lambda(x,t)$ is another scalar field. The field $\vec{\sigma}(x,t)$ maps the two–dimensional space–time into the N–dimensional internal manifold whose coordinates are $\sigma_k(x,t)$. The Euler Lagrange equations fo σ_k and λ are

$$\sigma_k'' - \ddot{\sigma}_k + 2\lambda\sigma_k = 0 \tag{3.2a}$$

$$(\sigma_k^2 - 1) = 0 \tag{3.2b}$$

The canonical momenta conjugate respectively to σ_k and λ are:

$$\pi_k := \partial\mathcal{L}^N / \partial\dot{\sigma}_k = \dot{\sigma}_k; \qquad k = 1,2,\ldots,N. \tag{3.3a}$$

$$p_\lambda := \partial\mathcal{L}^N / \partial\dot{\lambda} = 0 \tag{3.3b}$$

The last equation (Eq. (3.3b)) implies that the theory possesses a primary constraint:

$$\chi_1 := p_\lambda \approx 0 \tag{3.4}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}^N$ is

$$\begin{aligned}\mathcal{H}_c^{\,N} &= \pi_k\dot{\sigma}_k + p_\lambda\dot{\lambda} - \mathcal{L}^N \\ &= \tfrac{1}{2}\,\pi_k^2 + \tfrac{1}{2}\,\sigma_k'^{\,2} - \lambda(\sigma_k^2 - 1)\end{aligned} \tag{3.5}$$

After including the primary constraint χ_1 in the canonical Hamiltonian density $\mathcal{H}_c^{\,N}$ with the help of a lagrange multiplier w, one can write the total Hamiltonian density $\mathcal{H}_T^{\,N}$ as

$$\mathcal{H}_T^{\,N} = \tfrac{1}{2}\,\pi_k^2 + \tfrac{1}{2}\,\sigma_k'^{\,2} - \lambda(\sigma_k^2 - 1) + p_\lambda w \tag{3.6}$$

demanding that the primary constraint χ_1 be preserved in the course of time, we obtain the secondary constraint

$$\chi_2 := \{\chi_1, \mathcal{H}_T^{\,N}\}_p = (\sigma_k^2 - 1) \approx 0 \tag{3.7}$$

The time evolution of χ_2 leads to a further constraint

$$\chi_3 := \{\chi_2, \mathscr{H}_T^{\,N}\}_p = 2\sigma_k\pi_k \approx 0 \qquad (3.8)$$

demanding that χ_3 be also maintained in the course of time one obtains a further constraint

$$\chi_4 := \{\chi_3, \mathscr{H}_T^{\,N}\}_p = (2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'') \approx 0 \qquad (3.9)$$

the time evolution of χ_4 does not, however, lead to any further constraint but instead leads to a condition involving the Lagrange multiplier w, namely:

$$4\sigma_k^2 w + 2\pi_k\sigma_k\sigma_k''' + 2\pi_k\sigma_k'' - 4\pi_k\sigma_k'\sigma_k'' + 16\,\lambda\sigma_k\pi_k = 0 \qquad (3.10)$$

The theory is thus seen to possess a set of four constraints χ_1, χ_2, χ_3 and χ_4.

The matrix of the Poisson brackets of the constraints χ_i namely, $T_{\alpha\beta}(z,z') := \{\chi_\alpha(z), \chi_\beta(z')\}_p$, is then calculated. The nonvanishing matrix elements of the matrix $T_{\alpha\beta}(z,z')$ are (the arguments of the field variables being suppressed):

$$T_{14} = -T_{41} = -4\sigma_k^2\delta(z-z') \qquad (3.11a)$$

$$T_{23} = -T_{32} = 4\sigma_k^2\delta(z-z') \qquad (3.11b)$$

$$T_{24} = -T_{42} = 8\sigma_k\pi_k\delta(z-z') \qquad (3.11c)$$

$$T_{34} = -T_{43} = [8\pi_k^2 - 16\lambda\sigma_k^2 - 4\sigma_k\sigma_k'']\,\delta(z-z') - [4\sigma_k^2]\,\delta''(z-z') \qquad (3.11d)$$

The matrix $T_{\alpha\beta}$ is seen to be nonsingular and therefore its inverse exists. The nonvanishing elements of the inverse of the matrix $T_{\alpha\beta}$ (i.e. the elements of the matrix $(T^{-1})_{\alpha\beta}$ are (the arguments of the field variables being suppressed again):

$$(T^{-1})_{12} = -(T^{-1})_{21} = \left[\frac{2\pi_k^2 - 4\lambda\sigma_k^2 - \sigma_k\sigma_k''}{4\sigma_k^2\sigma_k^2}\right]\delta(z-z') - \left[\frac{1}{4\sigma_k^2}\right]\delta''(z-z') \qquad (3.12a)$$

$$(T^{-1})_{13} = -(T^{-1})_{31} = \left[\frac{-\sigma_k\pi_k}{2\sigma_k^2\sigma_k^2}\right]\delta(z-z') \tag{3.12b}$$

$$(T^{-1})_{14} = -(T^{-1})_{41} = \left[\frac{1}{4\sigma_k^2}\right]\delta(z-z') \tag{3.12c}$$

$$(T^{-1})_{23} = -(T^{-1})_{32} = \left[\frac{-1}{4\sigma_k^2}\right]\delta(z-z') \tag{3.12d}$$

Finally, the nonvanishing equal–time Dirac brackets of the gauge–non–invariant theory described by $\mathcal{L}^N$ are obtained as

$$\{\pi_\ell(x),\pi_m(y)\}_D = -\frac{1}{\sigma_k^2}\left[\sigma_\ell(x)\pi_m(y) - \pi_\ell(x)\sigma_m(y)\right]\,\delta(x-y) \tag{3.13}$$

$$\{\sigma_\ell(x),\pi_m(y)\}_D = \left[\delta_{\ell m} - \frac{\sigma_\ell(x)\,\sigma_m(y)}{\sigma_k^2}\right]\delta(x-y) \tag{3.14}$$

which are seen to be in agreement with the results obtained in Ref. [1].

3B. The Wess–Zumino Term

In constructing a gauge–invariant model corresponding to $\mathcal{L}^N$ (3.1), we calculate the Wess–Zumino term for $\mathcal{L}^N$. For this following the Stuckelberg method [10], we enlarge the Hilbert space of the quantum theory defined by $\mathcal{L}^N$, and introduce a new field θ (called the Wess–Zumino field [9,10]), through the following redefinition of fields σ_k and λ in the original Lagrangian density $\mathcal{L}^N$ (3.1) (the motivation for which comes from the gauge transformations (3.24)).

$$\sigma_k \longrightarrow \Sigma_k = \sigma_k - \theta; \quad \lambda \longrightarrow \Lambda = \lambda + \dot{\theta} \tag{3.15}$$

