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Lee–Yang Measures

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Dedicated to the memory of Ansgar Schnizer
who was such a wonderful friend

Abstract The relation between the zeros of partition functions and the coefficients of low–density expansions, and the convergence properties of the latter, are used to show the existence of the limiting distribution of zeros in the thermodynamic limit. The limiting set of zeros can be identified as the support of a measure whose moments are the coefficients of the low–density expansion. The measure is not uniquely defined by these moments. Applications include a general class of lattice models with fermions.

1 Motivation

The importance of the behaviour of the zeros of the finite volume partition function as the volume tends to infinity for the theory of phase transitions was revealed in the pioneering work of Lee and Yang [LY]. The Lee–Yang theorem states that the partition function of the nearest–neighbour Ising model, which for a system on a finite lattice is a polynomial in the fugacity z, can only have zeros for $|z| = 1$, and thereby rules out nonanalyticities in the free energy density in the thermodynamic limit for $|z| \neq 1$. Analogous statements have

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been shown for more general models [AR], and the use of this type of theorems is in general to rule out phase transitions in regions which remain free of zeros in the thermodynamic limit.

A natural question is about the converse of this statement, namely what happens in those regions of the complex plane where the zeros are, and how the zeros cause the phase transitions and determine their properties. The idea is that phase transitions are at those points where the zeros pinch the real axis in the thermodynamic limit [LY], and furthermore that also accumulation points of the zeros in the complex plane away from the real axis may influence the behaviour of correlations. One of the motivations for this is the very simple relationship between the zeros and the free energy density in finite volume. In order to see how this nice relationship between the zeros and the free energy density carries over to the thermodynamic limit, it is necessary to specify in which sense the set of zeros has a thermodynamic limit. The formula for the free energy in finite volume indicates that only points where the zeros accumulate in this limit are of importance, that is, isolated zeros do not influence thermodynamic behaviour. One problem which one might encounter when thinking in terms of accumulation points is that the sets of zeros of the partition functions of two different volumes are not parts of one sequence of complex numbers, but may not have any points at all in common, so that one has to deal with the convergence of the entire sets. The natural way to do this is to look at the measures concentrated on the sets of zeros because the formulas for the free energy density and its derivatives are also very simple in terms of these measures and suggest that the correct – and also the most useful – definition of the set of zeros in the thermodynamic limit is that of the support of a measure.

This measure has been constructed for the Ising model [LY ,Be], using the relationship between the zeros and the coefficients of the fugacity (or low–density) expansion. In that particular case, one can use the Lee–Yang theorem to conclude that the support of the measure must be a subset of the unit circle. It then follows that the coefficients of the fugacity expansion are Fourier coefficients of this measure, and the existence of the measure can be proven from the knowledge of these coefficients. Moreover, standard theorems about the Fourier transformation imply that the measure is determined uniquely by the series expansion coefficients. Further physical properties of the measure have been derived in approximations using the finite number of expansion coefficients which are available in practice [Be].

In this paper I show under much weaker assumptions that a limiting measure for the zeros exists, and that it can be identified as the support of a measure $\mathcal{L}$ from which the free energy density can be obtained by a simple formula. The main idea of this treatment is simple: it is well–known that in finite volume the coefficients of the low–density expansion are simple functions of the zeros, and that therefore in finite volume the zeros determine the expansion coefficients and vice versa. But for virtually any system of interest, it is also well–known from the theory of cluster expansions [Se] that the low–density expansion has a positive radius of convergence, and that its coefficients converge to the infinite–volume ones in the thermodynamic limit. Since these convergence properties pose a restriction on the zeros, it suggests itself to try to recover the limiting distribution of zeros from these coefficients. In fact, given the existence of $\mathcal{L}$, the latter simply turn out to be moments of $\mathcal{L}$. Once these observations are made, it is not difficult to construct limiting measures from these moments.

This construction is based only on properties of the expansion coefficients and thus works without any additional input about the zeros like the Lee–Yang theorem which was used in the Ising model case. It turns out, however, that the information contained in the infinite–volume series expansion coefficients alone is not sufficient to guarantee uniqueness, but that one can construct measures which reproduce them as moments but do not reveal anything about the actual limiting behaviour of the zeros. The reason for this is that the expansion coefficients are not all the moments of the measure but only the "holomorphic" ones. There are other moments which are not determined by the low–density expansion and therefore the latter is in general not sufficient to show that they converge for a given sequence of volumes tending to infinity. However, one can use a compactness argument to show that they have a thermodynamic limit for subsequences. For a given subsequence, the limiting measure is then uniquely determined, but since different sequences may produce different limits, uniqueness cannot be shown. This is only possible with further input, either by determination of the just mentioned additional moments, or by information about the support of the measure, as in the Ising model.

The measures defined by the zeros in finite volume converge to the thus constructed limiting measure in the sense that convergence holds when both sides are applied to continuous functions. To obtain the free energy density, not only continuous functions, but logarithms have to be integrated, and even more singular functions for its derivatives. I can show the convergence of these finite–volume quantities to those obtained from the limiting measure in regions where the zeros do not accumulate. The investigation of this point reveals a more subtle problem: the points where the zeros accumulate "so strongly" that one can see this behaviour when testing the measure with a continuous function constitute the support of the infinite–volume measure, however, there may be regions where accumulation is not sufficient for this, but still strong enough to prevent convergence of the derivative of the finite volume free energy density to the analytic continuation defined by the measure. It will also require further information to rule out – or learn more about – this kind of behaviour.

While stronger assumptions may give stronger results, the merit of the method used here is that the limiting measures are shown to exist and thus can be used under rather weak assumptions and without any detailed knowledge. Moreover, the supports of these measures contain only points where the zeros really accumulate, so that no artificial singularities are introduced by this, admittedly indirect, reconstruction of the measure from its moments. It should also be noted that the idea to do this is not that far–fetched: the resummation techniques used to predict critical behaviour from series expansions also aim at determining the limiting behaviour of the zeros, from the (for practical studies, truncated) series expansion. This observation also shows that the uniqueness problem mentioned above has a very practical aspect: its solution is equivalent to finding a resummation method which is known to converge to the thermodynamic function associated to the limiting measure of the zeros.

As an application I show that a simple inequality for the lowest coefficient of the polynomial implies that the support of the measure must contain points outside of a certain disk. This restricts the region where the low–density expansion converges or reproduces the free energy density. In case there is no phase transition on the real axis, I derive a sum rule which links the integrated susceptibility to the distance of the support of the measure to the real axis. Finally, I show that a class of lattice fermionic models which comprises those of use in lattice gauge theory satisfy the assumptions needed to construct the measure. The low–density expansion is in that case the hopping parameter expansion, and its resummations play a central role in the analysis of these systems. A technical innovation is a norm on Grassmann algebras which makes activity estimates for cluster expansions very simple.

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2 Construction

Let $d \geq 1$, and assume a statistical mechanical system, that is an assignment $(\Lambda, M) \mapsto Z_\Lambda(M)$, where Λ runs through the finite subsets of $\mathbb{Z}^d$ and $M \in \mathbb{C}$, to be given, such that the partition function in volume Λ is a polynomial in M of degree a multiple of $|\Lambda|$:

Assumption 1: There is $a_o \in \mathbb{N}$ such that for all $\Lambda \subset \mathbb{Z}^d$

$$Z_\Lambda(M) = \sum_{n=0}^{L} p_n(\Lambda) M^n \tag{2.1}$$

with $p_n(\Lambda) \in \mathbb{C}$ and $L = a_o|\Lambda|$.

Examples for systems of that kind are the Ising model, with M the fugacity, or models of lattice gauge theory with fermions, with M the mass parameter (or monomer activity), which will be considered below. The thermodynamic limit , written briefly as $\Lambda \to \infty$, is to be understood as limit of a sequence of finite volumes $(\Lambda_k)_{k \in \mathbb{N}}$ in the sense that for every arbitrary large finite volume $\Lambda_o \subset \mathbb{Z}^d$ there is a $K \in \mathbb{N}$ such that for all $k \geq K$, $\Lambda_k \supset \Lambda_o$. In the setting here, convergence in the sense of Van Hove is not needed.

Introducing the zeros $\lambda_i \in \mathbb{C}$ of the partition function, Z_Λ can be factorized as

$$Z_\Lambda(M) = p_L(\Lambda) \prod_{i=1}^{L} (M - \lambda_i). \tag{2.2}$$

The highest coefficient of the polynomial will not play a role and it can always be divided out, so without loss, $p_L(\Lambda) = 1$ (it should be noted, however, that in applications where there is more than one coupling, p_L may depend on the other couplings). The free energy density and its derivative can be expressed in terms of the zeros as

$$f_\Lambda(M) = \frac{1}{|\Lambda|} \log Z_\Lambda(M) = \frac{1}{|\Lambda|} \sum_{i=1}^{L} \log(M - \lambda_i) \tag{2.3}$$

and

$$X_\Lambda(M) = \frac{\partial f_\Lambda}{\partial M} = \frac{1}{|\Lambda|} \sum_{i=1}^{L} \frac{1}{M - \lambda_i}. \tag{2.4}$$

Taking out the highest power of M as well and expanding in powers of $1/M$,

$$f_\Lambda(M) - a_o \log M = - \sum_{n \geq 1} \frac{1}{n} a_n(\Lambda) M^{-n}, \tag{2.5}$$

where the right side is to be understood for the moment as a formal power series in M^{-1}. From (2.3), get

$$a_n(\Lambda) = \frac{1}{|\Lambda|} \sum_{i=1}^{L} \lambda_i{}^n \tag{2.6}$$

and from (2.4), with $a_o(\Lambda) = a_o$,

$$X_\Lambda(M) = \sum_{n=0}^{\infty} a_n(\Lambda) M^{-n-1}. \tag{2.7}$$

The simple relation (2.6) between the coefficients of the low–density expansion and the zeros of Z_Λ is particularly useful if convergence of the former is known to hold.

Assumption 2:

i) (Convergence of the coefficients) There is a sequence $(a_n)_{n \in \mathbb{N}}$ such that for any sequence of Λ tending to infinity, $a_n(\Lambda) \to a_n$ as $\Lambda \to \infty$.

ii) (Uniform convergence of the series) There is $M_o \geq 0$ and $\Lambda_o \subset \mathbb{Z}^d$ ($|\Lambda_o| < \infty$) such that for all Λ with $\Lambda \supset \Lambda_o$ and all M with $|M| \geq M_o$ the expansion (2.7) converges absolutely.

A system $(\Lambda, M) \mapsto Z_\Lambda(M)$ which fulfills Assumptions 1 and 2 is called a *Lee–Yang system.* If nothing else is stated, it is assumed that the given system is a Lee–Yang system and that $\Lambda \supset \Lambda_o$. The constant M_o could be put to one by suitable rescaling of M, but it does not hurt to keep it.

Whenever a convergent cluster expansion exists at large M, Assumption 2 holds (see also Section 5).

Remark: For a Lee–Yang system $(\Lambda, M) \mapsto Z_\Lambda(M)$

1. the limiting free energy density

$$f(M) = \lim_{\Lambda \to \infty} f_\Lambda(M) \tag{2.8}$$

exists, $f(M) - a_o \log M$ is analytic on $\{M \in \mathbb{C} : |M| \geq M_o\}$, and given by the convergent expansion

$$f(M) - a_o \log M = -\sum_{n \geq 1} \frac{a_n}{n} M^{-n}, \tag{2.9}$$

in particular, f is independent of the sequence of volumes that tended to infinity.

