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On the Algebraic Structure of Gravity with Torsion Including Weyl Symmetry

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Abstract. The BRST transformations for gravity with torsion including Weyl symmetry are discussed by using the so-called Maurer-Cartan horizontality conditions. Also the coupling of scalar matter fields to gravity is incorporated in this analysis. With the help of an operator δ which allows to decompose the exterior space-time derivative as a BRST commutator we solve the Wess-Zumino consistency condition corresponding to invariant Lagrangians and anomalies for the cases with and without Weyl symmetry.

Contents:

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1 Introduction

In the discussion of the unification of all fundamental interactions, gauge field theories play a central role. Electroweak theory and quantum chromodynamic (QCD) are examples of Yang-Mills gauge theories [1, 2] associated with non-abelian Lie groups. In that way gravity is introduced as a gauge theory which is associated with local Lorentz invariance [3].

The symmetry content of a field theoretic model is usually described by Ward identities (WI) leading to functional differential equations for the various Green's functions generated by the corresponding generating functionals [4, 5]. Sometimes, the transition from the classical to the quantized level modifies these Ward identities by non-trivial contributions (anomalies), expressing the fact, that the original symmetry of the classical model is broken at the quantum level.

An anomaly is usually defined as the gauge variation of the connected vacuum functional in the presence of external gauge fields. When an anomaly occurs, this variation does not vanish and the vacuum functional is not gauge invariant.

The most famous anomaly is the Adler-Bell-Jackiw (ABJ) anomaly [6, 7, 8] which describes the breaking term in the axial vector current divergence equation. This anomaly is needed to discuss successfully the $\pi^0 \rightarrow 2\gamma$ decay.

In connection with conformal field theories of gravity, Weyl anomalies are of great interest. It is well-known that due to the existence of Weyl anomalies the Weyl symmetry, which is valid at the classical level, is broken in the presence of quantum corrections. It is apparent that in the discussion about the quantum conformal structure of a theory one needs the identification of all the Weyl anomalies [9, 10] and Weyl invariants in arbitrary space-time dimensions.

Therefore, in order to discuss anomalies one needs a tool for a characterization. This may be achieved in a very compact manner with the help of the Wess-Zumino (WZ) consitency condition [11] in the context of the Becchi-Rouet-Stora-Tyupin (BRST) formalism [12]. This BRST scheme is an elegant and powerful instrument for the consistent discussion of gauge symmetries in quantum field theory, and in addition this concept is available for a large class of gauge field models whose classical symmetries have an algebra which closes. In particular this BRST formalism allows to characterize the classical action and possible anomalies as BRST invariant local functionals of the basic fields.

In order to describe the general procedure applicable to any gauge field model one starts with the one-particle-irreducible (1PI) vertex functional given by

$$\Gamma(\phi_{cl}) = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \phi_{cl}(x_1) ... \phi_{cl}(x_n) \langle 0 | T(\phi(x_1) ... \phi(x_n)) | 0 \rangle^{1PI} , \tag{1.1}$$

where $\langle 0 | T(\phi(x_1) ... \phi(x_n)) | 0 \rangle^{1PI}$ denotes the vacuum expectation value of the quantum fields $\phi(x)$. The classical sources ϕ_{cl} are test functions for the functional (1.1). In perturbation theory one can make a loop expansion for $\Gamma(\phi_{cl})$, i.e. the vertex functional can be written as a formal power series in $\hbar$:

$$\Gamma(\phi_{cl}) = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}(\phi_{cl}) \ . \tag{1.2}$$

At the classical level, in the so-called tree approximation, one gets

$$\Gamma^{(0)}(\phi_{cl}) = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \phi_{cl}(x_1) ... \phi_{cl}(x_n) \langle 0|T(\phi(x_1)...\phi(x_n))|0\rangle_{tree}^{1PI} = \Gamma_{cl}(\phi_{cl}) \ , \tag{1.3}$$

where $\langle 0|T(\phi(x_1)...\phi(x_n))|0\rangle_{tree}^{1PI}$ collects now all possible tree graphs, i.e. graphs without radiative corrections. In this approximation the zero-loop order corresponds to the classical action. In order to simplify notation one substitutes $\phi_{cl} \to \phi$ in the functionals (1.1) and (1.3).

For the discussion of the symmetry content of the field model one introduces the Ward identity operator W_s in its global form

$$W_s = \int d^N x \ \delta_s \phi(x) \frac{\delta}{\delta\phi(x)} \ , \tag{1.4}$$

belonging to an arbitrary infinitesimal symmetry transformation $\delta_s\phi(x)$ characterized by a local parametric function $\varepsilon(x)$

$$\delta_s \phi(x) = \varepsilon(x) \mathcal{P}(\phi) \ , \tag{1.5}$$

whereby for general reason $\mathcal{P}(\phi)$ may be linear or non-linear in ϕ. Applying the WI-operator (1.4) to (1.3) one gets, for the case that (1.5) is a symmetry of the model, the following global WI:

$$W_s \Gamma_{cl}(\phi) = 0 \ . \tag{1.6}$$

The corresponding local WI may be obtained from (1.6) by functional differentiation with respect to $\varepsilon(x)$.

The presence of radiative corrections is now governed by the renormalized action principle [13, 14] leading in general to a modified WI for the full vertex functional (1.2)

$$W_s \Gamma(\phi) = \Delta \cdot \Gamma(\phi) = \hbar \Delta(\phi) + \mathcal{O}(\hbar^2) \ , \tag{1.7}$$

where Δ is an integrated well-defined quantum insertion of definite dimensions [13, 14] and $\Delta(\phi)$ is an integrated local polynomial in the fields and their derivatives. For the search of anomalies it is enough to limit ourselves to the one-loop order.

Following the general procedure for the discussion of non-invariant counterterms an anomaly occurs if the correction Δ cannot be expressed as a variation of the underlying symmetry and therefore it is not possible to absorb $\Delta(\phi)$ by an appropriate counterterm in order to get

$$W_s \widehat{\Gamma}(\phi) = 0 \ . \tag{1.8}$$

To clarify this point, we assume that $\Delta(\phi)$ can be written as

$$\Delta(\phi) = a(\phi) + W_s b(\phi) \ , \tag{1.9}$$

where $a(\phi)$ and $b(\phi)$ are integrated local polynomials and where $a(\phi)$ cannot be expressed as $W_s\widehat{a}(\phi)$ for any integrated local polynomial $\widehat{a}(\phi)$. The second term in (1.9) can be absorbed in a redefined $\widehat{\Gamma}(\phi)$, but the first term leads to an anomaly

$$W_s\widehat{\Gamma}(\phi) = W_s(\Gamma(\phi) - \hbar b(\phi)) = \hbar a(\phi) \ . \tag{1.10}$$

The use of the BRST scheme demands now that the given symmetry is converted into a BRST symmetry. This can be achieved by replacing the infinitesimal parametric functions $\varepsilon(x)$ by anticommuting ghost fields $c(x)$ with ghost number one, which leads to a nilpotent symmetry operator, the so-called BRST operator s. The search of BRST invariant Lagrangians and possible anomalies is then reduced to solve the following cohomology problem

$$sa = 0 \quad , \quad a \neq s\widehat{a} \ , \tag{1.11}$$

where s is the nilpotent BRST operator and a is the breaking term of eq.(1.10). In particular, the BRST formalism allows now to characterize classical actions and anomalies as BRST invariant functionals. Especially, an action is a BRST invariant functional with ghost number zero while an anomaly corresponds to a BRST invariant functional with ghost number one. Eq.(1.11) is now the Wess-Zumino consistency relation within the BRST formalism and restricts strongly the possible solutions of a. For further need we are using the concepts of differential forms, where d denotes the exterior space-time derivative $d = dx^\mu \partial_\mu$ and where a is described by an integrated volume form (N-form) in N space-time dimension

$$a = \int \mathcal{A} \ , \tag{1.12}$$

with $\mathcal{A}$ local polynomial. The condition (1.11) implies the following local equation

$$s\mathcal{A} + d\mathcal{Q} = 0 \ , \tag{1.13}$$

where $\mathcal{Q}$ is some local polynomial. The exterior space-time derivative d and the BRST operator s fulfill

$$s^2 = d^2 = sd + ds = 0 \ . \tag{1.14}$$

$\mathcal{A}$ is said non-trivial if

$$\mathcal{A} \neq s\hat{\mathcal{A}} + d\hat{\mathcal{Q}} \ , \tag{1.15}$$

with $\hat{\mathcal{A}}$ and $\hat{\mathcal{Q}}$ local polynomials. In this case the integral of $\mathcal{A}$ on space-time, $\int \mathcal{A}$, identifies a cohomology class of the BRST operator s and, according to its ghost number, it corresponds to an invariant Lagrangian (ghost number zero) or to an anomaly (ghost number one).

The local equation (1.13), due to the relations (1.14) and to the familiar algebraic Poincaré Lemma [15, 16]

$$d\Omega = 0 \quad \Longleftrightarrow \quad \Omega = d\widehat{\Omega} + d^N x \ \mathcal{L} + const \ , \tag{1.16}$$

is easily seen to generate a tower of descent equations

$$
\begin{aligned}
&sQ + dQ^1 = 0 \ , \\
&sQ^1 + dQ^2 = 0 \ , \\
&sQ^2 + dQ^3 = 0 \ , \\
&\quad \cdots\cdots \\
&\quad \cdots\cdots \\
&sQ^{k-1} + dQ^k = 0 \ , \\
&sQ^k = 0 \ ,
\end{aligned}
\tag{1.17}
$$

with Q^i local field polynomials. The index i describes the grading of the local polynomials Q^i (see Section 3).

As it has been well-known for several years, these equations can be solved by using a transgression procedure based on the so-called *Russian formula* [17, 18, 19, 20, 21, 22, 23, 24, 25]. More recently an alternative way of finding non-trivial solutions of the ladder (1.17) has been proposed by S.P. Sorella and has been successfully applied to the study of the Yang-Mills gauge anomalies [26]. The method is based on the introduction of an operator δ which allows the expression of the exterior derivative d as a BRST commutator, i.e.:

$$
d = -[s, \delta] \ . \tag{1.18}
$$

One easily verifies that, once the decomposition (1.18) has been found, successive applications of the operator δ on the polynomial Q^k which solves the last equation of the tower (1.17) give an explicit non-trivial solution for the higher cocycles $Q^{k-1}, \ldots.., Q^1, Q$, and $\mathcal{A}$.

Actually, the decomposition (1.18) represents one of the most interesting features of the topological field theories [27, 28] and of the bosonic string and superstring in the Beltrami and Super-Beltrami parametrization [29]. A remarkable fact is also that solving the last equation of the tower (1.17) is a problem of local BRST cohomology instead of a modulo-d one. One sees then that, due to the operator δ, the study of the cohomology of s modulo d is essentially reduced to the study of the local cohomology of s which, in turn, can be systematically analyzed by using the powerful technique of the spectral sequences [30]. Actually, as proven in [31], the solutions obtained by utilizing the decomposition (1.18) turn out to be completely equivalent to that based on the *Russian formula*, i.e. they differ only by trivial cocycles.

The aim of this work is twofold. In a first step it will be demonstrated that the decomposition (1.18) can be successfully applied to gravity including local Lorentz rotations, diffeomorphisms, and Weyl transformations, and that it holds also in the presence of torsion. In the second step, we will see that the operator δ gives an elegant and straightforward way of classifying the cohomology classes of the full BRST operator in any space-time dimension. In particular, the eq.(1.18) will allow for a cohomological interpretation of the cosmological constant, of Lagrangians for pure Einstein gravity and generalizations including also torsion. Additionally, Chern-Simons terms, gravitational and Weyl anomalies are considered. The last point is devoted to the discussion of the coupling of a scalar matter field to gravity with and without Weyl symmetry.

This work is a continuation of a previous one [32], where the decomposition (1.18) was shown to hold in the case of pure Lorentz transformations involving only the Lorentz connection ω and the Riemann tensor R and without taking into account the explicit presence of the vielbein e and of the torsion T. More recently the decomposition (1.18) was used to investigate the cohomological problem of gravity with torsion [33].

The further analysis is based on the geometrical formalism introduced by L. Baulieu and J. Thierry-Mieg [18, 20] which allows to reinterpret the BRST transformations as Maurer-Cartan horizontality conditions. In particular, this formalism turns out to be very useful in the case of gravity [18, 20], since it naturally includes the torsion. In addition, it allows to formulate the diffeomorphism transformations as local translations in the tangent space by means of the introduction of the ghost field $\eta^a = \xi^\mu e^a_\mu$ where ξ^μ denotes the usual diffeomorphism ghost and e^a_μ is the vielbein.

We recall also that the BRST formulation of gravity with torsion has already been proposed by [34, 35] in order to study the quantum aspects of gravity. In particular, the authors of [35] discussed a four dimensional torsion Lagrangian, with $GL(4, R)$ as the gauge group, which is able to reproduce the Einstein gravity in the low energy limit. These BRST transformations could be taken as the starting point for a purely cohomological algebraic analysis without any reference to a particular Lagrangian. Furthermore, our choice of adopting the Maurer-Cartan formalism is due to the fact that when combined with the introduction of the translation ghost η^a it will give us the possibility of a fully tangent space formulation of gravity.

This step, as we shall see in details, will allow to introduce the decomposition (1.18) in a very simple way and will produce an elegant and compact formula (see Section 5) for expressing the whole solution of the BRST descent equations, our aim is that of giving a cohomological interpretation of the gravitational Lagrangians and of the anomalies in any space-time dimension with and without Weyl symmetry. Moreover, the explicit presence of the torsion T and of the translation ghost η^a gives the possibility of introducing an algebraic BRST setup which turns out to be different from that obtained from the analysis of Brandt et al. [25], where similar techniques have been used.

Finally, we stress that the main purposes of the present work are dedicated on one hand to discuss the solutions of the local gravitational cohomology problem without Weyl symmetry

$$sa(e, \omega, R, T) = 0 \ , \tag{1.19}$$

and on the other hand to find solutions of the full local cohomology problem including Weyl symmetry

$$sa(e, \omega, A, R, T, F) = 0 \ , \tag{1.20}$$

where the vielbein field e, the Lorentz connection ω, the abelian Weyl gauge field A, the Riemann tensor R, the torsion T, and the Weyl curvature F are treated as unquantized classical fields, as done in [32], which when coupled to some matter fields (scalars or fermions) give rise to an effective action whose quantum expansion reduces to the one-loop order.

The work is organized as follows: In the several parts of Section 2 we briefly mention some basic elements, respectively in Section 2.1 gravity with torsion, in Section 2.2 Weyl transformations, in Section 2.3 the powerful BRST formalism, and finally in Section 2.4 the elegant technique of differential forms. After this, we will introduce in Section 3 the so-called Maurer-Cartan horizontality conditions for gravity with torsion, for Weyl transformations, and for scalar matter fields. In particular, in Section 3.3 the BRST transformations for local Lorentz rotations, diffeomorphisms, and Weyl transformations are derived in a complete tangent space formalism. In Section 4 the operator δ is introduced and we show how it can be used to solve the descent equations (1.17). Section 5 is dedicated to the study of some explicit examples, like the cosmological constant, the Einstein and the generalized torsion Lagrangians as well as the Chern-Simons terms, the gravitational and Weyl anomalies, and the matter field Lagrangians. Section 6 deals with the geometrical meaning of the decomposition (1.18) and some detailed calculations can be found in the final appendices, respectively important commutator relations in the tangent space and the determinant of the vielbein in connection with the ε tensor.

2 Basic elements

2.1 Gravitational fields

It is well-known that the gauge transformations associated with gravity are general coordinate transformations or diffeomorphisms, i.e. arbitrary reparametrizations of the N-dimensional space-time, and local Lorentz rotations in a tangent space of dimension N [2]. A diffeomorphism can be written as

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu(x) \ , \tag{2.1}$$

where $\xi^\mu(x)$ is an infinitesimal parameter of the general coordinate transformation. A scalar field $\varphi(x)$ has the property

$$\bar{\varphi}(\bar{x}) = \varphi(x) \ , \tag{2.2}$$

and can be expanded over a point x as follows:

$$\varphi(\bar{x}) = \varphi(x + \xi(x)) = \varphi(x) + \xi^\mu(x)\partial_\mu\varphi(x) + \mathcal{O}(\xi)^2 \ . \tag{2.3}$$

From eqs.(2.2) and (2.3) one can read off the transformation of a scalar field under general coordinate transformations or diffeomorphisms

$$\delta_D\varphi = \varphi(x) - \varphi(\bar{x}) = -\xi^\mu\partial_\mu\varphi = \mathcal{L}_\xi\varphi \ , \tag{2.4}$$

[2]As usual, Latin and Greek indices refer to the tangent space and to the euclidean space-time. Both the world indices and the local Lorentz indices take values from 1 to N.

where $\mathcal{L}_\xi$ denotes the Lie derivative [36]. A covariant (contravariant) vector field V_μ (V^μ) transforms under diffeomorphisms as

$$\begin{aligned}\delta_D V_\mu &= -\xi^\lambda \partial_\lambda V_\mu - (\partial_\mu \xi^\lambda) V_\lambda = \mathcal{L}_\xi V_\mu \ , \\ \delta_D V^\mu &= -\xi^\lambda \partial_\lambda V^\mu + (\partial_\lambda \xi^\mu) V^\lambda = \mathcal{L}_\xi V^\mu \ ,\end{aligned} \tag{2.5}$$

and analogous for tensors of higher ranks.

Now we consider local Lorentz rotations in the tangent space which act on the fields according to

$$\phi(x) \rightarrow \phi(x) + \frac{1}{2}\epsilon^{ab}(x) M_{ab} \phi(x) \ , \tag{2.6}$$

where M_{ab} are the generators of the local Lorentz rotation group $SO(N)$ in N dimensions in the representation appropriate to the field $\phi(x)$ and ϵ^{ab} are the infinitesimal parameters of the transformation. The generators M_{ab} and parameters ϵ^{ab} are antisymmetric in (ab)

$$M_{ab} = -M_{ba} \quad , \quad \epsilon^{ab} = -\epsilon^{ba} \ . \tag{2.7}$$

An antihermitian representation of the generators M_{ab} is given by

$$(M_{ab})^c{}_d = \delta^c_a \eta_{bd} - \delta^c_b \eta_{ad} \ , \tag{2.8}$$

which satisfies the commutator algebra

$$[M_{ab}, M_{cd}] = -\eta_{ac} M_{bd} + \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac} \ . \tag{2.9}$$

With the help of eq.(2.8) follows from eq.(2.6) the transformation of a contravariant tangent space vector field V^a under local Lorentz rotations

$$\begin{aligned}\delta_L V^a &= \frac{1}{2}\epsilon^{mn} (M_{mn})^a{}_b V^b \\ &= \frac{1}{2}\epsilon^{mn} (\delta^a_m \eta_{nb} - \delta^a_n \eta_{mb}) V^b \\ &= \epsilon^a{}_b V^b \ ,\end{aligned} \tag{2.10}$$

and analogous for a covariant vector field V_a

$$\delta_L V_a = \epsilon_a{}^b V_b \ . \tag{2.11}$$

Remark that a scalar field φ has no tangent space index and therefore does not transform under local Lorentz rotations

$$\delta_L \varphi = 0 \ . \tag{2.12}$$

Now one can define a Lorentz covariant derivative D_μ by introducing a gauge field associated with the local Lorentz rotations (2.6) which is called the Lorentz connection field $\omega^{ab}{}_\mu$

$$D_\mu \phi = \partial_\mu \phi + \frac{1}{2}\omega^{ab}{}_\mu M_{ab} \phi \ , \tag{2.13}$$

The covariant derivative transforms according to eq.(2.6) as

$$\delta_L(D_\mu\phi) = \frac{1}{2}\epsilon^{ab}M_{ab}(D_\mu\phi) \ , \tag{2.14}$$

which determines with eq.(2.9) the transformation property of the Lorentz connection field under local Lorentz rotations

$$\delta_L\omega^{ab}{}_\mu = -\partial_\mu\epsilon^{ab} - \omega^a{}_{c\mu}\epsilon^{cb} + \omega^b{}_{c\mu}\epsilon^{ca} \ . \tag{2.15}$$

Remark that the Lorentz connection field $\omega^{ab}{}_\mu$ carries both world and tangent space indices. The presence of a world index is characteristic for gauge fields, since these fields are introduced in order to compensate for the effects caused by the local gauge invariance transformations.

