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On the Covering Property in Physical Theories

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Abstract. A class of quantum logics (orthomodular atomic lattices) is defined and studied. The obtained results suggest that the covering property acts as a selection criterion for theories in the sense that, given a set of orthomodular atomic lattices, those which may be theories for a fixed set of systems may be selected by verifying if they have the covering property or not. A relation between physical meanings of the covering property, lattice - algebraic operations and the commutativity relation is also suggested.

1 Introduction

An important problem which appears when a physical theory is considered to be an orthomodular atomic lattice having the covering property is to find physical interpretations for its basic properties [1]. In this work we will define and study a class of orthomodular atomic lattices - called in our text quantum logics - which permits to discuss some interesting problems concerning the structure of physical theories. We have in view the fact that the mathematical properties of this class of quantum logics makes much easier the understanding of difficulties appearing in the interpretation of the axioms of physical theories. To be more precise, the difficulties which will be discussed refer to the interpretation of the covering property, of the lattice-algebraic operations ("meet" and "join") and of the commutativity relation as describing the empirical compatibility of "yes-no" experiments (tests). The quantum logics studied in our paper suggest a possible connection between these objects and also some ways for obtaining their physical interpretation. It is important to mention here that orthomodularity and atomicity may be considered as being physically meaningful properties of physical theories [2-9].This justifies the fact that we discuss the mentioned problems by starting with a class of atomic orthomodular lattices.

In Paragraph 2 we will define a class - denoted by $\mathbb{T}$ - of orthomodular lattices and will prove some interesting results concerning it which will be the basis of our physical discussion. In this discussion will be used also the following transparent facts (part of them being, in a sense, proved):

(a) any observable - considered as an independent on any theory object - may be described by an appropriate Boolean algebra [8];

(b) the only relation on an arbitrarily given orthomodular atomic lattice L which might describe the empirical compatibility of tests ("yes-no" experiments) represented by the elements of L is the commutativity relation on L, [5];

(c) two observables which are empirically compatible (incompatible) must be also compatible (incompatible) in any physically admissible theory.

Concerning point (c) we must notice that in its formulation we understand by **observable a given model of that observable**, i.e. Boolean algebra. In other words, if the Boolean algebra B_ω is considered to be the model of the observable ω, then in any theory L the observable ω must be represented by a Boolean sublattice of L which is isomorphic to B_ω. At first sight the example of observables position and impulse contradicts the statement (c) because the mentioned observables are compatible in classical mechanics and incompatible in quantum mechanics. In fact this phenomenon appears because these two theories use different models for the observables in question, i.e. a classical model in classical mechanics and a nonclassical model in quantum mechanics [8].

In Paragraph 3 we will discuss the "physical" implications of the properties of the quantum logics -elements of $\mathbb{T}$ - studied in Paragraph 2. The most important conclusions of this discussion are the following:

(i) The commutativity relation cannot describe correctly the empirical compatibility in all elements of $\mathbb{T}$.
(ii) As a consequence of point (i) the lattice-algebraic operations have not a clear physical interpretation in all elements of $\mathbb{T}$.
(iii) The covering property seems to be a criterion for selecting the "good" theories from a given family of quantum logics, i.e. those quantum logics which may describe a given class of physical systems. This fact, if it is true, appears to be related with the correct description of the empirical compatibility by the commutativity relation in a quantum logic.

2 A class of theories describing systems of two species of particles

We will define first the class $\mathbb{T}$ of quantum logics which was mentioned in Introduction. The elements of $\mathbb{T}$ are possible theories for describing systems of two species of particles, so that we will use often the term **theory** instead of quantum logic.

Let us consider Π_1, Π_2 two species of structureless particles. We characterize each of the species Π_i $(i = 1, 2)$ by an observable B_i - corresponding to the physical quantity "number of particles of the species Π_i" - which is a Boolean algebra isomorphic to the Boolean algebra of all subsets of $\mathbf{N} = \{0, 1, 2,...\}$. We may write, for instance, $B_i = \{(A,i);\ A \subseteq \mathbf{N}\}$, define the order on B_i by $(A,i) \leq_i (A',i) \Leftrightarrow A \subseteq A'$ and the orthocomplementation by $(A,i)^{\perp} = (\mathbf{N} - A, i)$.