Performing the changes (3.15) in $\mathcal{L}^N$ (3.1), we obtain the modified Lagrangian density (ignoring total space and time derivatives) as:

$$\mathcal{L}^{I} = \mathcal{L}^{N} + \mathcal{L}^{WZ} \tag{3.16a}$$

with

$$\mathcal{L}^{WZ} = \frac{1}{2}\,\partial_{\mu}\,\theta\partial^{\mu}\theta - \partial_{\mu}\sigma_{k}\partial^{\mu}\theta + \dot{\theta}(\sigma_{k}^{2}-1) - (\lambda+\dot{\theta})\,\theta(2\sigma_{k}-\theta)$$

$$= \frac{1}{2}\,\dot{\theta}^{2} - \frac{1}{2}\,\theta'^{\,2} - \dot{\sigma}_{k}\dot{\theta} + \sigma_{k}'\theta' + \dot{\theta}(\sigma_{k}^{2}-1) - (\lambda+\dot{\theta})\,\theta(2\sigma_{k}-\theta) \tag{3.16b}$$

where $\mathcal{L}^{WZ}$ is the appropriate Wess–Zumino term corresponding to $\mathcal{L}^{N}$. We shall see later that $\mathcal{L}^{I}$ describes a gauge–invariant theory. Infact, we will be able to recover the physical content of the gauge–non–invariant theory described by $\mathcal{L}^{N}$ under some special choice of gauge [15,19]. The Euler–Lagrange equations obtained from $\mathcal{L}^{I}$ (3.16) are:

$$(\ddot{\theta}-\theta'') - (\ddot{\sigma}_{k}-\sigma_{k}'') + 2(\sigma_{k}-\theta)(\lambda+\dot{\theta}) = 0 \tag{3.17a}$$

$$(\theta''-\ddot{\theta}) + (\ddot{\sigma}_{k}-\sigma_{k}'') - 2(\sigma_{k}-\theta)(\dot{\sigma}_{k}+\lambda) = 0 \tag{3.17b}$$

$$(\sigma_{k}^{2}-1) - \theta(2\sigma_{k}-\theta) = 0 \tag{3.17c}$$

3C Hamiltonian Formulation of the Gauge–Invariant Theory (Model A)

The canonical momenta for the gauge–invariant theory described by $\mathcal{L}^{I}$ are

$$p_{\lambda} = \frac{\partial\mathcal{L}^{I}}{\partial\dot{\lambda}} = 0; \tag{3.18a}$$

$$\pi_{k} = \frac{\partial\mathcal{L}^{I}}{\partial\dot{\sigma}_{k}} = (\dot{\sigma}_{k}-\dot{\theta}) \tag{3.18b}$$

$$\pi_{\theta} = \frac{\partial\mathcal{L}^{I}}{\partial\dot{\theta}} = [(\dot{\theta}-\dot{\sigma}_{k})+(\sigma_{k}^{2}-1) - \theta(2\sigma_{k}-\theta)]$$

$$= [-\pi_{k} + (\sigma_{k}^{2}-1) - \theta(2\sigma_{k}-\theta)] \tag{3.18c}$$

Eqs. (3.18) imply that $\mathcal{L}^{I}$ possesses two primary constraints:

$$\psi_1 := p_\lambda \approx 0; \quad \psi_2 := [\pi_\theta + \pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)] \approx 0 \tag{3.19}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}^I$ is

$$\begin{aligned}\mathcal{H}_c^{\,I} &= \pi_k \dot{\sigma}_k + \pi_\theta \dot{\theta} + p_\lambda \dot{\lambda} - \mathcal{L}^I \\ &= \tfrac{1}{2}\pi_k^2 + \tfrac{1}{2}\sigma_k'^{\,2} + \tfrac{1}{2}\theta'^{\,2} - \sigma_k'\theta' - \lambda(\sigma_k^2 - 1) + \lambda\theta\,(2\sigma_k - \theta)\end{aligned} \tag{3.20}$$

After including the primary constraints ψ_1, ψ_2 in the canonical Hamiltonian density $\mathcal{H}_c^{\,I}$ with the help of Lagrange multipliers u and v, one can write the total Hamiltonian density $\mathcal{H}_T^{\,I}$ as

$$\mathcal{H}_T^{\,I} = \mathcal{H}_c^{\,I} + p_\lambda u + [\pi_\theta + \pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)]\, v \tag{3.21}$$

The Hamilton's equations obtained from the total Hamiltonian $H_T^{\,I} = \int \mathcal{H}_T^{\,I} dx$ are:

$$\dot{\sigma}_k = \frac{\partial H_T^{\,I}}{\partial \pi_k} = (\pi_k + v); \qquad -\dot{\pi}_k = \frac{\partial H_T^{\,I}}{\partial \sigma_k} = [-\sigma_k'' + \theta'' - 2(\sigma_k - \theta)(\lambda + v)] \tag{3.22a}$$

$$\dot{\lambda} = \frac{\partial H_T^{\,I}}{\partial p_\lambda} = u; \qquad -\dot{p}_\lambda = \frac{\partial H_T^{\,I}}{\partial \lambda} = [-(\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)] \tag{3.22b}$$

$$\dot{\theta} = \frac{\partial H_T^{\,I}}{\partial \pi_\theta} = v; \qquad -\dot{\pi}_\theta = \frac{\partial H_T^{\,I}}{\partial \theta} = [\sigma_k'' - \theta'' + 2(\sigma_k - \theta)(\lambda + v)] \tag{3.22c}$$

$$\dot{u} = \frac{\partial H_T^{\,I}}{\partial p_u} = 0; \qquad -\dot{p}_u = \frac{\partial H_T^{\,I}}{\partial u} = p_\lambda \tag{3.22d}$$

$$\dot{v} = \frac{\partial H_T^{\,I}}{\partial p_v} = 0; \qquad -\dot{p}_v = \frac{\partial H_T^{\,I}}{\partial v} = [\pi_\theta + \pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)] \tag{3.22e}$$

These are the equations of motion of the theory that preserve the constraints of the theory ψ_1 and ψ_2 in the course of time.

Demanding that the primary constraint ψ_1 be preserved in the course of time, we obtain the secondary constraint

$$\psi_3 = \{\psi_1, \mathcal{H}_T^{\,I}\} = (\sigma_k^2-1) - \theta(2\sigma_k-\theta) \approx 0 \tag{3.23}$$

The preservation of ψ_3 for all time does not give rise to any further constraints. The preservation of ψ_2 for all time also does not yield any further constraints. The theory is thus seen to possess three constraints ψ_1, ψ_2, ψ_3. Also the matrix of the Poisson brackets of the constraints ψ_i is seen to be singular implying that the constraints ψ_i form a set of first–class constraints [12], and that the theory described by $\mathcal{L}^I$ is a gauge–invariant theory. In fact, the Lagrangian density $\mathcal{L}^I$ is seen to be invariant under the time–dependent gauge transformations:

$$\delta\sigma_k = \beta(x,t); \quad \delta\pi_k = 0; \quad k = 1,2,...,N. \tag{3.24a}$$

$$\delta\lambda = -\dot{\beta}(x,t); \quad \delta\theta = \beta(x,t); \quad \delta p_\lambda = 0; \quad \delta\pi_\theta = 0 \tag{3.24b}$$

where $\beta(x,t)$ is an arbitrary function of the coordinates. The action $S^I = \int \mathcal{L}^I \, dxdt$ is therefore gauge–invariant. In quantizing the theory using Dirac's procedure [12], we convert the first–class constraints of the theory into second–class ones by imposing, arbitrarily, some additional constraints on the system in the form of gauge–fixing conditions. We go to a special gauge given by $\theta = 0$, and accordingly choose the gauge–fixing conditions of the theory as [15]):

$$\mathcal{G}_1 = 2\sigma_k\pi_k - (\pi_\theta + \pi_k) \approx 0 \tag{3.25a}$$
$$\mathcal{G}_2 = 2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'' \approx 0 \tag{3.25b}$$
$$\mathcal{G}_3 = \theta \approx 0 \tag{3.25c}$$

With the gauge–fixing conditions (3.25) the total set of constraints of the theory becomes

$$\xi_1 = \psi_1 = p_\lambda \approx 0 \tag{3.26a}$$
$$\xi_2 = \psi_3 = (\sigma_k^2-1) - \theta(2\sigma_k-\theta) \approx 0 \tag{3.26b}$$
$$\xi_3 = \mathcal{G}_1 = 2\sigma_k\pi_k - (\pi_\theta+\pi_k) \approx 0 \tag{3.26c}$$
$$\xi_4 = \mathcal{G}_2 = 2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'' \approx 0 \tag{3.26d}$$
$$\xi_5 = \psi_2 = (\pi_\theta+\pi_k) - (\sigma_k^2-1) + \theta(2\sigma_k-\theta) \approx 0 \tag{3.26e}$$
$$\xi_6 = \mathcal{G}_3 = \theta \approx 0 \tag{3.26f}$$