Up to now the derivative (2.13) is covariant under local Lorentz rotations but not covariant under diffeomorphisms. In other words the variation of $D_\mu\phi$ does not only depend on the parameters $\xi^\mu(x)$ taken at the same space-time point, but it also depends on their values at neighbouring points. This is reflected in the presence of a derivative of $\xi^\mu(x)$ in the transformation law

$$\delta(D_\mu\phi) = \frac{1}{2}\epsilon^{ab}M_{ab}(D_\mu\phi) - \xi^\nu\partial_\nu(D_\mu\phi) - (\partial_\mu\xi^\nu)(D_\nu\phi) \ , \tag{2.16}$$

with

$$\delta = \delta_D + \delta_L \ . \tag{2.17}$$

From eqs.(2.15) and (2.16) one gets the transformation of the Lorentz connection field $\omega^{ab}{}_\mu$ under diffeomorphisms

$$\delta_D\omega^{ab}{}_\mu = -\xi^\nu\partial_\nu\omega^{ab}{}_\mu - (\partial_\mu\xi^\nu)\omega^{ab}{}_\nu \ , \tag{2.18}$$

which is the usual transformation law of a covariant vector field under general coordinate transformations (2.5). To remove $\partial_\mu\xi^\nu$ in the transformation rule (2.16) one introduces another field whose variation can compensate for the undesired term. Since $D_\mu\phi$ still carries a world index the new type of field should convert world indices into local Lorentz indices. This leads to the introduction of the field E^μ_a, which can be used to define a fully covariant derivative $D_a\phi$ in the following way

$$D_a\phi = E^\mu_a D_\mu\phi \ . \tag{2.19}$$

This derivative no longer carries a world index and transforms therefore as a vector field under local Lorentz rotations and as a scalar field under diffeomorphisms

$$\begin{aligned}\delta(D_a\phi) &= -\xi^\mu\partial_\mu(D_a\phi) + \epsilon_a{}^b(D_b\phi) \\ &= -\xi^\mu\partial_\mu(D_a\phi) + \frac{1}{2}\epsilon^{mn}(M_{mn})_a{}^b(D_b\phi) \ ,\end{aligned} \tag{2.20}$$

which determines the transformation behaviour of E^μ_a under diffeomorphisms and local Lorentz rotations

$$\delta E^\mu_a = \epsilon_a{}^b E^\mu_b - \xi^\lambda\partial_\lambda E^\mu_a + (\partial_\lambda\xi^\mu)E^\lambda_a \ . \tag{2.21}$$

The field E_a^μ not only may be regarded as the gauge field of general coordinate transformations but also may be seen in a geometric context. Namely the N contravariant vector fields E_a^μ specify the basis vectors of the linear tangent space at each point of a N-dimensional Riemannian space-time manifold. This implies that E_a^μ is non-singular and has an inverse e_μ^a defined by

$$e_\mu^a E_b^\mu = \delta^a{}_b \ ,$$
$$e_\nu^a E_a^\mu = \delta_\nu{}^\mu \ . \tag{2.22}$$

The standard nomenclature is to call e_μ^a the vielbein field and E_a^μ the inverse vielbein field. Under diffeomorphisms and local Lorentz rotations the vielbein field transforms as

$$\delta e_\mu^a = \epsilon^a{}_b e_\mu^b - \xi^\lambda \partial_\lambda e_\mu^a - (\partial_\mu \xi^\lambda) e_\lambda^a \ . \tag{2.23}$$

In this context the group of local Lorentz rotations is called the structure group or tangent space group, which rotates the local tangent space frame. The existence of the vielbein field allows the introduction of a covariant metric tensor on the Riemannian space-time manifold which is symmetric

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab} = g_{\nu\mu}(x) \ , \tag{2.24}$$

where η_{ab} is a Lorentz invariant tensor, which can be used for local measurements of distances and angles in space-time.

As usual one extends the concept of covariance under diffeomorphisms and local Lorentz rotations to quantities that carry world indices. The construction of a covariant derivative $\mathcal{D}_\mu$ for world covariant vector fields X_μ is straightforward. By using the inverse vielbein field one convert the world indices to tangent space indices, then apply the covariant derivative (2.13), and with the help of the vielbein field one reconvert the Lorentz indices into world indices. Hence one has

$$\mathcal{D}_\mu X_\nu = e_\nu^a D_\mu (E_a^\rho X_\rho) \ ,$$
$$\mathcal{D}_\mu X^\nu = E_a^\nu D_\mu (e_\rho^a X^\rho) \ . \tag{2.25}$$

The same argument can be applied to define covariant derivatives acting directly on world tensors. Eqs.(2.25) can be rewritten in the form

$$\mathcal{D}_\mu X_\nu = D_\mu X_\nu - \Gamma_{\mu\nu}{}^\rho X_\rho \ ,$$
$$\mathcal{D}_\mu X^\nu = D_\mu X^\nu + \Gamma_{\mu\rho}{}^\nu X^\rho \ , \tag{2.26}$$

with the so-called affine connection $\Gamma_{\mu\nu}{}^\rho$ defined by

$$\Gamma_{\mu\nu}{}^\rho = -e_\nu^a D_\mu E_a^\rho = (D_\mu e_\nu^a) E_a^\rho \ . \tag{2.27}$$

Note, that the covariant derivatives in (2.26) and the affine connection contain the Lorentz connection field $\omega^{ab}{}_\mu$. The representation (2.27) for the affine connection satisfies the metric postulate, which asserts that the metric and thus the vielbein field is covariantly constant. Indeed it is straightforward to verify that (2.27) is equivalent to

$$\mathcal{D}_\mu e^a_\nu = 0 \; . \tag{2.28}$$

The affine and Lorentz connection fields are therefore not independent; furthermore it is easy to show with eq.(2.24) that

$$\mathcal{D}_\mu g_{\nu\rho} = 0 \; , \tag{2.29}$$

from which one deduces that the affine connection must satisfy following identity

$$\Gamma_{\mu\nu}{}^\sigma g_{\sigma\rho} + \Gamma_{\mu\rho}{}^\sigma g_{\nu\sigma} = \partial_\mu g_{\nu\rho} \; . \tag{2.30}$$

The so-called curvature or Riemann tensor R and torsion tensor T are found by using the Ricci identity [36], which relates these tensors to commutators of covariant derivatives

$$[D_m, D_n] \equiv \frac{1}{2} R^{ab}{}_{mn}(\omega) M_{ab} - T^a_{mn}(e,\omega) D_a \; . \tag{2.31}$$

Evaluation of the left-hand side of eq.(2.31) shows that the Riemann tensor of the local Lorentz rotations and the torsion tensor are given by

$$R^{ab}{}_{\mu\nu}(\omega) = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu \; , \tag{2.32}$$

$$T^a_{\mu\nu}(e,\omega) = D_\mu e^a_\nu - D_\nu e^a_\mu \; . \tag{2.33}$$

According to eq.(2.28) and the definition (2.33) the torsion tensor T corresponds to the antisymmetric part of the affine connection

$$\Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho = E^\rho_a T^a_{\mu\nu} \; . \tag{2.34}$$

Alternatively one may also compute the Ricci identity for covariant derivatives (2.26) applied on a world vector

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] X_\rho \equiv \frac{1}{2} R^{ab}{}_{\mu\nu} M_{ab} X_\rho - T^\tau_{\mu\nu} \mathcal{D}_\tau X_\rho - R^\sigma{}_{\rho\mu\nu} X_\sigma \; . \tag{2.35}$$

With the relation

$$[D_\mu, D_\nu] = \frac{1}{2} R^{ab}{}_{\mu\nu} M_{ab} \; , \tag{2.36}$$

and eq.(2.34) one obtains the curvature tensor

$$R^\sigma{}_{\rho\mu\nu}(\Gamma) = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \partial_\nu \Gamma_{\mu\rho}{}^\sigma + \Gamma_{\mu\rho}{}^\tau \Gamma_{\nu\tau}{}^\sigma - \Gamma_{\nu\rho}{}^\tau \Gamma_{\mu\tau}{}^\sigma \; . \tag{2.37}$$

Remark that, however, $R(\Gamma)$ is not an independent object. By evaluating

$$\mathcal{D}_{[\mu}\mathcal{D}_{\nu]} e^a_\rho = \mathcal{D}_\mu \mathcal{D}_\nu e^a_\rho - \mathcal{D}_\nu \mathcal{D}_\mu e^a_\rho = 0 \; , \tag{2.38}$$

one verifies that

$$R^\sigma{}_{\rho\mu\nu} = R^a{}_{b\mu\nu} e^b_\rho E^\sigma_a \; . \tag{2.39}$$

Taking cyclic permutations of the triple commutator of covariant derivatives, which satisfies the Jacobi identity

$$[D_\mu,[D_\nu,D_\rho]]+[D_\nu,[D_\rho,D_\mu]]+[D_\rho,[D_\mu,D_\nu]]=0\ , \tag{2.40}$$

and taking cyclic permutations of the covariant derivative applied on the definition of the torsion (2.33)

$$D_{[\mu}T^a_{\nu\rho]}=D_{[\mu}D_\nu e^a_{\rho]}\ , \tag{2.41}$$

yields the well-known Bianchi identities

$$D_{[\mu}R^{ab}{}_{\nu\rho]}=0\ , \tag{2.42}$$

$$D_{[\mu}T^a_{\nu\rho]}=R^a{}_{b[\mu\nu}e^b_{\rho]}\ . \tag{2.43}$$

The combination of (2.30) and (2.34) now fully determines the affine connection in terms of the metric field and the torsion tensor

$$\Gamma_{\mu\nu}{}^\rho=\{{}^{\rho}_{\mu\nu}\}+E^\rho_a e_{\nu b}K^{ab}_\mu\ , \tag{2.44}$$

where $\{{}^{\rho}_{\mu\nu}\}$ denoting the Christoffel symbol which is symmetric in $(\mu\nu)$

$$\{{}^{\rho}_{\mu\nu}\}=\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu})\ , \tag{2.45}$$

and K^{ab}_μ the contorsion tensor which is antisymmetric in (ab)

$$K^{ab}_\mu=\frac{1}{2}E^{a\rho}E^{\nu b}(T^c_{\mu\nu}e_{\rho c}+T^c_{\rho\mu}e_{\nu c}-T^c_{\nu\rho}e_{\mu c})\ . \tag{2.46}$$

The value of the Lorentz connection field corresponding to eq.(2.44) is

$$\omega^{ab}{}_\mu=\omega^{ab}{}_\mu(e)+K^{ab}_\mu\ , \tag{2.47}$$

with

$$\omega^{ab}{}_\mu(e)=\frac{1}{2}e^c_\mu(-\Omega^{ab}{}_c+\Omega^b{}_c{}^a+\Omega_c{}^{ab})\ , \tag{2.48}$$

and $\Omega^{ab}{}_c$ the coefficients of anholonomity which measures the non-commutativity of the vielbein basis

$$\Omega_{ab}{}^c=E^\mu_a E^\nu_b(\partial_\mu e^c_\nu-\partial_\nu e^c_\mu)\ . \tag{2.49}$$

In the absence of torsion the Bianchi identitiy (2.43) takes the simple form

$$R^{ab}{}_{[\mu\nu}e_{\rho]b}=0\ . \tag{2.50}$$

In this case the Lorentz connection field $\omega^{ab}{}_\mu$ is given by eq.(2.48)

$$\omega^{ab}{}_\mu=\omega^{ab}{}_\mu(e)\ , \tag{2.51}$$

and the affine connection $\Gamma_{\mu\nu}{}^\rho$ is nothing but the Christoffel symbol

$$\Gamma_{\mu\nu}{}^\rho=\{{}^{\rho}_{\mu\nu}\}\ . \tag{2.52}$$

This implies the symmetry of the Riemann tensor under cyclic permutations of the indices

$$R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0 \ , \tag{2.53}$$

and the pair exchange symmetry of the Riemann tensor

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho} \ . \tag{2.54}$$

By contracting the Riemann tensor one can construct two further objects, the Ricci tensor $R^a{}_\mu$

$$R^a{}_\mu = R^{ab}{}_{\mu\nu} E^\nu_b \ , \tag{2.55}$$

which is in the case of vanishing torsion symmetric

$$R^a{}_\mu = R_\mu{}^a \ , \tag{2.56}$$

and the Riemann scalar or curvature scalar R

$$R = R^a{}_\mu E^\mu_a = R^{ab}{}_{\mu\nu} E^\mu_a E^\nu_b \ . \tag{2.57}$$

2.2 Weyl transformations

In this section we will introduce the familiar Weyl transformations and the connection to diffeomorphisms and local Lorentz rotations. A local Weyl transformation [37] is a local rescaling of the metric field $g_{\mu\nu}$ according to

$$g_{\mu\nu}(x) \to e^{\mathcal{W}(g_{\mu\nu})\Omega(x)} g_{\mu\nu}(x) \ , \tag{2.58}$$

where $\mathcal{W}(g_{\mu\nu})$ is the so-called Weyl weight of the metric tensor and $\Omega(x)$ is the parameter for the Weyl transformation. As usual, the Weyl weight for the metric tensor $g_{\mu\nu}$ is fixed to the value two

$$\mathcal{W}(g_{\mu\nu}) = 2 \ . \tag{2.59}$$

The infinitesimal transformation is then given by

$$\delta_W g_{\mu\nu} = 2\Omega g_{\mu\nu} \ , \tag{2.60}$$

and analogous for an arbitrary field ϕ, with an appropriate Weyl weight, one has

$$\delta_W \phi = \mathcal{W}(\phi)\Omega\phi \ . \tag{2.61}$$

The determinant of the metric field, defined as

$$g = det(g_{\mu\nu}) \ , \tag{2.62}$$

has now the Weyl weight

$$\mathcal{W}(g) = 2N \ , \tag{2.63}$$

where N denotes the space-time dimension. From eq.(2.24) one can read off the following identity for the determinant of the vielbein field e^a_μ

$$e = \sqrt{g} \quad , \quad e = det(e^a_\mu) \ , \tag{2.64}$$

which implies the Weyl weight for the determinant of the vielbein field

$$\mathcal{W}(e) = N \ . \tag{2.65}$$

With this setup the Weyl weights for the following basic fields are then fixed to

$$\begin{aligned} \mathcal{W}(e^a_\mu) &= +1 \ , \\ \mathcal{W}(E^\mu_a) &= -1 \ , \\ \mathcal{W}(g_{\mu\nu}) &= +2 \ , \\ \mathcal{W}(g^{\mu\nu}) &= -2 \ , \end{aligned} \tag{2.66}$$

and the corresponding Weyl transformations are given by

$$\begin{aligned} \delta_W e^a_\mu &= \Omega e^a_\mu \ , \\ \delta_W E^\mu_a &= -\Omega E^\mu_a \ , \\ \delta_W g_{\mu\nu} &= 2\Omega g_{\mu\nu} \ , \\ \delta_W g^{\mu\nu} &= -2\Omega g^{\mu\nu} \ , \end{aligned} \tag{2.67}$$

respectively

$$\delta_W e = \delta_W \sqrt{g} = N\Omega\sqrt{g} \ . \tag{2.68}$$

Remark that the Weyl weights are additive. The standard procedure to find the Weyl weight for a scalar field φ is to assume, that the action of a scalar field should have Weyl weight zero under a global Weyl transformation. From

$$\Gamma = \frac{1}{2}\int d^N x \sqrt{g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \ , \tag{2.69}$$

one gets therefore the Weyl weight for a scalar field

$$\mathcal{W}(\varphi) = -\frac{N-2}{2} \ , \tag{2.70}$$

and the Weyl transformation of a scalar field

$$\delta_W \varphi = -\frac{N-2}{2}\Omega\varphi \ . \tag{2.71}$$

Remark also that the partial derivative ∂_μ and the flat metric η_{ab} have Weyl weights zero. The Weyl transformations for the remaining fields are determined by the definitions of the Riemann tensor (2.32) and the torsion (2.33) and are given by

$$\begin{aligned} \delta_W {\omega^{ab}}_\mu &= 0 \ , \\ \delta_W T^a_{\mu\nu} &= \Omega T^a_{\mu\nu} \ , \\ \delta_W {R^{ab}}_{\mu\nu} &= 0 \ . \end{aligned} \tag{2.72}$$

However, the action of the scalar field (2.69) is not generally Weyl invariant under a *local* infinitesimal Weyl transformation. Only in two space-time dimensions one can find a Weyl invariance:

$$\begin{aligned}\delta_W\Gamma &= \frac{1}{2}\delta_W \int d^Nx\sqrt{g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ &= -(\frac{N-2}{2})\int d^Nx\sqrt{g}g^{\mu\nu}(\partial_\mu\Omega)\varphi\partial_\nu\varphi \ .\end{aligned} \tag{2.73}$$

In order to keep the Weyl invariance in arbitrary space-time dimensions we introduce a gauge field A_μ for the Weyl transformations and the corresponding Weyl covariant derivative

$$\nabla_\mu\phi = \partial_\mu\phi + \mathcal{W}(\phi)A_\mu\phi \ . \tag{2.74}$$

As usual, the covariant derivative of a field ϕ should transform as the field ϕ (see eq.(2.61))

$$\delta_W(\nabla_\mu\phi) = \mathcal{W}(\phi)\Omega(\nabla_\mu\phi) \ , \tag{2.75}$$

which implies the proper transformation of the Weyl gauge field

$$\delta_W A_\mu = -\partial_\mu\Omega \ . \tag{2.76}$$

The correct Weyl invariant scalar field action can be found by replacing the partial derivative ∂_μ in (2.69) by the Weyl covariant derivative (2.74):

$$\Gamma = \frac{1}{2}\int d^Nx\sqrt{g}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \ . \tag{2.77}$$

Now one can easily proof that the action (2.77) is Weyl invariant in any space-time dimension. From (2.77) follows

$$\begin{aligned}\delta_W\Gamma &= \frac{1}{2}\delta_W \int d^Nx\sqrt{g}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \\ &= \frac{1}{2}\int d^Nx[(\delta_W\sqrt{g}g^{\mu\nu})\nabla_\mu\varphi\nabla_\nu\varphi + 2\sqrt{g}g^{\mu\nu}(\delta_W\nabla_\mu\varphi)\nabla_\nu\varphi] \ ,\end{aligned} \tag{2.78}$$

which leads with the help of (2.67),(2.68) and (2.75) to the invariance of the action:

$$\delta_W\Gamma = 0 \ . \tag{2.79}$$

We remark that in the case of gravity with vanishing torsion exists a second possibility to construct a Weyl invariant scalar field action without using the Weyl gauge field [38]:

$$\Gamma = \frac{1}{2}\int d^Nx\sqrt{g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha R\varphi^2] \ , \tag{2.80}$$

with the coupling constant

$$\alpha = \frac{1}{4}\frac{N-2}{N-1} \ . \tag{2.81}$$

The Weyl transformation of the Riemann scalar R for vanishing torsion is given by

$$\delta_W R = -2\Omega R + 2(N-1)\Box\Omega \ . \tag{2.82}$$

Now it is straightforward to check that (2.80) is Weyl invariant. But in the presence of torsion the Weyl transformation of the Riemann scalar R is changed according to

$$\delta_W R = -2\Omega R \ , \tag{2.83}$$

and the action (2.80) is no more Weyl invariant!