The Boolean product $C = B_1 \times B_2$ is obviously a theory having B_1, B_2 as observables, [8]. It is not difficult to observe that C is a theory which, like phenomenological thermodynamics, works with physical quantities supposed to depend on the number of particles of the species Π_1 and Π_2 only, the values of other possible parameters being fixed. For the sake of convenience we will identify C with the set of all subsets of $\mathbf{N} \times \mathbf{N}$. We will consider also other theories which are able to describe - at least in principle - systems of particles of the species Π_1, Π_2. These theories are strongly connected with C in the sense that any such theory is isomorphic to a so-called quasisublattice of C having B_1, B_2 as observables.

> **Definition 1. Any subset of C which is an orthomodular atomic lattice with the order and orthocomplementation inherited from C, will be called a quasisublattice of C.**

If L is a quasisublattice, we will denote by $a \vee_L b$, $a \wedge_L b$, $(a,b)K_L$ the join, meet and commutativity/compatibility of $a,b \in L \subseteq C$ in L, respectively [13]. The simplest example of quasisublattice interesting for us is $S = \{A \times \mathbf{N};\ A \subseteq \mathbf{N}\} \cup \{\mathbf{N} \times A;\ A \subseteq \mathbf{N}\}$. The elements of S which are of the type $\{n\} \times \mathbf{N}$ or $\mathbf{N} \times \{n\}$, $n \in \mathbf{N}$, will be called lines. The elements of $\mathbf{N} \times \mathbf{N}$ will be called points. The set of all atoms of an orthomodular lattice U will be denoted by $\Omega(U)$. We define now the family of theories denoted by $\mathbb{T}$.

> **Definition 2. $T \in \mathbb{T}$ if T is isomorphic to a quasisublattice $L \subseteq C$ having the following two properties:**
>
> **(i) $S \subseteq L$;**
>
> **(ii) for any $\alpha \in \Omega(L)$ there exists a line l such that $\alpha \subseteq l$.**

It is obvious that condition (i) states the fact that B_1, B_2 are observables of the theory T. Concerning condition (ii), we may say that it has a more or less technical character. Besides, for our purposes, it is obviously not necessary to consider quasisublattices which are not elements of $\mathbb{T}$.

It is interesting to see if there exists a physical motivation for considering an example like $\mathbb{T}$. And such a motivation exists. It is based on the fact that any test consists in a set of logically equivalent propositions and a set of contacts (ideal measurements) for measuring these propositions [5]. Taking into account this fact, we might assume that not all elements of C correspond to tests, i.e. to measurable propositions. For instance, we may have reasons to accept that it is impossible to determine by the same experiment the exact number of particles of species Π_1, Π_2 existing in a system. In this case the one-element sets $\{(n,m)\}$, $n,m \in \mathbf{N}$, do not correspond to tests and they cannot appear in a theory for systems of species Π_1, Π_2, so that a theory from $\mathbb{T} - \{C\}$ might be used instead of C.

We will prove now a set of mathematical results concerning quantum logics which are elements of $\mathbb{T}$.

Proposition 1. $(\{n\}\times \mathbf{N},\ \mathbf{N}\times\{m\})K_L$, **if and only if** $\{(n,m)\}\in L$.

Proof. Suppose first that $(\{n\}\times \mathbf{N},\ \mathbf{N}\times\{m\})K_L$. Obviously, $\{n\}\times \mathbf{N} \ \wedge_L\ \mathbf{N}\times\{m\}$ is $\{(n,m)\}$ or $\varnothing$. The second situation is excluded since from
$\{n\}\times \mathbf{N} = (\{n\}\times \mathbf{N}\wedge_L \mathbf{N}\times\{m\})\ \vee_L\ (\{n\}\times \mathbf{N}\wedge_L(\mathbf{N}\times\{m\})^{\perp})$ we would obtain
$\{n\}\times \mathbf{N} = \{n\}\times \mathbf{N}\wedge_L(\mathbf{N}\times\{m\})^{\perp}$, that is $\{n\}\times \mathbf{N} \subseteq \mathbf{N}\times(\mathbf{N}-\{m\})$, which is impossible.