2. The zeros of Z_Λ satisfy

$$|\lambda_i| \leq M_o. \tag{2.10}$$

Proof: 1. Vitali's theorem, applied to $(f_\Lambda)_\Lambda$. 2. Since f_Λ is bounded on $\{M \in \mathbb{C} : |M| \geq M_o$, $Z_\Lambda(M) = \exp(|\Lambda| f_\Lambda(M)$ is nonzero. ■

Notation: For $\varepsilon > 0$ and $z_o \in \mathbb{C}$, call $D_\varepsilon(z_o) = \{z \in \mathbb{C} : |z - z_o| < \varepsilon\}$, and $\bar{D}_\varepsilon(z_o) = \{z \in \mathbb{C} : |z - z_o| \leq \varepsilon\}$. If $z_o = 0$, abbreviate $D_\varepsilon(0) = D_\varepsilon$, similarly for $\bar{D}$. For $m, n \in \mathbb{N}_o$, the monomials $p_{mn} : \bar{D}_{M_o} \to \mathbb{C}$ are defined as

$$p_{mn}(z, \bar{z}) := \bar{z}^m z^n. \tag{2.11}$$

The space of continuous functions $\mathcal{C} = \{f : \bar{D}_{M_o} \to \mathbb{C} : f \text{ continuous }\}$ is a Banach space with $\|\cdot\|_\infty$, and $p_{mn} \in \mathcal{C}$. The following subspaces of $\mathcal{C}$ will be needed in what follows: $\mathcal{A} = \{f \in \mathcal{C} : f|_{D_{M_o}} \text{ is analytic }\}$, $\mathcal{H} = \{\text{Re } f : f \in \mathcal{A}\}$, $\mathcal{P} = \text{span } \{p_{mn} : m, n \in \mathbb{N}_o\}$, $\mathcal{Q} = \text{span } \{p_{on} : n \in \mathbb{N}_o\}$, $\mathcal{Q}^* = \text{span } \{p_{no} : n \in \mathbb{N}_o\}$, $\mathcal{R} = \{\text{Re } q : q \in \mathcal{Q}\}$. Under completion with respect to $\|\cdot\|_\infty$, $\overline{\mathcal{P}} = \mathcal{C}$, $\overline{\mathcal{Q}} = \mathcal{A}$, $\overline{\mathcal{R}} = \mathcal{H}$.

Define $\mathcal{N}_\Lambda = \{z \in \mathbb{C} : Z_\Lambda(z) = 0\}$, $\mu_\Lambda(\lambda)$ as the multiplicity of λ as a zero of Z_Λ, i.e. how often it appears in the sequence $(\lambda_1, \ldots \lambda_L)$ and a measure

$$\mathcal{L}_\Lambda = \frac{1}{|\Lambda|} \sum_{\lambda \in \mathcal{N}_\Lambda} \mu_\Lambda(\lambda) \delta_\lambda^{(2)} \tag{2.12}$$

where $\delta_\lambda^{(2)}$ is the two–dimensional delta–distribution concentrated at $\lambda \in \mathbb{C} \cong \mathbb{R}^2$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{L}_\Lambda(p_{on}) &= a_n(\Lambda) \\ \mathcal{L}_\Lambda(p_{no}) &= \overline{a_n(\Lambda)}, \end{aligned} \tag{2.13}$$

so the expansion coefficients are moments of $\mathcal{L}_\Lambda$. Moreover, for all $M \notin \mathcal{N}_\Lambda$, (2.3) and (2.4) can be restated as

$$f_\Lambda(M) = \int d\mathcal{L}_\Lambda(z) \log(M - z) \tag{2.14}$$

and

$$X_\Lambda(M) = \int d\mathcal{L}_\Lambda(z) \frac{1}{M - z}, \tag{2.15}$$

and the convergence of the expansions in $1/M$ of these integrals (uniformly in Λ) can be traced to the fact that due to the second point in the above remark, the support of $\mathcal{L}_\Lambda$ is a subset of $\overline{D_{M_0}}$ for all Λ.

Remark: The infinite–volume measure will be defined using the limits of the $a_n(\Lambda)$, therefore it is important to note that the $a_n(\Lambda)$ are only a part of the moments

$$b_{mn}(\Lambda) = \mathcal{L}_\Lambda(p_{mn}) = \frac{1}{|\Lambda|} \sum_{i=1}^{L} \bar{\lambda}_i^m \lambda_i^{\,n} \tag{2.16}$$

of $\mathcal{L}_\Lambda$. This is the source of the non–uniqueness mentioned in the Motivation. However, in finite volume the $b_{mn}(\Lambda)$ can easily be obtained from the $a_n(\Lambda)$ because $\mathcal{L}_\Lambda$ is a discrete measure: given the $a_n(\Lambda)$, the coefficients of the polynomial for Z_Λ are reconstructed from $Z_\Lambda = \exp(|\Lambda| f_\Lambda)$, i.e. explicitly

$$p_n(\Lambda) = \sum_{l=1}^{n} \frac{|\Lambda|^l}{l!} \sum_{\substack{m_1,\dots,m_l \geq 1 \\ m_1+\dots+m_l=n}} \prod_{i=1}^{l} \frac{a_{m_i}(\Lambda)}{m_i!}, \tag{2.17}$$

and it then suffices to note that the zeros of a polynomial are determined if its coefficients are known.

If $|\lambda_i| = M_1$ for all i, as is the case in the Ising model, this problem is absent because then all the moments $b_{mn}(\Lambda)$ are multiples of the $a_n(\Lambda)$ or their complex conjugates.

Theorem 1 Let $(\Lambda, M) \mapsto Z_\Lambda(M)$ be a Lee–Yang system, then there is a unique bounded $\mathbb{R}$–linear functional $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ such that for all $h \in \mathcal{H}$, $\mathcal{L}_\Lambda(h) \to \mathcal{L}(h)$ as $\Lambda \to \infty$.

Proof: Define $\mathcal{L} : \mathcal{Q} + \mathcal{Q}^* \to \mathbb{C}$ by $\mathcal{L}(p_{on}) = a_n$, $\mathcal{L}(p_{no}) = \overline{a_n}$, and the requirement that it be linear. For $r \in \mathcal{R}$, there is $q \in \mathcal{Q}$ such that $r = q + \bar{q}$, so $\mathcal{L}(r) = (\mathcal{L}(q) + \mathcal{L}(\bar{q})) \in \mathbb{R}$, that is, the restriction to $\mathcal{R}$ is a real–valued linear functional. Writing $r = \sum_{n=0}^{N}(c_n p_{on} + \bar{c}_n p_{no})$, for all Λ

$$|(\mathcal{L} - \mathcal{L}_\Lambda)(r)| \le 2\sum_{n=0}^{N} |c_n| \cdot |a_n - a_n(\Lambda)|, \tag{2.18}$$

and $\mathcal{L}_\Lambda(r) \to \mathcal{L}(r)$ for any sequence of Λ that tends to infinity. Since by (2.12) $\mathcal{L}_\Lambda$ is a positive functional for all Λ, the same is true for the limit $\mathcal{L}$, so for a $q \in \mathcal{R}$ which is pointwise nonnegative, $\mathcal{L}(q) \ge 0$. The remaining part of the argument is now almost standard [RS1]: since $\mathcal{R}$ is a space of polynomials, $1 \in \mathcal{R}$, and

$$\begin{aligned} 0 &\le \|p\|_\infty - p \in \mathcal{R} \\ 0 &\le \|p\|_\infty + p \in \mathcal{R}, \end{aligned} \tag{2.19}$$

so the linearity and positivity of $\mathcal{L}$ imply that

$$|\mathcal{L}(p)| \le \|p\|_\infty \mathcal{L}(1). \tag{2.20}$$

$\mathcal{L}(1) = a_o$, so $\mathcal{L}$ is a bounded linear functional on $\mathcal{R}$ with norm $\|\mathcal{L}\| = a_o$. By the Hahn–Banach theorem, there is a unique (since $\overline{\mathcal{R}} = \mathcal{H}$) extension to a functional $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ with the same norm. Finally, since $\mathcal{R}$ is dense in $\mathcal{H}$ and the norms of $\mathcal{L}$ and $\mathcal{L}_\Lambda$ are bounded by a_o, convergence $\mathcal{L}_\Lambda(h) \to \mathcal{L}(h)$ holds for all $h \in \mathcal{H}$. ■

The method of proof required no knowledge of the n–dependence of the convergence $a_n(\Lambda) \to a_n$.

Corollary: There is a measure ρ on S^1 such that for all $h \in \mathcal{H}$,

$$\mathcal{L}(h) = \int d\rho(\theta) h(M_o e^{i\theta}). \tag{2.21}$$

For all M with $|M| > M_o$

$$X(M) = \int d\rho(\theta) \frac{1}{M - M_o e^{i\theta}}. \tag{2.22}$$

Proof: The Poisson integral [Ho] establishes an isomorphism $\mathcal{H} \to C(S^1, \mathbb{R})$ between $h \in \mathcal{H}$ and its boundary values, which are continuous functions on the circle. So $\mathcal{L}$ induces a bounded linear functional on the continuous functions on S^1, which by the Riesz–Markov theorem [RS1] is a measure. If $|M| > M_o$, the integral on the right side of (2.22) can be expanded in powers of $1/M$, and the coefficients are

$$\tilde{a}_n = M_o{}^n \int_0^{2\pi} d\rho(\theta) e^{in\theta} = \mathcal{L}(r_n) + i\mathcal{L}(s_n), \tag{2.23}$$

where $r_n = \text{Re}\,(z^n)$ and $s_n = \text{Im}\,(z^n)$ are in $\mathcal{H}$. By definition of $\mathcal{L}$, $\tilde{a}_n = a_n$. ■

While very simple, Theorem 1 and the Corollary are only of limited use because they do not reveal anything about the actual location of the zeros, as is clear from the proof of the Corollary: harmonic functions are determined by their boundary values. Another way to produce a measure on the disk instead of its boundary would be, of course, to just use the Hahn–Banach theorem to extend $\mathcal{L}$, as defined on $\mathcal{R}$, to all of $\mathcal{C}$. Again, not only uniqueness, but also any connection to the finite volume measures is lost in this procedure. While uniqueness cannot be shown here, the connection to the finite–volume measures can be maintained by using a different argument, which requires knowledge about the additional moments b_{nm} for both n and m nonzero.

Lemma 1: If the zeros stay in $\overline{D_{M_o}}$ (as is the case for Lee–Yang systems), any sequence $\Lambda \to \infty$ has subsequences $S = (\Lambda_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathcal{L}_{\Lambda_k}(p_{mn}) = \lim_{k\to\infty} b_{mn}(\Lambda_k) = b^{(S)}_{mn} \tag{2.24}$$

exists for all $m, n \in \mathbb{N}_o$.

Proof: A compactness argument, which can be implemented as follows: The power series

$$g_\Lambda(x,y) = \sum_{m,n\geq 0} b_{mn}(\Lambda) x^m y^n \tag{2.25}$$

converges absolutely and uniformly for $|x| < M_o{}^{-1}$ and $|y| < M_o{}^{-1}$, so $(g_\Lambda)_\Lambda$ is a normal family of functions on $D_{1/M_o}{}^2$. Therefore any sequence of Λ which tends to infinity has a subsequence $(\Lambda_k)_{k\in\mathbb{N}}$ for which $(g_{\Lambda_n})_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $D_{1/M_o}{}^2$ Therefore the limit function is analytic, and it is easy to see from the uniform convergence that the limits of the $b_{mn}(\Lambda_k)$ exist and are given by the expansion coefficients of the limiting function. ■

Theorem 2: Let $(\Lambda, M) \mapsto Z_\Lambda(M)$ be a Lee–Yang system, and let a sequence of Λ tending to infinity be given.

1. For any given subsequence $S = (\Lambda_k)_{k\in\mathbf{N}}$ for which the limit $b_{mn}^{(S)}$ exists, there is a unique measure $\mathcal{L}^{(S)}$ supported in $\overline{D_{M_o}}$ such that $b_{mn}^{(S)} = \mathcal{L}^{(S)}(p_{mn})$. $\mathcal{L}_{\Lambda_n} \to \mathcal{L}^{(S)}$ as a measure, that is, for all $\varphi \in \mathcal{C}$, $\mathcal{L}_{\Lambda_n}(\varphi) \to \mathcal{L}^{(S)}(\varphi)$ as $n \to \infty$.
2. For all such subsequences S, the formula

$$X^{(S)}(M) = \int d\mathcal{L}^{(S)}(z)\frac{1}{M-z} \tag{2.26}$$

defines a continuation of X to $\mathbb{C}\setminus$ supp $\mathcal{L}^{(S)}$, and the so defined function $X^{(S)}$ is analytic on $\mathbb{C}\setminus$ supp $\mathcal{L}^{(S)}$.