The commutator of two covariant derivatives is related to the Weyl field strength

$$[\nabla_\mu, \nabla_\nu]\phi = \mathcal{W}(\phi)F_{\mu\nu}\phi \ . \tag{2.84}$$

The corresponding Weyl curvature $F_{\mu\nu}$ is then given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{2.85}$$

which is invariant under Weyl transformations

$$\delta_W F_{\mu\nu} = 0 \ , \tag{2.86}$$

and obeys the Bianchi identity for the Weyl curvature

$$\partial_{[\mu}F_{\nu\rho]} = 0 \ . \tag{2.87}$$

In order to incorporate also Weyl transformations with diffeomorphisms and local Lorentz rotations one introduces the following covariant derivative

$$\mathbf{D}_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu M_{ab} + \mathcal{W}A_\mu \ . \tag{2.88}$$

The composition of the symmetry transformations, i.e. diffeomorphisms, local Lorentz rotations, and Weyl transformations, is done by

$$\delta = \delta_D + \delta_L + \delta_W \ , \tag{2.89}$$

which leads to the following set of transformations for the basic fields

$$\begin{aligned}
\delta\varphi &= -\xi^\lambda\partial_\lambda\varphi - \frac{N-2}{2}\Omega\varphi \ , \\
\delta e^a_\mu &= \epsilon^a{}_b e^b_\mu - \xi^\lambda\partial_\lambda e^a_\mu - (\partial_\mu\xi^\lambda)e^a_\lambda + \Omega e^a_\mu \ , \\
\delta\omega^a{}_{b\mu} &= -\partial_\mu\epsilon^a{}_b + \epsilon^a{}_c\omega^c{}_{b\mu} - \epsilon^c{}_b\omega^a{}_{c\mu} - \xi^\lambda\partial_\lambda\omega^a{}_{b\mu} - (\partial_\mu\xi^\lambda)\omega^a{}_{b\lambda} \ , \\
\delta A_\mu &= -\xi^\lambda\partial_\lambda A_\mu - (\partial_\mu\xi^\lambda)A_\lambda - \partial_\mu\Omega \ , \\
\delta T^a_{\mu\nu} &= \epsilon^a{}_b T^b_{\mu\nu} - \xi^\lambda\partial_\lambda T^a_{\mu\nu} - (\partial_\mu\xi^\lambda)T^a_{\lambda\nu} - (\partial_\nu\xi^\lambda)T^a_{\mu\lambda} + \Omega T^a_{\mu\nu} \ , \\
\delta R^a{}_{b\mu\nu} &= \epsilon^a{}_c R^c{}_{b\mu\nu} - \epsilon^c{}_b R^a{}_{c\mu\nu} - \xi^\lambda\partial_\lambda R^a{}_{b\mu\nu} - (\partial_\mu\xi^\lambda)R^a{}_{b\lambda\nu} - (\partial_\nu\xi^\lambda)R^a{}_{b\mu\lambda} \ , \\
\delta F_{\mu\nu} &= -\xi^\lambda\partial_\lambda F_{\mu\nu} - (\partial_\mu\xi^\lambda)F_{\lambda\nu} - (\partial_\nu\xi^\lambda)F_{\mu\lambda} \ .
\end{aligned} \tag{2.90}$$

The commutators of the covariant derivatives (2.31) and (2.36) can be generalized to

$$[\mathbf{D}_\mu, \mathbf{D}_\nu]\phi = \frac{1}{2}R^{ab}{}_{\mu\nu}M_{ab}\phi + \mathcal{W}(\phi)F_{\mu\nu}\phi \ , \tag{2.91}$$

and

$$[\mathbf{D}_m, \mathbf{D}_n]\phi = \frac{1}{2}R^{ab}{}_{mn}M_{ab}\phi + \mathcal{W}(\phi)F_{mn}\phi - T^a_{mn}\mathbf{D}_a\phi \ , \tag{2.92}$$

where the Riemann tensor $R^{ab}{}_{\mu\nu}$, given by eq.(2.32), and the Weyl curvature $F_{\mu\nu}$, given by eq.(2.85), are unchanged, but the torsion tensor field $T^a_{\mu\nu}$ is now modified, due to the including of Weyl transformations, according to

$$\begin{aligned} T^a_{\mu\nu} &= \mathbf{D}_\mu e^a_\nu - \mathbf{D}_\nu e^a_\mu \ , \\ &= \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu + \omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu \ , \\ &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu + A_\mu e^a_\nu - A_\nu e^a_\mu \ . \end{aligned} \tag{2.93}$$

2.3 BRST transformations

The BRST formalism is an elegant and powerful tool for the consistent describtion of gauge symmetries in quantum field theory. The standard procedure is to substitute the infinitesimal parameters of the several symmetry transformations eqs.(2.1), (2.6),(2.58) for the corresponding anticommuting Faddeev-Popov ghosts

$$\begin{aligned} \xi^\mu &\rightarrow \xi^\mu \ , \\ \epsilon^{ab} &\rightarrow \theta^{ab} \ , \\ \Omega &\rightarrow \sigma \ , \end{aligned} \tag{2.94}$$

where ξ^μ, θ^{ab}, and σ denoting the diffeomorphism ghost[3], the Lorentz ghost, and the Weyl ghost. From the antisymmetry of the parameter ϵ^{ab} for the local Lorentz rotations it follows immediately that also the Lorentz ghost is antisymmetric

$$\theta^{ab} = -\theta^{ba} \ . \tag{2.95}$$

All above ghosts have ghost number one. Further the several symmetry operations are then expressed by a nilpotent anticommuting operator s which is called the BRST operator

$$s = s_D + s_L + s_W \ . \tag{2.96}$$

The BRST operator increase the ghost number by one. The BRST transformations of the ghosts are constructed in a way that all transformations are nilpotent. For all the basic

[3]Both, for the parameter of diffeomorphisms and for the diffeomorphism ghost, we use the same symbol.

fields mentioned so far one has now the following BRST transformations

$$\begin{aligned}
s\varphi &= -\xi^\lambda\partial_\lambda\varphi - \frac{N-2}{2}\sigma\varphi \ , \\
se^a_\mu &= \theta^a{}_b e^b_\mu - \xi^\lambda\partial_\lambda e^a_\mu - (\partial_\mu\xi^\lambda)e^a_\lambda + \sigma e^a_\mu \ , \\
sE^\mu_a &= -\theta^b{}_a E^\mu_b - \xi^\lambda\partial_\lambda E^\mu_a + (\partial_\lambda\xi^\mu)E^\lambda_a - \sigma E^\mu_a \ , \\
s\omega^a{}_{b\mu} &= -\partial_\mu\theta^a{}_b + \theta^a{}_c\omega^c{}_{b\mu} - \theta^c{}_b\omega^a{}_{c\mu} - \xi^\lambda\partial_\lambda\omega^a{}_{b\mu} - (\partial_\mu\xi^\lambda)\omega^a{}_{b\lambda} \ , \\
sA_\mu &= -\xi^\lambda\partial_\lambda A_\mu - (\partial_\mu\xi^\lambda)A_\lambda - \partial_\mu\sigma \ , \\
sT^a_{\mu\nu} &= \theta^a{}_b T^b_{\mu\nu} - \xi^\lambda\partial_\lambda T^a_{\mu\nu} - (\partial_\mu\xi^\lambda)T^a_{\lambda\nu} - (\partial_\nu\xi^\lambda)T^a_{\mu\lambda} + \sigma T^a_{\mu\nu} \ , \\
sR^a{}_{b\mu\nu} &= \theta^a{}_c R^c{}_{b\mu\nu} - \theta^c{}_b R^a{}_{c\mu\nu} - \xi^\lambda\partial_\lambda R^a{}_{b\mu\nu} - (\partial_\mu\xi^\lambda)R^a{}_{b\lambda\nu} - (\partial_\nu\xi^\lambda)R^a{}_{b\mu\lambda} \ , \\
sF_{\mu\nu} &= -\xi^\lambda\partial_\lambda F_{\mu\nu} - (\partial_\mu\xi^\lambda)F_{\lambda\nu} - (\partial_\nu\xi^\lambda)F_{\mu\lambda} \ ,
\end{aligned} \tag{2.97}$$

and

$$\begin{aligned}
s\xi^\mu &= -\xi^\lambda\partial_\lambda\xi^\mu \ , \\
s\theta^a{}_b &= \theta^a{}_c\theta^c{}_b - \xi^\lambda\partial_\lambda\theta^a{}_b \ , \\
s\sigma &= -\xi^\lambda\partial_\lambda\sigma \ ,
\end{aligned} \tag{2.98}$$

implying the nilpotency of the BRST operator

$$s^2 = 0 \ . \tag{2.99}$$

Furthermore the BRST operator commutes with the partial derivative

$$[s, \partial_\mu] = 0 \ . \tag{2.100}$$

More generally, the consistent treatment of any gauge field model demands to fix the gauge, in order to guarantee the existence of the corresponding gauge field propagator. This is achieved with a Lagrange multiplier field B, which forms together with an antighost field $\bar{c}$ a so-called BRST-doublet

$$\begin{aligned}
s\bar{c} &= B \ , \\
sB &= 0 \ .
\end{aligned} \tag{2.101}$$

The dependence of this BRST-doublet within the cohomological problem (1.11) is managed by a useful theorem [25] showing its triviality.

2.4 Differential forms

For the sake of clarity and completeness, this section is devoted to give a brief review of some properties and definitions of the well-known and useful formalism of differential forms. Differential forms simply provide an exceedingly compact notation for vectors and tensors on an arbitrary manifold.

A scalar function $f(x)$ is called a zero form. One defines the differential of the zero form f as the one form

$$df \equiv \frac{\partial f}{\partial x^\mu} dx^\mu \ , \tag{2.102}$$

where in N dimensions the index μ runs from 1 to N. Thus, the exterior derivative d can be written as

$$d \equiv dx^\mu \partial_\mu \ , \tag{2.103}$$

which increases the form degree by one and which is a nilpotent operator

$$d^2 = 0 \ . \tag{2.104}$$

The nilpotency of d is automatically guaranteed because of the vanishing commutator of two partial derivatives

$$[\partial_\mu, \partial_\nu] = 0 \ . \tag{2.105}$$

Given a vector function ϕ_μ one constructs the one form ϕ as follows:

$$\phi \equiv \phi_\mu dx^\mu \ . \tag{2.106}$$

The exterior derivative d of the one form (2.106) is defined as

$$d\phi = \frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) dx^\mu \wedge dx^\nu \ , \tag{2.107}$$

where the so-called wedge product is given by

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \ . \tag{2.108}$$

Therefore, $d\phi$ gives the curl of ϕ.

In general, given a completely antisymmetric tensor with p indices $\omega_{\mu_1 \mu_2 \dots \mu_p}$ one defines a p-form as

$$\omega = \frac{1}{p!} \omega_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \ . \tag{2.109}$$

Clearly, in N dimensions, one cannot have p-forms with $p \geq N$ which do not vanish identically. In order to simplify the notation, one omits the wedge product symbol and one simply regards dx^μ as an anticommuting Grassmann object.

To illustrate the use of forms, we look at the Yang-Mills theory, where the gauge field is the one form

$$A = A_\mu dx^\mu \ . \tag{2.110}$$

Remark, that here $A_\mu = A^a_\mu \lambda^a$ with generators λ^a, and so A is at the same time a form and a matrix. The curvature associated with the one form A is a two form:

$$F = dA + AA \ . \tag{2.111}$$

Then the Bianchi identity, expressed in terms of forms, is nothing but the nilpotency of d

$$dF = (dA)A - A(dA) \ , \tag{2.112}$$

$$[A, F] = A(dA) - (dA)A \ . \tag{2.113}$$

With the definition of the covariant derivative one has

$$DF = dF + [A, F] = 0 \ . \tag{2.114}$$

Now we can reformulate the results of the previous sections in the calculus of differential forms. The corresponding basic fields are given by the one forms $(e^a, \omega^a{}_b, A)$, e^a, $\omega^a{}_b$, and A being respectively the vielbein, the Lorentz connection, and the Weyl gauge field

$$\begin{aligned} e^a &= e^a_\mu dx^\mu \ , \\ \omega^a{}_b &= \omega^a{}_{b\mu} dx^\mu = \omega^a{}_{bm} e^m \ , \\ A &= A_\mu dx^\mu = A_m e^m \ , \end{aligned} \tag{2.115}$$

and the two forms $(T^a, R^a{}_b, F)$, T^a, $R^a{}_b$, and F denoting the torsion, the Riemann tensor, and the Weyl curvature

$$\begin{aligned} T^a &= \frac{1}{2} T^a_{\mu\nu} dx^\mu dx^\nu = de^a + \omega^a{}_b e^b + A e^a = De^a \ , \\ R^a{}_b &= \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu dx^\nu = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b \ , \\ F &= \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA \ , \end{aligned} \tag{2.116}$$

where

$$D = d + \omega + A \tag{2.117}$$

is the covariant derivative (2.88). The remaining zero form fields are the scalar field φ, the ghost field for diffeomorphisms ξ^μ, the Lorentz ghost field $\theta^a{}_b$, and the Weyl ghost field σ. From eq.(2.116) one easily obtains the Bianchi identities

$$\begin{aligned} DT^a &= dT^a + \omega^a{}_b T^b + AT^a = R^a{}_b e^b + F e^a, \\ DR^a{}_b &= dR^a{}_b + \omega^a{}_c R^c{}_b - \omega^c{}_b R^a{}_c = 0 \ , \\ DF &= dF = 0 \ . \end{aligned} \tag{2.118}$$

Furthermore, one has the anticommutator relation

$$\{s, d\} = sd + ds = 0 \ . \tag{2.119}$$

3 Maurer-Cartan horizontality conditions

The aim of this section is to derive the given set of BRST transformations, defined by the eqs.(2.97)-(2.98), from Maurer-Cartan horizontality conditions [18, 20]. In a first step this geometrical formalism is used to discuss the simpler case of non-abelian Yang-Mills theory [39].

3.1 Yang-Mills case

The BRST transformations of the one form gauge connection $A^a = A^a_\mu dx^\mu$ and the zero form ghost field c^a are given by

$$sA^a = dc^a + f^{abc}c^bA^c \ , \qquad sc^a = \frac{1}{2}f^{abc}c^bc^c \ , \tag{3.1}$$

with

$$s^2 = 0 \ , \tag{3.2}$$

where f^{abc} are the structure constants of the corresponding gauge group[4]. As usual, the adopted grading is given by the sum of the form degree and of the ghost number. In this sense, the fields A^a and c^a are both of degree one, their ghost number being respectively zero and one. A p-form with ghost number q will be denoted by Ω^q_p, its total grading being $(p+q)$. The two form field strength F^a is given by

$$F^a = \frac{1}{2}F^a_{\mu\nu}dx^\mu dx^\nu = dA^a + \frac{1}{2}f^{abc}A^bA^c \ , \tag{3.3}$$

and

$$dF^a = f^{abc}F^bA^c \ , \tag{3.4}$$

is its Bianchi identity. In order to reinterpret the BRST transformations (3.1) as a Maurer-Cartan horizontality condition we introduce the combined gauge-ghost field

$$\widetilde{A}^a = A^a + c^a \ , \tag{3.5}$$

and the generalized nilpotent differential operator

$$\widetilde{d} = d - s \ , \quad \widetilde{d}^2 = 0 \ . \tag{3.6}$$

Notice that both $\widetilde{A}^a$ and $\widetilde{d}$ have degree one. The nilpotency of $\widetilde{d}$ in (3.6) just implies the nilpotency of s and d, and furthermore fulfills the anticommutator relation (2.119).

Let us introduce also the degree-two field strenght $\widetilde{F}^a$:

$$\widetilde{F}^a = \widetilde{d}\widetilde{A}^a + \frac{1}{2}f^{abc}\widetilde{A}^b\widetilde{A}^c \ , \tag{3.7}$$

which, from eq.(3.6), obeys the generalized Bianchi identity

$$\widetilde{d}\widetilde{F}^a = f^{abc}\widetilde{F}^b\widetilde{A}^c \ . \tag{3.8}$$

The Maurer-Cartan horizontality condition reads then

$$\widetilde{F}^a = F^a \ . \tag{3.9}$$

Now it is very easy to check that the BRST transformations (3.1) can be obtained from the horizontality condition (3.9) by simply expanding $\widetilde{F}^a$ in terms of the elementary fields A^a and c^a and collecting the terms with the same form degree and ghost number. In addition, we remark also the equality leading to the generalized Bianchi identity

$$\widetilde{d}\widetilde{F}^a - f^{abc}\widetilde{F}^b\widetilde{A}^c = dF^a - f^{abc}F^bA^c = 0 \ . \tag{3.10}$$

[4]Notice that here the indices $a, b, c, \ldots$ are denoting the gauge group indices.

3.2 Gravitational case with Weyl symmetry

To write down the gravitational Maurer-Cartan horizontality conditions for the model described in the previous section one introduces a further ghost, as done in [18, 20], the local translation ghost η^a having ghost number one and tangent space indices. As explained in [20], the field η^a represents the ghost of local translations in the tangent space. See also the discussion of [40] based on an affine approach to gravity.

The local translation ghost η^a can be related [20] to the ghost of local diffeomorphism ξ^μ by the relation

$$\xi^\mu = E^\mu_a \eta^a \quad , \quad \eta^a = \xi^\mu e^a_\mu \, , \tag{3.11}$$

where E^μ_a denotes the inverse of the vielbein e^a_μ, i.e.

$$\begin{aligned} e^a_\mu E^\mu_b &= \delta^a_b \, , \\ e^a_\mu E^\nu_a &= \delta^\nu_\mu \, . \end{aligned} \tag{3.12}$$

Proceeding now as for the Yang-Mills case, one defines the nilpotent differential operator $\tilde{d}$ of degree one:

$$\tilde{d} = d - s \, , \tag{3.13}$$

and the generalized vielbein-ghost field $\tilde{e}^a$, the extended Lorentz connection $\tilde{\omega}^a{}_b$, and the generalized Weyl gauge field $\tilde{A}$

$$\begin{aligned} \tilde{e}^a &= e^a + \eta^a \, , \\ \tilde{\omega}^a{}_b &= \widehat{\omega}^a{}_b + \theta^a{}_b \, , \\ \tilde{A} &= \widehat{A} + \sigma \, , \end{aligned} \tag{3.14}$$

where $\widehat{\omega}^a{}_b$ and $\widehat{A}$ are given by

$$\begin{aligned} \widehat{\omega}^a{}_b &= \omega^a{}_{bm} \tilde{e}^m = \omega^a{}_b + \omega^a{}_{bm} \eta^m \, , \\ \widehat{A} &= A_m \tilde{e}^m = A + A_m \eta^m \, , \end{aligned} \tag{3.15}$$

with the zero forms $\omega^a{}_{bm}$ [5] and A_m defined by the expansion of the zero form connection $\omega^a{}_{b\mu}$ and the zero form Weyl gauge field A_μ in terms of the vielbein e^a_μ, i.e.:

$$\begin{aligned} \omega^a{}_{b\mu} &= \omega^a{}_{bm} e^m_\mu \, , \\ A_\mu &= A_m e^m_\mu \, . \end{aligned} \tag{3.16}$$

As it is well-known, the last formulas stem from the fact that the vielbein formalism allows to transform locally the space-time indices of an arbitrary tensor $\mathcal{N}_{\mu\nu\rho\sigma\ldots}$ into flat tangent space indices $\mathcal{N}_{abcd\ldots}$ by means of the expansion

$$\mathcal{N}_{\mu\nu\rho\sigma\ldots} = \mathcal{N}_{abcd\ldots} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma \ldots \, . \tag{3.17}$$

[5] Remark that the zero form $\omega^a{}_{bm}$ does not possess any symmetric or antisymmetric property with respect to the lower indices (bm).