The matrix of the Poisson brackets of the constraints ξ_i namely $R_{\alpha\beta}(z,z') := \{\xi_\alpha(z),\xi_\beta(z')\}_p$ is then calculated. The nonvanishing matrix elements of the matrix $R_{\alpha\beta}(z,z')$ (with the arguements of the field variables being suppressed) are:

$$R_{14} = -R_{41} = -4\sigma_k^2\ \delta(z-z') \tag{3.27a}$$
$$R_{23} = -R_{32} = 4\sigma_k\ (\sigma_k-\theta)\ \delta(z-z') \tag{3.27b}$$
$$R_{24} = -R_{42} = 8\pi_k\ (\sigma_k-\theta)\ \delta(z-z') \tag{3.27c}$$
$$R_{34} = -R_{43} = \{[8\pi_k^2 - (2\sigma_k-1)\ (8\lambda\sigma_k+2\sigma_k'')]\ \delta(z-z') - [2\sigma_k\ (2\sigma_k-1)]\ \delta''(z-z')\} \tag{3.27d}$$
$$R_{35} = -R_{53} = [2\pi_k + 4\sigma_k\ (\sigma_k-\theta)]\ \delta(z-z') \tag{3.27e}$$
$$R_{36} = -R_{63} = \delta(z-z') \tag{3.27f}$$
$$R_{45} = -R_{54} = \{[8\lambda\sigma_k + 2\sigma_k'' + 8\pi_k\ (\sigma_k-\theta)]\ \delta(z-z') + 2\sigma_k\ \delta''(z-z')\} \tag{3.27g}$$
$$R_{56} = -R_{65} = -\delta(z-z') \tag{3.27h}$$

The matrix $R_{\alpha\beta}$ is clearly nonsingular and therefore its inverse exists. The nonvanishing elements of the inverse of the matrix $R_{\alpha\beta}$ (i.e. the elements of the matrix $(R^{-1})_{\alpha\beta}$ are (with the arguments of the field variables being suppressed once again):

$$(R^{-1})_{12} = -(R^{-1})_{21} = \left[\frac{2\pi_k^2-4\lambda\sigma_k^2-\sigma_k\sigma_k''-2\sigma_k\pi_k+2\theta\pi_k}{\sigma_k^2\ 4\sigma_k(\sigma_k-\theta)}\right]\delta(z-z') - \left[\frac{1}{4\sigma_k(\sigma_k-\theta)}\right]\delta''(z-z') \tag{3.28a}$$

$$(R^{-1})_{13} = -(R^{-1})_{31} = \left[\frac{-\pi_k}{2\sigma_k^2\ \sigma_k}\right]\delta(z-z') \tag{3.28b}$$

$$(R^{-1})_{14} = -(R^{-1})_{41} = \left[\frac{1}{4\sigma_k^2}\right]\delta(z-z') \tag{3.28c}$$

$$(R^{-1})_{15} = -(R^{-1})_{51} = \left[\frac{-\pi_k}{2\sigma_k^2\ \sigma_k}\right]\delta(z-z') \tag{3.28d}$$

$$(R^{-1})_{16} = -(R^{-1})_{61} = \left[\frac{4\lambda\sigma_k-\sigma_k''}{2\sigma_k^2}\right]\delta(z-z') + \left[\frac{\pi_k^2}{\sigma_k^2\ \sigma_k}\right]\delta(z-z') - \left[\frac{1}{2\sigma_k}\right]\delta''(z-z') \tag{3.28e}$$

$$(R^{-1})_{23} = -(R^{-1})_{32} = \left[\frac{-1}{4\sigma_k(\sigma_k-\theta)}\right]\delta(z-z') \tag{3.28f}$$

$$(R^{-1})_{25} = -(R^{-1})_{52} = \left[\frac{-1}{4\sigma_k(\sigma_k-\theta)}\right]\delta(z-z') \tag{3.28g}$$

$$(R^{-1})_{26} = -(R^{-1})_{62} = \left[1+\frac{2\pi_k}{4\sigma_k(\sigma_k-\theta)}\right]\delta(z-z') \tag{3.28h}$$

$$(R^{-1})_{56} = -(R^{-1})_{65} = \delta(z-z') \tag{3.28i}$$

with

$$\int dz\, R(x,z)\, R^{-1}(z,y) = 1_{6\times 6}\,\delta(x-y) \tag{3.29}$$

The nonvanishing equal–time Dirac brackets of the gauge–invariant theory described by $\mathcal{L}^I$ under the gauge (3.25) are finally obtained (with the arguments of the field variables being suppressed) as:

$$\{\pi_\ell(x),\pi_m(y)\}_D = \left[\frac{\theta(\pi_m-\pi_\ell) - (\sigma_\ell\pi_m-\pi_\ell\sigma_m)}{\sigma_k(\sigma_k-\theta)}\right]\delta(x-y) \tag{3.30a}$$

$$\approx \left[\frac{-(\sigma_\ell\pi_m - \pi_\ell\sigma_m)}{\sigma_k^2}\right]\delta(x-y) \tag{3.30b}$$

$$\{\sigma_\ell(x),\pi_m(y)\}_D = \left[\delta_{\ell m} + \left[\frac{\theta\sigma_\ell-\sigma_\ell\sigma_m}{\sigma_k(\sigma_k-\theta)}\right]\right]\delta(x-y) \tag{3.31a}$$

$$\approx \left[\delta_{\ell m} - \frac{\sigma_\ell\sigma_m}{\sigma_k^2}\right]\delta(x-y) \tag{3.31b}$$

$$\{\pi_\theta(x),\pi_\theta(y)\}_D = \left[\frac{2\sigma_k^2(\sigma_k-\theta) + \sigma_k\pi_k}{\sigma_k^2}\right]\delta(x-y) \tag{3.32a}$$

$$\approx \left[\frac{2\sigma_k^2\sigma_k + \sigma_k\pi_k}{\sigma_k^2}\right]\delta(x-y) \tag{3.32b}$$

$$\{\lambda(x),\pi_\theta(y)\}_D = \left[\frac{-4\lambda\sigma_k}{\sigma_k^2}\right]\delta(x-y) \tag{3.33}$$

The results for the Dirac brackets expressed by (3.30a) – (3.32a), are valid throughout the phase space of the gauge–invariant theory $\mathcal{L}^{I}$ (3.16), except for the constraint surface or on the submanifold of the constraints, where they are given by (3.30b) – (3.32b). This is a simple consequence of the fact that the constraints vanish identically on the constraint surface or on the submanifold of the constraints. On the other hand the result for the Dirac bracket expressed by (3.33) is valid throughout the phase space of the gauge–invariant theory.

Further the relations (3.30) – (3.33), together with $\mathcal{H}_{c}^{I}$ (3.20) under the gauge (3.25), reproduce precisely the quantum system described by $\mathcal{L}^{N}$ (3.1) [5]. The gauge (3.25) translates the gauge–invariant version of the theory described by $\mathcal{L}^{I}$ into the gauge–non–invariant one described by $\mathcal{L}^{N}$. A comparison of (3.30) – (3.33) and (3.13) – (3.14) reveals that (3.30b) – (3.31b) coincide completely with (3.13) and (3.14) as they should. The additional commutators (3.32) and (3.33) express merely the dependence on θ and π_{θ}. In fact, as explained in Sec. 2, the physical Hilbert spaces of the two theories ($\mathcal{L}^{I}$ and $\mathcal{L}^{N}$) are the same. Also as observed in Sec. 2, the addition of the Wess–Zumino term ($\mathcal{L}^{WZ}$) to the theory (i.e. to $\mathcal{L}^{N}$) enlarges only the unphysical part of the full Hilbert space of the theory $\mathcal{L}^{N}$, without modifying the physical content of the theory.