3. Any two measures associated to different subsequences coincide on the analytic and harmonic functions.
4. $X^{(S)} = X^{(S')}$ on the connected component C_o of $\mathbb{C} \setminus ($ supp $\mathcal{L}^{(S)} \cup$ supp $\mathcal{L}^{(S')}$ which contains $\{M : |M| > M_o\}$.

Proof: 1.The construction is similar to the one in the proof of Theorem 1, only that now $\mathcal{L}^{(S)}$ is defined on $\mathcal{P}$ by $\mathcal{L}^{(S)}(p_{mn}) = b_{mn}^{(S)}$ and the requirement of linearity. Again, since $\mathcal{L}_{\Lambda_n}(p) \to \mathcal{L}^{(S)}(p)$ for all $p \in \mathcal{P}$, $\mathcal{L}^{(S)}$ is a positive linear functional on $\mathcal{P}$. Since $1 \in \mathcal{P}$, $\mathcal{L}^{(S)}$ is bounded and thus can be extended uniquely to a bounded linear form on $\mathcal{C} = \overline{\mathcal{P}}$. By the Riesz–Markov–Theorem, $\mathcal{L}^{(S)}$ is a measure supported in $\overline{D}_{M_o}$. Convergence of $\mathcal{L}_{\Lambda_n}(\varphi) \to \mathcal{L}^{(S)}(\varphi)$ for $\varphi \in \mathcal{C}$ holds because the norms of $\mathcal{L}_{\Lambda_k}$ and $\mathcal{L}^{(S)}$ are bounded by a_o and convergence holds on $\mathcal{P}$.

2. supp $\mathcal{L}^{(S)}$ is compact, therefore for any $M \notin$ supp $\mathcal{L}^{(S)}$, there is a neighbourhood U_M of $\mathcal{L}^{(S)}$ such that the distance of M to U_M is positive. Therefore the map $z \mapsto \Phi_M(z) = \frac{1}{M-z}$ is continuous in z for all $z \in U$, and the integral is well–defined. Also, Φ_M is analytic at M for all $z \in U_M$, so the integral is analytic as well. It is a continuation of X by construction.

3. Let $f \in \mathcal{Q}$, $f(z) = \sum_{n=0}^{N} f_n z^n$, then $\mathcal{L}^{(S)}(f) = \sum_{n=0}^{N} f_n a_n = \mathcal{L}^{(S')}(f)$ because the limit of the $a_n(\Lambda)$ is independent of the chosen sequence. Since $\mathcal{L}^{(S)}$ and $\mathcal{L}^{(S')}$ are continuous and $\overline{\mathcal{Q}} = \mathcal{A}$, they agree on $\mathcal{A}$. A similar argument works for $\mathcal{R}$ and $\mathcal{H}$.

4. For all $|M| > M_o$, $X^{(S)}(M)$ and $X^{(S')}(M)$ agree by (3) because $\Phi_M(z)$ is analytic on D_{M_o}, so they must be equal on C_o by the identity theorem. ■

Remark: There may be many subsequences with different limits for the $b_{mn}(\Lambda)$, but for a given subsequence there is no arbitrariness in the measure. If convergence of the $b_{mn}(\Lambda)$ to a limit which is independent of the chosen sequence is known by some different argument, the limiting measure is unique. This is true if the b_{mn} are determined by the a_n, e.g. if all zeros are on the unit circle (as in the case of the nearest–neighbour Ising model), or a conformal image of it. If supp $\mathcal{L}$ is not known, the b_{mn} are not determined by the a_n.

To see the relation of this measure to the behaviour of the zeros in the thermodynamic limit, it is useful to distinguish two kinds of accumulation points.

Definition: For a finite set $\Lambda \subset \mathbb{Z}^d$, $\varepsilon > 0$ and $M \in \mathbb{C}$, let

$$T_\Lambda(\varepsilon, M) = \sum_{\lambda \in \mathcal{N}_\Lambda \cap D_\varepsilon(M)} \mu_\Lambda(\lambda) = \Lambda \mathcal{L}_\Lambda(D_\varepsilon(M)). \tag{2.27}$$

Let $S = (\Lambda_k)_{k\in\mathbb{N}}$ be a sequence tending to infinity. $M \in \mathbb{C}$ is called an accumulation point of the zeros for S if for all $\varepsilon > 0$, $\limsup_{k\to\infty} T_{\Lambda_k}(\varepsilon, M) = \infty$. The set of accumulation points is denoted by A. An accumulation point M is called weak if there is $\varepsilon_0 > 0$ such that $\frac{1}{|\Lambda_k|} T_{\Lambda_k}(\varepsilon_0, M) \to 0$ as $k \to \infty$, and strong otherwise. M is called an isolated limit point of zeros if it is not an accumulation point, but if for all $\varepsilon > 0$ the set $\{k \in \mathbb{N} : T_{\Lambda_k}(\varepsilon, M) > 0\}$ is infinite.

Remark: A is compact. For any $\varepsilon > 0$, the number of isolated limit points in $\mathbb{C} \setminus \{z : d(z, A) \le \varepsilon\}$ is finite.

Theorem 3: Let $(\Lambda, M) \mapsto Z_\Lambda(M)$ be a Lee–Yang system, $S = (\Lambda_k)_{k\in\mathbb{N}}$ be a sequence tending to infinity for which the b_{nm} converge, and $M \in \mathbb{C}$. Then

1. M is a strong accumulation point for S if and only if $M \in$ supp $\mathcal{L}^{(S)}$.
2. If M neither an accumulation point of zeros nor an isolated limit point of zeros, $X_{\Lambda_k}(M) \to X^{(S)}(M)$ as $k \to \infty$.

Proof: 1. Call the set of all strong accumulation points Σ. Let $M \in$ supp $\mathcal{L}^{(S)}$, but assume that $M \notin \Sigma$. Then there is $\varepsilon_0 > 0$ such that $\frac{1}{|\Lambda_k|} T_{\Lambda_k}(\varepsilon_0, M) \to 0$ as $k \to \infty$, and thus for any continuous function φ with $\varphi\,|_{\mathbb{C}\setminus D_{\varepsilon_0}(M)} = 0$,

$$|\mathcal{L}_{\Lambda_k}(\varphi)| \le \|\varphi\|_\infty \frac{1}{|\Lambda_k|} T_{\Lambda_k}(\varepsilon_0, M) \to 0 \tag{2.28}$$

as $k \to \infty$. Since $\mathcal{L}_{\Lambda_k}(\varphi) \to \mathcal{L}^{(S)}(\varphi)$ as $k \to \infty$, this implies that $\mathcal{L}^{(S)}(\varphi) = 0$ for all continuous φ supported in $D_{\varepsilon_0}(M)$, which contradicts $M \in$ supp $\mathcal{L}^{(S)}$. So, supp $\mathcal{L}^{(S)} \subset \Sigma$. Now assume that there exists $M \in \Sigma \setminus$ supp $\mathcal{L}^{(S)}$. Since supp $\mathcal{L}^{(S)}$ is compact, there is $\varepsilon > 0$ such that $\overline{D_\varepsilon}(M) \cap$ supp $\mathcal{L}^{(S)} = \emptyset$, and a continuous function $\varphi : \mathbb{C} \to [0,1]$ such that $\varphi\,|_{D_{\varepsilon/2}(M)} = 1$ and $\varphi\,|_{\mathbb{C}\setminus D_\varepsilon(M)} = 0$. Consequently, $\mathcal{L}^{(S)}(\varphi) = 0$. But since $M \in \Sigma$, there is $c_\varepsilon > 0$ and a subsequence $\left(\Lambda_{k_p}\right)_{p\in\mathbb{N}}$ such that for all $p \in \mathbb{N}$, $\frac{1}{|\Lambda_{k_p}|} T_{\Lambda_{k_p}}(\varepsilon/2, M) \ge c_\varepsilon$, and so $\mathcal{L}_{\Lambda_{k_p}}(\varphi) \ge c_\varepsilon$ for all p, hence $\liminf_{p\to\infty} \mathcal{L}_{\Lambda_{k_p}}(\varphi) \ge c_\varepsilon$. Since $\lim_{k\to\infty} \mathcal{L}_{\Lambda_k}(\varphi)$ exists, this implies $\lim_{k\to\infty} \mathcal{L}_{\Lambda_k}(\varphi) \ge c_\varepsilon$ which again contradicts convergence to $\mathcal{L}^{(S)}(\varphi)$.

2. By assumption, $M \notin$ supp $\mathcal{L}^{(S)}$ by (1), and there are $K > 0$ and $\varepsilon > 0$ such that for all $k \ge K$, $\mathcal{N}_{\Lambda_k} \cap D_{2\varepsilon}(M) = \emptyset$. Again, since supp $\mathcal{L}^{(S)}$ is compact, ε can be chosen so small

that also $\overline{D_{2\varepsilon}(M)} \cap$ supp $\mathcal{L}^{(S)} = \emptyset$. Now take a continuous function $\varphi : \mathbb{C} \to [0,1]$ such that $\varphi \mid_{\overline{D_\varepsilon(M)}} = 0$ and $\varphi \mid_{\mathbb{C}\setminus D_{2\varepsilon}(M)} = 1$, and define

$$\Psi_M(z) = \varphi(z)\frac{1}{M-z}. \tag{2.29}$$

Then $\Psi_M(z) = 1/(M-z)$ on supp $\mathcal{L}^{(S)}$, and so $\mathcal{L}^{(S)}(\Psi_M) = X^{(S)}(M)$. Also, by construction, $\mathcal{L}_{\Lambda_k}(\Psi_M) = X_{\Lambda_k}(M)$, and Ψ_M is continuous on $\mathbb{C}$. Therefore, by Theorem 2,

$$X_{\Lambda_k}(M) = \mathcal{L}_{\Lambda_k}(\Psi_M) \underset{k\to\infty}{\longrightarrow} \mathcal{L}^{(S)}(\Psi_M) = X^{(S)}(M) \quad \blacksquare \tag{2.30}$$

Definition: A limiting measure $\mathcal{L}^{(S)}$ associated to a sequence S for which $b_{mn}(\Lambda)$ converges is called a *Lee–Yang–measure* of the system $(\Lambda, M) \mapsto Z_\Lambda(M)$.

3 Discussion

The isolated limit points have no influence on the thermodynamics of the system: although the finite–volume functions $X_\Lambda(M)$ may have poles at or around these points, the sum of their residues vanishes as $\Lambda \to \infty$ because there is only a finite number of such poles in a given neighbourhood of the isolated limiting point. Since the set can have accumulation points (in the usual sense) only in A, $X^{(S)}$ defines the unique analytic continuation to these points.

The support of the infinite–volume measure $\mathcal{L}^{(S)}$ consists of all the strong accumulation points of zeros. Thus, while the weak accumulation points escape the support of the measure, no artificial singularities are generated, and the non–analyticities of the thermodynamic functions in supp $\mathcal{L}^{(S)}$ are really caused by the accumulation of zeros. Isolated weak accumulation points do not influence the thermodynamics of the system because convergence of the finite–volume functions to the ones defined by the measure holds arbitrarily closely to any of these points and so there cannot be a singularity there. However, as the following example shows, weak accumulation points need not be isolated, but can fill regions, and these regions need not contain any strong accumulation points, so they need not intersect with supp $\mathcal{L}^{(S)}$.