Vice versa one has

$$\mathcal{N}_{abcd\dots} = \mathcal{N}_{\mu\nu\rho\sigma\dots} E^{\mu}_a E^{\nu}_b E^{\rho}_c E^{\sigma}_d \cdots . \tag{3.18}$$

According to eqs.(2.116), the generalized torsion field, the generalized Riemann tensor, and the generalized Weyl curvature are given by

$$\begin{aligned} \tilde{T}^a &= \tilde{d}\tilde{e}^a + \tilde{\omega}^a{}_b \tilde{e}^b + \tilde{A}\tilde{e}^a = \widetilde{D}\tilde{e}^a , \\ \tilde{R}^a{}_b &= \tilde{d}\tilde{\omega}^a{}_b + \tilde{\omega}^a{}_c \tilde{\omega}^c{}_b , \\ \tilde{F} &= \tilde{d}\tilde{A} , \end{aligned} \tag{3.19}$$

and are easily seen to obey the generalized Bianchi identities

$$\begin{aligned} \widetilde{D}\tilde{T}^a &= \tilde{d}\tilde{T}^a + \tilde{\omega}^a{}_b \tilde{T}^b + \tilde{A}\tilde{T}^a = \tilde{R}^a{}_b \tilde{e}^b + \tilde{F}\tilde{e}^a , \\ \widetilde{D}\tilde{R}^a{}_b &= \tilde{d}\tilde{R}^a{}_b + \tilde{\omega}^a{}_c \tilde{R}^c{}_b - \tilde{\omega}^c{}_b \tilde{R}^a{}_c = 0 , \\ \widetilde{D}\tilde{F} &= \tilde{d}\tilde{F} = 0 , \end{aligned} \tag{3.20}$$

with

$$\widetilde{D} = \tilde{d} + \tilde{\omega} + \tilde{A} \tag{3.21}$$

the generalized covariant derivative.

With these definitions the Maurer-Cartan horizontality conditions for gravity (with non-vanishing torsion) in the presence of a scalar field may be expressed in the following way: *$\tilde{e}$ and all its generalized covariant exterior differentials can be expanded over $\tilde{e}$ with classical coefficients,*

$$\tilde{e}^a = \delta^a_b \tilde{e}^b \equiv horizontal , \tag{3.22}$$

$$\tilde{T}^a(\tilde{e}, \tilde{\omega}) = \frac{1}{2} T^a_{mn}(e, \omega) \tilde{e}^m \tilde{e}^n \equiv horizontal , \tag{3.23}$$

$$\tilde{R}^a{}_b(\tilde{\omega}) = \frac{1}{2} R^a{}_{bmn}(\omega) \tilde{e}^m \tilde{e}^n \equiv horizontal , \tag{3.24}$$

$$\tilde{F}(\tilde{A}) = \frac{1}{2} F_{mn}(A) \tilde{e}^m \tilde{e}^n \equiv horizontal , \tag{3.25}$$

$$\widetilde{D}\varphi = (D_m\varphi)\tilde{e}^m = (\partial_m\varphi - \frac{N-2}{2} A_m\varphi)\tilde{e}^m \equiv horizontal . \tag{3.26}$$

Through eq.(3.17), the zero forms T^a_{mn}, $R^a{}_{bmn}$, and F_{mn} are defined by the vielbein expansion of the two forms of the torsion, the Riemann tensor, and the Weyl curvature of eq.(2.116),

$$\begin{aligned} T^a &= \frac{1}{2} T^a_{mn} e^m e^n , \\ R^a{}_b &= \frac{1}{2} R^a{}_{bmn} e^m e^n , \\ F &= \frac{1}{2} F_{mn} e^m e^n , \end{aligned} \tag{3.27}$$

and the zero form of the covariant derivative D_m is given by (see also later)

$$D = e^m D_m . \tag{3.28}$$

Notice also that eqs.(3.15) are nothing but the horizontality conditions for the Lorentz connection and the Weyl gauge field expressing the fact that $\widehat{\omega}$ and $\widehat{A}$ themselfs can be expanded over $\tilde{e}$.

Eqs.(3.22)-(3.26) define the Maurer-Cartan horizontality conditions for the gravitational case in the presence of scalar fields and, when expanded in terms of the elementary fields $(e^a, \omega^a{}_b, A, \eta^a, \theta^a{}_b, \sigma)$, give the nilpotent BRST transformations corresponding to the diffeomorphism transformations, the local Lorentz rotations, and the Weyl transformations.

For a better understanding of this point let us first discuss in detail the horizontality conditions (3.26) for the scalar field and (3.23) for the torsion. Making use of eqs.(3.13), (3.14), (3.15) and of the definition (3.19), one verifies that eq.(3.26) gives

$$d\varphi - s\varphi - \frac{N-2}{2}A\varphi - \frac{N-2}{2}A_m\eta^m\varphi - \frac{N-2}{2}\sigma\varphi = (D_m\varphi)e^m + (D_m\varphi)\eta^m \ , \tag{3.29}$$

and eq.(3.23) leads to

$$\begin{aligned} & de^a - se^a + d\eta^a - s\eta^a + \omega^a{}_b e^b + \theta^a{}_b e^b \\ & + \omega^a{}_b\eta^b + \theta^a{}_b\eta^b + \omega^a{}_{bm}\eta^m e^b + \omega^a{}_{bm}\eta^m\eta^b \\ & + Ae^a + A_m\eta^m e^a + \sigma e^a + A\eta^a + A_m\eta^m\eta^a + \sigma\eta^a \\ & = \frac{1}{2}T^a_{mn}e^m e^n + T^a_{mn}e^m\eta^n + \frac{1}{2}T^a_{mn}\eta^m\eta^n \ , \end{aligned} \tag{3.30}$$

from which, collecting the terms with the same form degree and ghost number, one easily obtains the BRST transformations for the scalar field φ, the vielbein e^a, and for the ghost η^a:

$$\begin{aligned} s\varphi &= -\eta^m\partial_m\varphi - \frac{N-2}{2}\sigma\varphi \ , \\ se^a &= d\eta^a + \omega^a{}_b\eta^b + \theta^a{}_b e^b + \omega^a{}_{bm}\eta^m e^b + A_m\eta^m e^a + \sigma e^a \\ &\quad + A\eta^a - T^a_{mn}e^m\eta^n \ , \\ s\eta^a &= \theta^a{}_b\eta^b + \omega^a{}_{bm}\eta^m\eta^b + A_m\eta^m\eta^a + \sigma\eta^a - \frac{1}{2}T^a_{mn}\eta^m\eta^n \ . \end{aligned} \tag{3.31}$$

These equations, when rewritten in terms of the variable ξ^μ of eq.(3.11), take the more familiar forms (see eqs.(2.97))

$$\begin{aligned} s\varphi &= -\xi^\lambda\partial_\lambda\varphi - \frac{N-2}{2}\sigma\varphi \ , \\ se^a_\mu &= \theta^a{}_b e^b_\mu + \mathcal{L}_\xi e^a_\mu + \sigma e^a_\mu \ , \\ s\xi^\mu &= -\xi^\lambda\partial_\lambda\xi^\mu \ , \end{aligned} \tag{3.32}$$

where $\mathcal{L}_\xi$ denotes the ordinary Lie derivative along the direction ξ^μ, i.e.

$$\mathcal{L}_\xi e^a_\mu = -\xi^\lambda\partial_\lambda e^a_\mu - (\partial_\mu\xi^\lambda)e^a_\lambda \ . \tag{3.33}$$

It is now apparent that eq.(3.31) represents the tangent space formulation of the usual BRST transformations corresponding to local Lorentz rotations, diffeomorphisms, and Weyl transformations for the scalar field φ, the one form vielbein field e^a, and the zero form translation ghost field η^a.

One sees then that the Maurer-Cartan horizontality conditions (3.22)-(3.26) together with eq.(3.19) carry in a very simple and compact form all the informations concerning the symmetry content with respect to the BRST formalism. It is quite easy indeed to expand eqs.(3.22)-(3.26) in terms of e^a and η^a and work out the BRST transformations of the remaining fields $(\omega^a{}_b, A, R^a{}_b, T^a, F, ...)$.

However, in view of the fact that we will use as fundamental variables the zero forms $(\omega^a{}_{bm}, A_m, R^a{}_{bmn}, T^a_{mn}, F_{mn})$ and the one form e^a rather than the one form Lorentz connection $\omega^a{}_b$, the one form Weyl gauge field A, and the two forms $R^a{}_b$, T^a, and F, let us proceed by introducing the partial derivative ∂_a of the tangent space. According to the formulas (3.17) and (3.18), the latter is defined by

$$\partial_a \equiv E^\mu_a \partial_\mu \ , \tag{3.34}$$

and

$$\partial_\mu = e^a_\mu \partial_a \ , \tag{3.35}$$

so that the intrinsic exterior differential d becomes

$$d = dx^\mu \partial_\mu = e^a \partial_a \ , \tag{3.36}$$

and analogous for the covariant derivative D

$$D = dx^\mu D_\mu = e^m D_m \ . \tag{3.37}$$

The introduction of the operator ∂_a and the use of the zero forms $(\omega^a{}_{bm}, A_m, R^a{}_{bmn}, T^a_{mn}, F_{mn})$ and the one form e^a allows for a complete tangent space formulation. This step, as we shall see later, turns out to be very useful in the analysis of the corresponding BRST cohomology. Moreover, as one can easily understand, the knowledge of the BRST transformations of the zero form sector $(\omega^a{}_{bm}, A_m, R^a{}_{bmn}, T^a_{mn}, F_{mn})$ together with the expansions (3.16), (3.27) and the equation (3.31) completely characterize the transformation law of the forms $(\omega^a{}_b, A, R^a{}_b, T^a, F)$.

For completeness, now we will discuss the remaining Maurer-Cartan horizontality conditions and we will derive the BRST transformations of the corresponding fields. From eq.(3.25) follows

$$\begin{aligned} &dA - sA + d(A_m\eta^m) - s(A_m\eta^m) + d\sigma - s\sigma \\ &= \frac{1}{2}F_{mn}e^m e^n + F_{mn}e^m\eta^n + \frac{1}{2}F_{mn}\eta^m\eta^n \ , \end{aligned} \tag{3.38}$$

from which, collecting again the terms with the same form degree and ghost number, one can read off the BRST transformations for the one form Weyl gauge field A and for the zero form ghost field $\widehat{\sigma}$:

$$
\begin{aligned}
sA &= d\widehat{\sigma} - F_{mn} e^m \eta^n \ , \\
s\widehat{\sigma} &= -\frac{1}{2} F_{mn} \eta^m \eta^n \ ,
\end{aligned}
\tag{3.39}
$$

where $\widehat{\sigma}$ is given by the combination

$$
\widehat{\sigma} = A_m \eta^m + \sigma \ . \tag{3.40}
$$

To determine the BRST transformation for the zero form Weyl gauge field A_m one needs the partial derivative of it in the tangent space. This can be found by the definition for the two form Weyl curvature F (2.116)

$$
\begin{aligned}
dA = F &= \frac{1}{2} F_{mn} e^m e^n \\
&= d(A_m e^m) = \frac{1}{2}(\partial_m A_n - \partial_n A_m) e^m e^n + A_m (de^m) \ .
\end{aligned}
\tag{3.41}
$$

By inserting the definition of the torsion two form T^a (2.116) above equation leads to

$$
\partial_m A_n - \partial_n A_m = F_{mn} - A_k T^k_{mn} - A_k \omega^k{}_{mn} + A_k \omega^k{}_{nm} \ . \tag{3.42}
$$

With the help of eq.(3.31) one easily finds from eq.(3.39) the BRST transformation of the zero form Weyl gauge field A_m according to

$$
sA_m = -\eta^k \partial_k A_m - \theta^k{}_m A_k - \partial_m \sigma - A_m \sigma \ , \tag{3.43}
$$

and by using the eqs.(3.31), (3.42), and (3.43) one gets the BRST transformation of the Weyl ghost σ from eq.(3.39)

$$
s\sigma = -\eta^k \partial_k \sigma \ . \tag{3.44}
$$

These equations, when rewritten in terms of the variable ξ^μ, take the form (see eqs.(2.97))

$$
\begin{aligned}
sA_\mu &= -\xi^\lambda \partial_\lambda A_\mu - (\partial_\mu \xi^\lambda) A_\lambda - \partial_\mu \sigma \ , \\
s\sigma &= -\xi^\lambda \partial_\lambda \sigma \ .
\end{aligned}
\tag{3.45}
$$

Finally, the Maurer-Cartan horizontality condition for the Riemann tensor (3.24) gives

$$
\begin{aligned}
& d\omega^a{}_b - s\omega^a{}_b + d(\omega^a{}_{bm}\eta^m) - s(\omega^a{}_{bm}\eta^m) + d\theta^a{}_b - s\theta^a{}_b \\
& + \omega^a{}_c \omega^c{}_b + \omega^a{}_c \theta^c{}_b + \theta^a{}_c \omega^c{}_b + \theta^a{}_c \theta^c{}_b \\
& + \omega^a{}_c \omega^c{}_{bm} \eta^m + \omega^a{}_{cm} \eta^m \omega^c{}_b + \omega^a{}_{cm} \eta^m \omega^c{}_{bn} \eta^n \\
& + \theta^a{}_c \omega^c{}_{bm} \eta^m + \omega^a{}_{cm} \eta^m \theta^c{}_b \\
& = \frac{1}{2} R^a{}_{bmn} e^m e^n + R^a{}_{bmn} e^m \eta^n + \frac{1}{2} R^a{}_{bmn} \eta^m \eta^n \ ,
\end{aligned}
\tag{3.46}
$$

from which the BRST transformations for the one form Lorentz connection field $\omega^a{}_b$ and for the zero form ghost field $\widehat{\theta}^a{}_b$ are found to

$$
\begin{aligned}
s\omega^a{}_b &= d\widehat{\theta}^a{}_b + \widehat{\theta}^a{}_c \omega^c{}_b - \widehat{\theta}^c{}_b \omega^a{}_c - R^a{}_{bmn} e^m \eta^n \ , \\
s\widehat{\theta}^a{}_b &= \widehat{\theta}^a{}_c \widehat{\theta}^c{}_b - \frac{1}{2} R^a{}_{bmn} \eta^m \eta^n \ ,
\end{aligned}
\tag{3.47}
$$

where $\widehat{\theta}^a{}_b$ is defined by the combination

$$\widehat{\theta}^a{}_b = \omega^a{}_{bm}\eta^m + \theta^a{}_b \,. \tag{3.48}$$

The partial derivative of the zero form Lorentz connection $\omega^a{}_{bm}$ follows from the definition of the two form Riemann tensor $R^a{}_b$ (2.116)

$$\begin{aligned} d\omega^a{}_b &= R^a{}_b - \omega^a{}_c\omega^c{}_b = \frac{1}{2}R^a{}_{bmn}e^m e^n - \omega^a{}_{cm}\omega^c{}_{bn}e^m e^n \\ &= d(\omega^a{}_{bn}e^n) = \frac{1}{2}(\partial_m\omega^a{}_{bn} - \partial_n\omega^a{}_{bm})e^m e^n + \omega^a{}_{bm}(de^m) \,, \end{aligned} \tag{3.49}$$

which leads to

$$\begin{aligned} \partial_m\omega^a{}_{bn} - \partial_n\omega^a{}_{bm} &= R^a{}_{bmn} - \omega^a{}_{cm}\omega^c{}_{bn} + \omega^a{}_{cn}\omega^c{}_{bm} - \omega^a{}_{bk}T^k_{mn} \\ &\quad + \omega^a{}_{bk}\omega^k{}_{nm} - \omega^a{}_{bk}\omega^k{}_{mn} + \omega^a{}_{bn}A_m - \omega^a{}_{bm}A_n \,. \end{aligned} \tag{3.50}$$

The BRST transformations for the zero form Lorentz connection field $\omega^a{}_{bm}$ and for the Lorentz ghost $\theta^a{}_b$ are then determined by

$$s\omega^a{}_{bm} = -\eta^k\partial_k\omega^a{}_{bm} - \partial_m\theta^a{}_b + \theta^a{}_c\omega^c{}_{bm} - \theta^c{}_b\omega^a{}_{cm} - \theta^k{}_m\omega^a{}_{bk} - \sigma\omega^a{}_{bm} \,, \tag{3.51}$$

$$s\theta^a{}_b = -\eta^k\partial_k\theta^a{}_b + \theta^a{}_c\theta^c{}_b \,. \tag{3.52}$$

By using the variable ξ^μ above BRST transformations correspond to (see eqs.(2.97))

$$\begin{aligned} s\omega^a{}_{b\mu} &= -\xi^\lambda\partial_\lambda\omega^a{}_{b\mu} - (\partial_\mu\xi^\lambda)\omega^a{}_{b\lambda} - \partial_\mu\theta^a{}_b + \theta^a{}_c\omega^c{}_{b\mu} - \theta^c{}_b\omega^a{}_{c\mu} \,, \\ s\theta^a{}_b &= -\xi^\lambda\partial_\lambda\theta^a{}_b + \theta^a{}_c\theta^c{}_b \,. \end{aligned} \tag{3.53}$$

The BRST transformations for the two forms $(T^a, R^a{}_b, F)$, the torsion, the Riemann tensor, and the Weyl curvature, can be worked out from the generalized Bianchi identities (3.20)

$$\begin{aligned} sT^a &= (dT^a_{mn})e^m\eta^n - T^a_{mn}e^m(d\eta^n) + \theta^a{}_bT^b + \sigma T^a + T^a_{mn}T^m\eta^n \\ &\quad - T^a_{mn}\omega^m_k e^k\eta^n + \omega^a{}_{bk}\eta^k T^b + \omega^a{}_b T^b_{mn}e^m\eta^n + A_k\eta^k T^a \\ &\quad - R^a{}_b\eta^b - R^a{}_{bmn}e^m\eta^n e^b - F\eta^a - F_{mn}e^m\eta^n e^a \,, \end{aligned} \tag{3.54}$$

$$\begin{aligned} sR^a{}_b &= (dR^a{}_{bmn})e^m\eta^n - R^a{}_{bmn}e^m(d\eta^n) + \theta^a{}_cR^c{}_b - \theta^c{}_bR^a{}_c \\ &\quad - R^a{}_{bmn}\omega^m_k e^k\eta^n + \omega^a{}_{ck}\eta^k R^c{}_b - \omega^c{}_{bk}\eta^k R^a{}_c + \omega^a{}_c R^c{}_{bmn}e^m\eta^n \\ &\quad - \omega^c{}_b R^a{}_{cmn}e^m\eta^n + R^a{}_{bmn}T^m\eta^n - R^a{}_{bmn}Ae^m\eta^n \,, \end{aligned} \tag{3.55}$$

$$\begin{aligned} sF &= (dF_{mn})e^m\eta^n - F_{mn}e^m(d\eta^n) + F_{mn}T^m\eta^n \\ &\quad - F_{mn}\omega^m_k e^k\eta^n - F_{mn}Ae^m\eta^n \,. \end{aligned} \tag{3.56}$$