Conversely, if $\{n,m\} \in L$, we have $\{n\}\times \mathbf{N}\wedge_L \mathbf{N}\times\{m\} = \{(n,m)\}$. By using the notations $a = \{n\}\times \mathbf{N}$, $b = \mathbf{N}\times\{m\}$, $c = \{(n,m)\}$, we define $a_1 = a\wedge_L c^{\perp}$, $b_1 = b\wedge c^{\perp}$.
Then: $a_1 \subseteq \{n\}\times \mathbf{N}\bigcap(\mathbf{N}\times \mathbf{N} - \{(n,m)\}) = \{n\}\times(\mathbf{N} - \{m\})$,
$b_1 \subseteq \mathbf{N}\times\{m\}\bigcap(\mathbf{N}\times \mathbf{N} - \{(n,m)\}) = (\mathbf{N} - \{n\})\times\{m\}$,
so that $(a_1,b_1)\perp$, $(a_1,c)\perp$ and $(b_1,c)\perp$. Finally, since $(a,c)K_L$, $(c,c^{\perp})K_L$, we may write $a_1 \vee_L c = (a\wedge c^{\perp})\vee_L c = a\vee_L c = a$, etc. It results $(\{n\}\times \mathbf{N},\ \mathbf{N}\times\{m\})K_L$,

Q.E.D.

By using similar technics, we may prove also

Proposition 2. If $a \subseteq \{n\} \times \mathbf{N}$, $a \in L$, $(n,m) \in a$, $\{(n,m)\} \notin L$, **then** a **and** $\mathbf{N} \times \{m\}$ **are not compatible.**

Proposition 3. If $a,b \in L$, $a,b \subseteq \{n\} \times \mathbf{N}$, **then** $a \vee_L b = a\bigcup b$.

Proof. Suppose that $a \vee_L b - a\bigcup b \ni (n,m)$. If $\{(n,m)\} \in L$, then we get the absurd conclusion that $\{(n,m)\} \subseteq a \vee_L b$ and $(\{(n,m)\}, a \vee_L b)\perp$. If $\{(n,m)\} \notin L$, then from Proposition 2 we get that $a \vee_L b$ and $N\times\{m\}$ are not compatible. On the other hand, $(N\times\{m\}, a)\perp$, $(N\times\{m\}, b)\perp$ and we obtain again a contradiction. It results that $a \vee_L b = a \bigcup b$,

Q.E.D.

Proposition 4. If $\alpha,\beta \in \Omega(L)$, $\alpha,\beta \subseteq \{n\}\times N$ and $\alpha \bigcap \beta \neq \emptyset$, then $\alpha = \beta$.

Proof. Suppose that $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$. Then $\alpha \vee_L \beta \notin \Omega(L)$ and, since L is orthomodular, we can find $\gamma \in L$, $(\gamma,\alpha)\perp$ such that $\alpha \vee_L \gamma = \alpha \vee_L \beta$. From Proposition 3 we get $\alpha \bigcup \gamma = \alpha \bigcup \beta$ and, since $\gamma \bigcap \alpha = \emptyset$, we find $\gamma \subseteq \beta$. On the other hand, $\alpha \bigcap \beta = \emptyset$, so that $\gamma \neq \beta$, which is absurd. Consequently $\alpha = \beta$,

Q.E.D.

Proposition 5. Any line has an unique orthogonal decomposition from atoms of L.

Proof. Let $A, A' \subseteq \Omega(L)$ be two orthogonal decomposition of a given line. If $\alpha \in A$, then it is obvious that there exists $\alpha' \in A'$ such that $\alpha \bigcap \alpha' \neq \emptyset$. From Proposition 4 we get $\alpha = \alpha'$, so that $A \subseteq A'$.

Q.E.D.

Let us consider the theories C and S. If $\mathbb{T}$ is considered an ordered by inclusion set, it is obvious that C and S are the largest, respectively the smallest elements of $\mathbb{T}$. It is also clear that, among theories of $\mathbb{T}$, C has the smallest and S the largest atoms. Finally, it may be easily verified that C satisfies and S does not satisfy the covering property. These facts suggest that the smaller the atoms of a theory $L \in \mathbb{T}$ are, the higher is the chance of L to satisfy the covering property. The next theorem reflects clearly enough that such an assumption might be true.

Theorem. Let $L \in \mathbb{T}$ be a theory such that there exists an atom $\alpha \in \Omega(L)$ containing more than two points. Then L has not the covering property.