For the later use for considering the BRST formulation of the gauge– invariant theory described by $\mathcal{L}^{I}$, we convert the total Hamiltonian density $\mathcal{H}_{T}^{I}$ into the first–order Lagrangian density:

$$\begin{aligned}\mathcal{L}_{I0}^{I} &= \pi_k\dot{\sigma}_k + p_\lambda\dot{\lambda} + \pi_\theta\dot{\theta} + p_u\dot{u} + p_v\dot{v} - \mathcal{H}_T^{I}\\ &= \pi_k\dot{\sigma}_k + p_u\dot{u} + p_v\dot{v} - \tfrac{1}{2}\pi_k^2 - \tfrac{1}{2}\sigma_k'^2 - \tfrac{1}{2}\theta'^2 + \sigma_k'\theta' + \lambda(\sigma_k^2-1) - \lambda\theta(2\sigma_k-\theta)\\ &\quad - [\pi_k - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)]\,\dot{\theta} \qquad (3.34)\end{aligned}$$

3D The BRST Formulation of the Gauge–Invariant Theory (Model A)

3D1 The BRST Invariance

We now rewrite the gauge–invariant theory of nonlinear sigma model (model A) as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge–invariant model and replace the notion of gauge transformation by a BRST transformation (as in Sec. 2, in terms of the new anti– commuting variables c and $\bar{c}$ and a commuting variable b) (with $\hat{\delta}^2 = 0$) such that:

$$\hat{\delta}\sigma_k = c, \quad \hat{\delta}\lambda = -\dot{c}, \quad \hat{\delta}\theta = c, \quad \hat{\delta}\pi_k = 0, \quad \hat{\delta}p_\lambda = 0, \quad \hat{\delta}\pi_\theta = 0; \tag{3.35a}$$

$$\hat{\delta}c = 0, \qquad \hat{\delta}\,\bar{c} = b, \qquad \hat{\delta}b = 0 \tag{3.35b}$$

The transformations for the Lagrange multiplier fields and their canonical momenta again need not be specified as they are not needed. We now define a BRST–invariant function of the dynamical variables to be a function $f(\pi_k, p_\lambda, \pi_\theta, p_b, \pi_c, \pi_{\bar{c}}, \sigma_k, \lambda, \theta, b, c, \bar{c})$ such that $\hat{\delta}f = 0$.

3D2 Gauge–Fixing in the BRST Formalism

Performing gauge–fixing in the BRST formalism implies adding to the first–order Lagrangian density (3.34) a trivial BRST–invariant function [14]. We thus write the quantum Lagrangian density (taking, e.g., a trivial BRST–invariant function as follows) [14,15]:

$$\begin{aligned}
\mathcal{L}_{BRST} &= \mathcal{L}^{I}_{IO} - \hat{\delta}[\,\bar{c}(\dot{\lambda} + \tfrac{1}{2}b - \sigma_k - \theta)] \\
&= \pi_k\dot{\sigma}_k + p_u\dot{u} + p_v\dot{v} - \tfrac{1}{2}\pi_k^2 - \tfrac{1}{2}\sigma_k'^2 - \tfrac{1}{2}\theta'^2 + \sigma_k'\theta' + \lambda(\sigma_k^2 - 1) \\
&\quad + \lambda(\theta^2 - 2\sigma_k\theta) - [\pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)]\,\dot{\theta} \\
&\qquad - \hat{\delta}[\,\bar{c}(\dot{\lambda} + \tfrac{1}{2}b - \sigma_k - \theta)]
\end{aligned} \tag{3.36a}$$

The last term in the above equation (3.36a) is the extra BRST–invariant gauge–fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathcal{L}_{BRST}$ (with one integration by parts):

$$\mathcal{L}_{BRST} = \pi_k \dot{\sigma}_k + p_u \dot{u} + p_v \dot{v} - \frac{1}{2}\pi_k^2 - \frac{1}{2}\sigma_k'^2 - \frac{1}{2}\theta'^2 + \sigma_k'\theta' + \lambda(\sigma_k^2 - 1) + \lambda(\theta^2 - 2\sigma_k\theta)$$
$$- [\pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)]\,\dot{\theta}$$
$$- \frac{1}{2}b^2 - b(\dot{\lambda} - \sigma_k - \theta) + \dot{\bar{c}}\dot{c} - 2\bar{c}c \qquad (3.36b)$$

Proceeding classically, the Euler–Lagrange equation for b reads:

$$-b = (\dot{\lambda} - \sigma_k - \theta) \qquad (3.37)$$

Also, the requirement $\hat{\delta}b = 0$ (cf. Eq. (3.35b)) implies:

$$-\hat{\delta}b = (\hat{\delta}\dot{\lambda} - \hat{\delta}\sigma_k - \hat{\delta}\theta) = 0 \qquad (3.38)$$

which in turn implies

$$-\ddot{c} = 2c \qquad (3.39)$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{BRST}$ with respect to $\bar{c}$. In introducing momenta we have to be careful in defining those for fermionic variables. Thus we define the bosonic momenta in the usual way so that

$$p_\lambda = \frac{\partial}{\partial \dot{\lambda}} \mathcal{L}_{BRST} = -b \qquad (3.40)$$

but for the fermionic momenta with directional derivatives, we set as before

$$\pi_c := \mathcal{L}_{BRST} \frac{\overleftarrow{\partial}}{\partial \dot{c}} = \dot{\bar{c}}\,; \quad \pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial \dot{\bar{c}}} \mathcal{L}_{BRST} = \dot{c} \qquad (3.41)$$

implying that the variable canonically conjugate to c is $\dot{\bar{c}}$ and the variable conjugate to $\bar{c}$ is $\dot{c}$. In forming the Hamiltonian density $\mathcal{H}_{BRST}$ from the Lagrangian density in the usual way we remember that the former has to be Hermitian. Then

$$\mathscr{H}_{BRST} = \pi_k \dot{\sigma}_k + \pi_\theta \dot{\theta} + p_\lambda \dot{\lambda} + p_u \dot{u} + p_v \dot{v} + \pi_c \dot{c} + \dot{\bar{c}} \pi_{\bar{c}} - \mathcal{L}_{BRST}$$

$$= \frac{1}{2} \pi_k^2 + \frac{1}{2} \sigma_k'^2 + \frac{1}{2} \theta'^2 - \sigma_k' \theta' - \lambda(\sigma_k^2 - 1) + \lambda\, \theta(2\sigma_k - \theta) + \frac{1}{2} p_\lambda^2 + p_\lambda(\sigma_k + \theta)$$
$$+ \pi_c \pi_{\bar{c}} + 2\bar{c}c \qquad (3.42)$$

We can check the consistency of (3.41) with (3.42) by looking at Hamilton's equations for the fermionic variables, i.e.

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial \pi_c} \mathscr{H}_{BRST}, \qquad \dot{\bar{c}} = \mathscr{H}_{BRST} \frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} \qquad (3.43)$$

Thus

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial \pi_c} \mathscr{H}_{BRST} = \pi_{\bar{c}}\,; \qquad \dot{\bar{c}} = \mathscr{H}_{BRST} \frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} = \pi_c \qquad (3.44)$$

in agreement with (3.41). Further, the Eqs. (2.49) – (2.52) hold in the present case also.