Example 1: Let $\alpha \in (0,1)$, denote $Q = [L^\alpha]$, where $[x] = \max\{n \in \mathbb{Z} : n \le x\}$, and define $Z_\Lambda(M) = \prod_{i=1}^{L}(M - \lambda_i)$, where $L = |\Lambda|$, as follows: for $i \in \{1, \ldots, Q\}$, $\lambda_i = \frac{i}{Q}$, and for $i \in \{Q+1, \ldots, L\}$, $\lambda_i = 2$. If $x \in [0,1]$ and $0 < \varepsilon < 1$, $[Q\varepsilon] - 1 \le T_\Lambda(\varepsilon, M) \le [2Q\varepsilon]$, so every $x \in [0,1]$ is a weak accumulation point, and 2 is the only strong one. This defines a Lee–Yang–system, and $X^{(S)}(M) = \frac{1}{M-2}$. But if $x \in [0,1] \cap \mathbb{Q}$, $x = m/n$, then x is a zero of

Z_L for every L for which Q is a multiple of n. Thus, there are sequences tending to infinity such that $X_\Lambda(x)$ is undefined for all Λ in the sequence, and thus there is no convergence on $[0,1] \cap \mathbb{Q}$ for these sequences.

It is easy to construct similar examples where entire disks are filled with weak accumulation points, and therefore, under the general assumptions stated here, the problem of non–convergence in regions that are away from supp $\mathcal{L}^{(S)}$, but contain weak accumulation points, cannot be ruled out. This point should be illuminated by studies of specific models. However, the above results show that what goes on in any region W of weak accumulation points has no influence on convergence or thermodynamic behaviour at any point $M \in \mathbb{C} \setminus \overline{W}$ (this is, after all, the reason why the weak accumulation points are not in the support of the measure). Therefore, only $\overline{W} \cap \mathbb{R}$ can be of interest for thermodynamics. If $\overline{W} \cap \mathbb{R}$ consists of isolated points only, the presence of weak accumulation points will only spoil the convergence of the finite–volume functions to the limiting ones at these points and thus be inessential.

By adapting the proof of the Corollary to Theorem 1, one can construct different measures that have the same expansion coefficients a_n: instead of taking a circle of radius M_o as support of the measure in the Corollary, one can repeat the construction for any curve which surrounds the zeros in all finite volumes and is such that the corresponding Poisson integral kernel is defined.

The following simple example illustrates that $\mathcal{L}$ cannot be determined from the series expansion for $X(M)$ alone.

Example 2: Call $|\Lambda| = L$ and set $Z_\Lambda(M) = M^L + {M_1}^L$. Then, in the infinite–volume limit,

$$X(M) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{M - M_1 e^{i\theta}} = \begin{cases} \frac{1}{M} & |M| > M_1 \\ 0 & |M| < M_1 \end{cases} . \tag{3.1}$$

Clearly, for any $M_1 < M_o$, the functions are the same on $\{|M| > M_o\}$, and from $a_n = \delta_{n,o}$ one cannot distinguish between measures supported on circles of different radius. Even a δ–function at zero gives the same a_n. Knowledge of the $b_{mn} = {M_1}^{2n} \delta_{mn}$ allows to distinguish between different radii, but the a_n do not determine M_1.

The uniqueness problem can be summarized as follows. Consider the statements

(a) all a_n are known
(b) all b_{mn} are known
(c) the support of $\mathcal{L}$ is known to be on the unit circle
(d) $\mathcal{L}$ can be determined uniquely.

Then

(i) (a) does not imply (d),as the above discussion shows.
(ii) (b) implies (d) by Theorem 2 (recall that $a_m = b_{om}$); uniqueness follows from the Stone–Weierstraß theorem $\overline{\mathcal{P}} = \mathcal{C}$.
(iii) (a) and (c) together imply (d).

As said before, (c) holds for the Ising model. (iii) extends to a case where supp $\mathcal{L}$ is part of a suitable curve.

Statement (i) is in accordance with theorems about the convergence of Padé approximants to the function they approximate because these theorems require some specific assumptions about the analytical structure of the function, e.g. that it is meromorphic in a certain disk, or a Herglotz function [Ba]. The analytic structure of the function $X(M)$ considered here is determined by $\mathcal{L}$ (see (2.26)), so further assumptions about its analytic structure are equivalent to further assumptions about $\mathcal{L}$ or at least its support.

The weak accumulation points cannot be detected by any resummation method based on the series expansion coefficients because the latter are not changed by the presence or absence of these points. If the zeros are all on some curve which intersects the real axis only in a set of isolated points, as is the case in the Ising model, $\overline{W} \cap \mathbb{R}$ is contained in this set and, by the above remarks, the weak accumulation points are then, if present at all, inessential for thermodynamics, as well as for resummation studies.

In general, the support of $\mathcal{L}$ need not be a curve and therefore a construction along the lines of (iii) may not be possible. However, (ii) can still be applied if the b_{mn} can be determined. It is therefore an important open problem to decide whether the b_{mn} can be obtained from a similar analysis of a statistical mechanical system with some additional interactions, or more complicated observables.

4 Application

In applications one often knows the free energy density f_Λ at $M = 0$, or at least a lower bound for it [Sa,GLS]. The partition function at $M = 0$ is the product of the zeros λ_i, so a lower bound for it implies that there must be λ_i outside a certain disk around zero. Since the function $X_\Lambda(M)$ has poles at every zero, this in turn implies an upper bound on the radius of convergence of its expansion in $1/M$ in finite volume [GLS]. The radius of convergence $1/\rho$ in infinite volume is of interest because at the point $M = \rho$ the expansion in $1/M$ breaks down, and there are models in which a phase transition happens at this point. It is, however, a bit subtler to show an analogous upper bound for the radius in the thermodynamic limit. This has two reasons: first, by (2.4), the residue at every pole of $X_\Lambda(M)$ is $1/|\Lambda|$, so it vanishes in the infinite–volume limit, and singularities can persist in this limit only where $O(|\Lambda|)$ zeros accumulate, that is, at the strong accumulation points. Second, the limiting measure will in most cases not be discrete and therefore the singularities of $X^{(S)}$ will not simply be poles.

The assumptions on $Z_\Lambda(0)$ stated above indeed imply that the support of the Lee–Yang measure contains points outside a certain disk D_1 centered at zero (Proposition 1). This implies an upper bound on the radius of convergence of the expansion of the infinite–volume function in $1/M$ if the measure has a discrete part or produces branch points in $X^{(S)}$. If the function has discontinuities but no branch points or poles, as in Example 2 of the previous section, the convergence radius is not restricted by supp $\mathcal{L}^{(S)} = \{M : |M| =$

$M_1\}$. However, in this example, the expansion in $1/M$ does not represent the function for $|M| < M_1$. The alternative that the expansion either diverges or fails to reproduce the actual thermodynamic function (i.e. the infinite–volume limit of X_Λ) inside the disk D_1 is shown to hold more generally in Proposition 2.

Let $\rho \geq 0$ be the inverse convergence radius of the expansion in $1/M$, i.e. the smallest number such that the expansion

$$X(M) = \sum_{n\geq 0} a_n M^{-n-1} \tag{4.1}$$

converges for all M with $|M| > \rho$. Whenever $X(M)$ is written without superscript S, it is to denote the analytic continuation of the function defined on $\mathbb{C} \setminus \overline{D_{M_0}}$ to $\mathbb{C} \setminus D_\rho(0)$ defined by (4.1). Also, let C_o be the connected component of $\mathbb{C} \setminus$ supp $\mathcal{L}^{(S)}$ which contains $\mathbb{C} \setminus \overline{D_{M_0}}$. The curves γ used below are understood to be rectifiable.

Proposition 1: Let $(\Lambda, M) \mapsto Z_\Lambda(M)$ be a Lee–Yang–system and assume that there is a sequence of volumes Λ tending to infinity such that

$$r = \liminf_{\Lambda\to\infty} |Z_\Lambda(0)|^{1/|\Lambda|} = \liminf_{\Lambda\to\infty} |p_o(\Lambda)|^{1/|\Lambda|} > 0. \tag{4.2}$$

(i) Let $s = r^{1/a_0}$. Then, for all limiting measures associated to subsequences $S = (\Lambda_k)_{k\in\mathbb{N}}$, supp $\mathcal{L}^{(S)} \setminus D_s(0) \neq \emptyset$, and $\mathcal{L}^{(S)}(\{0\}) = 0$.

(ii) Let $\mathcal{L}_d{}^{(S)}$ be the discrete part of $\mathcal{L}^{(S)}$. If there is $\zeta \in$ supp $\mathcal{L}_d{}^{(S)} \cap \overline{C_o} \setminus D_s(0)$, then $\rho \geq s$.

Proof: By Lemma 1, the given sequence has subsequences for which the $b_{mn}(\Lambda)$ converge. For every such subsequence S, $\liminf_S |Z_\Lambda(0)|^{1/|\Lambda|} \geq r > 0$ as well. Pick one of these subsequences, $S = (\Lambda_k)_{k\in\mathbb{N}}$. In terms of the zeros, $Z_\Lambda(0) = p_o(\Lambda) = \prod_{\lambda\in\mathcal{N}_\Lambda} (-\lambda)^{\mu_\Lambda(\lambda)}$. By hypothesis, for $\varepsilon > 0$ there is $k_o \in \mathbb{N}$ such that for all $k \geq k_o$, $|p_o(\Lambda_k)|^{1/|\Lambda_k|} > r - \varepsilon$. Choose $\varepsilon < r$, then

$$\frac{1}{|\Lambda_k|} \log |p_o(\Lambda_k)| = \frac{1}{|\Lambda_k|} \sum_{\lambda\in\mathcal{N}_{\Lambda_k}} \mu_{\Lambda_k}(\lambda) \log|\lambda| = \int d\mathcal{L}_{\Lambda_k}(z) \log|z| > \log(r-\varepsilon). \tag{4.3}$$

For $n \in \mathbb{N}$, define $\chi_n(z) = \max\{\log \frac{1}{n}, \log|z|\}$, then for all $z \in \overline{D_{M_0}}$ and all $n \in \mathbb{N}$, $-\log n \leq \chi_n(z) \leq \log M_o$ and $\chi_n(z) \geq \chi_{n+1}(z)$. For all $z \neq 0$, $\chi_n(z) \to \log|z|$ as $n \to \infty$. Also, for all $\varepsilon \in (0, r)$,

$$\mathcal{L}_{\Lambda_k}(\chi_n) = \int d\mathcal{L}_{\Lambda_k}(z)\chi_n(z) \geq \int d\mathcal{L}_{\Lambda_k}(z) \log|z| > \log(r-\varepsilon). \tag{4.4}$$

Since $\chi_n \in \mathcal{C}$, for all n, $\mathcal{L}_{\Lambda_k}(\chi_n) \to \mathcal{L}^{(S)}(\chi_n)$ as $k \to \infty$, and so

$$\mathcal{L}^{(S)}(\chi_n) \geq \log r \tag{4.5}$$

follows for all $n \in \mathbb{N}$ from (4.4) by taking $\varepsilon \to 0$, and therefore $\mathcal{L}^{(S)}(\{0\})$ must be zero. Thus, calling $\chi(z) = \log|z|$, $\chi_n \searrow \chi$ a.e. ($\mathcal{L}^{(S)}$), and by (4.5) and the monotone convergence theorem, $\int d\mathcal{L}^{(S)}\chi$ exists and $\int d\mathcal{L}^{(S)}\chi \geq \log r$. If supp $\mathcal{L}^{(S)}$ were a subset of D_s, then $\int d\mathcal{L}^{(S)}(z) \log|z| < a_0 \log s = \log r$ because $\log|z| < \log s$ there, so (i) must hold. To see (ii), note that by construction, the expansion (4.1) in $1/M$ for X is the same as for $X^{(S)}$, as given by (2.26). So, on C_0, $X = X^{(S)}$. Every $z \in$ supp ${\mathcal{L}_d}^{(S)}$ produces a pole in $X^{(S)}$. If ρ were less than s, X would be bounded for all M with $|M| \geq s$. But the limit of $|X^{(S)}(M)|$ as $M \to \zeta$ is infinite. ■

Remark: 1. It suffices to show the assumptions made in Proposition 1 for a single sequence tending to infinity, because the expansion coefficients a_n are independent of the chosen sequence S, and because only a lower bound and not equality is stated.

