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Quantum Logic Without Negation

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Abstract. The algebraic tools based on generating semigroups are suggested to describe property lattices possessing the relation of exclusivity rather than the operation of negation. The reduction to standard situation of orthocomplemented and orthomodular lattices is described. As an example of non-orthocomplementable property lattice that of a hypothetical "topologymeter" is studied.

Introduction

The quantum logical approach to quantum mechanics suggests to consider the collection of properties of a physical system as primary object in the mathematical description of the system.

Usually the collection of properties is assumed to form an orthocomplemented orthomodular lattice [1]. It was shown by Piron [2] that under certain additional algebraic assumptions this lattice is isomorphic to the projector lattice of a Hilbert space. The algebraic machinery provided by quantum logic (see [1] for an account) allows to deal with a more general class of collections of properties not necessarily representable by projectors.

These algebraic activities are nevertheless called quantum *logic* since the lattice operations are interpreted as logical connectives. Namely, the lattice joins, meets and orthocomplements are thought of as disjunction, conjunction and negation, respectively. Although, the recent treatise of Garola [3] shows that, unlike situations described by classical mechanics, the difference must be drawn between true (or potentially true) and testable properties.

Within the conventional quantum mechanics all properties are described by closed projectors in the state space of a system and are testable. Passing to a more general situation and keeping the same logical approach, one could adopt the existence of systems whose lattices of *testable* properties may not be representable by projectors. Moreover, it could be even adopted that there is no operation of negation on properties (an example will be adduced in section 4).

It is just the case that will be tackled in this paper: the collection $\mathcal{L}$ of properties of an object will be considered primary essence given *ab initio*. It will be assumed that $\mathcal{L}$ possesses the structure of complete atomistic coatomistic (CAC) lattice. The algebraic tools based on generating semigroups [4] (which are, in turn, the generalization of Foulis semigroups of orthomodular lattices [5]) are suggested to represent these lattices. To link this machinery with the well-known quantum logical picture, the following properties of $\mathcal{L}$ are expressed in terms of generating semigroups:

- orthocomplementability (section 2)
- orthomodularity (section 3)

Finally, the example of a "device" whose property lattice admits no negation and thus can not be described by standard quantum logical means is adduced in section 4. This is a hypothetical "topologymeter" whose pointer is labelled by the topologies of a finite space [7], [8].

1 Basic Definitions and Results

The notions of CAC lattice and generating semigroup are introduced in this section. The restoring theorem for Rees matrix semigroups is adduced.

Definition. An *atom* of a lattice L is a minimal proper element, denote the set of atoms by V:

$$v \in V \text{ means } \forall a \in L a \leq v \quad \text{implies} \quad a = 0 \text{ or } a = v$$

dually, the set Λ of coatoms is defined:

$$\lambda \in \Lambda \text{ means } \forall a \in L \lambda \leq a \quad \text{implies} \quad a = I \text{ or } a = \lambda$$

where $0, I$ are the least and the greatest elements of L, respectively.

Definition. A lattice L is called *CAC* whenever it is:

- **C). Complete.** *all meets and joins do exist* $\forall A \subseteq L \exists b \in L \mid b = \vee A$
- **A). Atomistic.**
$$\forall a \in L a = \vee\{v \in V \mid v \leq a\} \tag{1.1}$$
- **C). Coatomistic.**
$$\forall a \in L a = \wedge\{\lambda \in \Lambda \mid a \leq \lambda\} \tag{1.2}$$

Let S be a semigroup with zero 0. For any subset $A \subseteq S$ its *left (right) annihilator* ${}^0A$ (resp., A^0) is:

$${}^0A = \{s \in S \mid \forall a \in A sa = 0\}$$

$$A^0 = \{t \in S \mid \forall a \in A at = 0\}$$

Denote by $\Gamma_L(S)$ (resp., $\Gamma_R(S)$) the collection of all left (resp., right) annihilators in S, that is, the collection of all subsets of S of the form ${}^0A$ (resp., A^0). Both $\Gamma_R(S)$ and $\Gamma_L(S)$ are complete lattices (I omit the symbol S when no ambiguity occurs). The partial order in these lattices is the set inclusion; the meets are set intersections. The pair of mappings $A \to A^0$ and $B \to^0 B$ establishes the canonical anti-isomorphism between $\Gamma_L(S)$ and $\Gamma_R(S)$. All these facts follow from the general polarity construction (see, e.g. [9]).