3D3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (3.35). According to its conventional definition, its commutators with Bose operators and its anti–commutators with Fermi operators in the present case satisfy:

$$[\sigma_k, Q] = [\lambda, Q] = [\theta, Q] = \dot{c} \qquad (3.45a)$$

$$[\pi_k, Q] = -[\pi_\theta, Q] = 2(c - \dot{c})(\sigma_k - \theta) \qquad (3.45b)$$

$$\{\bar{c}, Q\} = p_\lambda + \pi_k + \pi_\theta - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta) \qquad (3.45c)$$

$$\{\dot{\bar{c}}, Q\} = -(\sigma_k^2 - 1) + \theta(2\sigma_k - \theta) \qquad (3.45c)$$

All other commutators and anti–commutators involving Q vanish. In view of (3.45), the BRST charge operator of the present gauge–invariant theory can be written as

$$Q = \int dx\Big[ic\{(\sigma_k^2-1) - \theta(2\sigma_k-\theta)\} - i\dot{c}\{p_\lambda + \pi_\theta + \pi_k - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big] \quad (3.46)$$

This equation implies that the set of states satisfying the conditions (3.19), and (3.23) belongs to the dynamically stable subspace of states $|\psi>$ satisfying $Q|\psi> = 0$, i.e., it belongs to the set of BRST–invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Eq. (3.39) (namely, $-\ddot{c} = 2c$). The solution of this equation gives the Heisenberg operator c(t) (and correspondingly $\bar{c}(t)$) as:

$$c(t) = e^{i\sqrt{2}t}\, B + e^{-i\sqrt{2}t}\, D\,; \quad \bar{c}(t) = e^{-i\sqrt{2}t}B^\dagger + e^{i\sqrt{2}t}D^\dagger\,; \quad (3.47)$$

which at time t = 0 imply

$$c \equiv c(0) = B + D, \quad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger \quad (3.48a)$$

$$\dot{c} \equiv \dot{c}(0) = i\sqrt{2}(B-D), \quad \dot{\bar{c}} \equiv \dot{\bar{c}}(0) = -i\sqrt{2}(B^\dagger - D^\dagger) \quad (3.48b)$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c},c\} = \{\dot{\bar{c}},\dot{c}\} = 0; \quad \{\dot{\bar{c}},c\} = i = -\{\dot{c},\bar{c}\} \quad (3.49)$$

one then obtains

$$B^2 + \{B,D\} + D^2 = B^{\dagger 2} + \{B^\dagger,D^\dagger\} + D^{\dagger 2} = 0 \quad (3.50a)$$

$$\{B,B^\dagger\} + \{D,D^\dagger\} + \{B,D^\dagger\} + \{B^\dagger,D\} = 0 \quad (3.50b)$$

$$\{B,B^\dagger\} + \{D,D^\dagger\} - \{B,D^\dagger\} - \{B^\dagger,D\} = 0 \quad (3.50c)$$

$$\{B,B^\dagger\} - \{D,D^\dagger\} - \{B,D^\dagger\} + \{D,B^\dagger\} = -1/\sqrt{2} \quad (3.50d)$$

$$\{B,B^\dagger\} - \{D,D^\dagger\} + \{B,D^\dagger\} - \{D,B^\dagger\} = -1/\sqrt{2} \quad (3.50e)$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = \{B,D\} = \{B^\dagger,D\} = \{B,D^\dagger\} = \{B^\dagger,D^\dagger\} = 0 \quad (3.51a)$$

$$\{B^\dagger,B\} = -\frac{1}{2\sqrt{2}}; \quad \{D^\dagger,D\} = +\frac{1}{2\sqrt{2}} \quad (3.51b)$$

We now let $|0>$ denote the fermionic vacuum for which

$$B|0> = D|0> = 0; \quad (3.52)$$

Defining $|0>$ to have norm one, (3.51b) implies

$$\left\langle 0|BB^\dagger|0\right\rangle = -\frac{1}{2\sqrt{2}}; \quad \left\langle 0|DD^\dagger|0\right\rangle = +\frac{1}{2\sqrt{2}} \quad (3.53)$$

so that

$$B^\dagger|0> \neq 0; \qquad D^\dagger|0> \neq 0 \quad (3.54)$$

As usual the theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{BRST}$ is, however, again irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators the Hamiltonian density is

$$\mathcal{H}_{BRST} = \frac{1}{2}\pi_k^2 + \frac{1}{2}\sigma_k'^2 + \frac{1}{2}\theta'^2 - \sigma_k'\theta' - \lambda(\sigma_k^2-1) + \lambda\,\theta(2\sigma_k-\theta) + \frac{1}{2}p_\lambda^2 + p_\lambda(\sigma_k+\theta)$$
$$+ 4(B^\dagger B + D^\dagger D) \quad (3.55)$$

and the BRST charge operator Q is

$$Q = \int dx\Big[+iB\Big[\{(\sigma_k^2-1) - \theta(2\sigma_k-\theta)\} - i\sqrt{2}\,\{p_\lambda+\pi_\theta+\pi_k-(\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big]$$
$$+ iD\Big[\{(\sigma_k^2-1) - \theta(2\sigma_k-\theta)\} + i\sqrt{2}\,\{p_\lambda+\pi_\theta+\pi_k-(\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big]\Big] \quad (3.56)$$

Now, because $Q|\psi> = 0$, the set of states annihilated by Q contains not only the set of states for which (3.19) and (3.23) hold but also additional states for which $B|\psi> = D|\psi> = 0$ and for which the conditions (3.19) and (3.23) do not hold. However, the Hamiltonian is also invariant under the anti–BRST transformation (in which the role of c and $-\bar{c}$ is interchanged) given by

$$\bar{\hat{\delta}}\sigma_k = \bar{c}, \quad \bar{\hat{\delta}}\lambda = \dot{\bar{c}}, \quad \bar{\hat{\delta}}\theta = -\bar{c}, \quad \bar{\hat{\delta}}\pi_k = 0, \quad \bar{\hat{\delta}}\pi_\theta = 0, \quad \bar{\hat{\delta}}p_\lambda = 0; \tag{3.57a}$$

$$\bar{\hat{\delta}}\bar{c} = 0, \quad \bar{\hat{\delta}}c = -b, \quad \bar{\hat{\delta}}b = 0 \tag{3.57b}$$

with generator or anti–BRST charge

$$\begin{aligned}\overline{Q} &= \int dx\Big[-i\bar{c}\{(\sigma_k^2-1) - \theta(2\sigma_k-\theta)\} + i\dot{\bar{c}}\,\{p_\lambda+\pi_\theta+\pi_k-(\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big] \\ &= \int dx\Big[-iB^\dagger\Big[\{(\sigma_k^2-1) - \theta(2\sigma_k-\theta)\} + i\sqrt{2}\,\{p_\lambda+\pi_\theta+\pi_k-(\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big] \\ &\quad -iD^\dagger\Big[\{(\sigma_k^2-1)- \theta(2\sigma_k-\theta)\} - i\sqrt{2}\,\{p_\lambda+\pi_\theta+\pi_k-(\sigma_k^2-1) + \theta(2\sigma_k-\theta)\}\Big]\Big]\end{aligned} \tag{3.58}$$

We again have $[Q,H] = 0$, and $[\overline{Q},H] = 0$, and we also impose the dual condition that both Q and $\overline{Q}$ annihilate physical states implying that

$$Q|\psi> = 0 \text{ and } \overline{Q}|\psi> = 0 \tag{3.59}$$

The states for which (3.19) and (3.23) hold strongly, satisfy both of these conditions and, in fact, are the only states satisfying both conditions since, although with (3.51)

$$4(B^\dagger B + D^\dagger D) = -4(BB^\dagger + DD^\dagger) \tag{3.60}$$

there are no states of this operator with $B^\dagger|0> = 0$ and $D^\dagger|0> = 0$ (cf. (3.54)), and hence no free eigenstates of the fermionic part of $\mathscr{H}_{BRST}$ which are annihilated by each of $B,B^\dagger,D,D^\dagger$. Thus the only states satisfying (3.59) are those satisfying the constraints (3.19) and (3.23).