Proof. Consider $\alpha \in \Omega(L)$, $\alpha \subseteq \{n\}\times N$ and $(n,m), (n,l), (n,k) \in \alpha$ $(m \neq l \neq k)$. The line $N\times\{m\}$ includes an atom β such that $(n,m) \in \beta$. The element $\alpha \vee_L \beta$ includes strictly the set $\alpha \bigcup \beta$. Indeed, the only atoms contained in $\alpha \bigcup \beta$ are α and β and since α, β are not orthogonal, $\alpha \vee_L \beta = \alpha \bigcup \beta$ would mean that $\{\alpha\}$ (or $\{\beta\}$) is an orthogonal decomposition into atoms of $\alpha \vee_L \beta$, which is absurd. It results that there exists $\beta_1 \in \Omega(L)$, $(\beta_1,\beta)\perp$, $\beta_1 \subseteq \alpha \vee_L \beta$. The relation $(\beta_1,\alpha)\perp$ cannot be true since in this situation we would obtain $\beta_1 = \emptyset$. Therefore $\beta_1 \bigcap \alpha \neq \emptyset$ and, since β_1 is contained in a line, we have two possibilities:

(i) $\beta_1 \subseteq \{n\} \times \mathbf{N}$;

(ii) β_1 is included in a line which is parallel with $\mathbf{N} \times \{m\}$ and has a nonempty intersection with α.

The situation (i) is excluded since $\beta_1 \cap \alpha \neq \emptyset \Rightarrow \beta_1 = \alpha$ (Proposition 4) and we know that $(\alpha,\beta)\perp$. It results that β_1 satisfies (ii), so that it contains exactly one point of α, for instance the point (n,l). Since $\beta_1, \beta \subseteq (\mathbf{N} \times \{k\})^\perp$, we get $\beta_1 \vee_L \beta \subseteq (\mathbf{N} \times \{k\})^\perp$. This means that $\{\beta_1, \beta\}$ is not an orthogonal decomposition of $\alpha \vee_L \beta$ and an atom $\beta_2 \ni (n,k)$, $\beta_2 \subseteq \alpha \vee_L \beta$, $(\beta_2, \beta)\perp$ must exist. It results that $\beta \subsetneq \beta \vee_L \beta_1 \subsetneq \alpha \vee_L \beta$ and L does not satisfy the covering property,

Q.E.D.

We will give now an example of such a theory whose atoms have not more than two points and which has the covering property.

Let us consider the quasisublattice $F \neq C$ having the following atoms: $\alpha = \{(0,0), (0,1)\}$, $\beta = \{(0,0), (1,0)\}$, $\gamma = \{(0,1), (1,1)\}$, $\delta = \{(1,0), (1,1)\}$, $\{(n,m)\}$ for all $n,m \neq 0, 1$. Obviously, if $a \in F$ contains two different atoms from the set $\{\alpha, \beta, \gamma, \delta\}$, then $H = \{(0,0), (0,1), (1,0), (1,1)\} \subseteq a$. The quasisublattice F will be defined as the set of all subsets of $\mathbf{N} \times \mathbf{N}$ which are of the form $\bigcup \sigma$, where $\sigma \subseteq \Omega(F)$ is a set of mutually orthogonal atoms. In order to prove that F is a quasisublattice, let us note that any $a \in F$ may be written in the form $a = A \cup P_a$, where $A \subseteq H$, $P_a \subseteq P \equiv \mathbf{N} \times \mathbf{N} - H$. For proving the implication $a \in F \Rightarrow a^\perp \in F$ it is sufficient to consider the case $a = \alpha \cup P_a$. Since $a^\perp = \alpha^\perp \cap P_a^\perp$, $\alpha^\perp = \gamma \cup P$ and $P_a^\perp = H \cup \bar{P}_a$ ($\bar{P}_a = P_a^\perp = H$), we get $a^\perp = \gamma \cup \bar{P}_a \in F$. On the other hand, we may write $a \vee_F b = (A \vee_F B) \cup (P_a \cup P_b)$, where $a = A \cup P_a$, $b = B \cup P_b$, $A \vee_F B = H$ if $A \neq B$ and $A \vee_F B = A$ if $A = B$. It is almost obvious that $a \vee_F b = \sup_F\{a,b\}$, so that F is indeed a quasisublattice. It may be proved also without difficulty that F has the covering property.