2. An example which satisfies (4.2) is given in [Sa,GLS]. The assumption on ${\mathcal{L}_d}^{(S)}$ made in Proposition 1 is sufficient but not necessary to get a bound on ρ. However, that some additional assumption about $\mathcal{L}^{(S)}$ is necessary to bound ρ from below can be seen from Example 2 in the previous section. In this example, $\rho = 0$ although supp $\mathcal{L}^{(S)}$ is the circle with radius M_1. On the other hand, the analytic continuation to the inside of the circle which the expansion defines, $X(M) = 1/M$, has nothing to do with the actual behaviour of the function. One may expect the alternative that either $\rho \geq s$ or the continuation X differs from $X^{(S)}$ to be true more generally. To further elaborate this point, the following statement is useful.

Lemma 2: If γ is a simple closed curve so that $\oint_\gamma dM \int d\mathcal{L}^{(S)}(z) \frac{1}{|M-z|}$ converges, then

$$\oint_\gamma X^{(S)}(M)dM = \mathcal{L}^{(S)}(\text{Int } \gamma), \tag{4.6}$$

where Int γ denotes the interior of γ.

Proof: By Fubini's theorem, the left side of (4.6) exists and the order of integrations can be exchanged, so

$$\begin{aligned}\oint_\gamma X^{(S)}(M)dM &= \int d\mathcal{L}^{(S)}(z) \oint_\gamma dM \frac{1}{M-z} = \\ &= \int d\mathcal{L}^{(S)}(z)\chi_{\text{Int}\gamma}(z) = \mathcal{L}^{(S)}(\text{Int } \gamma).\end{aligned} \tag{4.7}$$

■

Remark: The assumption in the lemma is fulfilled for all γ which stay away from supp $\mathcal{L}^{(S)}$, but it also holds at least in those parts of supp $\mathcal{L}^{(S)}$ where $\mathcal{L}^{(S)}$ has a bounded density.

Proposition 2: For $0 \le s < M_o$, let $\tilde{C}_s$ be the connected component of $C_o \setminus \overline{D}_s(0)$ that contains $\mathbb{C} \setminus \overline{D_{M_o}}$.
1. If, for some $s \in (0, M_o)$, there is a simple closed curve γ in $\tilde{C}_s$ around a part Σ of supp $\mathcal{L}^{(S)}$, which, in $\mathbb{C} \setminus \overline{D}_s(0)$, is nullhomotopic, then $\rho \ge s$.
2. Let $z_o \in$ supp $\mathcal{L}^{(S)}$, $0 < s < |z_o|$, and $0 < r < |z_o| - s$ such that for almost all $\varphi \in [0, 2\pi]$, $z_o + re^{i\varphi} \in \tilde{C}_s$, and that $\oint\limits_{|M-z_o|=r} dM \int d\mathcal{L}^{(S)}(z) \frac{1}{|M-z|} < \infty$. Then $\rho \ge s$.
3. If there is another connected component C_1 of $\mathbb{C} \setminus$ supp $\mathcal{L}^{(S)}$, and $C_1 \setminus D_\rho(0)$ is not empty and has a connected component R that contains no weak accumulation points, but contains a simple closed curve γ which, in the set $\mathbb{C} \setminus D_\rho(0)$, is homotopic to the circle with radius M_o, then $X(M) = X^{(S)}(M)$ can hold only for a subset of R that has no limit point in R.

Proof: 1. Assume that $\rho < s$. By the identity theorem, $X^{(S)}(M) = X(M)$ for all $M \in \tilde{C}_s$, and so their integrals along γ are the same as well. But since γ is nullhomotopic in $\mathbb{C} \setminus \overline{D}_s(0)$, $\oint\limits_\gamma X(M)dM = 0$, whereas by the Lemma, $\oint\limits_\gamma X^{(S)}(M)dM = \mathcal{L}^{(S)}(\Sigma) > 0$.
2. Again, assume $\rho < s$, then, by the identity theorem, for all $M \in \tilde{C}_s$, $X(M) = X^{(S)}(M)$. Since $\overline{D}_r(z_o) \subset \mathbb{C} \setminus \overline{D}_s(0)$, $\oint\limits_{|M-z_o|=r} X(M)dM = 0$. Since the integrands are equal a.e., the integral of $X^{(S)}$ also exists, and $\int\limits_0^{2\pi} X^{(S)}(z_o + re^{i\varphi})d\varphi = 0$. But, by hypothesis, Lemma 2 applies, and so $\mathcal{L}^{(S)}(D_r(z_o)) = 0$, which contradicts $z_o \in$ supp $\mathcal{L}^{(S)}$.
3. R is connected, and both X and $X^{(S)}$ are analytic on R. Thus, if the set of points $M \in R$ where $X(M) = X^{(S)}(M)$ has a limit point in R, $X = X^{(S)}$ on R. Then, by homotopy invariance of the integral over X,

$$\begin{aligned}\oint\limits_\gamma X^{(S)}(M)dM = \oint\limits_\gamma X(M)dM = \oint\limits_{|M|=M_o+1} X(M)dM = \\ = \oint\limits_{|M|=M_o+1} X^{(S)}(M)dM = a_o\end{aligned} \tag{4.8}$$

by definition of $\mathcal{L}^{(S)}$. On the other hand, in finite volume, the integral of X_Λ over γ (or a slight deformation of γ which avoids isolated limit points) is always less than a_o because by the residue theorem

$$\oint\limits_{|M|=M_o+1} X_\Lambda(M)dM - \oint\limits_\gamma X_\Lambda(M)dM = \frac{1}{|\Lambda|}\mathcal{L}_\Lambda(\mathbb{C} \setminus \text{ Int } \gamma) \tag{4.9}$$

and the right side has a positive limit as $\Lambda \to \infty$ because there are strong accumulation points of zeros in $\mathbb{C} \setminus$ Int γ. Since there are no weak accumulation points in R, $X_\Lambda(M) \underset{\Lambda\to\infty}{\longrightarrow} X^{(S)}(M)$ for all $M \in R$, and so $\oint_\gamma X^{(S)}(M)dM < a_o$ which contradicts (4.8). ∎

Remark: Note again that, in 3., it is indeed $X^{(S)}$ which is the limit of X_Λ in R as $\Lambda \to \infty$, so 3. really shows that the expansion does not represent the thermodynamic limit of the physical function. Example 2 is a special case of 3., and 2. is true e.g. if the measure has support on a curve which ends at z_o, so that X has a branch point there. If this curve is also such that a neighbourhood of it can be mapped conformally to a neighbourhood of the real line, a proposition in [RS4] can be applied to see that $X^{(S)}$ is either discontinuous or singular at the points in supp $\mathcal{L}^{(S)}$ (non–analyticity of $X^{(S)}$ in supp $\mathcal{L}^{(S)}$ also follows for general supp $\mathcal{L}^{(S)}$ from an argument using Lemma 2 if the measure is absolutely continuous or discrete).

Sum Rule: Let $a_o = 2$ and assume that for all $M \in \mathbb{R}$ and all finite $\Lambda \subset \mathbb{Z}^d$, $Z_\Lambda(M) > 0$, so that all zeros are off the real axis and come in complex conjugate pairs. Define

$$\theta_M(z) = \frac{1}{(M-z)(M-\overline{z})}. \tag{4.10}$$

Let $M \notin \mathcal{N}_\Lambda$, then

$$\mathcal{L}_\Lambda(\theta_M) = \frac{1}{|\Lambda|}\sum_{k=1}^{L}\frac{1}{(M-\lambda_k)(M-\overline{\lambda_l})} = \frac{1}{2i}\frac{1}{|\Lambda|}\sum_{k=1}^{L}\frac{1}{\text{Im }\lambda_k}\left(\frac{1}{M-\lambda_k}-\frac{1}{M-\overline{\lambda_k}}\right). \tag{4.11}$$

$\mathcal{L}_\Lambda(\theta_M)$ is meromorphic in M and decays as $|M|^{-2}$ for $|M| \to \infty$. Therefore $\int_{\mathbf{R}} dM \mathcal{L}_\Lambda(\theta_M)$ exists and can be evaluated as a residue,

$$\int_{-\infty}^{\infty} dM \mathcal{L}_\Lambda(\theta_M) = \frac{\pi}{|\Lambda|}\sum_{k=1}^{L}\frac{1}{|\text{Im }\lambda_k|} \tag{4.12}$$

Proposition 3: Let $(\Lambda, M) \mapsto Z_\Lambda(M)$ be a Lee–Yang–system for which $Z_\Lambda(M) > 0$ for all $M \in \mathbb{R}$ and all finite $\Lambda \subset \mathbb{Z}^d$. Assume that there is $\varepsilon > 0$ such that there are no accumulation points of zeros of Z_Λ in $U_\varepsilon = \{z : |\text{Im } z| < \varepsilon\}$, that is, that there is no phase transition on the real axis. Then the thermodynamic limit of (4.12) yields an upper bound for the integrated susceptibility:

$$\int_{-\infty}^{\infty} |\frac{\partial^2 f^{(S)}}{\partial M^2}| dM \leq \pi \int d\mathcal{L}^{(S)}(z) \frac{1}{|\mathrm{Im}\ z|} \leq \pi a_o \min_{z \in \mathrm{supp}\ \mathcal{L}^{(S)}} \frac{1}{|\mathrm{Im}\ z|} \leq \frac{\pi a_o}{\varepsilon} \tag{4.13}$$

Proof: Without loss it can be assumed that U_ε does not contain any isolated limit points either. From Theorem 3, supp $\mathcal{L}^{(S)} \cap U_\varepsilon = \emptyset$, so for all $z \in$ supp $\mathcal{L}^{(S)}$, $|\mathrm{Im}\ z|^{-1} \leq \varepsilon^{-1}$, and $\int d\mathcal{L}^{(S)}(z)|\mathrm{Im}\ z|^{-1}$ exists. By a similar argument as in the proof of Theorem 3, θ_M can be deformed in U_ε such that the resulting function is continuous and still gives $\mathcal{L}_\Lambda(\theta_M)$ and $\mathcal{L}^{(S)}(\theta_M)$ when $\mathcal{L}_\Lambda$ and $\mathcal{L}^{(S)}$ are applied to it, and therefore convergence $\mathcal{L}_\Lambda(\theta_M) \rightarrow \mathcal{L}^{(S)}(\theta_M)$ as $\Lambda \rightarrow \infty$ holds. A similar argument works for the integral over $z \mapsto |\mathrm{Im}\ z|^{-1}$. So (4.12) has a thermodynamic limit, and it remains to note that for $M \notin$ supp $\mathcal{L}^{(S)}$

$$\frac{\partial^2 f^{(S)}}{\partial M^2} = -\int d\mathcal{L}^{(S)}(z) \frac{1}{(M-z)^2}, \tag{4.14}$$

and thus for $M \in \mathbb{R}$

$$|\frac{\partial^2 f^{(S)}}{\partial M^2}| \leq \int d\mathcal{L}^{(S)}(z) \frac{1}{|M-z|^2} = \mathcal{L}^{(S)}(\theta_M). \quad \blacksquare \tag{4.15}$$

5 Illustration

In this section I show that a general class of lattice models with fermions are Lee–Yang systems. The variable M is the mass, and the expansion parameter $\kappa = 1/M$ is usually called the hopping parameter. The partition function on a finite lattice is a polynomial in M because of the nilpotency of the Grassmann variables. Assumption 2 is verified by showing that there is a convergent cluster expansion for large $|M|$.