Also, the states for which $p_\lambda|\psi> = 0$, $[(\pi_\theta+\pi_k) - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)]|\psi> = 0$ and $[(\sigma_k^2-1) - \theta(2\sigma_k-\theta)]|\psi> = 0$ satisfy both of these conditions (3.59) and, in fact, are the only states satisfying both of these conditions (3.59) because in view of (2.57), one can not have simultaneously $c, \dot{c}$ and $\bar{c}, \dot{\bar{c}}$ applied to $|\psi>$ to give zero. Thus the only states satisfying (3.59) are those that satisfy the constraints of the theory (3.19) and (3.23), and they belong to the set of BRST–invariant and anti–BRST–invariant states.

As in the preceeding section, one can also understand the above point in terms of annihilation and creation operators of the theory in the following way. The condition $Q|\psi> = 0$ implies that the set of states annihilated by Q contains not only the states for which $p_\lambda|\psi> = 0$, $[(\pi_\theta+\pi_k) - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)]|\psi> = 0$ and $[(\sigma_k^2-1) - \theta(2\sigma_k-\theta)]|\psi> = 0$, but also additional states for which $B|\psi> = D|\psi> = 0$, but $p_\lambda|\psi> \neq 0$, $[(\pi_\theta+\pi_k) - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)]|\psi> \neq 0$ and $[(\sigma_k^2-1) - \theta(2\sigma_k-\theta)]|\psi> \neq 0$. However, $\bar{Q}|\psi> = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which $p_\lambda|\psi> = 0$, $[(\pi_\theta+\pi_k) - (\sigma_k^2-1) + \theta(2\sigma_k-\theta)]|\psi> = 0$ and $[(\sigma_k^2-1) - \theta(2\sigma_k-\theta)]|\psi> = 0$, simply because $B^\dagger|\psi> \neq 0$ and $D^\dagger|\psi> \neq 0$. Thus in this alternative way also one finds that the states satisfying $Q|\psi> = \bar{Q}|\psi> = 0$ (i.e. (3.59)) are only those that satisfy the constraints of the theory (3.19) and (3.23) and also that these states belong to the set of BRST–invariant and anti–BRST–invariant states.

4. The Gauge–Invariant O(N) Non–Linear Sigma Model (Model B)

4A. Construction of Gauge–Invariant model (Model B) using Mitra–Rajaraman Method and its Hamiltonian Formulation

Mitra and Rajaraman [2] have constructed a gauge–invariant version of the gauge–non–invariant O(N) nonlinear sigma model $\mathcal{L}^N$ (3.1) considered in Sec. 3 (using their procedure of gauge–invariant reformulation [2] described by the total Hamiltonian density [2]:

$$\begin{aligned}\mathcal{H}_T^{\,I} &= \mathcal{H}_T^{\,N} - \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma}.\vec{\pi}) \\ &= \frac{1}{2}\pi_k^2 + \frac{1}{2}\sigma_k'^2 - \lambda(\sigma_k^2-1) + p_\lambda u - \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma}.\vec{\pi}) \qquad (4.1)\end{aligned}$$

and the associated first–order Lagrangian density [2]:

$$\mathcal{L}_{I0}^{I} = \pi_k\dot{\sigma}_k - \frac{1}{2}\pi_k^2 + \frac{1}{2}\sigma_k'^2 + \lambda(\sigma_k^2-1) + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma}.\vec{\pi}) + p_\lambda(\dot{\lambda}-u) \qquad (4.2)$$

Now (4.1) and (4.2) define the new gauge–invariant theory, namely, the model B. The Hamilton's equations obtained from the total Hamiltonian $H_T^{\,I} = \int dx \mathcal{H}_T^{\,I}$ are:

$$\dot{\sigma}_k = \frac{\partial H_T^{\,I}}{\partial \pi_k} = (\pi_k - \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right] 2\sigma_k); \quad -\dot{\pi}_k = \frac{\partial H_T^{\,I}}{\partial \sigma_k} = -\sigma_k'' - 2\lambda\sigma_k - \frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_k^2\sigma_k^2}\,[\sigma_k^2\pi_k - \sigma_k(\vec{\sigma}\cdot\vec{\pi})] \tag{4.3a}$$

$$\dot{\lambda} = \frac{\partial H_T^{\,I}}{\partial p_\lambda} = u; \quad -\dot{p}_\lambda = \frac{\partial H_T^{\,I}}{\partial \lambda} = -(\sigma_k^2 - 1) \tag{4.3b}$$

$$\dot{u} = \frac{\partial H_T^{\,I}}{\partial p_u} = 0; \quad -\dot{p}_u = \frac{\partial H_T^{\,I}}{\partial u} = p_\lambda \tag{4.3c}$$

The second–order Lagrangian density corresponding to $\mathcal{H}_T^{\,I}$ with the help of (4.3) could be written as [2]:

$$\mathcal{L}^I = \frac{1}{2}\,\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1) + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](2\sigma_k\dot{\sigma}_k) \tag{4.4}$$

As observed in Ref. [2], it is not possible to eliminiate $\vec{\pi}$ in the last term of (4.4). In view of this the authors of Ref. [2], introduce a new field (called η here) defined by [2]:

$$\eta := \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right] \tag{4.5}$$

In view of (4.5), the gauge–invariant second–order Lagrangian density equivalent to (4.4) could now be written as [2]:

$$\mathcal{L}^I = \frac{1}{2}\,\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1) + \eta(2\sigma_k\dot{\sigma}_k) \tag{4.6}$$

In the following, we would, however, work only with the gauge–invariant model B defined by (4.1) and (4.2) (and not with (4.6)).

The gauge–invariant model B defined by (4.1) and (4.2) is seen to possess two first–class constraints [2]:

$$\chi_1 = p_\lambda \approx 0 ; \tag{4.7}$$
$$\chi_2 = \sigma_k^2 - 1 \approx 0 \tag{4.8}$$

where χ_1 is a primary and χ_2 is a secondary constraint. The matrix of the Poisson brackets of χ_i is seen to be a 2×2 null matrix implying that $\mathcal{L}_{IO}^{I}$ describes a bonafide (pure) gauge– invariant theory. Further, $\mathcal{L}_{IO}^{I}$ is seen to be invariant up to a total divergence [2]:

$$\delta\mathcal{L}_{IO}^{I} = \frac{d}{dt}\,[\beta(\sigma_k^2 - 1)] \tag{4.9}$$

under the time–dependent gauge transformations [2]:

$$\delta\sigma_k = 0, \quad \delta\lambda = \dot{\alpha}(x,t), \quad \delta\eta = \alpha(x,t), \tag{4.10a}$$

$$\delta\pi_k = 2\sigma_k\alpha(x,t), \quad \delta p_\lambda = 0, \tag{4.10b}$$

where $\alpha(x,t)$ is an arbitrary function of the coordinates. The corresponding first–order action is therefore gauge–invariant.

In order to quantize the gauge–invariant theory using Dirac's procedure [12], we convert the set of first–class constraints of the theory χ_i into a set of second–class constraints, by imposing, arbitrarily, some additional constraints on the system as gauge–fixing conditions. One acceptable set of gauge–fixing conditions under which the above theory could be quantized is [2]:

$$\nu_1 = 2\sigma_k\pi_k \approx 0; \tag{4.11a}$$
$$\nu_2 = (2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'') \approx 0 \tag{4.11b}$$

Corresponding to the above choice of gauge–fixing conditions, one obtains the following set of constraints

$$\rho_1 = \chi_1 = p_\lambda \approx 0 \tag{4.12a}$$
$$\rho_2 = \chi_2 = (\sigma_k^2 - 1) \approx 0 \tag{4.12b}$$
$$\rho_4 = \nu_1 = 2\sigma_k\pi_k \approx 0 \tag{4.12c}$$

$$\rho_4 = \nu_2 = (2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'') \approx 0 \tag{4.12d}$$