Let $\mathcal{L}$ be a complete lattice. A semigroup S is said to be *generating* for $\mathcal{L}$ whenever the left annihilator lattice $\Gamma_L(S)$ is isomorphic to $\mathcal{L}$:

$$\mathcal{L} \simeq \Gamma_L(S)$$

The class of semigroups within which the generating one for an arbitrary CAC lattice can be found is the class of Rees matrix semigroups over a trivial group. Recall the necessary definitions [10].

Definition. Let V,Λ be two non-empty sets. A *Rees matrix* over the trivial group is any $V \times \Lambda$ matrix $A : V \times \Lambda \to \{0, 1\}$ having **at most** one non-zero entry.

So, any non-zero Rees matrix A can be unambiguously described by pointing out this non-zero entry:

$$A = 1_{v\lambda}$$

The zero matrix 0 is also assumed to be Rees.

Now fix up an arbitrary (not Rees, in general) $V \times \Lambda$ 0, 1-matrix S.

Definition. A *Rees semigroup* $\mathcal{T}(V, \Lambda, S)$ over the trivial group with the *sandwich matrix* S is the collection of all Rees $V \times \Lambda$ matrices with the product defined as

$$A * B = ASB \tag{1.3}$$

In matrix terms the definition (2.1) turns to

$$1_{v\lambda} * a_{v'\lambda'} = S_{\lambda v} a_{v\lambda'} = \begin{cases} 1_{v\lambda'} & \text{if } P(\lambda, v') = 1 \\ 0 & \text{otherwise} \end{cases}$$

The zero matrix 0 is the zero element of $\mathcal{T}(V, \Lambda, S)$. So, whenever the sets V, Λ and the sandwich matrix $S(v, \lambda)$ are fixed up, the semigroup $\mathcal{T}(V, \Lambda, S)$ is unambiguously defined.

Now let $\mathcal{L}$ be an arbitrary CAC lattice. Denote by V (resp., Λ) the set of atoms (resp., coatoms) of $\mathcal{L}$.

Theorem 1. Let $\mathcal{T}(V, \Lambda, S)$ be the defined above Rees semigroup with the following sandwich matrix S:

$$S(\lambda, v) = \begin{cases} 0 & \text{, if } v \leq \lambda \text{ (considered elements of } \mathcal{L}) \\ 1 & \text{otherwise} \end{cases} \tag{1.4}$$

Then $\mathcal{T}(V, \Lambda, S)$ is the left generating semigroup for $\mathcal{L}$:

$$\mathcal{L} \simeq \Gamma_L(\mathcal{T}(V, \Lambda, S))$$

Proof is in [4].

Remark. The suggested representation of lattices by annihilators is, of course, not unique. The constructions of such sort arise from the generalization of the techniques proposwed by Foulis [5] when the partially ordered semigroup with unit is associated with an ortholattice. The general account of semigroups of such sort (poe-semigroups) can be found in [6].

2 Orthocomplementation

The criterion of orthocomplementability of CAC lattices in terms of Rees generating semigroups is established in this section.

A lattice $\mathcal{L}$ with the greatest element I and the least element 0 is said to be *orthocomplemented* if $\mathcal{L}$ is endowed by an operation $(\cdot)^{\perp} : \mathcal{L} \to \mathcal{L}$ such that for any $a, b \in \mathcal{L}$

$$a^{\perp\perp} = a \; ; \quad a \vee a^{\perp} = I \; ; \quad a \wedge a^{\perp} = 0 \tag{2.1}$$

$$(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp} \; ; \quad (a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$$

and *orthocomplementable* if such mapping $(\cdot)^{\perp}$ can be defined (perhaps, not uniquely, see the example below).

In the case when $\mathcal{L}$ is CAC lattice, the operation $(\cdot)^{\perp}$ induces the pair of one-to-one mappings o, ω between the sets of atoms and coatoms:

$$o : v \mapsto vo = v^{\perp} \in \mathcal{L} \; ; \quad \omega : \lambda \mapsto \lambda\omega = \lambda^{\perp} \in V \tag{2.2}$$

The obvious necessary condition for a CAC lattice $\mathcal{L}$ to be orthocomplementable is that the sets V and Λ must have equal cardinality: $\mid V \mid = \mid \Lambda \mid$, or, equivalently, the pair of 1-1 mappings $i : V \to \Lambda, j : \Lambda \to V$ must exist such that for all v, λ

$$vij = v \quad \text{and} \quad \lambda ji = \lambda$$

Then the sandwich matrix S is associated with the $V \times V$ matrix P defined as:

$$P(u, v) = S(ui, v) \tag{2.3}$$

which restores S:

$$S(\lambda, v) = P(\lambda j, v)$$

The following theorem yields the sufficient condition of orthocomplementability. Let L be a CAC lattice with $\mid V \mid = \mid \Lambda \mid$, S be the sandwich matrix for the Rees generating semigroup for L and P be (2.3).

