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Inter-Relations of Solvable Potentials

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Abstract. Solvable Natanzon potentials in nonrelativistic quantum mechanics are known to group into two disjoint classes depending on whether the Schrödinger equation can be reduced to a hypergeometric or a confluent hypergeometric equation. All the potentials within each class are connected via point canonical transformations. We establish a connection between the two classes with appropriate limiting procedures and redefinition of parameters, thereby inter-relating all known solvable potentials.

It is well known that the Natanzon potentials[1] are exactly solvable in nonrelativistic quantum mechanics. These potentials are of two types corresponding to whether the Schrödinger equation can be reduced to either a hypergeometric or a confluent hypergeometric equation. Those that lead to a hypergeometric equation (confluent hypergeometric equation) will be called type-I (type-II) potentials. It has been shown[2, 3, 4] that the members within each class can be mapped into each other by point canonical transformations (PCT); however, members of these two different classes cannot be connected by a PCT. Since a hypergeometric differential equation reduces to a confluent hypergeometric one under appropriate limits, it is reasonable to expect that the potentials of the above mentioned two classes can also be connected by a similar procedure. The purpose of this note is to establish a connection between specifically chosen potentials in each class. A convenient choice is the so called shape invariant potentials[5] which form a distinguished class in the sense that their spectra can be determined entirely by an algebraic procedure, akin to that of the harmonic oscillator, without ever referring to the underlying differential equations. We provide a list of mappings that connect shape invariant type-I potentials to type-II potentials. In Fig. 1, we depict inter-relations among all known shape invariant potentials.

Before proceeding further, it is worth reviewing point canonical transformations in nonrelativistic quantum mechanics. We consider a time-independent Schrödinger equation with a potential function $V(\alpha_i; x)$ that depends upon several parameters α_i (we will use $\hbar = 2m = 1$):

$$\left[-\frac{d^2}{dx^2} + V(\alpha_i; x) - E(\alpha_i)\right] \psi(\alpha_i; x) = 0. \tag{0.1}$$

Under a point canonical transformation which replaces the independent variable x by z ($x = f(z)$) and transforms the wave function $\left[\psi(\alpha_i; x) = v(z)\,\tilde{\psi}(\alpha_i; z)\right]$, the Schrödinger equation transforms into:

$$-\frac{d^2\tilde{\psi}}{dz^2} - \left\{\frac{2v'}{v} - \frac{f''}{f'}\right\}\frac{d\tilde{\psi}}{dz} + \left\{f'^2\left[V(\alpha_i; f(z)) - E(\alpha_i)\right] + \left(\frac{f''v'}{f'v} - \frac{v''}{v}\right)\right\}\tilde{\psi} = 0. \tag{0.2}$$

Requiring the first derivative term to be absent gives $v(z) = C\sqrt{f'(z)}$. This then leads to another Schrödinger equation with a new potential.

$$\left[-\frac{d^2}{dz^2} + \left\{f'^2\left[V(\alpha_i; f(z)) - E(\alpha_i)\right] + \frac{1}{2}\left(\frac{3f''^2}{2f'^2} - \frac{f'''}{f'}\right)\right\}\right]\tilde{\psi}(\alpha_i; z) = 0. \tag{0.3}$$

In general, this is not an eigenvalue equation, unless $\{f'^2\left(V(\alpha_i; f(z)) - E(\alpha_i)\right)\}$ has a term independent of z, which will act like the energy term for the new Hamiltonian. This condition constrains allowable choices for the function $f(z)$. For a general potential $V(\alpha_i; f(z))$, many choices for $f(z)$ are still possible that would give rise to Schrödinger type eigenvalue equations, and thus, if we have one solvable model, we can generate many others from it.

Ref.[3] contains a list of functions $f(z)$ that relate all shape invariant potentials of type-I (type-II) to the Scarf (harmonic oscillator) potential. In the following, we will present two examples where suitable limits take one beyond class barriers, and connect type-I potentials to those of type-II. In particular, we shall exhibit the limiting procedures that convert (a) the Scarf potential into the harmonic oscillator potential; and (b) the generalized Pöschl-Teller into either the Morse or the harmonic oscillator potentials. In Table I, we provide additional examples of limiting procedures and redefinition of parameters.

Scarf potential to harmonic oscillator:
The Scarf potential, given by

$$V_{Scarf}(x) = -A^2 + (A^2 + B^2 - A\alpha)\sec^2(\alpha x) - B(2A - \alpha)\tan(\alpha x)\sec(\alpha x)$$

goes into the three-dimensional harmonic oscillator potential (HO)

$$V_{HO} = \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - \left(l + \frac{3}{2}\right)\omega$$

after a shift of origin $x \to \left(r - \frac{\pi}{2\alpha}\right)$, a redefinition of parameters $A \to \left(\frac{\omega}{\alpha} + \alpha\frac{(l+1)}{2}\right)$, $B \to \left(\frac{\omega}{\alpha} - \alpha\frac{(l+1)}{2}\right)$, and then taking the limit $\alpha \to 0$.