Convergence of this expansion is well–known for the standard models of lattice gauge theory [Se]. In most of these cases the fermionic action is bilinear (four–fermion terms can be linearized by introduction of auxiliary fields), the fermions can then be integrated over, and the resulting determinant can be expanded in $1/M$. I choose a different strategy and estimate the Grassmann integral directly in terms of a norm on the Grassmann algebra. The product inequality for this norm allows for a simple proof of the activity estimates for finite–range interactions which may be arbitrary polynomials in the fermionic variables, for which the introduction of auxiliary fields would be very complicated, if viable at all. The norms used to estimate the fermionic integrals are defined in the Appendix.

Let $R > 0$ and Δ be a finite set, $|\Delta| = D$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, and denote by $\mathcal{G}_\Lambda$ the Grassmann algebra with generators $\psi_\alpha(x)$, $\bar{\psi}_\alpha(x)$, $\alpha \in \Delta$, $x \in \Lambda$. The set $\Delta \times \Lambda$ can be assumed to be ordered in some way and then the definitions of the Appendix apply directly. The interaction S of the fermions consists of two parts

$$S = MS_0 + S_1 \tag{5.1}$$

$$S_0 = \sum_{x\in\Lambda} \sum_{\alpha,\beta\in\Delta} \bar\psi_\alpha(x) W_{\alpha\beta} \psi_\beta(x) \tag{5.2}$$

with $W \in GL(D,\mathbb{C})$ satisfying $\|W^{-1}\| = 1$, where $\|\cdot\|$ is the norm for matrices defined in the Appendix. Call $S_R(\Lambda) = \{X \subset \Lambda : \text{ diam } X \le R\}$ (here diam $X = \max\{|x-y| : x,y \in X\}$).

$$S_1 = \sum_{X\in S_R(\Lambda)} \sum_{\substack{p\in 2\mathbb{N}\\ p\ge|X|}} \sigma_{p,X}, \tag{5.3}$$

$$\sigma_{p,X} = \sum_{\substack{P,P'\subset X\times\Delta\\ P_1\cup P'_1 = X, |P|+|P'|=p}} F_{PP'}(\Phi) \bar\psi^P \psi^{P'} \tag{5.4}$$

(for $P = (Y,C) \subset X \times \Delta$, $P_1 = Y$) and $\Phi \in \mathcal{B}$, where $\mathcal{B}$ denotes a set of parameters (part of which may be random variables which will be integrated over afterwards), $F_{PP'}(\Phi)$ is only assumed to be locally bounded for the moment. Define

$$Z^0_\Lambda = \int_{\mathcal{G}_\Lambda} d\psi d\bar\psi\, e^{MS_0}, \tag{5.5}$$

for $\varphi \in \mathcal{G}_\Lambda$

$$\langle\varphi\rangle_{0,\Lambda} = Z^{0\,-1}_\Lambda \int_{\mathcal{G}_\Lambda} d\psi d\bar\psi\, e^{MS_0}\varphi, \tag{5.6}$$

and

$$\mathcal{Z}_\Lambda(\Phi) = \langle e^{S_1}\rangle_{0,\Lambda}. \tag{5.7}$$

Denoting $Q = 1 \otimes W$, $\|Q^{-1}\| = |M|^{-1} = q$, $\langle\varphi\rangle_{0,\Lambda} = \langle\varphi\rangle_Q$ with $\langle\cdot\rangle_Q$ given in the Appendix, and it is clear from an expansion of $\mathcal{Z}_\Lambda(\Phi)$ in powers of M that Assumption 1 is fulfilled, with $a_0 = D$.

Polymer expansion for $\mathcal{Z}_\Lambda(\Phi)$: Define polymers $\gamma = \{(p_1,X_1),\dots,(p_r,X_r)\}$, where $r \in N$, $X_i \in S_R(\Lambda)$, $p_i \in 2\mathbb{N}$ with $|X_i| \le p_i \le 2D|X_i|$, and where $X_1,\dots,X_r$ are such that the graph $G(X_1,\dots,X_r)$ is connected ($G(X_1,\dots,X_r)$ is defined as the graph with vertex set $\{1,\dots,r\}$ such that the line (ij) is in $G(X_1,\dots,X_r)$ if and only if $X_i \cap X_j \neq \emptyset$). The set of all polymers is called $\Gamma_R(\Lambda)$, and for $\gamma = \{(p_1,X_1),\dots,(p_r,X_r)\} \in \Gamma_R(\Lambda)$, supp $\gamma = X_1 \cup \ldots \cup X_r$, and $|\gamma| = \sum_{i=1}^{r} p_i$. Then, with $\rho_{p,X} = e^{\sigma_{p,X}} - 1$,

$$\begin{aligned} e^{S_1} &= \prod_{X\in S_R(\Lambda)} \prod_{\substack{p\in 2\mathbb{N}\\ |X|\le p\le 2|X|D}} (1+\rho_{p,X}) = \\ &= \sum_{\Gamma\subset\Gamma_R(\Lambda)} \prod_{\gamma\in\Gamma} a(\gamma) \prod_{\gamma\neq\gamma'} (1+g(\gamma,\gamma')) \end{aligned} \tag{5.8}$$

where $a(\gamma) = \prod_{(p,X)\in\gamma} \rho_{p,X}$, and $g(\gamma,\gamma') = -1$ if the supports of γ and γ' are not disjoint, zero otherwise. The activity of the polymer γ is

$$z(\gamma) = \langle a(\gamma)\rangle_{o,\Lambda} \tag{5.9}$$

and by definition of the polymers, $\mathcal{Z}_\Lambda(\Phi)$ can be written in the standard form of a polymer expansion

$$\mathcal{Z}_\Lambda(\Phi) = \sum_{\Gamma\subset\Gamma_R(\Lambda)} \prod_{\gamma\in\Gamma} z(\gamma) \prod_{\gamma\neq\gamma'} (1+g(\gamma,\gamma')). \tag{5.10}$$

Consequently,

$$S_\Lambda(\Phi) = \frac{1}{|\Lambda|}\log \mathcal{Z}_\Lambda(\Phi) = \sum_{n\geq 1}\frac{1}{n!}\sum_{\gamma_1,\dots,\gamma_n\in\Gamma_R(\Lambda)} \mathcal{U}(\gamma_1,\dots,\gamma_n)\prod_{i=1}^n z(\gamma_i), \tag{5.11}$$

with the Ursell functions $\mathcal{U}$ [Se], and convergence of the expansion for $S_\Lambda(\Phi)$ can be shown if activity and entropy estimates hold.

Lemma 2:

1. (Entropy estimate) Let $E_\Lambda(\gamma_o,s) = |\{\gamma\in\Gamma_R(\Lambda) : g(\gamma,\gamma_o) = -1, |\gamma| = s\}$. then there are $s_o\in\mathbb{N}$ and $K>0$ such that for all $s\geq s_o$ and all Λ,

   $$E_\Lambda(\gamma_o,s) \leq |\text{ supp }\gamma_o|K^s \leq |\gamma_o|K^s. \tag{5.12}$$

   More concretely, $K\leq 6\cdot 2^{V/2}$, where $V = |\{x\in\mathbb{Z}^d : |x|\leq 2R\}|$ is the volume of a ball of radius $2R$ in $\mathbb{Z}^d$.

2. (Activity estimate) Let

   $$\mu(\Phi) = \sup_{X\in S_R(\Lambda)} \max_{P,P'\subset X\times\Delta} |F_{PP'}(X,\Phi)| < \infty, \tag{5.13}$$

   without loss, assume $\mu(\Phi)\geq 1$. If $|M| > 4^D$, for all $\gamma\in\Gamma_R(\Lambda)$,

   $$|z(\gamma)| \leq \left(\frac{\mu(\Phi)e^{\mu(\Phi)}4^D}{|M|}\right)^{|\gamma|}. \tag{5.14}$$

Proof: 1. See Appendix B. 2. Recall $q = |M|^{-1}$. Since $\mu(\Phi)<\infty$, by definition $(A.8)$ of $\|\cdot\|_q$,

$$\begin{aligned}\|\sigma_{p,X}\|_q &\leq \sum_{\substack{P,P'\subset X\times\Delta\\ P_1\cup P'_1=X,|P|+|P'|=p}} |F_{PP'}(X,\Phi)|q^{|P|+|P'|} \leq \\ &\leq q^p\mu(\Phi)\sum_{\substack{P,P'\subset X\times\Delta\\ P_1\cup P'_1=X,|P|+|P'|=p}} 1 \leq (4^Dq)^p\mu(\Phi).\end{aligned} \tag{5.15}$$

Now by (A.21) and repeated application of (A.9),

$$
\begin{aligned}
|\langle a(\gamma)\rangle_{o,\Lambda}| \le \|a(\gamma)\|_q \le \prod_{(p,X)\in\gamma} \|e^{\sigma_{p,X}} - 1\|_q \le \\
\le \prod_{(p,X)\in\gamma} \left(e^{\|\sigma_{p,X}\|_q} - 1\right) \le \prod_{(p,X)\in\gamma} \|\sigma_{p,X}\|_q e^{\|\sigma_{p,X}\|_q},
\end{aligned}
\tag{5.16}
$$

where $|e^\alpha - 1| \le |\alpha| e^{|\alpha|}$ was used in the last step. Let $\gamma = \{(p_1, X_1), \dots, (p_r, X_r)\}$. If $|M| > 4^D$, $\sum (4^D q)^{p_i} \le r \le \sum p_i = |\gamma|$, so

$$
|z(\gamma)| \le (\mu(\Phi) e^{\mu(\Phi)})^r (4^D q)^{\sum p_i} \le (\mu(\Phi) e^{\mu(\Phi)} 4^D q)^{|\gamma|}. \quad \blacksquare \tag{5.17}
$$

Corollary 2: Let $\mathcal{K} \subset \mathcal{B}$ such that for all Λ

$$
\mathcal{M} = \sup_{\Phi \in \mathcal{K}} \mu(\Phi) < \infty. \tag{5.18}
$$

Then there is $M_1(\mathcal{M}) > 0$ such that for all $\Phi \in \mathcal{K}$ and all Λ, the expansion (5.11) converges absolutely for all $|M| > M_1(\mathcal{M})$, and in this region, $\lim_{\Lambda \to \infty} S_\Lambda(\Phi)$ exists, is independent of the chosen sequence and an analytic function of M. Assumption 2 holds, so $(\Lambda, M) \mapsto \mathcal{Z}_\Lambda(\Phi)(M)$ is a Lee–Yang–System. Moreover, $S_\Lambda(\Phi)$ is analytic in any parameter on which $F_{PP'}(\Phi)$ depends analytically.

Proof: For all $\Phi \in \mathcal{K}$, $|M| > 4^D$, $|z(\gamma)| \le (\mathcal{M} e^{\mathcal{M}} 4^D / |M|)^{|\gamma|}$. Therefore convergence uniformly in Λ is a standard consequence [Se] of Lemma 2. Since the polymers have bounded supports, no activity eventually depends on Λ any more as $\Lambda \to \infty$, so the chosen sequence is inessential. Existence of the limit function and the analyticity statements follow from Vitali's theorem. To complete the proof of Assumption 2, note that the uniform convergence implies that the coefficients of the expansion in $1/M$ in finite volume converge to those in infinite volume. ■

This generalizes the statement that the expansion in $1/M$ of the fermion determinant, viewed as effective action from integrating out fermions with a bilinear interaction, converges. For gauge fields $\Phi = U$ coupled to the fermions via the discretized covariant derivative [Se], boundedness (5.18) holds globally for all U. For bosonic fields coupled to the fermions by Yukawa interactions, $F_{PP'}(\Phi) \sim \Phi$ and so boundedness holds on compact subsets.