3 Comments

The theorem proved in Paragraph 2 suggests that a theory which correctly describes the considered systems must satisfy the covering property. Indeed, it has been seen that any quantum logic from $\mathbb{T}$ may describe, in principle, the systems constituted from particles of two species. Therefore, it appears as necessary to find some criterions able to select those elements of $\mathbb{T}$ which are "good" theories in the sense that they correctly describe at least some essential facts concerning the mentioned systems. In our opinion the theorem from Paragraph 2 suggests that the covering property might be such a criterion and we will adopt this hypothesis (it is known that, in spite of some attempts to interpret the covering property, its meaning still remains quite unclear, [10]). Once this hypothesis accepted, we have to establish if the covering property is able to select only "good" theories from $\mathbb{T}$.

The example F constructed in Paragraph 2 leads to the conclusion that the covering property is not a sufficient criterion for choosing a "good" theory from the family $\mathbb{T}$. In other words, there are theories from $\mathbb{T}$ - i.e. orthomodular atomic lattices satisfying the covering property - which do not describe any system of particles of the species Π_1, Π_2. This fact results from the simple observation that the observables B_1, B_2 are compatible in C and incompatible in F, which contradicts the acception (c) from Introduction. Although such a result was to be expected, it is nontrivial. Indeed, given $T \in \mathbb{T}$ a theory, it is, in principle, impossible to decide if T describes a system from the considered class, simply because we have no possibility "to control" all these systems. Similarly, it is impossible to decide if the pair (B_1, B_2) (or another arbitrarily given pair of observables) is compatible or not. Usually, in such cases we take into account the available experimental information concerning the observables in question and postulate their compatibility or incompatibility.

Our reasoning does not depend on these difficulties. Indeed, we prove that there are two theories $C, F \in \mathbb{T}$, each of them satisfying the covering property and such that at least one of them is not correct: if B_1, B_2 are supposed to be compatible (incompatible), then at least F(C) is not correct. Besides it is clear, even if nonexplicitly stated, that the images of the observables B_1, B_2 in theories of the family $\mathbb{T}$ are obtained by using the same "physical rules", which confirms once more the correctness of our conclusion.

In what follows the hypothesis that any "good" theory must satisfy the covering property will be used for discussing some interesting problems concerning compatibility relation and "meets" and "joins" in physical theories.

It has been seen that there are tests $a,b \in S \subseteq C$ which are incompatible in S. On the other hand, the tests a,b are obviously compatible in C. Taking account of the point (c), it follows that K_S or K_C do not describe correctly the empirical compatibility. Since S does not satisfy the covering property, we draw the conclusion that K_S is a wrong description for the empirical compatibility. It is natural to ask ourselves what is the origin of the fact that, although K_S is the only relation which might describe the empirical compatibility on S, it is not able to do this. Since $(a,b)K_S \Leftrightarrow a = (a \wedge_S b) \vee_S (a \wedge_S b^{\perp})$, it seems normal to accept that, in S, the join " $\vee_S$" and the meet " $\wedge_S$" are not physically meaningful for all pairs of elements of the theory S.

This is an interesting and important conclusion, which deserves a special attention. It is clear that the physical meaning of joins and meets of tests is in close connection with compatibility. Indeed, if we know that the tests of a theory are empirically compatible, then there are no problems to define for them a physically meaningful join/meet and admit that they are elements of any theory having those tests as elements, [5]. In the other cases, the meaning of the existence of meets and joins becomes quite obscure. In such situations it seems that the lattice operations " $\wedge$" and " $\vee$" are merely technically useful objects. Taking account of the relation which seems to exist between compatibility and the covering property, we might assume that, if a theory has the covering property, then the meets and joins are physically meaningful objects.

Unfortunately this assumption cannot be true since we know that $F \subset C$, F and C satisfy the covering property but in F there are incompatible pairs of tests so that at least in one of the theories F, C the empirical compatibility is not correctly described. Consequently, even if a theory has the covering property, we cannot affirm that its meets and joins have a physical meaning. A possible solution of this dilemma is to assume that any physically admissible theory - i.e. a theory which describes correctly a nonempty set of systems - is a subtheory of a classical theory (a Boolean atomic algebra), i.e. it is isomorphic to a quasisublattice of a classical theory (see Definition 1, where C may be changed by any other theory, classical or not). If this is so, then it results that incompatibility is simply a consequence of the fact that we unconsciously ignore certain tests when we try to describe physical systems in terms of nonclassical theories.