To calculate out the corresponding zero forms $(T^a_{mn}, R^a{}_{bmn}, F_{mn})$ one needs the partial derivative in the tangent space of these fields. From the Bianchi identities (2.118) one gets the complete antisymmetrized relations

$$\partial_k T^a_{mn} + \partial_m T^a_{nk} + \partial_n T^a_{km} = R^a{}_{kmn} + R^a{}_{mnk} + R^a{}_{nkm}$$

$$
\begin{aligned}
&+ \delta^a_k F_{mn} + \delta^a_m F_{nk} + \delta^a_n F_{km} \\
&- \omega^a{}_{bk} T^b_{mn} - \omega^a{}_{bm} T^b_{nk} - \omega^a{}_{bn} T^b_{km} \\
&+ A_k T^a_{mn} + A_m T^a_{nk} + A_n T^a_{km} \\
&- T^a_{lk} T^l_{mn} - T^a_{lm} T^l_{nk} - T^a_{ln} T^l_{km} \\
&+ T^a_{lk} \omega^l{}_{nm} + T^a_{ln} \omega^l{}_{mk} + T^a_{lm} \omega^l{}_{kn} \\
&- T^a_{lk} \omega^l{}_{mn} - T^a_{lm} \omega^l{}_{nk} - T^a_{ln} \omega^l{}_{km} , \qquad (3.57)
\end{aligned}
$$

$$
\begin{aligned}
\partial_k R^a{}_{bmn} + \partial_m R^a{}_{bnk} + \partial_n R^a{}_{bkm} &= -\omega^a{}_{ck} R^c{}_{bmn} - \omega^a{}_{cm} R^c{}_{bnk} - \omega^a{}_{cn} R^c{}_{bkm} \\
&+ \omega^c{}_{bk} R^a{}_{cmn} + \omega^c{}_{bm} R^a{}_{cnk} + \omega^c{}_{bn} R^a{}_{ckm} \\
&- R^a{}_{blk} T^l_{mn} - R^a{}_{blm} T^l_{nk} - R^a{}_{bln} T^l_{km} \\
&+ R^a{}_{blk} \omega^l{}_{nm} + R^a{}_{bln} \omega^l{}_{mk} + R^a{}_{blm} \omega^l{}_{kn} \\
&- R^a{}_{blk} \omega^l{}_{mn} - R^a{}_{blm} \omega^l{}_{nk} - R^a{}_{bln} \omega^l{}_{km} \\
&+ 2A_k R^a{}_{bmn} + 2A_m R^a{}_{bnk} + 2A_n R^a{}_{bkm} , \qquad (3.58)
\end{aligned}
$$

$$
\begin{aligned}
\partial_k F_{mn} + \partial_m F_{nk} + \partial_n F_{km} &= -F_{lk} T^l_{mn} - F_{lm} T^l_{nk} - F_{ln} T^l_{km} \\
&+ F_{lk} \omega^l{}_{nm} + F_{ln} \omega^l{}_{mk} + F_{lm} \omega^l{}_{kn} \\
&- F_{lk} \omega^l{}_{mn} - F_{lm} \omega^l{}_{nk} - F_{ln} \omega^l{}_{km} \\
&+ 2A_k F_{mn} + 2A_m F_{nk} + 2A_n F_{km} . \qquad (3.59)
\end{aligned}
$$

Inserting eqs.(3.57)-(3.59) into eqs.(3.54)-(3.56) leads to the BRST transformations of the zero form torsion field T^a_{mn}, the zero form Riemann tensor $R^a{}_{bmn}$, and the zero form Weyl curvature F_{mn}

$$
\begin{aligned}
sT^a_{mn} &= -\eta^k \partial_k T^a_{mn} + \theta^a{}_b T^b_{mn} - \theta^k{}_m T^a_{kn} - \theta^k{}_n T^a_{mk} - \sigma T^a_{mn} , \qquad (3.60) \\
sR^a{}_{bmn} &= -\eta^k \partial_k R^a{}_{bmn} + \theta^a{}_c R^c{}_{bmn} - \theta^c{}_b R^a{}_{cmn} - \theta^k{}_m R^a{}_{bkn} \\
&- \theta^k{}_n R^a{}_{bmk} - 2\sigma R^a{}_{bmn} , \qquad (3.61) \\
sF_{mn} &= -\eta^k \partial_k F_{mn} - \theta^k{}_m F_{kn} - \theta^k{}_n F_{mk} - 2\sigma F_{mn} . \qquad (3.62)
\end{aligned}
$$

These equations, when rewritten in terms of the variable ξ^μ, take the form (see eqs.(2.97))

$$
\begin{aligned}
sT^a_{\mu\nu} &= -\xi^\lambda \partial_\lambda T^a_{\mu\nu} - (\partial_\mu \xi^\lambda) T^a_{\lambda\nu} - (\partial_\nu \xi^\lambda) T^a_{\mu\lambda} + \theta^a{}_b T^b_{\mu\nu} + \sigma T^a_{\mu\nu} , \\
sR^a{}_{b\mu\nu} &= -\xi^\lambda \partial_\lambda R^a{}_{b\mu\nu} - (\partial_\mu \xi^\lambda) R^a{}_{b\lambda\nu} - (\partial_\nu \xi^\lambda) R^a{}_{b\mu\lambda} + \theta^a{}_c R^c{}_{b\mu\nu} - \theta^c{}_b R^a{}_{c\mu\nu} , \\
sF_{\mu\nu} &= -\xi^\lambda \partial_\lambda F_{\mu\nu} - (\partial_\mu \xi^\lambda) F_{\lambda\nu} - (\partial_\nu \xi^\lambda) F_{\mu\lambda} . \qquad (3.63)
\end{aligned}
$$

Notice that, contrary to the case of the usual space-time derivative ∂_μ, the operator ∂_a does not commute with the BRST operator s or with the exterior derivative d due to the explicit presence of the vielbein e^a (see Appendix A for the detailed calculations). One has:

$$
\begin{aligned}
[s, \partial_m] = (&\partial_m \eta^k - \theta^k{}_m - T^k_{mn} \eta^n - \omega^k{}_{mn} \eta^n + \omega^k{}_{nm} \eta^n \\
&+ A_m \eta^k - A_n \eta^n \delta^k_m - \sigma \delta^k_m) \partial_k , \qquad (3.64)
\end{aligned}
$$

and

$$[d, \partial_m] = (T^k_{mn} e^n + \omega^k{}_{mn} e^n - \omega^k{}_{nm} e^n - A_m e^k + A_n e^n \delta^k_m - (\partial_m e^k))\partial_k \ . \tag{3.65}$$

Also the commutator of two tangent space derivatives does not vanish

$$[\partial_m, \partial_n] = -(T^k_{mn} + \omega^k{}_{mn} - \omega^k{}_{nm} - A_m \delta^k_n + A_n \delta^k_m)\partial_k \ . \tag{3.66}$$

Nevertheless, taking into account the vielbein transformation (3.31), one consistently verifies that

$$\{s, d\} = 0 \quad , \quad d^2 = 0 \quad , \quad s^2 = 0 \ . \tag{3.67}$$

3.3 BRST transformations and Bianchi identities

The last section is devoted to collect, as a summary for the reader, the whole set of BRST transformations and the Bianchi identities which emerge from the Maurer-Cartan horizontality conditions (3.22)-(3.26) and from eqs.(3.19), (3.20) for each form sector and ghost number.

- **Form sector two, ghost number zero** $(T^a, R^a{}_b, F)$

$$\begin{aligned}
sT^a &= (dT^a_{mn}) e^m \eta^n - T^a_{mn} e^m d\eta^n + \theta^a{}_b T^b + \sigma T^a + T^a_{mn} T^m \eta^n \\
&\quad - T^a_{mn} \omega^m{}_k e^k \eta^n + \omega^a{}_{bk} \eta^k T^b + \omega^a{}_b T^b_{mn} e^m \eta^n + A_k \eta^k T^a \\
&\quad - R^a{}_b \eta^b - R^a{}_{bmn} e^m \eta^n e^b - F \eta^a - F_{mn} e^m \eta^n e^a \ , \\
sR^a{}_b &= (dR^a{}_{bmn}) e^m \eta^n - R^a{}_{bmn} e^m d\eta^n + \theta^a{}_c R^c{}_b - \theta^c{}_b R^a{}_c \\
&\quad - R^a{}_{bmn} \omega^m{}_k e^k \eta^n + \omega^a{}_{ck} \eta^k R^c{}_b - \omega^c{}_{bk} \eta^k R^a{}_c + \omega^a{}_c R^c{}_{bmn} e^m \eta^n \\
&\quad - \omega^c{}_b R^a{}_{cmn} e^m \eta^n + R^a{}_{bmn} T^m \eta^n - R^a{}_{bmn} A e^m \eta^n \ , \\
sF &= (dF_{mn}) e^m \eta^n - F_{mn} e^m d\eta^n + F_{mn} T^m \eta^n \\
&\quad - F_{mn} \omega^m{}_k e^k \eta^n - F_{mn} A e^m \eta^n \ .
\end{aligned} \tag{3.68}$$

For the Bianchi identities one has

$$\begin{aligned}
DT^a &= dT^a + \omega^a{}_b T^b + AT^a = R^a{}_b e^b + F e^a \ , \\
DR^a{}_b &= dR^a{}_b + \omega^a{}_c R^c{}_b - \omega^c{}_b R^a{}_c = 0 \ , \\
DF &= dF = 0 \ .
\end{aligned} \tag{3.69}$$

- **Form sector one, ghost number zero** $(e^a, \omega^a{}_b, A)$

$$\begin{aligned}
se^a &= d\eta^a + \omega^a{}_b\eta^b + \theta^a{}_b e^b + \omega^a{}_{bm}\eta^m e^b + A_m\eta^m e^a + \sigma e^a \\
&\quad + A\eta^a - T^a_{mn} e^m \eta^n \ , \\
s\omega^a{}_b &= (d\omega^a{}_{bm})\eta^m + \omega^a{}_{bm} d\eta^m + d\theta^a{}_b + \omega^a{}_{cm}\eta^m\omega^c{}_b + \theta^a{}_c\omega^c{}_b \\
&\quad - \omega^c{}_{bm}\eta^m\omega^a{}_c - \theta^c{}_b\omega^a{}_c - R^a{}_{bmn} e^m\eta^n \ , \\
sA &= (dA_m)\eta^m + A_m d\eta^m + d\sigma - F_{mn} e^m \eta^n \ . \qquad (3.70)
\end{aligned}$$

The exterior derivatives of these fields are given by the definitions of the two form field strengths (2.116)

$$\begin{aligned}
d\omega^a{}_b &= R^a{}_b - \omega^a{}_c\omega^c{}_b \ , \\
de^a &= T^a - \omega^a{}_b e^b - Ae^a \ , \\
dA &= F \ . \qquad (3.71)
\end{aligned}$$

- **Form sector zero, ghost number zero** $(\varphi, \omega^a{}_{bm}, A_m, T^a_{mn}, R^a{}_{bmn}, F_{mn})$

$$\begin{aligned}
s\varphi &= -\eta^m\partial_m\varphi - \frac{N-2}{2}\sigma\varphi \ , \\
s\omega^a{}_{bm} &= -\eta^k\partial_k\omega^a{}_{bm} - \partial_m\theta^a{}_b + \theta^a{}_c\omega^c{}_{bm} - \theta^c{}_b\omega^a{}_{cm} - \theta^k{}_m\omega^a{}_{bk} - \sigma\omega^a{}_{bm} \ , \\
sA_m &= -\eta^k\partial_k A_m - \theta^k{}_m A_k - \partial_m\sigma - A_m\sigma \ , \\
sT^a_{mn} &= -\eta^k\partial_k T^a_{mn} + \theta^a{}_b T^b_{mn} - \theta^k{}_m T^a_{kn} - \theta^k{}_n T^a_{mk} - \sigma T^a_{mn} \ , \\
sR^a{}_{bmn} &= -\eta^k\partial_k R^a{}_{bmn} + \theta^a{}_c R^c{}_{bmn} - \theta^c{}_b R^a{}_{cmn} - \theta^k{}_m R^a{}_{bkn} \\
&\quad - \theta^k{}_n R^a{}_{bmk} - 2\sigma R^a{}_{bmn} \ , \\
sF_{mn} &= -\eta^k\partial_k F_{mn} - \theta^k{}_m F_{kn} - \theta^k{}_n F_{mk} - 2\sigma F_{mn} \ . \qquad (3.72)
\end{aligned}$$

The Bianchi identities (3.69) are projected on the zero form torsion T^a_{mn}, the zero form Riemann tensor $R^a{}_{bmn}$, and on the zero form Weyl curvature F_{mn} to give

$$\begin{aligned}
dT^a_{mn} &= (\partial_k T^a_{mn})e^k \\
&= (R^a{}_{kmn} + R^a{}_{mnk} + R^a{}_{nkm} \\
&\quad + \delta^a_k F_{mn} + \delta^a_m F_{nk} + \delta^a_n F_{km} \\
&\quad - \omega^a{}_{bk} T^b_{mn} - \omega^a{}_{bm} T^b_{nk} - \omega^a{}_{bn} T^b_{km} \\
&\quad + A_k T^a_{mn} + A_m T^a_{nk} + A_n T^a_{km} \\
&\quad - T^a_{lk} T^l_{mn} - T^a_{lm} T^l_{nk} - T^a_{ln} T^l_{km} \\
&\quad + T^a_{lk}\omega^l{}_{nm} + T^a_{ln}\omega^l{}_{mk} + T^a_{lm}\omega^l{}_{kn} \\
&\quad - T^a_{lk}\omega^l{}_{mn} - T^a_{lm}\omega^l{}_{nk} - T^a_{ln}\omega^l{}_{km} \\
&\quad - \partial_m T^a_{nk} - \partial_n T^a_{km})e^k \ , \\
dR^a{}_{bmn} &= (\partial_k R^a{}_{bmn})e^k \\
&= (-\omega^a{}_{ck} R^c{}_{bmn} - \omega^a{}_{cm} R^c{}_{bnk} - \omega^a{}_{cn} R^c{}_{bkm}
\end{aligned}$$

$$
\begin{aligned}
&+ \omega^c{}_{bk} R^a{}_{cmn} + \omega^c{}_{bm} R^a{}_{cnk} + \omega^c{}_{bn} R^a{}_{ckm} \\
&- R^a{}_{blk} T^l_{mn} - R^a{}_{blm} T^l_{nk} - R^a{}_{bln} T^l_{km} \\
&+ R^a{}_{blk} \omega^l{}_{nm} + R^a{}_{bln} \omega^l{}_{mk} + R^a{}_{blm} \omega^l{}_{kn} \\
&- R^a{}_{blk} \omega^l{}_{mn} - R^a{}_{blm} \omega^l{}_{nk} - R^a{}_{bln} \omega^l{}_{km} \\
&+ 2A_k R^a{}_{bmn} + 2A_m R^a{}_{bnk} + 2A_n R^a{}_{bkm} \\
&- \partial_m R^a{}_{bnk} - \partial_n R^a{}_{bkm}) e^k \,, \\
dF_{mn} &= (\partial_k F_{mn}) e^k \\
&= (-F_{lk} T^l_{mn} - F_{lm} T^l_{nk} - F_{ln} T^l_{km} \\
&+ F_{lk} \omega^l{}_{nm} + F_{ln} \omega^l{}_{mk} + F_{lm} \omega^l{}_{kn} \\
&- F_{lk} \omega^l{}_{mn} - F_{lm} \omega^l{}_{nk} - F_{ln} \omega^l{}_{km} \\
&+ 2A_k F_{mn} + 2A_m F_{nk} + 2A_n F_{km} \\
&- \partial_m F_{nk} - \partial_n F_{km}) e^k \,. \qquad (3.73)
\end{aligned}
$$

One has also the equations

$$
\begin{aligned}
d\omega^a{}_{bm} &= (\partial_n \omega^a{}_{bm}) e^n \\
&= -(R^a{}_{bmn} - \omega^a{}_{cm} \omega^c{}_{bn} + \omega^a{}_{cn} \omega^c{}_{bm} \\
&- \omega^a{}_{bk} T^k_{mn} + \omega^a{}_{bk} \omega^k{}_{nm} - \omega^a{}_{bk} \omega^k{}_{mn} \\
&+ \omega^a{}_{bn} A_m - \omega^a{}_{bm} A_n - \partial_m \omega^a{}_{bn}) e^n \,, \\
dA_m &= (\partial_n A_m) e^n \\
&= -(F_{mn} - A_k T^k_{mn} - A_k \omega^k{}_{mn} \\
&+ A_k \omega^k{}_{nm} - \partial_m A_n) e^n \,. \qquad (3.74)
\end{aligned}
$$

- **Form sector zero, ghost number one** $(\eta^a, \theta^a{}_b, \sigma)$

  The BRST transformations for the ghost fields are given by

$$
\begin{aligned}
s\eta^a &= \theta^a{}_b \eta^b + \omega^a{}_{bm} \eta^m \eta^b + A_m \eta^m \eta^a + \sigma \eta^a - \frac{1}{2} T^a_{mn} \eta^m \eta^n \,, \\
s\theta^a{}_b &= \theta^a{}_c \theta^c{}_b - \eta^k \partial_k \theta^a{}_b \,, \\
s\sigma &= -\eta^k \partial_k \sigma \,. \qquad (3.75)
\end{aligned}
$$

- **Commutator relations for the tangent space derivative** ∂_m

  The following commutator relations are valid:

$$
\begin{aligned}
[s, \partial_m] &= (\partial_m \eta^k - \theta^k{}_m - T^k_{mn} \eta^n - \omega^k{}_{mn} \eta^n + \omega^k{}_{nm} \eta^n \\
&+ A_m \eta^k - A_n \eta^n \delta^k_m - \sigma \delta^k_m) \partial_k \,, \\
[d, \partial_m] &= (T^k_{mn} e^n + \omega^k{}_{mn} e^n - \omega^k{}_{nm} e^n \\
&- A_m e^k + A_n e^n \delta^k_m - (\partial_m e^k)) \partial_k \,, \qquad (3.76)
\end{aligned}
$$

and

$$
[\partial_m, \partial_n] = -(T^k_{mn} + \omega^k{}_{mn} - \omega^k{}_{nm} - A_m \delta^k_n + A_n \delta^k_m) \partial_k \,. \qquad (3.77)
$$

- **Algebra between s and d**

  From the above transformations it follows:

$$s^2 = 0 \ , \quad d^2 = 0 \ , \tag{3.78}$$

  and

$$\{s, d\} = 0 \ . \tag{3.79}$$

Let us conclude this section by making two remarks. The first one concerns the role played by the torsion T in the BRST transformations. We emphasize that, as one can see from eqs.(3.68)-(3.75), a fully tangent space formulation of the gravitational algebra can be obtained only when the torsion is explicitly present.

The second remark is related to the use of the variable η^a. Observe that, when expressed in terms of η^a, the BRST transformation of the vielbein e^a in (3.70) starts with a term linear in the fields (i.e. the term $d\eta^a$). This feature makes the analogy between gravitational and gauge theories more transparent. Moreover, it suggests that one may compute the local cohomology of the gravitational BRST operator s [41] without expanding the vielbein e^a around a flat background, as shown in [16].

4 Descent equations and decomposition

The discussion of invariant Lagrangians and anomalies implies to find non-trivial solutions of the so-called Wess-Zumino consistency condition [11] formulated in terms of the BRST transformations

$$sa = 0 \ , \tag{4.1}$$

where s is the nilpotent BRST operator and a is an integrated local field polynomial in the space of differential forms [19]. The use of the space of polynomials of forms is not a restriction on the generality of the solutions of the consistency equation, as recently proven by M. Dubois-Violette et al. [19]. Non-trivial solutions of (4.1) are given by the descent-equations technique [16, 18, 19, 24, 25, 42]. Setting

$$a = \int \mathcal{A} \ , \tag{4.2}$$

condition (4.1) translates into the local equation

$$s\mathcal{A} + d\hat{\mathcal{A}} = 0 \ , \tag{4.3}$$

where $\mathcal{A}$ are some local polynomials and $d = dx^\mu \partial_\mu$ denotes the exterior space-time derivative. $\mathcal{A}$ is said non-trivial if

$$\mathcal{A} \neq s\mathcal{B} + d\hat{\mathcal{B}} \ , \tag{4.4}$$

with $\mathcal{B}$ and $\hat{\mathcal{B}}$ local polynomials. The volume form (N-form) $\mathcal{A}$ with a given ghost number G is denoted by $\mathcal{A}_N^G$. The local equation (4.3) reads then

$$s\mathcal{A}_N^G + d\mathcal{A}_{N-1}^{G+1} = 0 \ . \tag{4.5}$$

In this case the integral of $\mathcal{A}$ on space-time, $\int \mathcal{A}$, identifies a cohomology class of the BRST operator s and, according to its ghost number G, it corresponds to an invariant Lagrangian ($G = 0$) or to an anomaly ($G = 1$).

