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On the spectrum of the harmonic oscillator with a δ-type perturbation

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Abstract. The eigenvalues of the Hamiltonian of the harmonic oscillator perturbed by an attractive point interaction are determined by means of its resolvent. The explicit form of such a resolvent is determined by means of a suitable limit procedure. The eigenvalues are then seen to be the solutions of a nonlinear algebraic equation.

1 Introduction

In this brief communication we intend to study the eigenvalues of the perturbed (quantum) harmonic oscillator whose Hamiltonian is given by

$$H = H_0 - \|v\|_1 \delta$$

where

$$H_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$$

(unperturbed harmonic oscillator), δ is the Dirac distribution and $v \geq 0$, $v \in L^1$. As remarked in [2], this problem is related to the quark physics at small distances.

As is well known (see for example [4.a, Appendix to sect.V.3]), the operator H_0 is *essentially self-adjoint* on the Schwartz space $\mathcal{S}(I\!R)$ and its eigenfunctions are

$$\Phi_n(x) = \frac{H_n(x) e^{\frac{-x^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}}$$

where H_n is the n^{th} Hermite polynomial. The corresponding eigenvalues are

$$E_n = (n + \frac{1}{2}), \quad n = 0, 1, \ldots.$$

We can find the equation determining the eigenvalues of the perturbed operator H by writing its resolvent explicitly. From the physical point of view these are the new *bound states* of the system.

Our goal is to produce explicit formulae providing the dependence of the eigenvalues of H on the strength of the perturbing term. Our results are related to the paper by Avakian et al. [2] in which a qualitative analysis is given. In the following we shall denote the class of compact operators whose eigenvalues constitute a sequence belonging to l_p by $\mathcal{T}_p$. $\mathcal{T}_1$ is the familiar space of *trace class* operators.

Throughout this paper we use the so-called Dirac notation according to which $\forall \psi \in L^2(\mathbb{R})$, $< \psi|$ is the functional associated to ψ according to the Riesz lemma. The symbol $|\psi >$ is to be regarded as the multiplication operator by ψ. Therefore $\frac{|\psi><\psi|}{\|\psi\|_2^2}$ is nothing else but the orthogonal projection onto the one-dimensional subspace spanned by ψ.

Furthermore, we shall often regard the resolvents as integral operators on $L^2(\mathbb{R})$. This leads us to use the notation $A(x, y)$ to denote the integral kernel of the operator A.

2 The resolvent of H

The explicit form of the resolvent of H can be found by starting with the resolvent of

$$H_\epsilon = H_0 - v_\epsilon \qquad \text{where} \quad v_\epsilon(x) = \frac{1}{\epsilon} v(\frac{x}{\epsilon})$$

and performing the limit $\epsilon \to 0_+$.

Such a scaling procedure is one of the main tools used in the rigorous treatment of quantum systems involving short range interactions (see for example [1]). The resolvent of H_ϵ is given by

$$(H_\epsilon - E)^{-1} = (H_0 - E)^{-\frac{1}{2}}(I - (H_0 - E)^{-\frac{1}{2}} v_\epsilon (H_0 - E)^{-\frac{1}{2}})^{-1}(H_0 - E)^{-\frac{1}{2}}$$

as follows from Tiktopoulos' formula (see [6]) for E sufficiently negative, where

$$(H_0 - E)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} \frac{|\Phi_l\rangle\langle\Phi_l|}{(l + \frac{1}{2} - E)^{\frac{1}{2}}}.$$

It is immediate to notice that $(H_0 - E)^{-1} \in \mathcal{T}_{1+\eta}$ and $(H_0 - E)^{-1/2} \in \mathcal{T}_{2+\eta}, \quad \forall \eta > 0$.

First we consider the operator

$$A_\epsilon(E) = (H_0 - E)^{-\frac{1}{2}} v_\epsilon (H_0 - E)^{-\frac{1}{2}}$$

whose properties are summarized in the following lemma.