Theorem 2. $\mathcal{L}$ is orthocomplementable if and only if there exists a permutation $\sigma : V \to V$ of the set of atoms such that $P(u, v\sigma)$ is the symmetric matrix with unit diagonal:

$$P(u\sigma, v) = P(v\sigma, u) \; ; \quad P(v\sigma, v) = 1 \tag{2.4}$$

Proof. ($\Rightarrow$). Let L is orthocomplementable, then a pair of mappings o, ω (2.2) exists. Due to (2.2), for any $u, v \in V$ $S(uo, v) = 0$ if and only if $u \leq v^\perp$, thus $S(uo, v) = S(vo, u)$. Define the permutation σ on V:

$$v\sigma = voj$$

Then $P(u\sigma, v) = P(uoj, v) = S(uo, v) = S(vo, u) = P(voj, u) = P(v\sigma, u)$. It follows from (2.1) that for no $v \in V$ $v \leq v^\perp$, hence for all v $S(vo, v) = 1$, thus $P(v\sigma, v) = 1$.

($\Leftarrow$). Let the permutation $\sigma : V \to V$ such that (2.4) holds exist. Define the mappings o and ω as:

$$vo = v\sigma i \; ; \quad \lambda\omega = \lambda j \sigma^{-1} \tag{2.5}$$

then $vo\omega = v$ and $\lambda\omega o = \lambda$. Now for each $a \in \mathcal{L}$ define:

$$a^\perp = \wedge\{vo \mid v \leq a\} \tag{2.6}$$

It follows immediately from (1.1) and (2.6) that $u \leq a^\perp$ if and only if $uo \geq a$, therefore

$$a^{\perp\perp} = \wedge\{vo \mid v \leq a^\perp\} = \wedge\{vo \mid vo \geq a\} = a$$

where the last identity holds by virtue of coatomicity of $\mathcal{L}$ (1.2). For any $a, b \in \mathcal{L}$

$$(a \wedge b)^\perp = \wedge\{vo \mid v \leq a \text{ and } v \leq b\}$$

Consider $a^\perp \vee b^\perp$. Since $\mathcal{L}$ is coatomistic

$$a^\perp \vee b^\perp = \wedge\{\lambda \mid \lambda \geq a^\perp \text{ and } \lambda \geq b^\perp\}$$

The mapping $o : V \to \Lambda$ (2.4) is one-to-one thus instead of λ we can range over vo, so

$$a^\perp \vee b^\perp = \wedge\{vo \mid vo \geq a^\perp \text{ and } vo \geq b^\perp\} = \wedge\{vo \mid v \leq a \text{ and } v \leq b\} = (a \wedge b)^\perp$$

Finally, consider $a \vee a^\perp = (a \wedge a^\perp)^\perp = \wedge\{vo \mid v \leq a \text{ and } v \leq a^\perp\}$. Although $v \leq a$ and $v \leq a^\perp$ implies $P(vo, v) = 0$ which contradicts (2.4). Thus $a \vee a^\perp = 1$ which completes

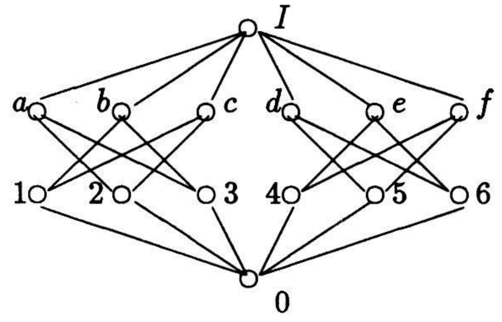

Figure 1: Hasse diagram

the proof since the remaining formulas (2.1) can be derived from $a^{\perp\perp} = a$, $a \vee a^{\perp} = 1$, and $a^{\perp} \vee b^{\perp} = (a \wedge b)^{\perp}$.