By applying the BRST operator s on the local equation (4.5), due to the relations

$$s^2 = d^2 = sd + ds = 0 \tag{4.6}$$

and to the algebraic Poincaré Lemma [15, 16]

$$d\Omega = 0 \iff \Omega = d\widehat{\Omega} + d^N x \ \mathcal{L} + const \ , \tag{4.7}$$

it follows for the next cocycle

$$s\mathcal{A}_{N-1}^{G+1} + d\mathcal{A}_{N-2}^{G+2} = 0 \ , \tag{4.8}$$

because $s\mathcal{A}_{N-1}^{G+1}$ is not a volume form nor a constant form. It is easily seen that repeated applications of the operator s generate a tower of descent equations

$$\begin{aligned}
&s\mathcal{A}_N^G + d\mathcal{A}_{N-1}^{G+1} = 0 \ , \\
&s\mathcal{A}_{N-1}^{G+1} + d\mathcal{A}_{N-2}^{G+2} = 0 \ , \\
&\quad \cdots\cdots \\
&\quad \cdots\cdots \\
&s\mathcal{A}_1^{G+N-1} + d\mathcal{A}_0^{G+N} = 0 \ , \\
&s\mathcal{A}_0^{G+N} = 0 \ ,
\end{aligned} \tag{4.9}$$

which ends with a zero form cocycle $\mathcal{A}_0^{G+N}$.

The main idea to solve the tower of descent equations (4.9) is based on the fact that the exterior derivative d can be written as a BRST commutator in the following sense

$$d = -[s, \delta] \ , \tag{4.10}$$

where δ is an operator which will be specified later. In general this operator decrease the ghost number g by one and increase the form degree p by one

$$\delta\mathcal{A}_p^g = \mathcal{A}_{p+1}^{g-1} \ . \tag{4.11}$$

4.1 Pure Yang-Mills

In order to elaborate the general idea of solving the tower (4.9) one starts again with the discussion of the pure Yang-Mills theory, as it was done in a recent paper of S.P. Sorella [26]. For this one introduces the operators δ and $\mathcal{G}$ defined by

$$\delta = -A^a \frac{\partial}{\partial c^a} + (F^a + \frac{1}{2} f^{abc} A^b A^c) \frac{\partial}{\partial dc^a} \ , \tag{4.12}$$

and

$$\mathcal{G} = -F^a \frac{\partial}{\partial c^a} + f^{abc} F^b A^c \frac{\partial}{\partial dc^a} \ . \tag{4.13}$$

It is easily verified that δ and $\mathcal{G}$ are respectively of degree zero and one and that the following algebraic relations hold:

$$d = -[s, \delta] \ , \tag{4.14}$$

$$2\mathcal{G} = [d, \delta] \ , \tag{4.15}$$

$$\{s, \mathcal{G}\} = 0 \quad , \quad \mathcal{G} = 0 \ , \tag{4.16}$$

$$\{d, \mathcal{G}\} = 0 \quad , \quad [\mathcal{G}, \delta] = 0 \ . \tag{4.17}$$

Notice that the closure of the algebra between d, s and δ requires the introduction of the operator $\mathcal{G}$. As an example, we discuss the three-dimensional Chern-Simons term, which is also relevant for the two-dimensional gauge anomaly. In this case the descent equations read:

$$\begin{aligned} s\Omega_3^0 + d\Omega_2^1 &= 0 \ , \\ s\Omega_2^1 + d\Omega_1^2 &= 0 \ , \\ s\Omega_1^2 + d\Omega_0^3 &= 0 \ , \\ s\Omega_0^3 &= 0 \ , \end{aligned} \tag{4.18}$$

where Ω_0^3 is the BRST invariant ghost monomial defined by

$$\Omega_0^3 = \frac{1}{3!} f^{abc} c^a c^b c^c \ . \tag{4.19}$$

Acting with the operator δ of eq.(4.12) on the last equation of the tower (4.18) one gets

$$[\delta, s]\Omega_0^3 + s\delta\Omega_0^3 = 0 \ , \tag{4.20}$$

which, using the decomposition (4.14), becomes

$$s\delta\Omega_0^3 + d\Omega_0^3 = 0 \ . \tag{4.21}$$

This equation shows that $\delta\Omega_0^3$ gives a solution for the cocycle Ω_1^2 in the tower (4.18). Acting again with δ on the eq.(4.21) and using the algebraic relations (4.14),(4.15) one has

$$s\frac{\delta\delta}{2}\Omega_0^3 - \mathcal{G}\Omega_0^3 + d\delta\Omega_0^3 = 0 \ . \tag{4.22}$$

Moreover, with the relations

$$\begin{aligned} \mathcal{G}\Omega_0^3 &= s\widehat{\Omega}_2^1 \ , \\ \widehat{\Omega}_2^1 &= F^a c^a \ , \end{aligned} \tag{4.23}$$

eq.(4.22) can be rewritten as:

$$s\left(\frac{\delta\delta}{2}\Omega_0^3 - \widehat{\Omega}_2^1\right) + d\delta\Omega_0^3 = 0 \ . \tag{4.24}$$

One sees that $(\frac{\delta\delta}{2}\Omega_0^3 - \widehat{\Omega}_2^1)$ gives a solution for Ω_2^1 modulo trivial contributions. To solve completely the tower (4.18) one have to apply once more the operator δ on the eq.(4.24). After a little algebra one gets:

$$s(\frac{\delta\delta\delta}{3!}\Omega_0^3 - \delta\widehat{\Omega}_2^1) + d(\frac{\delta\delta}{2}\Omega_0^3 - \widehat{\Omega}_2^1) = 0 \,, \tag{4.25}$$

which shows that the cocycle Ω_3^0 can be identified with $(\frac{\delta\delta\delta}{3!}\Omega_0^3 - \delta\widehat{\Omega}_2^1)$. It is apparent then how repeated applications of the operator δ on the zero form cocycle Ω_0^3 and the use of the operator $\mathcal{G}$ in the tower give in a simple way a solution of the descent equations. Summarizing, the solution of the descent equations (4.18) is given by

$$\Omega_3^0 = \frac{1}{3!}\delta\delta\delta\Omega_0^3 - \delta\widehat{\Omega}_2^1 \,, \tag{4.26}$$

$$\Omega_2^1 = \frac{1}{2}\delta\delta\Omega_0^3 - \widehat{\Omega}_2^1 \,, \tag{4.27}$$

$$\Omega_1^2 = \delta\Omega_0^3 \,, \tag{4.28}$$

where

$$\widehat{\Omega}_2^1 = F^a c^a \,, \tag{4.29}$$

$$\Omega_0^3 = \frac{1}{3!} f^{abc} c^a c^b c^c \,, \tag{4.30}$$

$$\Omega_1^2 = -\frac{1}{2} f^{abc} A^a c^b c^c = (dc^a)c^a - s(A^a c^a) \,, \tag{4.31}$$

$$\Omega_2^1 = \frac{1}{2} f^{abc} A^a A^b c^c - F^a c^a = -(dA^a)c^a \,, \tag{4.32}$$

$$\Omega_3^0 = F^a A^a - \frac{1}{3!} f^{abc} A^a A^b A^c \,. \tag{4.33}$$

One sees then, that (4.32) and (4.33) give respectively the two-dimensional gauge anomaly (modulo a d-coboundary) and the three-dimensional Chern-Simons term.

4.2 Gravity with torsion

Now one can apply the same technique to solve with an appropriate decomposition (1.18) the ladder for the gravitational case. In a first step one neglects the Weyl symmetry to simplify the matter. For this purpose one defines the operator δ as

$$\delta = -e^a \frac{\delta}{\delta\eta^a} \,, \tag{4.34}$$

or in terms of the basic fields

$$\begin{aligned} \delta\eta^a &= -e^a \,, \\ \delta\phi &= 0 \quad \text{for} \quad \phi = (\varphi, \omega, e, R, T, \theta) \,. \end{aligned} \tag{4.35}$$

It is easy to verify that δ is of degree zero and that, together with the BRST operator s, it obeys the following algebraic relations:

$$[s, \delta] = -d \ , \tag{4.36}$$

and

$$[d, \delta] = 0 \ . \tag{4.37}$$

One sees from eq.(4.36) that the operator δ allows to decompose the exterior derivative d as a BRST commutator. This property, as already shown in [26], gives an elegant and simple procedure for solving the descent equations (4.9).

The BRST transformations for gravity with torsion but without Weyl transformations are easily found by setting the corresponding Weyl fields, namely the Weyl gauge field A, the Weyl ghost σ, and the Weyl curvature F, to zero. Furthermore, for this case one has also to modify the projected Bianchi identities (3.73) and the commutator relations (3.76)-(3.77). This leads to the following summarized result:

- **Form sector two, ghost number zero** $(T^a, R^a{}_b)$

$$\begin{aligned}
sT^a &= (dT^a_{mn})e^m\eta^n - T^a_{mn}e^m d\eta^n + \theta^a{}_bT^b + T^a_{mn}T^m\eta^n \\
&\quad - T^a_{mn}\omega^m{}_k e^k\eta^n + \omega^a{}_{bk}\eta^k T^b + \omega^a{}_b T^b_{mn}e^m\eta^n \\
&\quad - R^a{}_b\eta^b - R^a{}_{bmn}e^m\eta^n e^b \ , \\
sR^a{}_b &= (dR^a{}_{bmn})e^m\eta^n - R^a{}_{bmn}e^m d\eta^n + \theta^a{}_cR^c{}_b - \theta^c{}_bR^a{}_c \\
&\quad - R^a{}_{bmn}\omega^m{}_k e^k\eta^n + \omega^a{}_{ck}\eta^k R^c{}_b - \omega^c{}_{bk}\eta^k R^a{}_c + \omega^a{}_cR^c{}_{bmn}e^m\eta^n \\
&\quad - \omega^c{}_bR^a{}_{cmn}e^m\eta^n + R^a{}_{bmn}T^m\eta^n \ .
\end{aligned} \tag{4.38}$$

For the Bianchi identities one has

$$\begin{aligned}
DT^a &= dT^a + \omega^a{}_bT^b = R^a{}_be^b \ , \\
DR^a{}_b &= dR^a{}_b + \omega^a{}_cR^c{}_b - \omega^c{}_bR^a{}_c = 0 \ .
\end{aligned} \tag{4.39}$$

- **Form sector one, ghost number zero** $(e^a, \omega^a{}_b)$

$$\begin{aligned}
se^a &= d\eta^a + \omega^a{}_b\eta^b + \theta^a{}_be^b + \omega^a{}_{bm}\eta^m e^b - T^a_{mn}e^m\eta^n \ , \\
s\omega^a{}_b &= (d\omega^a{}_{bm})\eta^m + \omega^a{}_{bm}d\eta^m + d\theta^a{}_b + \omega^a{}_{cm}\eta^m\omega^c{}_b + \theta^a{}_c\omega^c{}_b \\
&\quad - \omega^c{}_{bm}\eta^m\omega^a{}_c - \theta^c{}_b\omega^a{}_c - R^a{}_{bmn}e^m\eta^n \ .
\end{aligned} \tag{4.40}$$

The exterior derivatives of these fields are given by the defintions of the two form field strengths

$$\begin{aligned}
de^a &= T^a - \omega^a{}_be^b \ , \\
d\omega^a{}_b &= R^a{}_b - \omega^a{}_c\omega^c{}_b \ .
\end{aligned} \tag{4.41}$$

- **Form sector zero, ghost number zero** $(\varphi, \omega^a{}_{bm}, R^a{}_{bmn}, T^a_{mn})$

$$
\begin{aligned}
s\varphi &= -\eta^m \partial_m \varphi \, , \\
s\omega^a{}_{bm} &= -\eta^k \partial_k \omega^a{}_{bm} - \partial_m \theta^a{}_b + \theta^a{}_c \omega^c{}_{bm} - \theta^c{}_b \omega^a{}_{cm} - \theta^k{}_m \omega^a{}_{bk} \, , \\
sT^a_{mn} &= -\eta^k \partial_k T^a_{mn} + \theta^a{}_k T^k_{mn} - \theta^k{}_m T^a_{kn} - \theta^k{}_n T^a_{mk} \, , \\
sR^a{}_{bmn} &= -\eta^k \partial_k R^a{}_{bmn} + \theta^a{}_c R^c{}_{bmn} - \theta^c{}_b R^a{}_{cmn} - \theta^k{}_m R^a{}_{bkn} - \theta^k{}_n R^a{}_{bmk} \, .
\end{aligned} \tag{4.42}
$$

From the equations (3.73) one gets

$$
\begin{aligned}
dT^a_{mn} &= (\partial_k T^a_{mn}) e^k \\
&= (R^a{}_{kmn} + R^a{}_{mnk} + R^a{}_{nkm} \\
&\quad - \omega^a{}_{bk} T^b_{mn} - \omega^a{}_{bm} T^b_{nk} - \omega^a{}_{bn} T^b_{km} \\
&\quad - T^a_{lk} T^l_{mn} - T^a_{lm} T^l_{nk} - T^a_{ln} T^l_{km} \\
&\quad + T^a_{lk} \omega^l{}_{nm} + T^a_{ln} \omega^l{}_{mk} + T^a_{lm} \omega^l{}_{kn} \\
&\quad - T^a_{lk} \omega^l{}_{mn} - T^a_{lm} \omega^l{}_{nk} - T^a_{ln} \omega^l{}_{km} \\
&\quad - \partial_m T^a_{nk} - \partial_n T^a_{km}) e^k \, , \\
dR^a{}_{bmn} &= (\partial_k R^a{}_{bmn}) e^k \\
&= (-\omega^a{}_{ck} R^c{}_{bmn} - \omega^a{}_{cm} R^c{}_{bnk} - \omega^a{}_{cn} R^c{}_{bkm} \\
&\quad + \omega^c{}_{bk} R^a{}_{cmn} + \omega^c{}_{bm} R^a{}_{cnk} + \omega^c{}_{bn} R^a{}_{ckm} \\
&\quad - R^a{}_{blk} T^l_{mn} - R^a{}_{blm} T^l_{nk} - R^a{}_{bln} T^l_{km} \\
&\quad + R^a{}_{blk} \omega^l{}_{nm} + R^a{}_{bln} \omega^l{}_{mk} + R^a{}_{blm} \omega^l{}_{kn} \\
&\quad - R^a{}_{blk} \omega^l{}_{mn} - R^a{}_{blm} \omega^l{}_{nk} - R^a{}_{bln} \omega^l{}_{km} \\
&\quad - \partial_m R^a{}_{bnk} - \partial_n R^a{}_{bkm}) e^k \, .
\end{aligned} \tag{4.43}
$$

In addition, one has also the equation

$$
\begin{aligned}
d\omega^a{}_{bm} &= (\partial_n \omega^a{}_{bm}) e^n \\
&= (-R^a{}_{bmn} + \omega^a{}_{cm} \omega^c{}_{bn} - \omega^a{}_{cn} \omega^c{}_{bm} \\
&\quad + \omega^a{}_{bk} T^k_{mn} - \omega^a{}_{bk} \omega^k{}_{nm} + \omega^a{}_{bk} \omega^k{}_{mn} + \partial_m \omega^a{}_{bn}) e^n \, .
\end{aligned} \tag{4.44}
$$

- **Form sector zero, ghost number one** $(\theta^a{}_b, \eta^a)$

$$
\begin{aligned}
s\eta^a &= \theta^a{}_b \eta^b + \omega^a{}_{bm} \eta^m \eta^b - \frac{1}{2} T^a_{mn} \eta^m \eta^n \, , \\
s\theta^a{}_b &= \theta^a{}_c \theta^c{}_b - \eta^k \partial_k \theta^a{}_b \, .
\end{aligned} \tag{4.45}
$$

- **Commutator relations for the tangent space derivative** ∂_m

The following commutator relations are valid:

$$
\begin{aligned}
[s, \partial_m] &= (\partial_m \eta^k - \theta^k{}_m - T^k_{mn} \eta^n - \omega^k{}_{mn} \eta^n + \omega^k{}_{nm} \eta^n) \partial_k \, , \\
[d, \partial_m] &= (T^k_{mn} e^n + \omega^k{}_{mn} e^n - \omega^k{}_{nm} e^n - (\partial_m e^k)) \partial_k \, ,
\end{aligned} \tag{4.46}
$$

and

$$[\partial_m, \partial_n] = -(T^k_{mn} + \omega^k{}_{mn} - \omega^k{}_{nm})\partial_k \ . \tag{4.47}$$

- **Algebra between s and d**

  From the above transformations it follows:

$$s^2 = 0 \ , \quad d^2 = 0 \ , \tag{4.48}$$

and

$$\{s, d\} = 0 \ . \tag{4.49}$$

4.3 Decomposition in the presence of Weyl symmetry

This section is dedicated to the description of the generalization including also the Weyl symmetry. Analogous to eq.(4.35) one introduces the operator δ defined as

$$\delta = -e^a \frac{\delta}{\delta\eta^a} \ , \tag{4.50}$$

or in terms of the basic fields

$$\begin{aligned} \delta\eta^a &= -e^a \ , \\ \delta\phi &= 0 \quad \text{for} \quad \phi = (\varphi, \omega, e, A, R, T, F, \theta, \sigma) \ . \end{aligned} \tag{4.51}$$

It is easy to verify that δ is of degree zero and that, together with the BRST operator s, it obeys the same algebraic relations as in the case without Weyl transformations[6]:

$$[s, \delta] = -d \ , \tag{4.52}$$

and

$$[d, \delta] = 0 \ . \tag{4.53}$$

As in the case for gravity with torsion and without Weyl symmetry the algebra is unchanged and the operator δ allows again the decomposition of the exterior derivative d as a BRST commutator.

In order to summarize the meaning of the δ-operator one has to take into consideration the following points:

- In the discussion of non-abelian gauge anomalies without gravity one is enforced to introduce also an additional operator $\mathcal{G}$ in order to have a closed algebra between the operators d, s, δ, and $\mathcal{G}$. This operator, already present in the work of Brandt et al. [25], generates together with the BRST operator s a new tower of descent equations which are easily disentangled by using the general results of the cohomology of s [26].

[6]Remark that contrary to the previous section now s denotes the full BRST operator defined in Section 3.3.

- The decomposition without the operator $\mathcal{G}$ is also present in topological field theories. In the Chern-Simons theory in three space-time dimensions (quantized in the Landau gauge or in the axial gauge) the operator δ corresponds to a *new linear vector-supersymmetry*, which allows to prove the finiteness of the underlying model in a very elegant manner [43, 44, 45, 46, 47]. This linear supersymmetry is also present in string and superstring theory as it was shown recently [48, 49].

- In our present considerations we have now shown that in gravity with torsion (with or without Weyl symmetry) the $\mathcal{G}$-operator is again absent. The meaning of the operator $\mathcal{G}$ is, up to our knowledge, not yet fully understood. More recently, we have some hints that also in topological Yang-Mills theory in four space-time dimensions the operator δ corresponds again to a linear vector-supersymmetry, with a vanishing $\mathcal{G}$-operator [41].