Lemma 2.1. *For any $E < 1/2$ we have $A_\epsilon(E) \in \mathcal{T}_1$. Furthermore,*

$$\lim_{\epsilon\to 0_+} A_\epsilon(E) = A_0(E)$$

(the limit is taken in the trace class norm topology) where

$$A_0(E) = \|v\|_1 |(H_0 - E)^{-\frac{1}{2}}(\cdot, 0)\rangle\langle (H_0 - E)^{-\frac{1}{2}}(0, \cdot)|$$

with $(H_0 - E)^{-\frac{1}{2}}(0, y) = \sum_{l=0}^{\infty} \frac{\Phi_l(0)\Phi_l(y)}{(l+\frac{1}{2}-E)^{\frac{1}{2}}}$.

Proof. Since $A_\epsilon(E)$ is positive its trace class norm is equal to its trace and can be estimated in the following way

$$\begin{aligned}\|(H_0 - E)^{-\frac{1}{2}} v_\epsilon (H_0 - E)^{-\frac{1}{2}}\|_{\mathcal{T}_1} &= \sum_{j,l,l'=0}^{\infty} \frac{(\Phi_j, \Phi_l)(\Phi_l, v_\epsilon \Phi_{l'})(\Phi_{l'}, \Phi_j)}{(l + \frac{1}{2} - E)^{\frac{1}{2}}(l' + \frac{1}{2} - E)^{\frac{1}{2}}} \\ &= \sum_{j=0}^{\infty} \frac{(\Phi_j, v_\epsilon \Phi_j)}{(j + \frac{1}{2} - E)} \\ &\leq \|v_\epsilon\|_1 \sum_{j=0}^{\infty} \frac{\|\Phi_j\|_\infty^2}{(j + \frac{1}{2} - E)} \\ &= \|v\|_1 \sum_{j=0}^{\infty} \frac{\|\Phi_j\|_\infty^2}{(j + \frac{1}{2} - E)} < \infty.\end{aligned}$$

The convergence of the series is due to the fact that $\|\Phi_j\|_\infty \approx (j+1)^{-\frac{1}{12}}$ (see [5]). This proves the first claim.

We can simplify the proof of the convergence in $\mathcal{T}_1$ by using [7, thm.2.21] according to which we need only prove that:

$$(w) \lim_{\epsilon\to 0_+} A_\epsilon(E) = A_0(E)$$

(this limit is performed in the weak operator topology) and

$$\lim_{\epsilon\to 0_+} \|A_\epsilon(E)\|_{\mathcal{T}_1} = \|A_0(E)\|_{\mathcal{T}_1}.$$

The former limit is a consequence of the convergence of v_ϵ to $\|v\|_1\delta$ in the distributional sense. For any $\phi, \psi \in L^2(\mathbb{R})$ we have:

$$\begin{aligned}(\psi, A_\epsilon(E)\phi) &= \int v_\epsilon(x)[(H_0 - E)^{-\frac{1}{2}}\bar{\psi}](x)[(H_0 - E)^{-\frac{1}{2}}\phi](x)dx \\ &= \int v(x)[(H_0 - E)^{-\frac{1}{2}}\bar{\psi}](\epsilon x)[(H_0 - E)^{-\frac{1}{2}}\phi](\epsilon x)dx.\end{aligned}$$

Taking account of the fact that the series $\sum_{l=0}^{\infty} \frac{\Phi_l(x)}{(l+\frac{1}{2}-E)^{\frac{1}{2}}}(\Phi_l, \phi)$ is uniformly convergent on $\mathbb{R}$, we can infer its continuity so that

$$\lim_{\epsilon \to 0_+} (\psi, A_\epsilon(E)\phi) = \|v\|_1[(H_0 - E)^{-\frac{1}{2}}\bar{\psi}](0)[(H_0 - E)^{-\frac{1}{2}}\phi](0)$$
$$= \|v\|_1(\psi, (H_0 - E)^{-\frac{1}{2}}(\cdot, 0))((H_0 - E)^{-\frac{1}{2}}(0, \cdot), \phi)$$
$$= (\psi, A_0(E)\phi)$$