Generalized Pöschl-Teller potential to Morse:
The generalized Pöschl-Teller potential (GPT)

$$V_{GPT}(r) = A^2 + (A^2 + B^2 + A\alpha)\,\mathrm{cosech}^2\,(\alpha r + \beta) - B(2A + \alpha)\mathrm{cosech}\,(\alpha r + \beta)\coth\,(\alpha r + \beta)$$

can be converted into two shape invariant potentials of type-II by taking appropriate limits. One obtains the Morse potential when $B \to \frac{1}{2}Be^{\beta}$, and one takes the limit $\beta \to \infty$. Alternatively, one gets the three dimensional harmonic oscillator potential when[6]

$$A \to \left(\frac{\omega}{\alpha} - \alpha\frac{(l+1)}{2}\right), \; B \to \left(\frac{\omega}{\alpha} + \alpha\frac{(l+1)}{2}\right), \; \alpha \to 0, \; \beta \to 0.$$

Here, it is worth noting that we have given straightforward routes for going from type-I to type-II potentials. Type-I potentials give rise to hypergeometric differential equation which has three regular singular points. Two of them merge in the limiting procedures stated above, and as expected one gets a confluent hypergeometric equation. The reverse procedure of going from type-II to type-I is not well defined. We also provide a figure with information on different limiting procedures and point canonical transformations that take type-I potentials among each other or reduce them to type-II potentials.

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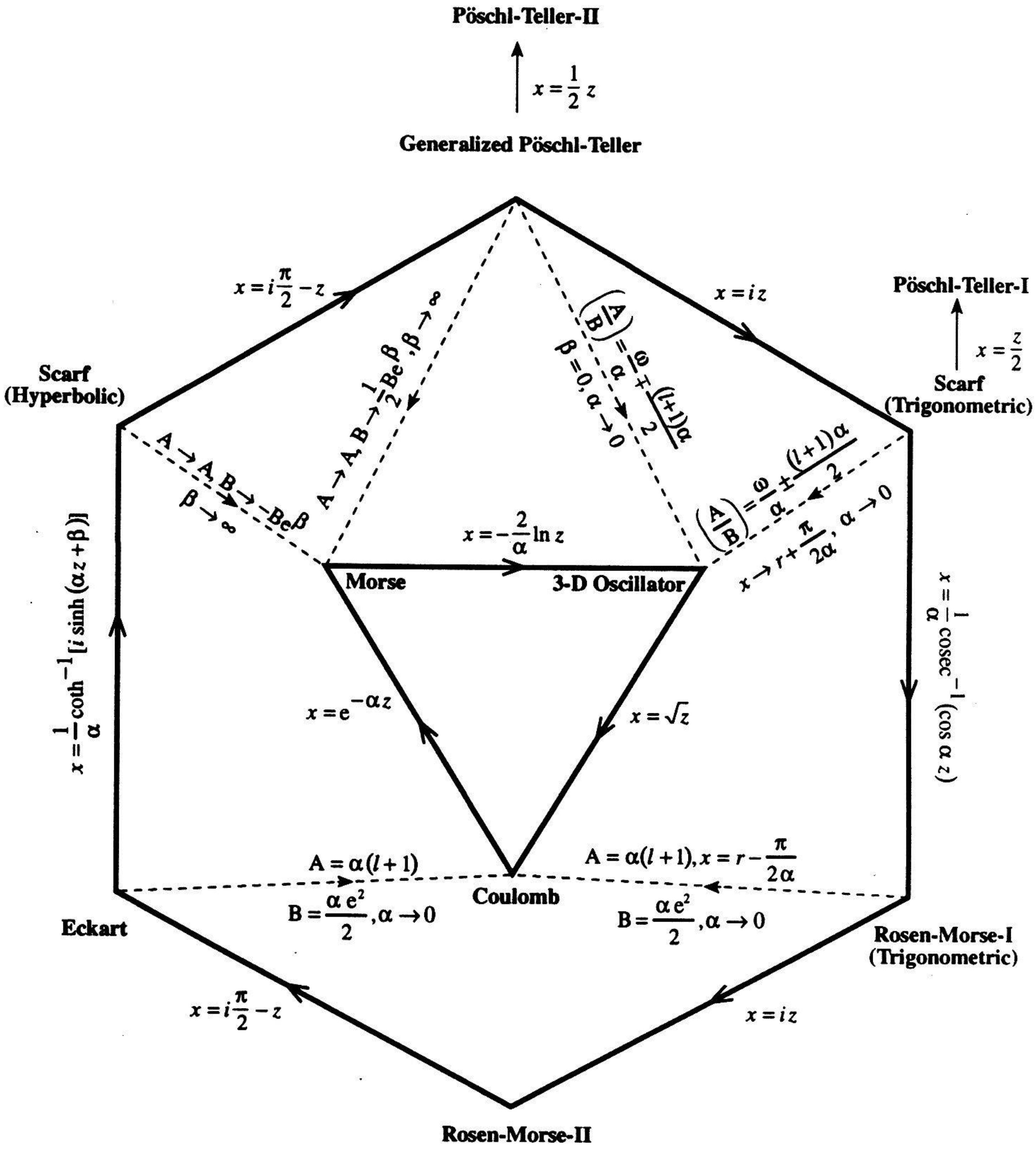

Figure 1.
Limiting procedures and point canonical transformations that take type-I potentials among each other or reduce them to type-II potentials. Potentials are same as those in Ref.[3].