The above set of constraints ρ_i is evidently identical with that of the set of constraints χ_i of the gauge–non–invariant theory $\mathcal{L}^N$ (cf. Sec. 3A). The nonvanishing equal–time Dirac brackets of the gauge–invariant theory under the gauge (4.11) are obtained to be identical with those of gauge–non–invariant theory $\mathcal{L}^N$ (3.1) and are given by (3.13) and (3.14). This is what one expects because the gauge–invariant system under the gauge (4.11) is equivalent to the system $\mathcal{L}^N$ (3.1) [2]. The main idea of the Mitra–Rajaraman method [2] lies in suitably modifying the total Hamiltonian and correspondingly the Lagrangian of a particular gauge–non–invariant theory possessing a set of second–class constraints (where at least one or more of the constraints are secondary) in such a way that all or some of the secondary constraints do not arise at all in the modified theory. The constraints of the modified theory obtained in this way then form a set of first–class constraints and consequently the resulting modified theory becomes a gauge–invariant theory. The secondary constraints which did not appear in the modified theory (but were otherwise present in the original gauge–non–invariant theory) could now be imposed on the modified (gauge–invariant) theory as gauge–fixing conditions, so that the total set of constraints again becomes a second–class set. The Dirac quantization of the modified gauge–invariant theory under such gauge–fixing conditions remains identical with that of the original gauge–non–invariant theory. Consequently the physical content of the modified gauge–invariant theory under such gauge–fixing conditions remains the same as that of the original gauge–non–invariant theory. The physical equivalence of the modified and the original theory is therefore transparent.

4B BRST Formulation of the Gauge–Invariant Theory (Model B)

4B1 The BRST Invariance

We now rewrite the gauge–invariant theory namely, the model B [2] as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we again enlarge the Hilbert space of our gauge–invariant model and replace the notion of gauge transformation by a BRST transformation (with c,$\bar{c}$ and b having the meanings as in Secs. 2 and 3) (with $\hat{\delta}^2 = 0$) such that

$$\hat{\delta}\sigma_k = 0, \quad \hat{\delta}\lambda = \dot{c}, \quad \hat{\delta}\pi_k = 2\sigma_k\, c, \quad \hat{\delta}p_\lambda = 0, \tag{4.13a}$$

$$\hat{\delta} c = 0, \qquad \hat{\delta}\,\bar{c} = b, \qquad \hat{\delta} b = 0 \tag{4.13b}$$

In this case also the transformations for the Lagrange multiplier field and its canonical momentum are not required to be specified because they are not needed. We now define a BRST–invariant function of the dynamical variables to be a function $f(\pi_k, p_\lambda, p_b, \pi_c, \pi_{\bar{c}}, \sigma_k, \lambda, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

4B2 Gauge–Fixing in the BRST Formalism

Performing gauge–fixing in the BRST formalism implies adding to the first–order Lagrangian density (4.2) a trivial BRST–invariant function. We thus write the quantum Lagrangian density (taking, e.g., a trivial BRST–invariant function as follows) [14,15]:

$$\begin{aligned} \mathcal{L}_{BRST} &= \mathcal{L}_{IO}^{I} + \hat{\delta}[\,\bar{c}(\dot{\lambda} + \tfrac{1}{2} b + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_k^2}\right])] \\ &= \pi_k \dot{\sigma}_k - \tfrac{1}{2}\pi_k^2 - \tfrac{1}{2}\sigma_k'^2 + \lambda(\sigma_k^2 - 1) + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma}\cdot\vec{\pi}) \\ &\quad + \hat{\delta}[\,\bar{c}(\dot{\lambda} + \tfrac{1}{2} b + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_k^2}\right])] \end{aligned} \tag{4.14}$$

The last term in the above equation (Eq. (4.14)) is the extra BRST–invariant gauge–fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathcal{L}_{BRST}$ (with one integration by parts):

$$\begin{aligned} \mathcal{L}_{BRST} &= \pi_k \dot{\sigma}_k - \tfrac{1}{2}\pi_k^2 - \tfrac{1}{2}\sigma_k'^2 + \lambda(\sigma_k^2 - 1) + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma}\cdot\vec{\pi}) + \frac{b^2}{2} + b\left[\dot{\lambda} + \frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_k^2}\right] \\ &\quad + \dot{\bar{c}}\dot{c} - 2\bar{c}c \end{aligned} \tag{4.15}$$

Proceeding classically, the Euler–Lagrange equation for b reads:

$$-b = \left[\dot{\lambda} + \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_k^2}\right]\right] \tag{4.16}$$

Also, the requirement $\hat{\delta} b = 0$ implies:

$$- \hat{\delta}\mathrm{b} = \left[\hat{\delta}\dot{\lambda} + \hat{\delta}\left[\frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_{\mathrm{k}}^2}\right]\right] = 0 \tag{4.17}$$

which in turn implies

$$- \ddot{\mathrm{c}} = 2\mathrm{c} \tag{4.18}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{\mathrm{BRST}}$ with respect to $\bar{\mathrm{c}}$. We define the bosonic momenta in the usual way so that

$$\mathrm{p}_{\lambda} = \frac{\partial}{\partial\dot{\lambda}}\mathcal{L}_{\mathrm{BRST}} = + \mathrm{b} \tag{4.19}$$

but for the fermionic momenta with directional derivatives, we again set

$$\pi_{\mathrm{c}} := \mathcal{L}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\dot{\mathrm{c}}} = \dot{\bar{\mathrm{c}}}\,; \quad \pi_{\bar{\mathrm{c}}} := \frac{\overrightarrow{\partial}}{\partial\dot{\bar{\mathrm{c}}}}\mathcal{L}_{\mathrm{BRST}} = \dot{\mathrm{c}} \tag{4.20}$$

implying that the variable canonically conjugate to c is $\dot{\bar{\mathrm{c}}}$ and the variable conjugate to $\bar{\mathrm{c}}$ is $\dot{\mathrm{c}}$. In forming the Hamiltonian density $\mathcal{H}_{\mathrm{BRST}}$ from the Lagrangian density in the usual way we remember that the former has to be Hermitian. Then

$$\begin{aligned}\mathcal{H}_{\mathrm{BRST}} &= \pi_{\mathrm{k}}\dot{\sigma}_{\mathrm{k}} + \mathrm{p}_{\lambda}\dot{\lambda} + \pi_{\mathrm{c}}\dot{\mathrm{c}} + \dot{\bar{\mathrm{c}}}\pi_{\bar{\mathrm{c}}} - \mathcal{L}_{\mathrm{BRST}} \\ &= \frac{1}{2}\pi_{\mathrm{k}}^2 + \frac{1}{2}\sigma_{\mathrm{k}}'^2 - \lambda(\sigma_{\mathrm{k}}^2 - 1) - \left[\frac{\vec{\sigma}\cdot\vec{\pi}}{2\sigma_{\mathrm{k}}^2}\right](\vec{\sigma}\cdot\vec{\pi}) - \frac{1}{2}\mathrm{p}_{\lambda}^2 - \mathrm{p}_{\lambda}\left[\frac{\vec{\sigma}\cdot\vec{\pi}}{\sigma_{\mathrm{k}}^2}\right] \\ &\quad + \pi_{\mathrm{c}}\pi_{\bar{\mathrm{c}}} + 2\bar{\mathrm{c}}\mathrm{c}\end{aligned} \tag{4.21}$$

We can again check the consistency of (4.20) with (4.21) by looking at Hamilton's equations for the fermionic variables, i.e.

$$\dot{\mathrm{c}} = \frac{\overrightarrow{\partial}}{\partial\pi_{\mathrm{c}}}\mathcal{H}_{\mathrm{BRST}}, \qquad \dot{\bar{\mathrm{c}}} = \mathcal{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\pi_{\bar{\mathrm{c}}}} \tag{4.22}$$

Thus

$$\dot{c} = \frac{\vec{\partial}}{\partial \pi_c} \mathscr{H}_{BRST} = \pi_{\bar{c}} ; \quad \dot{\bar{c}} = \mathscr{H}_{BRST} \frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} = \pi_c \tag{4.23}$$

are again in agreement with (4.20). The Eqs. (2.49) – (2.52) hold again in the present case.