If the Φ are random variables themselves, and their interaction is such that it admits a convergent expansion as well, convergence for the combined polymer expansion can be shown if μe^μ can be controlled using the interaction of the Φ. For brevity, the arguments will only be sketched here for some models of interest in lattice gauge theory. The following are Lee–Yang systems:

1. Models with gauge fields $\Phi = U$, $U : \Lambda_1 \to \mathcal{G}$, where Λ_1 denotes the set of links of Λ, and $\mathcal{G}$ is a compact group, in the "compact formulation", where the integration measure of the U_l is the Haar measure on $\mathcal{G}$. The interaction between fermions and gauge fields is as usual,

$$S_{gf} = \sum_{\substack{l\in\Lambda_1 \\ l=(x,\mu)}} \left(\bar\psi(x) T_l^{(+)} \otimes P(U_l)\psi(x+e_\mu) + \bar\psi(x+e_\mu) T_l^{(-)} \otimes P(U_l)^{-1}\psi(x) \right), \tag{5.19}$$

where $P(U_l)$ is the parallel transporter, P denoting a unitary representation of $\mathcal{G}$ under which the ψ transform, the matrix functions $T_l^{(\varepsilon)}$ are assumed to be bounded in l. The interaction of the gauge fields is βS_g where S_g is a bounded function of U, e.g. the Wilson plaquette action [Se]. Then $\mu(U) \le \sup_{l\in\Lambda_1} \left(\|T_l^{(+)}\| + \|T_l^{(-)}\| \right)$ is independent of U. Therefore, if $|\beta|$ is small enough and $|M|$ is large enough, activity estimates hold for all polymers in the combined expansion.

2. QED in noncompact formulation, that is, with integration over real gauge fields A_l and a suitable gauge fixing term. $U_l = e^{iA_l}$, so the bound for μ is the same as in the previous case. This model cannot be treated by an ordinary polymer expansion because the photon is massless, but since S_{gf} is gauge invariant, a combination of Corollary 2 and the renormalization group in the form of [BY] can be used to extend the proof that analyticity in $1/M$ holds uniformly in the volume, as given in [DH] using the methods of [BY], to models with non–bilinear gauge–invariant interactions of the fermions.

3. Scalar fields coupled to the fermions by Yukawa–type interactions. As remarked above, $\mu(\Phi) \sim |\Phi|$ in that case, but it is possible to split the Higgs potential V into two parts, $V = V_0 + V_1$, such that V_0 can be used to control μe^{μ} and convergence of the expansion can be shown if the boson hopping parameter is small enough and the other couplings are suitable.

Appendix A

Let $L \geq 1$, $\underline{L} = \{1, \ldots, L\}$ and $\mathcal{G}$ be the Grassmann algebra over $\mathbb{C}$ generated by $\psi_1, \ldots, \psi_L$ and $\bar{\psi}_1, \ldots, \bar{\psi}_L$. Every $I \subset \underline{L}$ defines an ordered sequence $i_1 < i_2 < \ldots < i_{|I|}$ such that $I = \{i_1, \ldots, i_{|I|}\}$, and, defining the monomials

$$\bar{\psi}^I = \prod_{r=1}^{|I|} \bar{\psi}_{i_r} \tag{A.1}$$

and

$$\psi^I = \prod_{r=1}^{|I|} \psi_{i_r}, \tag{A.2}$$

for every $f \in \mathcal{G}$ there is a finite sequence of complex numbers $(\phi_{IJ}(f))_{I,J \subset \underline{L}}$ such that

$$f = \sum_{I,J \subset \underline{L}} \phi_{IJ}(f) \bar{\psi}^I \psi^J. \tag{A.3}$$

The map $\phi : \mathcal{G} \to \mathbb{C}^{(2^{2L})}$, $f \mapsto \phi_{IJ}(f)$ is a vector–space isomorphism. In the notation of ordered sequences, this representation of f reads

$$f = \sum_{k,l=0}^{L} \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_l}} \phi_{\{i_1,\ldots,i_k\}\{j_1,\ldots,j_l\}}(f) \prod_{r=1}^{|I|} \bar{\psi}_{i_r} \prod_{r=1}^{|J|} \psi_{i_r}. \tag{A.4}$$

The *Grassmann integral* is the linear map $\int_{\mathcal{G}} d\psi d\bar{\psi} : \mathcal{G} \to \mathbb{C}$:,

$$f \mapsto \int_{\mathcal{G}} d\psi d\bar{\psi} f = (-1)^{L(L-1)/2} \phi_{\underline{L}\underline{L}}(f). \tag{A.5}$$

For $Q \in GL(L, \mathbb{C})$ the fermionic bilinear form with covariance $Q^{-1} = C$ is

$$(\bar{\psi}, Q\psi) = \sum_{i,j \in \{1,\ldots,L\}} \bar{\psi}_i Q_{ij} \psi_j. \tag{A.6}$$

and the associated expectation value is

$$\langle f \rangle_Q = \frac{1}{\det Q} \int_{\mathcal{G}} d\psi d\bar{\psi} e^{(\bar{\psi}, Q\psi)} f. \tag{A.7}$$

The purpose of the following is to introduce a norm $\|.\|$ on $\mathcal{G}$ such that for all $f \in \mathcal{G}$ $\langle f \rangle_Q \leq \|f\|$.

Definition: Let $q > 0$.

$$\|f\|_q = \sum_{I,J \subset \underline{L}} |\phi_{IJ}(f)| q^{\frac{|I|+|J|}{2}} = \sum_{i,j=0}^{L} q^{\frac{i+j}{2}} \sum_{|I|=i,|J|=j} |\phi_{IJ}(f)|. \tag{A.8}$$

Proposition: For all $q > 0$ $(\mathcal{G}, \|\cdot\|_q)$ is a Banach algebra, in particular, for all $f \in \mathcal{G}$ and $g \in \mathcal{G}$,

$$\|fg\|_q \leq \|f\|_q \|g\|_q. \tag{A.9}$$

Proof: Obviously, $\mathcal{G}$ is a $\mathbb{C}$–algebra, and for $q > 0$, $\|\cdot\|_q$ is a norm on $\mathcal{G}$. Since $\mathcal{G} \simeq \mathbb{C}^{2^{2L}}$ as a vector space, it is complete. To see the product inequality $(A.9)$, let $f, g \in \mathcal{G}$. Then

$$\phi_{MN}(fg) = \sum_{\substack{I \dot\cup I' = M \\ J \dot\cup J' = N}} \varepsilon(I, I', J, J') \phi_{IJ}(f) \phi_{I'J'}(g), \tag{A.10}$$

where the condition of disjoint union comes from nilpotency and $\varepsilon(I, I', J, J') \in \{-1, 1\}$ is a sign factor caused by anticommutativity when reordering the product. Applying the triangle inequality and summing over all M with $|M| = m$ and all N with $|N| = n$,

$$\|fg\|_q \leq \sum_{m,n=0}^{L} q^{\frac{m+n}{2}} \sum_{I,I',J,J'}{}' |\phi_{IJ}(f)| \cdot |\phi_{I'J'}(g)| \tag{A.11}$$

where the prime on the sum means that I, J, I', J' are restricted to $I \cap I' = \emptyset$, $|I \cup I'| = m$, and $J \cap J' = \emptyset$, $|J \cup J'| = n$. On the other hand,

$$\|f\|_q \|g\|_q = \sum_{m,n=0}^{L} q^{\frac{m+n}{2}} \sum_{\substack{|I|+|I'|=m \\ |J|+|J'|=n}} |\phi_{IJ}(f)| \cdot |\phi_{I'J'}(g)|, \tag{A.12}$$

and $(A.9)$ follows because the restriction in the sum $(A.11)$ is stronger than the one in $(A.12)$. ∎

Lemma:

$$\langle f \rangle_Q = \frac{1}{\det Q} \int_{\mathcal{G}} d\psi d\bar{\psi} e^{(\bar{\psi}, Q\psi)} f = \sum_{l=0}^{L} \sum_{|I|=|J|=l} \phi_{IJ}(f) \epsilon_{IJ} \det \Gamma_{JI}(C) \tag{A.13}$$

where $\Gamma_{JI}(C) \in M(l, \mathbb{C})$ has the coefficients $(\Gamma_{JI}(C))_{kl} = C_{j_k i_l}$ and $\epsilon_{IJ} \in \{-1, 1\}$.

Proof: Obviously, no term with $|I| \neq |J|$ can contribute to the integral, and by linearity,

$$\langle f \rangle_Q = \sum_{l=0}^{L} \sum_{I,J} \phi_{IJ}(f)\, R_{IJ} \tag{A.14}$$

with

$$R_{IJ} = \frac{1}{\det Q} \int_{\mathcal{G}} d\psi d\bar{\psi} e^{(\bar{\psi}, Q\psi)} \bar{\psi}^I \psi^J. \tag{A.15}$$

With sources $S \in M(L, \mathbb{C})$ for the bilinears

$$R_{IJ} = \left(\prod_{k=1}^{l} \frac{\partial}{\partial S_{i_k j_k}} \right) R(S) \mid_{S=0}, \tag{A.16}$$

where

$$R(S) = \frac{1}{\det Q} \int_{\mathcal{G}} d\psi d\bar{\psi} e^{(\bar{\psi}, (Q+S)\psi)} = \det(1 + CS). \tag{A.17}$$

Now, by definition of I_{IJ}, for all permutations $\pi, \rho \in S_l$

$$R_{\pi(I), \rho(J)} = \varepsilon(\pi) \varepsilon(\rho) R_{IJ}, \tag{A.18}$$

(here $I \subset \underline{L}$ is identified with the ordered sequence $i_1, \ldots, i_{|I|}$ which it defines and ε is the sign of the permutation), also by $(A.17)$, R_{IJ} is a function of $\Gamma_{JI}(C)$ alone, which is multilinear and of modulus at most one if $C = 1$. Up to a sign, it must therefore be the determinant of $\Gamma_{JI}(C)$. ■

Then

$$\begin{aligned} \det A &= \sum_{\pi \in S_n} \varepsilon(\pi) \prod_{i=1}^{n} A_{\pi(i),i} \leq \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{\pi(i),i} \leq \\ &\leq \sum_{\phi: \{1,\ldots,n\} \to \{1,\ldots,n\}} \prod_{i=1}^{n} A_{\phi(i),i} = \\ &= \prod_{i=1}^{n} (\sum_{j=1}^{n} A_{j,i}) \leq \|A\|^n, \end{aligned} \tag{A.19}$$

and $\|\tilde{A}\| \leq \|A\|$ for any submatrix $\tilde{A}$ of A, so minors of A are also bounded by $\|A\|^n$.

Proposition: For all $f \in \mathcal{G}$,

$$\langle f \rangle_Q \leq \|f\|_{\|Q^{-1}\|} \tag{A.20}$$

Proof: Insert $(A.19)$ into $(A.13)$. ■

Remark: (1) In a translationally invariant lattice theory with covariance C,

$$\|C\| = \sum_{y \in \Lambda} C_{xy} = C\hat{\ }(0), \tag{A.21}$$

the Fourier transform of the absolute value of the propagator at zero.

(2) For general $A \in M(n, \mathbb{C})$, Hadamard's inequality [Ha], $|\det A| \leq n^{n/2}|A|^n$ where $|A| = \sup_{i,j} |A_{ij}|$ gives the best possible bound for the determinant of A. The bound by $\|A\|$ is in general much worse because in the proof of $\det A \leq \|A\|$ all signs have been neglected (thus the same statement will be true if the determinant is replaced by a permanent). Actually, for the matrices which saturate Hadamard's inequality, $\|A\| = nA$. But the matrices encountered in the case of the hopping expansion are diagonal, so for them $\|A\| = A$ and $\|A\|$ is preferable because it obeys the product inequality $(A.9)$.