4.4 General solution of the tower

Now one solves explicitely the tower of descent equations for the gravitational case with and without Weyl symmetry. Therefore, without Weyl symmetry, we expect that for ghost number G and form degree N, N being the dimension of the space-time, the Ω_N^G are local polynomials in the fields $(\varphi, e^a, \omega^a{}_{bm}, T^a_{mn}, R^a{}_{bmn}\eta^a, \theta^a{}_b)$ and their derivatives, whereas for the case with Weyl symmetry one has to add the Weyl gauge field A_m, the corresponding Weyl ghost σ, the Weyl curvature F_{mn}, and derivatives of them. The tower of descent equations is given by

$$
\begin{aligned}
&s\Omega_N^G + d\Omega_{N-1}^{G+1} = 0 \ , \\
&s\Omega_{N-1}^{G+1} + d\Omega_{N-2}^{G+2} = 0 \ , \\
&\quad \cdots\cdots \\
&\quad \cdots\cdots \\
&s\Omega_1^{G+N-1} + d\Omega_0^{G+N} = 0 \ , \\
&s\Omega_0^{G+N} = 0 \ ,
\end{aligned}
\tag{4.54}
$$

with $(\Omega_{N-1}^{G+1}, ..., \Omega_1^{G+N-1}, \Omega_0^{G+N})$ local polynomials which, without loss of generality, will be always considered as irreducible elements, i.e. they cannot be expressed as the product of several factorized terms. In particular, the ghost numbers $G = (0,1)$ correspond to an invariant gravitational Lagrangian and to an anomaly, respectively.

Thanks to the operator δ and to the algebraic relations (4.36)-(4.37), in order to find a solution of the ladder (4.54) it is sufficient to solve only the last equation for the zero form Ω_0^{G+N}. It is easy to check that, once a non-trivial solution for Ω_0^{G+N} is known, the higher cocycles Ω_q^{G+N-q}, $(q = 1, ..., N)$ are obtained by repeated applications of the operator δ on Ω_0^{G+N}, i.e.

$$
\Omega_q^{G+N-q} = \frac{\delta^q}{q!}\Omega_0^{G+N} \quad , \quad q = 1, ..., N \quad , \quad G = (0,1) \ . \tag{4.55}
$$

Therefore, the solution of the last equation of the tower (4.54) is reduced to a problem of *local* BRST cohomology instead of a modulo-d one. It is well-known indeed that, once a particular solution of the descent equations (4.54) has been obtained, i.e. eq.(4.55), the search of the most general solution becomes essentially a problem of local BRST cohomology.

The complete general solutions of the local cohomological problem

$$s\Omega_0^{G+N} = 0 \tag{4.56}$$

are not yet obtained [41], but it is rather simple to discuss some interesting examples. This will be done in the next section.

5 Some examples

This section is devoted to apply the previous algebraic setup and to discuss some explicit examples. Especially, we draw our attention to the cohomological origin of the cosmological constant, the Einstein Lagrangians, and the generalized curvature Lagrangians. In addition, we discuss Lagrangians with torsion, Chern-Simons terms and anomalies. In a last step we investigate the scalar field Lagrangians in the presence of gravity, the Weyl anomalies, and the Weyl invariant scalar field Lagrangians. The analysis will be carried out for any space-time dimension, i.e. the Lorentz group will be assumed to be $SO(N)$ with N arbitrary.

Remark that in the following Sections 5.1-5.6 we will discuss the gravitational case with torsion but without Weyl symmetry. Therefore, in this sections one has to use the BRST transformations summarized in Section 4.2. In Sections 5.7 and 5.8 also Weyl symmetry is included, and there one has to use the full BRST transformations of Section 3.3.

5.1 The cosmological constant

The simplest local BRST invariant polynomial which one can define is

$$\Omega_0^N = \frac{(-1)^N}{N!} \varepsilon_{a_1 a_2 .....a_N} \eta^{a_1} \eta^{a_2} .....\eta^{a_N} \ , \tag{5.1}$$

with $\varepsilon_{a_1 a_2 .....a_N}$ the totally antisymmetric invariant tensor of $SO(N)$. Taking into account that in a N-dimensional space-time the product of $(N+1)$ ghost fields η^a automatically vanishes, it is easily checked that Ω_0^N identifies a cohomology class of the BRST operator, i.e.

$$s\Omega_0^N = 0 \quad , \quad \Omega_0^N \neq s\widehat{\Omega}_0^{N-1} \ . \tag{5.2}$$

For the case $G = 0$ the zero form cocycle (5.1) corresponds to the invariant Lagrangian Ω_N^0

$$\Omega_N^0 = \frac{\delta^N}{N!}\Omega_0^N = \frac{1}{N!}\varepsilon_{a_1a_2.....a_N} e^{a_1}e^{a_2}.....e^{a_N} , \tag{5.3}$$

which is easily identified with the $SO(N)$ cosmological constant. One sees thus that the cohomological origin of the cosmological constant (5.3) relies on the cocycles (5.1). With the help of Appendix B one can rewrite eq.(5.3) to the more familiar form

$$\Omega_N^0 = e^1.....e^N = e\ d^Nx . \tag{5.4}$$

5.2 Einstein Lagrangians

In this case, using the zero form curvature $R^{ab}{}_{mn}$, one gets for the cocycle Ω_0^N $(N > 2)$:

$$\Omega_0^N = \frac{1}{2}\frac{(-1)^N}{(N-2)!}\varepsilon_{a_1a_2.....a_N} R^{a_1a_2}{}_{mn}\eta^m\eta^n\eta^{a_3}.....\eta^{a_N} , \tag{5.5}$$

to which it corresponds the term

$$\begin{aligned}\Omega_N^0 &= \frac{\delta^N}{N!}\Omega_0^N \\ &= \frac{1}{2}\frac{1}{(N-2)!}\varepsilon_{a_1a_2.....a_N} R^{a_1a_2}{}_{mn}e^m e^n e^{a_3}.....e^{a_N} \\ &= \frac{1}{(N-2)!}\varepsilon_{a_1a_2.....a_N} R^{a_1a_2}e^{a_3}.....e^{a_N} .\end{aligned} \tag{5.6}$$

Expression (5.6) is nothing but the Einstein Lagrangian for the case of $SO(N)$. Using the result given in the Appendix B one gets

$$\begin{aligned}\Omega_N^0 &= \frac{1}{2}\frac{1}{(N-2)!}\varepsilon_{a_1a_2.....a_N} R^{a_1a_2}{}_{mn}\varepsilon^{mna_3.....a_N}e^1.....e^N \\ &= \frac{1}{2}eR^{a_1a_2}{}_{mn}(\delta^m_{a_1}\delta^n_{a_2} - \delta^n_{a_1}\delta^m_{a_2})\ d^Nx \\ &= eR^{mn}{}_{mn}\ d^Nx = eR\ d^Nx .\end{aligned} \tag{5.7}$$

Notice also that for the case of $SO(2)$ the zero form cocycle Ω_0^2

$$\Omega_0^2 = \frac{1}{2}\varepsilon_{ab}R^{ab}{}_{mn}\eta^m\eta^n \tag{5.8}$$

turns out to be BRST-exact:

$$\Omega_0^2 = -s(\varepsilon_{ab}\omega^{ab}{}_m\eta^m + \varepsilon_{ab}\theta^{ab}) . \tag{5.9}$$

As it is well-known, this implies that the two dimensional Einstein Lagrangian

$$\Omega_2^0 = \varepsilon_{ab}R^{ab} \tag{5.10}$$

is d-exact, i.e.

$$\Omega_2^0 = d(\varepsilon_{ab}\omega^{ab}) . \tag{5.11}$$

5.3 Generalized curvature Lagrangians

A straightforward generalization of the Einstein Lagrangians (5.6) is to replace any pair of vielbeins with the two form R^{ab}. Therefore, one gets another set of gravitational Lagrangians containing higher powers of the Riemann tensor.

To give an example, let us consider the zero form cocycle

$$\begin{aligned}\Omega_0^{2N} &= \frac{1}{(2N)!}\frac{1}{2^N}(\varepsilon_{a_1a_2a_3a_4.....a_{(2N-1)}a_{(2N)}}R^{a_1a_2}{}_{b_1b_2}R^{a_3a_4}{}_{b_3b_4}.....R^{a_{(2N-1)}a_{(2N)}}{}_{b_{(2N-1)}b_{(2N)}})\\ &\times(\eta^{b_1}\eta^{b_2}\eta^{b_3}\eta^{b_4}.....\eta^{b_{(2N-1)}}\eta^{b_{(2N)}})\ .\end{aligned} \tag{5.12}$$

Using eq.(4.55), for the corresponding invariant Lagrangian one gets

$$\begin{aligned}\Omega_{2N}^0 &= \frac{\delta^{(2N)}}{(2N)!}\Omega_0^{2N}\\ &= \frac{1}{(2N)!}\frac{1}{2^N}(\varepsilon_{a_1a_2a_3a_4.....a_{(2N-1)}a_{(2N)}}R^{a_1a_2}{}_{b_1b_2}R^{a_3a_4}{}_{b_3b_4}.....R^{a_{(2N-1)}a_{(2N)}}{}_{b_{(2N-1)}b_{(2N)}})\\ &\times(e^{b_1}e^{b_2}e^{b_3}e^{b_4}.....e^{b_{(2N-1)}}e^{b_{(2N)}})\\ &= \frac{1}{(2N)!}\varepsilon_{a_1a_2a_3a_4.....a_{(2N-1)}a_{(2N)}}R^{a_1a_2}R^{a_3a_4}.....R^{a_{(2N-1)}a_{(2N)}}\ .\end{aligned} \tag{5.13}$$

As an explicit example we analyze the case of $SO(4)$ with the cocycle

$$\begin{aligned}\Omega_4^0 &= \frac{1}{4!}\frac{1}{4}\varepsilon_{a_1a_2a_3a_4}R^{a_1a_2}{}_{b_1b_2}R^{a_3a_4}{}_{b_3b_4}e^{b_1}e^{b_2}e^{b_3}e^{b_4}\\ &= \frac{1}{4!}\frac{1}{4}\varepsilon_{a_1a_2a_3a_4}\varepsilon^{b_1b_2b_3b_4}R^{a_1a_2}{}_{b_1b_2}R^{a_3a_4}{}_{b_3b_4}\ e\ d^4x\\ &= \frac{1}{4!}(R^{ab}{}_{cd}R^{cd}{}_{ab}+4R^{ab}{}_{bd}R^{cd}{}_{ca}+R^{ab}{}_{ab}R^{cd}{}_{cd})\ e\ d^4x\\ &= \frac{1}{4!}e(R^{ab}{}_{cd}R^{cd}{}_{ab}-4R^{a}{}_{d}R^{d}{}_{a}+R^2)\ d^4x\ .\end{aligned} \tag{5.14}$$

Above expression is nothing else but the Euler density [10].

5.4 Lagrangians with torsion

It is known that, for special values of the space-time dimension N, i.e. $N=(4M-1)$ with $M\geq 1$, there is the possibility of defining non-trivial invariant Lagrangians which explicitly contain the torsion [18].
Let us begin by considering first the simpler case of $SO(3)$ $(M=1)$. By making use of the zero form T^a_{mn}, one has for the cocycle Ω_0^3 the following expression[7]

$$\Omega_0^3=\frac{1}{2}T^a_{mn}\eta^m\eta^n\eta_a\ , \tag{5.15}$$

[7]Tangent space indices are rised and lowered with the flat metric g_{ab}, $\eta_a=g_{ab}\eta^b$.

from which one gets the three dimensional torsion Lagrangian

$$\Omega_3^0 = \frac{\delta^3}{3!}\Omega_0^3 = -\frac{1}{2}T^a_{mn}e^m e^n e_a = -T^a e_a \ . \tag{5.16}$$

We remark that this term, known also as the *translational Chern-Simons term*, has been already discussed by several authors [50] and gives rise to interesting gravitational models. As shown by [50] it can be naturally included together with the three dimensional topological Chern-Simons term of Deser-Jackiw-Templeton [51] and the cosmological constant into the Einstein action. The resulting model is characterized by the presence of a massive graviton moving in a space of constant curvature.

Generalizing to the case of $SO(4M-1)$ with $(M > 1)$, one finds

$$\begin{aligned}\Omega_0^{4M-1} &= \frac{1}{2(2M-1)}(T_{k\,m_1m_2}R^k{}_{a_1m_3m_4}R^{a_1}{}_{a_2m_5m_6}\ldots.R^{a_{(2M-3)}}{}_{a_{(2M-2)}m_{(4M-3)}m_{(4M-2)}}) \\ &\times (\eta^{m_1}\eta^{m_2}\ldots.\eta^{m_{(4M-3)}}\eta^{m_{(4M-2)}}\eta^{a_{(2M-2)}}) \ ,\end{aligned} \tag{5.17}$$

which yields the following Lagrangians including also torsion

$$\begin{aligned}\Omega_{4M-1}^0 &= -\frac{1}{2(2M-1)}(T_{k\,m_1m_2}R^k{}_{a_1m_3m_4}R^{a_1}{}_{a_2m_5m_6}\ldots.R^{a_{(2M-3)}}{}_{a_{(2M-2)}m_{(4M-3)}m_{(4M-2)}}) \\ &\times (e^{m_1}e^{m_2}\ldots.e^{m_{(4M-3)}}e^{m_{(4M-2)}}e^{a_{(2M-2)}}) \\ &= -T_kR^k{}_{a_1}R^{a_1}{}_{a_2}\ldots.R^{a_{(2M-3)}}{}_{a_{(2M-2)}}e^{a_{(2M-2)}} \ .\end{aligned} \tag{5.18}$$

Let us also mention the possibility of defining invariant Lagrangians with torsion terms which are polynomial in T^a_{mn}. These Lagrangians exist in any space-time dimension and are easily obtained from the $SO(N)$ zero form cocycle

$$\Omega_0^N = \frac{(-1)^N}{N!}\varepsilon_{a_1a_2\ldots..a_N}\eta^{a_1}\eta^{a_2}\ldots..\eta^{a_N}\mathcal{P}(T) \ , \tag{5.19}$$

where $\mathcal{P}(T)$ is a scalar polynomial in the torsion as, for instance [52] (see also [53] for generalization),

$$\mathcal{P}(T) = T^a_{mn}T^{mn}_a \ . \tag{5.20}$$

The corresponding invariant Lagrangians containing only torsion are then given by

$$\Omega_N^0 = \frac{\delta^N}{N!}\Omega_0^N = \frac{1}{N!}\varepsilon_{a_1a_2\ldots..a_N}e^{a_1}e^{a_2}\ldots..e^{a_N}\mathcal{P}(T) \ . \tag{5.21}$$

5.5 Chern-Simons terms and anomalies

For what concerns the Chern-Simons terms and the Lorentz and diffeomorphism anomalies we recall that, as mentioned in the introduction, a systematic analysis based on the decomposition (4.36) has been recently carried out by [32].

Let us remark, however, that the decomposition found in [32] gives rise to a commutation relation between the operators δ and d which, contrary to the present case (see eq.(4.37)), does not vanish (see also Section 4.1). This implies the existence of a further operator $\mathcal{G}$ of degree one which has to be taken into account in order to solve the ladder (4.54).

Actually, the existence of the operator $\mathcal{G}$ relies on the fact that the decomposition of the exterior differential d found in [32] does not take into account the explicit presence of the vielbein e^a and of the torsion T^a. It holds for a functional space whose basic elements are built up only with the Lorentz connection $\omega^a{}_b$ and the Riemann tensor $R^a{}_b$, this choice being sufficient to characterize all known Lorentz anomalies and related second family diffeomorphism cocycles [23, 54].

It is remarkable then to observe that the algebra between s, δ, and d gets simpler only when the vielbein e^a and the torsion T^a are naturally present. Let us emphasize indeed that the particular elementary form of the operator δ in eq.(4.35) is due to the use of the tangent space ghost η^a whose introduction requires explicitly the presence of the vielbein e^a.

For the sake of clarity and to make contact with the results obtained in [32], let us discuss in details the construction of the $SO(3)$ Chern-Simons term. In this case the tower (4.54) takes the form

$$
\begin{aligned}
&s\Omega_3^0 + d\Omega_2^1 = 0 \ , \\
&s\Omega_2^1 + d\Omega_1^2 = 0 \ , \\
&s\Omega_1^2 + d\Omega_0^3 = 0 \ , \\
&s\Omega_0^3 = 0 \ ,
\end{aligned} \tag{5.22}
$$

where, according to eq.(4.55),

$$
\begin{aligned}
\Omega_1^2 &= \delta\Omega_0^3 \ , \\
\Omega_2^1 &= \frac{\delta^2}{2!}\Omega_0^3 \ , \\
\Omega_3^0 &= \frac{\delta^3}{3!}\Omega_0^3 \ .
\end{aligned} \tag{5.23}
$$

In order to find a solution for Ω_0^3 we use the redefined Lorentz ghost

$$
\widehat{\theta}^a{}_b = \omega^a{}_{bm}\eta^m + \theta^a{}_b \ , \tag{5.24}
$$

which, from eq.(4.35), transforms as

$$
\delta\widehat{\theta}^a{}_b = -\omega^a{}_b \ . \tag{5.25}
$$

For the cocycle Ω_0^3 one gets then

$$
\Omega_0^3 = \frac{1}{3}\widehat{\theta}^a{}_b\widehat{\theta}^b{}_c\widehat{\theta}^c{}_a - \frac{1}{2}R^a{}_{bmn}\eta^m\eta^n\widehat{\theta}^b{}_a \ , \tag{5.26}
$$

from which Ω_1^2, Ω_2^1, and Ω_3^0 are computed to be

$$\Omega_1^2 = -\omega^a{}_b\widehat{\theta}^b{}_c\widehat{\theta}^c{}_a + R^a{}_{bmn}e^m\eta^n\widehat{\theta}^b{}_a + \frac{1}{2}R^a{}_{bmn}\eta^m\eta^n\omega^b{}_a \,, \tag{5.27}$$

$$\Omega_2^1 = \omega^a{}_b\omega^b{}_c\widehat{\theta}^c{}_a - R^a{}_b\widehat{\theta}^b{}_a - R^a{}_{bmn}e^m\eta^n\omega^b{}_a \,, \tag{5.28}$$

$$\Omega_3^0 = R^a{}_b\omega^b{}_a - \frac{1}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a \,. \tag{5.29}$$

In particular, expression (5.29) gives the familiar $SO(3)$ Chern-Simons gravitational term. Finally, let us remark that the cocycle Ω_2^1 of eq.(5.28), when referred to $SO(2)$, reduces to the expression

$$\Omega_2^1 = -(d\omega^a{}_b)\theta^b{}_a \,, \tag{5.30}$$

which directly gives the two dimensional Lorentz anomaly. Analogous, for the zero form cocycle Ω_0^5 in $SO(5)$ one gets

$$\begin{aligned}\Omega_0^5 = & -\frac{1}{10}\widehat{\theta}^a{}_b\widehat{\theta}^b{}_c\widehat{\theta}^c{}_d\widehat{\theta}^d{}_e\widehat{\theta}^e{}_a + \frac{1}{4}R^a{}_{bmn}\eta^m\eta^n\widehat{\theta}^b{}_c\widehat{\theta}^c{}_d\widehat{\theta}^d{}_a \\ & - \frac{1}{4}R^a{}_{bmn}\eta^m\eta^n R^b{}_{ckl}\eta^k\eta^l\widehat{\theta}^c{}_a \,,\end{aligned} \tag{5.31}$$

which leads to the five dimensional Chern-Simons term

$$\begin{aligned}\Omega_5^0 = & \frac{1}{10}\omega^a{}_b\omega^b{}_c\omega^c{}_d\omega^d{}_e\omega^e{}_a - \frac{1}{4}R^a{}_{bmn}e^m e^n\omega^b{}_c\omega^c{}_d\omega^d{}_a \\ & + \frac{1}{4}R^a{}_{bmn}e^m e^n R^b{}_{ckl}e^k e^l\omega^c{}_a \\ = & \frac{1}{10}\omega^a{}_b\omega^b{}_c\omega^c{}_d\omega^d{}_e\omega^e{}_a - \frac{1}{2}R^a{}_b\omega^b{}_c\omega^c{}_d\omega^d{}_a + R^a{}_b R^b{}_c\omega^c{}_a \,.\end{aligned} \tag{5.32}$$

5.6 Scalar field Lagrangians

In order to couple a massless scalar field to gravity without Weyl symmetry one now considers the following zero form cocycle Ω_0^N in a N-dimensional space-time

$$\Omega_0^N = \frac{1}{2}\frac{1}{N!}\varepsilon_{a_1a_2.....a_N}\eta^{a_1}\eta^{a_2}.....\eta^{a_N}(D_m\varphi)(D^m\varphi) \,, \tag{5.33}$$

where the covariant derivative reduces to the ordinary one

$$\Omega_0^N = \frac{1}{2}\frac{1}{N!}\varepsilon_{a_1a_2.....a_N}\eta^{a_1}\eta^{a_2}.....\eta^{a_N}(\partial_m\varphi)(\partial^m\varphi) \,. \tag{5.34}$$

Taking into account the truncated BRST transformation

$$s(\partial_m\varphi) = -\theta^k{}_m(\partial_k\varphi) - \eta^k\partial_k(\partial_m\varphi) \,, \tag{5.35}$$

it can be easily checked that Ω_0^N is BRST invariant, i.e.

$$s\Omega_0^N = 0 \ . \tag{5.36}$$

For the case $G = 0$ one gets the invariant Lagrangian Ω_N^0

$$\Omega_N^0 = \frac{\delta^N}{N!}\Omega_0^N = \frac{1}{2}\frac{(-1)^N}{N!}\varepsilon_{a_1a_2.....a_N}e^{a_1}e^{a_2}.....e^{a_N}(\partial_m\varphi)(\partial^m\varphi) \ , \tag{5.37}$$

which is easily recognized to coincide with the $SO(N)$ scalar field Lagrangian

$$\begin{aligned}\Omega_N^0 &= \frac{1}{2}d^Nx \ e(\partial_m\varphi)(\partial^m\varphi) \\ &= \frac{1}{2}d^Nx \ \sqrt{g}(\partial_\mu\varphi)(\partial^\mu\varphi) \ .\end{aligned} \tag{5.38}$$

Notice that above scalar field Lagrangian is invariant under diffeomorphisms and local Lorentz rotations, but not invariant under Weyl transformations. This case will be studied in Section 5.8.