In fact such a point of view means nothing but to consider, in a sense, the existence of the so-called hidden variables. This very attractive hypothesis is not true. Indeed, it has been proved that, at least in this language, hidden variables do not exist: a well known result affirms that orthomodular lattices of projectors in Hilbert spaces of dimension greater than 3 cannot be "embedded" as quasisublattices into a classical theory [14]. We will present in Appendix a quite simple proof of this important result, which uses other technics than the above mentioned work [14]. Since it is well known that the Hilbert-space theory describes correctly a lot of experimental facts, it becomes clear from this theorem that the problem of relations between compatibility, existence of joins and meets and the covering property remains still open. Nevertheless, it seems to be true that such relations exist and meets, joints and commutativity become physical objects only in those theories which satisfy the covering property.

Appendix

Let L, L' be two theories. We say that L is a subtheory of L' if there exists φ: L $\rightarrow$ L' a mapping having the following properties:

(s1) $a \leq b \leftrightarrow \varphi(a) \leq \varphi(b)$;

(s2) **$\varphi(a^{\perp}) = \varphi(a)^{\perp}$;**

(s3) **if $(a_i)_{i\in I}$ is a family of mutually compatible elements of L such that $\bigvee_i a_i$ exist, then $\bigvee_i \varphi(a_i)$ exists in L' and $\varphi(\bigvee_i a_i) = \bigvee_i \varphi(a_i)$.**

It is easy to prove that a mapping which satisfies (s1)-(s3) is injective and "conserves" the compatibility, i.e. $(a,b)K_L \Rightarrow (\varphi(a), \varphi(b))K_{L'}$.

Consider now L an orthomodular atomic lattice having the covering property. We will assume also that L $\neq$ {0,1}, is irreducible and satisfies the following requirements.

(t1) **for any $\alpha \in \Omega(L)$ there exists a unique state $p:L \to [0,1]$ such that $p(\alpha) = 1$;**

(t2) **if p is a state on L and $\alpha,\beta \in \Omega(L)$, then $p(\alpha) = p(\beta) = 1 \Rightarrow \alpha = \beta$.**

The unique state taking the value 1 on the atom $\alpha \in \Omega(L)$ will be denoted by δ_α. The following important **lemma** is true: **for any $a \in L$ there exists $\beta \in \Omega(L)$, $\beta \leq a$ such that $\delta_\alpha(a) = \delta_\alpha(\beta)$**, [15].

We will prove now that **L cannot be a subtheory of a classical theory**. Indeed, let $\varphi:L \to B$ a mapping satisfying (s1)-(s3) and B a classical theory. Let $\alpha \in \Omega(L)$ be an arbitrarily fixed atom. Then there exists $\beta \in \Omega(B)$ such that $\beta \leq \varphi(\alpha)$. We see that the composition $\delta_\beta \circ \varphi$ is a state on L which takes the value 1 on α, so that from (t1) we get $\delta_\alpha = \delta_\beta \circ \varphi$. Moreover, since all possible values of δ_β are 0 and 1, the state δ_α has the same property. If $\alpha' \in \Omega(L)$, $\alpha' \neq \alpha$, then $\delta_\alpha(\alpha') = 0$. Indeed $\delta_\alpha(\alpha') \neq 0 \Rightarrow \delta_\alpha(\alpha') = 1 \overset{(t2)}{\Longrightarrow} \alpha = \alpha'$. Therefore, $\alpha \neq \alpha' \Rightarrow \delta_\alpha(\alpha') = 0 \Rightarrow \delta_\alpha(\alpha'^{\perp}) = 1 \overset{lemma}{\Longrightarrow} \exists \gamma \in \Omega(L)$, $\gamma \leq \alpha'^{\perp}$, $\delta_\alpha(\gamma) = 1 \Rightarrow \alpha = \gamma \Rightarrow (\alpha,\alpha')\perp$. By using this result we get immediately $(\alpha,a)K_L$ for all $\alpha \in L$ and, since $\alpha \neq (0,1)$, L is reducible, which is impossible.

It is easy to verify that, given $\mathcal{H}$ a Hilbert space, dim $\mathcal{H} \geq 3$, the lattice of all its projectors satisfies (t1), (t2), so that it cannot be a subtheory of a classical theory.

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