Appendix B

Introducing the notation $1(E) = 1$ for E true and $1(E) = 0$ for E false and remembering the definition of polymers,

$$\begin{aligned} E_\Lambda(\gamma_o, s) = & \sum_{\gamma \in \Gamma_R(\Lambda)} 1(g(\gamma, \gamma_o) = -1) 1(|\gamma| = s) = \\ & \leq \sum_{n \geq 1} \sum_{\substack{p_1, \ldots, p_n \in 2\mathbb{N} \\ p_1 + \ldots + p_n = s}} \epsilon_\Lambda(\gamma_o, p_1, \ldots, p_n), \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} \epsilon_\Lambda(\gamma_o, p_1, \ldots, p_n) = & \frac{1}{n!} \sum_{\substack{X_1, \ldots, X_n \in S_R(\Lambda) \\ |X_i| \leq p_i}} 1\left((X_1 \cup \ldots \cup X_n) \cap \text{ supp } \gamma_o \neq \emptyset\right) \\ & \times 1(G(X_1, \ldots, X_n) \text{ connected }). \end{aligned} \tag{B.2}$$

$(X_1 \cup \ldots \cup X_n) \cap \text{ supp } \gamma_o \neq \emptyset \Longrightarrow \exists j : X_j \cap \text{ supp } \gamma_o \neq \emptyset$, and, denoting the set of all tree graphs on $\{1, \ldots, n\}$ by $\mathcal{T}_n$, $G(X_1, \ldots, X_n)$ connected $\Longrightarrow \exists T \in \mathcal{T}_n : T \subset G(X_1, \ldots, X_n)$, so

$$\epsilon_\Lambda(\gamma_o, p_1, \ldots, p_n) \leq \frac{1}{n!} \sum_{T \in \mathcal{T}_n} \eta_\Lambda(T, \gamma_o, p_1, \ldots, p_n), \tag{B.3}$$

$$\begin{aligned}\eta_\Lambda(T, \gamma_o, p_1, \ldots, p_n) = \sum_{j=1}^{n} \sum_{\substack{X_1, \ldots, X_n \in \mathcal{S}_R(\Lambda) \\ |X_i| \leq p_i}} & 1(X_j \cap \text{ supp } \gamma_o \neq \emptyset) \\ & \times 1(G(X_1, \ldots, X_n) \supset T).\end{aligned} \tag{B.4}$$

The characteristic function in the last expression can be rewritten as

$$1(G(X_1, \ldots, X_n) \supset T) = \prod_{(vw) \in T} 1(X_v \cap X_w \neq \emptyset). \tag{B.5}$$

Inserting this, $\eta_\Lambda(T, \gamma_o, p_1, \ldots, p_n)$ can now be estimated by "stripping off the leaves of the tree" [Br]. Let $n \geq 2$, fix $T \in \mathcal{T}_n$ and $j \in \{1, \ldots, n\}$, let $d_1, \ldots, d_n$ be the incidence numbers of T, and order the vertices according to their distance from j: $V_k = \{i \in \{1, \ldots, n\} : d(i,j) = k\}$, where $d(i,j)$ is the number of steps necessary to go from i to j over bonds of T, and let $W_{i,k+1} = \{l \in V_{k+1} : (il) \in T\}$. Then $V_k = \dot{\bigcup\limits_{i \in V_{k-1}}} W_{i,k}$, for $w \in V_k$, $|W_{w,k}| = d_w - 1 + \delta_{k1}$, and

$$\{1, \ldots, n\} = \dot{\bigcup_{k=0}^{\bar{k}-1}} \dot{\bigcup_{i \in V_k}} W_{i,k+1}. \tag{B.6}$$

Define $\mathcal{V}_k = \bigcup\limits_{l=k}^{\bar{k}} V_k$, $N = |\{x \in \mathbb{Z}^d : |x| \leq 2R\}$, $B = 2^N$,

$$P_k = \prod_{v \in \mathcal{V}_k} (Bp_v)^{d_v - 1}, \tag{B.7}$$

and

$$\vartheta_k = \sum_{\substack{(X_v)_{v \in \mathcal{V}_k} \\ |X_v| \leq p_v, X_v \in \mathcal{S}_R(\Lambda)}} \prod_{\substack{(vw) \in T \\ v, w \in \mathcal{V}_{k-1}}} 1(X_v \cap X_w \neq \emptyset), \tag{B.8}$$

then the sum over the X can be estimated using the following inequality: for all $k \in \{1, \ldots, \bar{k}\}$,

$$\vartheta_k \leq P_k \prod_{w \in V_{k-1}} (B|X_w|)^{d_w - 1 + \delta_{k,1}}. \tag{B.9}$$

Proof of $(B.9)$: Induction downwards in k. Let $k = \bar{k}$. Then

$$\begin{aligned}\vartheta_{\bar{k}} = & \sum_{\substack{(X_v)_{v \in V_{\bar{k}}} \\ |X_v| \leq p_v, X_v \in \mathcal{S}_R(\Lambda)}} \prod_{v \in V_{\bar{k}-1}} \prod_{w \in W_{v,\bar{k}}} 1(X_v \cap X_w \neq \emptyset) \leq \\ & \leq \prod_{v \in V_{\bar{k}-1}} \prod_{w \in W_{v,\bar{k}}} \sum_{\substack{X_w \in \mathcal{S}_R(\Lambda) \\ |X_w| \leq p_w}} 1(X_v \cap X_w \neq \emptyset).\end{aligned} \tag{B.10}$$

For $t \geq 0$

$$\sum_{\substack{X_w \in S_R(\Lambda) \\ |X_w| \leq p_w}} 1(X_v \cap X_w \neq \emptyset) |X_w|^t \leq \sum_{x \in X_v} \sum_{r=1}^{p_w} r^t \sum_{\substack{X_w \in S_R(\Lambda) \\ |X_w| \leq r}} 1(X_w \ni x) \leq \\ \leq |X_v| {p_w}^t \sum_{r=1}^{p_w} \binom{N}{r} \leq |X_v| {p_w}^t B. \tag{B.11}$$

Thus

$$\vartheta_{\bar{k}} \leq \prod_{v \in V_{\bar{k}-1}} \prod_{w \in W_{v,\bar{k}}} |X_v| {p_w}^{\circ} B = \prod_{v \in V_{\bar{k}-1}} (B|X_v|)^{|W_{v,\bar{k}}|}, \tag{B.12}$$

which shows $(B.9)$ for $k = \bar{k}$ because $|W_{v,\bar{k}}| = d_v - 1$, and $d_v = 1$ for all $v \in V_{\bar{k}}$. Now let $1 \leq k \leq \bar{k} - 1$ and assume $(B.9)$ to be true for $l = k+1$. Since $k+1 \geq 2$, $\delta_{k+1,1} = 0$. By definition and $(B.9)$,

$$\begin{aligned} \vartheta_k = & \sum_{\substack{(X_v)_{v \in V_k} \\ |X_v| \leq p_v, X_v \in S_R(\Lambda)}} \prod_{\substack{(v,w) \in T \\ v \in V_{k-1}, w \in V_k}} 1(X_v \cap X_w \neq \emptyset) \vartheta_{k+1} \leq \\ \leq & \sum_{\substack{(X_v)_{v \in V_k} \\ |X_v| \leq p_v, X_v \in S_R(\Lambda)}} \prod_{\substack{(v,w) \in T \\ v \in V_{k-1}, w \in V_k}} 1(X_v \cap X_w \neq \emptyset) P_{k+1} \prod_{w \in V_k} (B|X_w|)^{d_w - 1} \leq \\ \leq & P_{k+1} \prod_{v \in V_{k-1}} \prod_{w \in W_{v,k}} \sum_{\substack{X_w \in S_R(\Lambda) \\ |X_w| \leq p_w}} 1(X_v \cap X_w \neq \emptyset) \left(B|X_w|\right)^{d_w - 1}. \end{aligned} \tag{B.13}$$

The summation over X is again done by $(B.11)$

$$\begin{aligned} \vartheta_k \leq & P_{k+1} \prod_{v \in V_{k-1}} \prod_{w \in W_{v,k}} \left(|X_v| B \left(B|X_w|\right)^{d_w - 1} \right) \leq \\ \leq & \left(\prod_{v \in V_{k-1}} (B|X_v|)^{|W_{v,k}|} \right) P_{k+1} \prod_{w \in V_k} (B p_w)^{d_w - 1} = \\ = & P_k \prod_{v \in V_{k-1}} (B|X_v|)^{d_v - 1 + \delta_{k1}}. \qquad \blacksquare \end{aligned} \tag{B.14}$$

Taking $(B.9)$ with $k=1$, and using again $(B.11)$ to estimate the last sum over X_j,

$$\begin{aligned}\eta_\Lambda(T,\gamma_o,p_1,\ldots,p_n) &\le \sum_{j=1}^{n} \sum_{\substack{X_j \cap \text{supp } \gamma_o \neq \emptyset \\ |X_j|\le p_j}} (B|X_j|)^{d_j} \prod_{v\in\mathcal{V}_1} (Bp_v)^{d_v-1} \le \\ &\le \sum_{j=1}^{n} |\text{ supp } \gamma_o| {p_j}^{d_j} B^{d_j+1} \prod_{v\in\mathcal{V}_1} (Bp_v)^{d_v-1} = \\ &= \sum_{j=1}^{n} |\text{ supp } \gamma_o| p_j B^2 \prod_{l=1}^{n} (Bp_l)^{d_l-1} \\ &= |\text{ supp } \gamma_o| s B^{n+2} \prod_{l=1}^{n} {p_l}^{d_l-1},\end{aligned} \tag{B.15}$$

where in the last step $\sum(d_i-1)=n$ was used. Summing over trees and using Cayley's theorem, get for $n\ge 2$

$$\begin{aligned}\epsilon_\Lambda(\gamma_o,p_1,\ldots,p_n) &\le |\text{ supp } \gamma_o| s \frac{B^{n+2}}{n!} \\ &\sum_{\substack{d_1,\ldots,d_n\ge 1 \\ d_1+\ldots+d_n=2n}} \frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)!} \prod_{i=1}^{n} {p_i}^{d_i-1} \le \\ &\le |\text{ supp } \gamma_o| s B^{n+2} \frac{(n-2)!}{n!} e^{\sum p_i} = \\ &= |\text{ supp } \gamma_o| s e^s \frac{B^{n+2}}{n(n-1)}.\end{aligned} \tag{B.16}$$

For $n=1$, $\gamma=\{(p,X)\}$ and $|\gamma|=p$ and

$$\epsilon_\Lambda(\gamma_o,p) = \sum_{X\in S_R(\Lambda),|X|\le p} 1(X\cap \text{ supp } \gamma_o \neq \emptyset) \le |\text{ supp } \gamma_o| B \le |\gamma_o| B^p. \tag{B.17}$$

With these estimates

$$E_\Lambda(\gamma_o,s) \le |\text{ supp } \gamma_o| \left(B + s e^s B^2 \sum_{n=2}^{s/2} \frac{B^n}{n(n-1)} m_{ns} \right) \tag{B.18}$$

where

$$m_{ns} = \sum_{\substack{p_1,\ldots,p_n\in 2\mathbb{N} \\ p_1+\ldots+p_n=s}} 1 = \oint_{0<|z|=\rho<1} \frac{dz}{2\pi i z^{s+1}} \frac{1}{(1-z^2)^{2n}} \le \frac{1}{\rho^s(1-\rho^2)^n} \tag{B.19}$$

for all $\rho \in (0,1)$. So

$$\begin{aligned} E_\Lambda(\gamma_o, s) &\le |\text{ supp } \gamma_o| \left(B + se^s\rho^{-s}B^2 \sum_{n=2}^{s/2} \frac{B^n}{n(n-1)}(1-\rho^2)^{-n} \right) \le \\ &\le |\text{ supp } \gamma_o| \left(B + se^s\rho^{-s}B^{2+s/2}(1-\rho^2)^{-s/2} \right) \le \\ &= |\text{ supp } \gamma_o|(B^2+B)s \left(\frac{e\sqrt{B}}{\rho\sqrt{1-\rho^2}} \right)^s . \end{aligned} \tag{B.20}$$

Choosing $\rho = 1/\sqrt{2}$, there is $s_o(B) \in \mathbb{N}$ such that for all $s \ge s_o$, Λ, γ_o

$$E_\Lambda(\gamma_o, s) \le |\text{ supp } \gamma_o| \left(6\sqrt{B} \right)^s . \tag{B.21}$$

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