Type-I Potential	Type-II Potential	Limits & Redef. of Parameters
Generalized Pöschl-Teller $V(r) = A^2 + \frac{(A^2+B^2+A\alpha)}{\sinh^2(\alpha r+\beta)} - \frac{B(2A+\alpha)\coth(\alpha r+\beta)}{\sinh(\alpha r+\beta)}$	**Harmonic Oscillator** $V(r) = \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - (l+\frac{3}{2})\omega$ $0 < r < \infty, \quad E_n = 2n\omega$	$A \to \left[\frac{\omega}{\alpha} - \alpha\left(\frac{l+1}{2}\right)\right]$ $B \to \left[\frac{\omega}{\alpha} + \alpha\left(\frac{l+1}{2}\right)\right]$ $\alpha \to 0,\ \beta \to 0$
$-\beta < \alpha r < \infty$ $E_n = A^2 - (A - n\alpha)^2$ $A < B$	**Morse Potential** $V(x) = A^2 + B^2 e^{-2\alpha x}$ $-2B\left(A + \frac{\alpha}{2}\right)e^{-\alpha x}$ $-\infty < x < \infty$ $E_n = A^2 - (A - n\alpha)^2$	$A \to A$ $B \to \frac{Be^{\beta}}{2}$ $r \to x$ $\beta \to \infty$
Scarf $V(x) = -A^2 + \frac{(A^2+B^2-A\alpha)}{\cos^2(\alpha x)} - \frac{B(2A-\alpha)\tan(\alpha x)}{\cos(\alpha x)}$ $-\frac{\pi}{2\alpha} < x < \frac{\pi}{2\alpha},\ A > B,$ $E_n = (A+n\alpha)^2 - A^2$	**Harmonic Oscillator** $V(r) = \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - (l+\frac{3}{2})\omega$ $0 < r < \infty$ $E_n = 2n\omega$	$A \to \left[\frac{\omega}{\alpha} + \alpha\left(\frac{l+1}{2}\right)\right]$ $B \to \left[\frac{\omega}{\alpha} - \alpha\left(\frac{l+1}{2}\right)\right]$ $x \to r + \frac{\pi}{2\alpha}$ $\alpha \to 0$
Scarf (Hyperbolic) $V(x) = A^2 + \frac{(-A^2+B^2-A\alpha)}{\cosh^2(\alpha x+\beta)} + \frac{B(2A+\alpha)\tanh(\alpha x+\beta)}{\cosh(\alpha x+\beta)}$ $-\infty < x < \infty,\ A > 0$ $E_n = A^2 - (A - n\alpha)^2$	**Morse Potential** $V(x) = A^2 + B^2 e^{-2\alpha x}$ $-2B\left(A + \frac{\alpha}{2}\right)e^{-\alpha x}$ $-\infty < x < \infty$ $E_n = A^2 - (A - n\alpha)^2$	$A \to A$ $B \to -\frac{Be^{\beta}}{2}$ $\beta \to \infty$
Eckart $V(r) = A^2 + \frac{B^2}{A^2} - 2B\coth\alpha r$ $+A(A-\alpha)\mathrm{cosech}^2\alpha r$ $0 < r < \infty,\ B > A^2,\ A > 0$ $E_n = A^2 - (A+n\alpha)^2$ $+\frac{B^2}{A^2} - \frac{B^2}{(A+n\alpha)^2}$	**Coulomb** $V(r) = -\frac{e^2}{r} + \frac{l(l+1)}{r^2}$ $+\frac{e^4}{4(l+1)^2}$ $0 < r < \infty$ $E_n = \frac{e^4}{4}\left(\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right)$	$A \to \alpha(l+1)$ $B \to \frac{\alpha}{2}e^2$ $\alpha \to 0$
Rosen-Morse I $V(x) = -A^2 + \frac{B^2}{A^2} + 2B\tan\alpha x$ $+A(A-\alpha)\sec^2\alpha x$ $-\frac{\pi}{2\alpha} < x < \frac{\pi}{2\alpha}$ $E_n = -A^2 + (A+n\alpha)^2$ $+\frac{B^2}{A^2} - \frac{B^2}{(A+n\alpha)^2}$	**Coulomb** $V(r) = -\frac{e^2}{r} + \frac{l(l+1)}{r^2}$ $+\frac{e^4}{4(l+1)^2}$ $0 < r < \infty$ $E_n = \frac{e^4}{4}\left(\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right)$	$A \to \alpha(l+1)$ $B \to \frac{\alpha}{2}e^2$ $x \to r - \frac{\pi}{2\alpha}$ $\alpha \to 0$

Table 1.
Limiting procedures and redefinition of parameters that relate type-I to type-II potentials.

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