4B3 The BRST Charge Operator

The BRST charge operator Q in this case is the generator of the BRST transformations (4.13). According to its conventional definition, its commutators with Bose operators and its anti–commutators with Fermi operators (in the present case satisfy):

$$[\pi_k, Q] = 2\sigma_k c ; \quad [\lambda, Q] = \dot{c} \tag{4.24a}$$

$$\{\bar{c}, Q\} = p_\lambda ; \quad \{\dot{\bar{c}}, Q\} = -(\sigma_k^2 - 1) \tag{4.24b}$$

All other commutators and anti–commutators involving Q vanish. In view of (4.24), the BRST charge operator of the present gauge–invariant theory can be written as

$$Q = \int dx \, \{ic(\sigma_k^2 - 1) - i\dot{c} p_\lambda\} \tag{4.25}$$

This equation implies that the set of states satisfying the condition $p_\lambda | \psi > = 0$ and $(\sigma_k^2 - 1) | \psi > = 0$ belongs to the dynamically stable subspace of states $| \psi >$ satisfying $Q | \psi > = 0$, i.e., it belongs to the set of BRST–invariant states. Also because the equation of motion (4.18) is identical with (3.39), the Eqs. (3.47) – (3.54) hold in the present case also.

In terms of annihilation and creation operators the Hamiltonian density is

$$\mathscr{H}_{BRST} = \frac{1}{2} \pi_k^2 + \frac{1}{2} \sigma_k'^2 - \lambda(\sigma_k^2 - 1) - \left[\frac{\vec{\sigma} \cdot \vec{\pi}}{2\sigma_k^2}\right](\vec{\sigma} \cdot \vec{\pi}) - \frac{1}{2} p_\lambda^2 - p_\lambda \left[\frac{\vec{\sigma} \cdot \vec{\pi}}{\sigma_k^2}\right] + 4(B^\dagger B + D^\dagger D) \tag{4.26}$$

and the BRST charge operator Q is

$$Q = \int dx\{+i[B(\sigma_k^2 - 1 - i\sqrt{2}p_\lambda) + D(\sigma_k^2 - 1 + i\sqrt{2}p_\lambda)]\} \qquad (4.27)$$

Now, because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which $p_\lambda = 0$ and $(\sigma_k^2-1) = 0$ but also additional states for which $B|\psi\rangle = D|\psi\rangle = 0$ with $p_\lambda \neq 0$ and $(\sigma_k^2-1) \neq 0$. However, the Hamiltonian is also invariant under the anti–BRST transformation (in which the role of c and $-\bar{c}$ is interchanged) given by

$$\hat{\bar{\delta}}\sigma_k = 0, \quad \hat{\bar{\delta}}\lambda = -\dot{\bar{c}}, \quad \hat{\bar{\delta}}\pi_k = -2\sigma_k\bar{c}, \quad \hat{\bar{\delta}}p_\lambda = 0, \qquad (4.28a)$$

$$\hat{\bar{\delta}}\bar{c} = 0, \quad \hat{\bar{\delta}}c = -b, \quad \hat{\bar{\delta}}b = 0 \qquad (4.28b)$$

with generator or anti–BRST charge

$$\bar{Q} = \int dx\{-i\bar{c}(\sigma_k^2-1) + i\dot{\bar{c}}p_\lambda\}$$

$$= \int dx\{-i[B^\dagger(\sigma_k^2 - 1 + i\sqrt{2}p_\lambda) + D^\dagger(\sigma_k^2 - 1 - i\sqrt{2}p_\lambda)]\} \qquad (4.29)$$

In this case also $[Q,H] = 0$ and $[\bar{Q},H] = 0$, and as in the previous cases, we again impose the dual condition that both Q and $\bar{Q}$ annihilate physical states implying that

$$Q|\psi\rangle = 0 \text{ and } \bar{Q}|\psi\rangle = 0 \qquad (4.30)$$

The states for which $p_\lambda = 0$ and $(\sigma_k^2-1) = 0$ satisfy both of these conditions and, in fact, are the <u>only</u> states satisfying both conditions since, although with (3.51)

$$4(B^\dagger B + D^\dagger D) = -4(BB^\dagger + DD^\dagger) \qquad (4.31)$$

there are no states of this operator with $B^\dagger|0\rangle = 0$ and $D^\dagger|0\rangle = 0$ (cf. (3.54)), and hence no free eigenstates of the fermionic part of $\mathcal{H}_{BRST}$ which are annihilated by each of $B, B^\dagger, D, D^\dagger$. Thus the only states satisfying (4.30) are those satisfying the constraints $p_\lambda = 0$ and $(\sigma_k^2 - 1) = 0$.

Also, the states for which $p_\lambda|\psi\rangle = 0$ and $(\sigma_k^2-1)|\psi\rangle = 0$ satisfy both of these conditions (4.30) and, in fact, are the only states satisfying both of these conditions (4.30) because in view of (2.57), one cannot have simultaneously $c, \dot{c}$ and $\bar{c}, \dot{\bar{c}}$ applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.30) are those that satisfy the constraints of the theory (4.8), and they belong to the set of BRST–invariant and anti–BRST–invariant states.

Once again, one can understand the above point in terms of annihilation and creation operators of the theory as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which $p_\lambda|\psi\rangle = 0$ and $(\sigma_k^2-1)|\psi\rangle = 0$, but also additional states for which $B|\psi\rangle = D|\psi\rangle = 0$, but $p_\lambda|\psi\rangle \neq 0$ and $(\sigma_k^2-1)|\psi\rangle \neq 0$. However, $\overline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which $p_\lambda|\psi\rangle = 0$ and $(\sigma_k^2-1)|\psi\rangle = 0$, simply because $B^\dagger|\psi\rangle \neq 0$ and $D^\dagger|\psi\rangle \neq 0$. Thus in this alternative way once again we see that the states satisfying $Q|\psi\rangle = \overline{Q}|\psi\rangle = 0$ (i.e. (4.30)) are only those that satisfy the constraints of the theory (4.8) and also that these states belong to the set of BRST–invariant and anti–BRST–invariant states.

5. Summary and Discussion

The transition to quantum mechanics is made in general, by the replacement of the Dirac brackets by the operator commutation relations [,], according to the Dirac quantization rule [12]:

$$\{A,B\}_D \longrightarrow (-i)\,[A,B] \tag{5.1}$$

where the classical dynamical variables A and B after quantization become quantum mechanical operators on some Hilbert space. In view of this, the equal–time commutators for the Klein–Gordon theory considered in Sec. 2, can be obtained immediately from the corresponding Dirac brackets by the above replacement (namely, using (5.1)).

For achieving the canonical quantization of the non–linear sigma model, we encounter the problem of operator ordering while going from Dirac brackets to commutation relations. This problem can be resolved, as explained in Ref. [1,18] by demanding that all the fields e.g., $\sigma_k(x)$, $\lambda(x)$, and $\theta(x)$; and all the canonical momenta e.g., $\pi_k(x)$, $p_\lambda(x)$ and $\pi_\theta(x)$ are now hermitian operators and that all the canonical commutation relations be consistent with the hermiticity of these operators [1,18].

In the usual Hamiltonian formulation of a gauge–invariant theory (like the ones considered in the present work) under some gauge–fixing conditions, one necessarily destroys the gauge–invariance of the theory. However, in the BRST formulation when we imbed a gauge–invariant theory into a BRST–invariant system, the new (BRST) symmetry which replaces the gauge invariance is maintained even under gauge–fixing and hence projecting any state onto the sector of BRST and anti–BRST–invariant states yields a theory which is isomorphic to the original gauge–invariant theory. The unitarity and consistency of the BRST–invariant theory described by $\mathcal{L}_{BRST}$ is guaranteed by the conservation and nilpotency of the BRST charge Q.

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