Note that the orthocomplementability does not assume the unique way of orthocomplementation, which can be figured out by considering the following example. Let $\mathcal{L}$ be the lattice with the Hasse diagram of Fig. 1.

Here we have $V = \{1,2,3,4,5,6\}$ and $\Lambda = \{a,b,c,d,e,f\}$. Provided the correspondence mapping i is:

$$i = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix}$$

the matrix P (2.3) will have the form:

$$P(u,v) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

It can be checked directly that the following permutations on V yield non-isomorphic orthocomplementations of $\mathcal{L}$:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix} \; ; \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

3 Orthomodularity

In this section the algorithm which tests the orthomodularity of an orthocomplemented lattice $\mathcal{L}$ is suggested.

First note that any exhaustive orthomodularity test needs ranging over elements of $\mathcal{L}$, that is, over subsets of V. It is because the orthomodularity of polar lattice is the second order property of the orthogonality relation on the set V [11]. The theoretical ground for the proposed orthomodularity test is the result obtained in [5]. In the form the most suitable in the sequel it is formulated as follows [12]:

Theorem 3. An ortholattice $(\mathcal{L},^\perp)$ is orthomodular if and only if for any $a \in \mathcal{L}$ the mapping $p_a : \mathcal{L} \to \mathcal{L}$ of the form

$$xp_a = (x \vee a^\perp) \wedge a \quad \text{(the postfix notation is used)}$$

satisfies the condition

$$\forall u, v \in V \quad up_a \leq v^\perp \quad \text{implies} \quad vp_a \leq u^\perp$$

The idea of the proposed test is to express $up_a \leq v^\perp$ as a binary relation, call it $\mathcal{D}_a$, on V, and then to convince that it is symmetric. Now let us build this relation using $\mathcal{P}$ considered Boolean matrix (since it contains only 0 and 1 entries). The Boolean arithmetics is:

- **Sum:** $0+1=1+0=1+1=1$; $0+0=0$
- **Product:** $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$; $1 \cdot 1 = 1$
- **Negation:** $\bar{1} = 0$; $\bar{0} = 1$

The following operations with Boolean matrices will be used:

- **Negation:** $\bar{\mathcal{A}}_{ik} = \overline{\mathcal{A}_{ik}}$
- **Sum:** $(\mathcal{A}+\mathcal{B})_{ik} = \mathcal{A}_{ik} + \mathcal{B}_{ik}$
- **Matrix product:** $(\mathcal{A}\mathcal{B})_{ik} = \sum_j \mathcal{A}_{ij}\mathcal{B}_{jk}$
- **Pointwise product:** $(\mathcal{A} \wedge \mathcal{B})_{ik} = \mathcal{A}_{ik} \cdot \mathcal{B}_{ik}$

and one more additional operation, denote it $\mathcal{A} \mapsto \mathcal{A}^0$, using the matrix $\mathcal{P}$:

$$\mathcal{A}^0 = \overline{\mathcal{A}\mathcal{P}} = \prod_j (\bar{\mathcal{A}}_{ij} + \bar{\mathcal{P}}_{jk})$$

Now, let a be an arbitrary element of $\mathcal{L}$. Since $\mathcal{L}$ is atomistic, a can be considered as the subset of V. Define the Boolean matrix $\mathcal{D}_a$ associated with a as follows:

$$\mathcal{D}_a = V \times \{v \in V \mid v \leq a\} \tag{3.1}$$

Lemma. The explicit expression for $\mathcal{D}_a$ is:

$$\mathcal{D}_a = (\mathcal{A} \wedge (\bar{\mathcal{P}} \wedge \mathcal{A})^0)^0 \tag{3.2}$$

Proof consists of stepwise development of the right side of the expression (3.2). The details of these techniques are in [12].

Orthomodularity test

- **1.** Let u, v range over the elements of V. For each $a \in \mathcal{L}$ build the matrix

$$\mathcal{A}_{uv} = \begin{cases} 1 & \text{if } v \leq a \\ 0 & \text{otherwise} \end{cases} = V \times a$$

  where a is considered subset of V.

- **2.** Build the matrix $\mathcal{D}_a$ (3.2).
- **3.** Check whether $\mathcal{D}_a$ is symmetric.
- **Criterion:** $\mathcal{L}$ is orthomodular if and only if for every $a \in \mathcal{L}$ the relation $\mathcal{D}_a$ was symmetric.