5.7 Weyl anomalies

In order to incorporate also the Weyl symmetry we use from now on the full BRST operator s, defined in Section 3.3. For a better understanding of the matter let us begin by discussing the simplest case, the Weyl anomaly in $SO(2)$. According to (4.54) we have therefore to solve the tower

$$\begin{aligned} s\Omega_2^1 + d\Omega_1^2 &= 0 \\ s\Omega_1^2 + d\Omega_0^3 &= 0 \\ s\Omega_0^3 &= 0 \ , \end{aligned} \tag{5.39}$$

where Ω_2^1 is denoting the corresponding anomaly. For the zero form cocycle Ω_0^3 one has

$$\Omega_0^3 = \frac{1}{2}\varepsilon_{ab}\eta^a\eta^b\sigma R \ , \tag{5.40}$$

with R as the Riemann scalar (2.57). One can easy verify with the BRST transformation of the Riemann scalar R, given by

$$sR = -\eta^k\partial_k R - 2\sigma R \ , \tag{5.41}$$

that above cocycle is BRST invariant, i.e.

$$s\Omega_0^3 = 0 \ . \tag{5.42}$$

By using eq.(4.55) one gets the well-known two dimensional Weyl anomaly

$$\begin{aligned}\Omega_2^1 &= \frac{\delta^2}{2!}\Omega_0^3 \\ &= \frac{1}{2}\varepsilon_{ab}e^a e^b\sigma R \ ,\end{aligned} \tag{5.43}$$

which can be rewritten to the more familiar form (see Appendix B)

$$\Omega_2^1 = e\sigma R \; d^2x \; . \tag{5.44}$$

The second example, which we will discuss now, is the four dimensional Weyl anomaly in $SO(4)$. One possible zero form cocycle Ω_0^5 is given by

$$\Omega_0^5 = \frac{1}{4!}\varepsilon_{abcd}\eta^a\eta^b\eta^c\eta^d\sigma R^2 \; , \tag{5.45}$$

which is BRST invariant, i.e.

$$s\Omega_0^5 = 0 \; . \tag{5.46}$$

This leads to the anomaly

$$\begin{aligned}\Omega_4^1 &= \frac{\delta^4}{4!}\Omega_0^5 \\ &= \frac{1}{4!}\varepsilon_{abcd}e^ae^be^ce^d\sigma R^2 \\ &= e\sigma R^2 \; d^4x \; . \end{aligned} \tag{5.47}$$

Two further possible zero form cocycles are given by

$$\Omega_0^5 = \frac{1}{4!}\varepsilon_{abcd}\eta^a\eta^b\eta^c\eta^d\sigma R_{mn}R^{mn} \; , \tag{5.48}$$

and

$$\Omega_0^5 = \frac{1}{4!}\varepsilon_{abcd}\eta^a\eta^b\eta^c\eta^d\sigma R_{mnkl}R^{mnkl} \; , \tag{5.49}$$

where the zero forms R_{mn} and R_{mnkl} denoting the Ricci tensor and the Riemann tensor with indices in the tangent space. The cocycles (5.48) and (5.49) are again BRST invariant:

$$s\Omega_0^5 = 0 \; . \tag{5.50}$$

From (5.48) and (5.49) one gets for the corresponding anomalies

$$\begin{aligned}\Omega_4^1 &= \frac{\delta^4}{4!}\Omega_0^5 \\ &= \frac{1}{4!}\varepsilon_{abcd}e^ae^be^ce^d\sigma R_{mn}R^{mn} \\ &= e\sigma R_{mn}R^{mn} \; d^4x \; , \end{aligned} \tag{5.51}$$

and

$$\begin{aligned}\Omega_4^1 &= \frac{\delta^4}{4!}\Omega_0^5 \\ &= \frac{1}{4!}\varepsilon_{abcd}e^ae^be^ce^d\sigma R_{mnkl}R^{mnkl} \\ &= e\sigma R_{mnkl}R^{mnkl} \; d^4x \; . \end{aligned} \tag{5.52}$$

From the variety of all possible cocycles in higher dimensions we quote only the simplest example for a zero form cocycle in $SO(2N)$ which has the following form:

$$\Omega_0^{2N+1} = \frac{1}{(2N)!}\varepsilon_{a_1a_2.....a_{2N}}\eta^{a_1}\eta^{a_2}.....\eta^{a_{2N}}\sigma R^N \ . \tag{5.53}$$

Of course, it can be easily checked that all these cocycles are BRST invariant, i.e.

$$s\Omega_0^{2N+1} = 0 \ . \tag{5.54}$$

The corresponding $2N$-dimensional Weyl anomalies are given by

$$\begin{aligned} \Omega_{2N}^1 &= \frac{\delta^{2N}}{(2N)!}\Omega_0^{2N+1} \\ &= \frac{1}{(2N)!}\varepsilon_{a_1a_2.....a_{2N}}e^{a_1}e^{a_2}.....e^{a_{2N}}\sigma R^N \\ &= e\sigma R^N \ d^{2N}x \ . \end{aligned} \tag{5.55}$$

5.8 Weyl invariant scalar field Lagrangians

In order to couple a massless scalar field to gravity including also Weyl symmetry we consider the following zero form cocycle Ω_0^N in a N-dimensional space-time

$$\Omega_0^N = \frac{1}{2}\frac{1}{N!}\varepsilon_{a_1a_2.....a_N}\eta^{a_1}\eta^{a_2}.....\eta^{a_N}(D_m\varphi)(D^m\varphi) \ , \tag{5.56}$$

where now the covariant derivative reduces to the Weyl covariant derivative (2.74)

$$\nabla_m\varphi = \partial_m\varphi - \frac{N-2}{2}A_m\varphi \ , \tag{5.57}$$

and one gets

$$\Omega_0^N = \frac{1}{2}\frac{1}{N!}\varepsilon_{a_1a_2.....a_N}\eta^{a_1}\eta^{a_2}.....\eta^{a_N}(\nabla_m\varphi)(\nabla^m\varphi) \ . \tag{5.58}$$

Notice, that from eq.(2.75) follows

$$s_w(\nabla_\mu\varphi) = -\frac{N-2}{2}\sigma(\nabla_\mu\varphi) \ , \tag{5.59}$$

where s_w denotes only the Weyl symmetry part of the full BRST operator. Therefore, one obtains for (5.57) the Weyl-BRST transformation

$$s_w(\nabla_m\varphi) = s_w(E_m^\mu\nabla_\mu\varphi) = -\frac{N}{2}\sigma(\nabla_m\varphi) \ . \tag{5.60}$$

Taking into account the full BRST transformation (including Weyl symmetry)

$$s(\nabla_m\varphi) = -\theta^k{}_m(\nabla_k\varphi) - \eta^k\partial_k(\nabla_m\varphi) - \frac{N}{2}\sigma(\nabla_m\varphi) \ , \tag{5.61}$$

it can be easily checked that Ω_0^N is BRST invariant, i.e.

$$s\Omega_0^N = 0 \ . \tag{5.62}$$

For the case $G = 0$ one gets the Weyl invariant Lagrangian Ω_N^0

$$\Omega_N^0 = \frac{\delta^N}{N!}\Omega_0^N = \frac{1}{2}\frac{(-1)^N}{N!}\varepsilon_{a_1 a_2 .....a_N} e^{a_1} e^{a_2} .....e^{a_N} (\nabla_m \varphi)(\nabla^m \varphi) \ , \tag{5.63}$$

which is easily recognized to coincide with the $SO(N)$ Weyl invariant scalar field Lagrangian

$$\begin{aligned} \Omega_N^0 &= \frac{1}{2} d^N x \ e(\nabla_m \varphi)(\nabla^m \varphi) \\ &= \frac{1}{2} d^N x \ \sqrt{g}(\nabla_\mu \varphi)(\nabla^\mu \varphi) \ . \end{aligned} \tag{5.64}$$

Notice that above scalar field Lagrangian is invariant under diffeomorphisms, local Lorentz rotations, and also invariant under Weyl transformations.

6 The geometrical meaning of the operator δ

Having discussed the role of the operator δ in finding explicit solutions of the descent equations (4.54), let us turn now to the study of its geometrical meaning. As we shall see, this operator turns out to possess a quite simple geometrical interpretation which will reveal an unexpected and so far unnoticed elementary structure of the ladder (4.54).

Let us begin by observing that all the cocycles Ω_p^{G+N-p} ($p = 0, ..., N$) entering the descent equations (4.54) are of the same degree (i.e. $(G + N)$), the latter being given by the sum of the ghost number and of the form degree.

We can collect then, following [31], all the Ω_p^{G+N-p} into a unique cocycle $\widehat{\Omega}$ of degree $(G + N)$ defined as

$$\widehat{\Omega} = \sum_{p=0}^{N} \Omega_p^{G+N-p} \ . \tag{6.1}$$

This expression, using eq.(4.55), becomes

$$\widehat{\Omega} = \sum_{p=0}^{N} \frac{\delta^p}{p!} \Omega_0^{G+N} \ , \tag{6.2}$$

where the cocycle Ω_0^{G+N}, according to its zero form degree, depends only on the set of zero form variables $(\omega^a{}_{bm}, R^a{}_{bmn}, T^a_{mn}, \theta^a{}_b, \eta^a)$ and their tangent space derivatives ∂_m. Taking into account that under the action of the operator δ the form degree and the ghost number are respectively raised and lowered by one unit and that in a space-time of dimension N

a $(N+1)$-form identically vanishes, it follows that eq.(6.2) can be rewritten in a more suggestive way as

$$\widehat{\Omega} = e^{\delta}\Omega_0^{G+N}(\eta^a, \theta^a{}_b, \omega^a{}_{bm}, R^a{}_{bmn}, T^a_{mn}) \ . \tag{6.3}$$

Let us make now the following elementary but important remark. As one can see from eq.(4.35), the operator δ acts as a translation on the ghost η^a with an amount given by $(-e^a)$. Therefor e^{δ} has the simple effect of shifting η^a into $(\eta^a - e^a)$. This implies that the cocycle (6.3) takes the form

$$\widehat{\Omega} = \Omega_0^{G+N}(\eta^a - e^a, \theta^a{}_b, \omega^a{}_{bm}, R^a{}_{bmn}, T^a_{mn}) \ . \tag{6.4}$$

This formula collects in a very elegant and simple expression the solution of the descent equations (4.54).

In particular, it states the important result that:

To find a non-trivial solution of the ladder (4.54) *it is sufficient to replace the variable* η^a *with* $(\eta^a - e^a)$ *in the zero form cocycle* Ω_0^{G+N} *which belongs to the local cohomology of the BRST operator s. The expansion of* $\Omega_0^{G+N}(\eta^a - e^a, \theta^a{}_b, \omega^a{}_{bm}, R^a{}_{bmn}, T^a_{mn})$ *in powers of the one form vielbein* e^a *yields then all the searched cocycles* Ω_p^{G+N-p}.

It is a simple exercise to check now that all the invariant Lagrangians and Chern-Simons terms computed in the previous section are indeed recovered by simply expanding the corresponding zero form cocycles Ω_0^{G+N} taken as functions of $(\eta^a - e^a)$.

Let us conclude by remarking that, up to our knowledge, expression (6.4) represents a deeper understanding of the algebraic properties of the gravitational ladder (4.54) and of the role played by the vielbein e^a and the associated ghost η^a.

7 Conclusion

The algebraic structure of gravity with torsion in the presence of Weyl symmetry has been analyzed in the context of the Maurer-Cartan horizontality formalism by introducing an operator δ which allows to decompose the exterior space-time derivative as a BRST commutator. Such a decomposition gives a simple and elegant way of solving the Wess-Zumino consistency condition corresponding to invariant Lagrangians and anomalies. The same technique can be applied to the study of the gravitational coupling of Yang-Mills gauge theories as well as to the characterization of the Weyl anomalies [41].

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Appendices:

Appendix A is devoted to demonstrate the computation of some commutators involving the tangent space derivative ∂_a introduced in Section 3. In the Appendix B one finds the definition of the determinant of the vielbein in connection with the ε tensor.

A Commutator relations

In order to find the commutator of two tangent space derivatives ∂_a, we make use of the fact that the usual space-time derivatives ∂_μ have a vanishing commutator:

$$[\partial_\mu, \partial_\nu] = 0 \ . \tag{A.1}$$

From

$$\partial_\mu = e_\mu^m \partial_m \tag{A.2}$$

one gets

$$\begin{aligned}
[\partial_\mu, \partial_\nu] = 0 &= [e_\mu^m \partial_m, e_\nu^n \partial_n] \\
&= e_\mu^m e_\nu^n [\partial_m, \partial_n] + e_\mu^m (\partial_m e_\nu^n) \partial_n - e_\nu^n (\partial_n e_\mu^m) \partial_m \\
&= e_\mu^m e_\nu^n [\partial_m, \partial_n] + (\partial_\mu e_\nu^k - \partial_\nu e_\mu^k) \partial_k \\
&= e_\mu^m e_\nu^n [\partial_m, \partial_n] + (T_{\mu\nu}^k - \omega^k{}_{n\mu} e_\nu^n + \omega^k{}_{m\nu} e_\mu^m - A_\mu e_\nu^k + A_\nu e_\mu^k) \partial_k \\
&= e_\mu^m e_\nu^n (T_{mn}^k + \omega^k{}_{mn} - \omega^k{}_{nm} - A_m \delta_n^k + A_n \delta_m^k) \partial_k \\
&\quad + e_\mu^m e_\nu^n [\partial_m, \partial_n] \ ,
\end{aligned} \tag{A.3}$$

so that

$$[\partial_m, \partial_n] = -(T_{mn}^k + \omega^k{}_{mn} - \omega^k{}_{nm} - A_m \delta_n^k + A_n \delta_m^k) \partial_k \ . \tag{A.4}$$

For the commutator of d and ∂_m we get

$$\begin{aligned}
[d, \partial_m] &= [e^n \partial_n, \partial_m] \\
&= -(\partial_m e^k) \partial_k - e^n [\partial_m, \partial_n] \\
&= -(\partial_m e^k) \partial_k + e^n (T_{mn}^k + \omega^k{}_{mn} - \omega^k{}_{nm} - A_m \delta_n^k + A_n \delta_m^k) \partial_k \ ,
\end{aligned} \tag{A.5}$$

and one has therefore

$$[d, \partial_m] = (T^k_{mn} e^n + \omega^k{}_{mn} e^n - \omega^k{}_{nm} e^n - A_m e^k + A_n e^n \delta^k_m - (\partial_m e^k))\partial_k \ . \tag{A.6}$$

Analogously, from

$$[s, \partial_\mu] = 0 \tag{A.7}$$

one easily finds

$$\begin{aligned}[s, \partial_m] &= (\partial_m \eta^k - \theta^k{}_m - \sigma \delta^k_m)\partial_k + \eta^n[\partial_m, \partial_n] \\ &= (\partial_m \eta^k - \theta^k{}_m - T^k_{mn}\eta^n - \omega^k{}_{mn}\eta^n + \omega^k{}_{nm}\eta^n \\ &\quad + A_m \eta^k - A_n \eta^n \delta^k_m - \sigma \delta^k_m)\partial_k \ . \end{aligned} \tag{A.8}$$

B Determinant of the vielbein and the ε tensor

The definition of the determinant of the vielbein e^a_μ is given by

$$e = det(e^a_\mu) = \frac{1}{N!}\varepsilon_{a_1 a_2 \ldots\ldots a_N} \varepsilon^{\mu_1 \mu_2 \ldots\ldots \mu_N} e^{a_1}_{\mu_1} e^{a_2}_{\mu_2} \ldots\ldots e^{a_N}_{\mu_N} \ . \tag{B.1}$$

One can easily verify that the BRST transformation of e reads

$$se = -\partial_\lambda(\xi^\lambda e) \ . \tag{B.2}$$

For the volume element one has

$$\begin{aligned} e^1 \ldots\ldots e^N &= \frac{1}{N!}\varepsilon_{a_1 \ldots\ldots a_N} e^{a_1} \ldots\ldots e^{a_N} \\ &= \frac{1}{N!}\varepsilon_{a_1 \ldots\ldots a_N} e^{a_1}_{\mu_1} \ldots\ldots e^{a_N}_{\mu_N} dx^{\mu_1} \ldots\ldots dx^{\mu_N} \\ &= \frac{1}{N!}\varepsilon_{a_1 \ldots\ldots a_N} \varepsilon^{\mu_1 \ldots\ldots \mu_N} e^{a_1}_{\mu_1} \ldots\ldots e^{a_N}_{\mu_N} dx^1 \ldots\ldots dx^N \\ &= e d^N x = \sqrt{g} d^N x \ , \end{aligned} \tag{B.3}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$

$$g = det(g_{\mu\nu}) \ . \tag{B.4}$$

The ε tensor has the usual norm

$$\varepsilon_{a_1 \ldots\ldots a_N} \varepsilon^{a_1 \ldots\ldots a_N} = N! \ , \tag{B.5}$$

and obeys the following relation under partial contraction of $(N-2)$ indices

$$\varepsilon_{a_1 \ldots\ldots a_N} \varepsilon^{m n a_3 \ldots\ldots a_N} = (N-2)!(\delta^m_{a_1}\delta^n_{a_2} - \delta^n_{a_1}\delta^m_{a_2}) \ , \tag{B.6}$$

and in general the contraction of two ε tensors is given by the determinant of δ tensors in the following way

$$\varepsilon_{a_1 \ldots\ldots a_N} \varepsilon^{b_1 \ldots\ldots b_N} = \begin{vmatrix} \delta^{b_1}_{a_1} & \delta^{b_2}_{a_1} & \ldots\ldots & \delta^{b_N}_{a_1} \\ \delta^{b_1}_{a_2} & \delta^{b_2}_{a_2} & \ldots\ldots & \delta^{b_N}_{a_2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \delta^{b_1}_{a_N} & \delta^{b_2}_{a_N} & \ldots\ldots & \delta^{b_N}_{a_N} \end{vmatrix} \ . \tag{B.7}$$

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