4 Topologymeter

The generating semigroup for the lattice of all topologies on a finite set considered property lattice of a hypothetical "topologymeter" is built in this section.

Let X be a finite set. To define a topology on X is to outline a collection τ of subsets of X called *open* such that

- T1). $\emptyset, X \in \tau$

- T2). $A, B \in \tau \Rightarrow A \cup B \in \tau$
- T3). $A, B \in \tau \Rightarrow A \cap B \in \tau$

All topologies are partially ordered with respect to the set inclusion (since their are *sets* of sets):

$$\tau_1 \leq \tau_2 \quad \text{means} \quad \forall A \subseteq X A \in \tau_1 \Rightarrow A \in \tau_2 \tag{4.1}$$

and τ_1 is said to be *weaker* (or *coarser*) than τ_2, while τ_2 is called *stronger* (or *finer*) than τ_1.

Moreover, the collection $T(X)$ of all topologies on the set X forms the lattice with respect to the partial order (4.1), and this lattice is CAC [8]. The arguments to consider $T(X)$ property lattice were analyzed in [7]. The object of this section is to construct the generating semigroup for $T(X)$.

Begin with the set of atoms of $T(X)$. These are weakest proper topologies. Each such topology contains exactly one open set. Thus the set of atoms $V(T(X))$, denote it by V, is in 1-1 correspondence with proper subsets of X:

$$V = \{v_A \mid A \subset X, A \neq \emptyset, X\}$$

where v_A is a topology containing exactly three sets: $v_A = \{\emptyset, A, X\}$

The cardinality of V is: $\text{card}V = 2^N - 2$, where N is the cardinality of the set X.

The coatoms of $T(X)$ are associated with all ordered pairs of elements of X [13], so

$$\Lambda = \Lambda(T(X)) = \{(u, v) \mid u, v \in X, u \neq v\}$$

where $(u, v) \in T(X)$ is the topology such that a set A is open in (u, v) if and only if $v \in A$ implies $u \in A$. Therefore the cardinality of the set of coatoms

$$\text{card}(\Lambda) = N(N-1) \neq \text{card}V = 2^N - 2$$

is not equal to that of the atoms (when $N \geq 4$). The strongest topologies are also associated with weakest proper partial orders on X [13]. The atomic topology v_A is weaker than (u, v) iff $v \notin A$ or $u \in A$. Therefore the sandwich matrix (1.4) has the form:

$$S_{uv,A} = \begin{cases} 0 & \text{if } v \notin A \text{ or } u \in A \\ 1 & \text{if } v \in A \text{ and } u \notin A \end{cases} = \chi_A(v)(1 - \chi_A(u))$$

where $\chi_A : X \to \{0, 1\}$ is the characteristic function of a subset A. Thus, the Rees generating semigroup for the topology lattice is defined.

The analog of the algebra of all linear operators in finite dimensional Hilbert space for the case of topologymeter will be the semigroup algebra spanned over the Rees semigroup $\mathcal{T}(T(X))$, denote it by $\mathcal{Q}$:

$$\mathcal{Q} = \{\sum c_{uv,A} \mid u, v \in X, u \neq v, A \subset X, A \neq \emptyset\}$$

where $c_{uv,A}$ are complex coefficients. As a quotient space, it has the finite dimensional linear space space $\mathcal{H}_V$ spanned over all proper subsets of X. Therefore, each topology on X is associated with a subspace of $\mathcal{H}_V$ spanned on basis vectors labelled by subsets forming this topology. I emphasize that this approach makes no difference between the subsets of different cardinality, and one can introduce the operator τ increasing cardinality [14] on basis vectors of $\mathcal{H}_V$ and the extend it by linearity on the whole $\mathcal{H}_V$. Besides, the superpositions of sets of different cardinality may be considered as fully fledged pure states associated with an observable which is complementary to the measuring of cardinality.

5 Concluding Remarks

In conclusion, I would like to dwell on some exotic features of systems described by the proposed machinery. As it was shown in section 4, where the topology lattice was considered property lattice of a "topologymeter". I emphasize that the algebras of observables suggested in section 4 possess no involution. When they are realized by operators, these operators will act from one space to another rather than in one state space. That means that negation-free systems need two in generally not isomorphic state spaces: the in- and out- ones. Thus, for such systems the duality between bra- and ket-vectors will not take place!

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