Finally, let us perform the limit on the trace class norm of $A_\epsilon(E)$. As we have just seen,

$$\|A_\epsilon(E)\|_{\mathcal{T}_1} = \sum_{j=0}^{\infty} \frac{(\Phi_j, v_\epsilon \Phi_j)}{(j + \frac{1}{2} - E)}.$$

By dominated convergence we get $\lim_{\epsilon \to 0_+} \|A_\epsilon(E)\|_{\mathcal{T}_1} = \|A_0(E)\|_{\mathcal{T}_1}$. □

Remark. The estimate of the trace class norm of $A_\epsilon(E)$ obtained in the proof of lemma 2.1 implies the existence of a uniform lower bound for the spectra of the operators H_ϵ, as can be seen by making use of the KLMN theorem [4.b,thm.X.17]. Although it is not important to determine such a quantity at this stage, we would like to stress that its existence will enable us to exploit a powerful result regarding the norm resolvent convergence of a net of uniformly semibounded self-adjoint operators in the next theorem.

We can now give the main result of this section.

Theorem 2.2. *Let $\{H_\epsilon\}_{\epsilon \in (0,1]}, H$ be the self-adjoint operators defined above. Then, $H_\epsilon \to H$ as $\epsilon \to 0_+$ in the norm resolvent sense and*

$$(H - E)^{-1} = (H_0 - E)^{-1} + \frac{|(H_0 - E)^{-1}(\cdot, 0)\rangle\langle(H_0 - E)^{-1}(0, \cdot)|}{\frac{1}{\|v\|_1} - (H_0 - E)^{-1}(0, 0)}. \tag{2.1}$$

Proof. Our strategy will be first to find the expression of $(H - E)^{-1}$ by means of the limit of $(H_\epsilon - E)^{-1}$ on a subinterval of the negative semiaxis where the resolvents can be expanded according to Tiktopoulos' formula because of their positivity. The reason for our doing so is that such an expansion leads to a much easier calculation of the limit itself than the one that would have to be used for a generic point of the resolvent set. The global nature of the results on the convergence of resolvents will later enable us to rewrite the expression in a suitable way on the whole resolvent set.

As a consequence of the KLMN theorem H_ϵ and H are self-adjoint operators with a common *form domain* (see [4.a,sect.VIII.6]) equal to that of H_0 and a common lower bound for their spectra (see the remark). This allows us to exploit [4.a,thm.VIII.25] in order to reduce the convergence of the resolvents to the one of the quadratic forms on the common form domain regarded as a Hilbert space with the norm

$$\|\psi\|_{|\gamma|}^2 = (\psi, H_0\psi) + |\gamma|\|\psi\|_2^2,$$

$\gamma < 0$ being the common lower bound of the spectra.

Then,

$$\frac{|(\psi, [\|v\|_1\delta - v_\epsilon]\psi)|}{\|\psi\|^2_{|\gamma|}} \le \sup_{f\in L^2(\mathbb{R})} \frac{|(f, [A_\epsilon(\gamma) - A_0(\gamma)]f)|}{\|f\|_2^2}$$
$$= \|A_\epsilon(\gamma) - A_0(\gamma)\|_\infty \le \|A_\epsilon(\gamma) - A_0(\gamma)\|_{\mathcal{T}_1}.$$

The last two steps are due to the definition of the norm of a self-adjoint operator and the fact that the trace class norm is the largest amongst the possible norms of a compact operator. Due to lemma 2.1 $\lim_{\epsilon\to 0_+} \|A_\epsilon(\gamma) - A_0(\gamma)\|_{\mathcal{T}_1} = 0$.

Hence, for any $E \in (-\infty, \gamma) \subset \rho(H) \cap \{\cap_\epsilon \rho(H_\epsilon)\}$ we have

$$(H-E)^{-1} = \lim_{\epsilon\to 0_+} (H_0 - E)^{-\frac{1}{2}}[I - A_\epsilon(E)]^{-1}(H_0 - E)^{-\frac{1}{2}}$$
$$= (H_0 - E)^{-\frac{1}{2}}[I - A_0(E)]^{-1}(H_0 - E)^{-\frac{1}{2}}.$$

It is easy to check that there exists $\bar{E} < 0$ such that, $\forall E < \bar{E}$ we have $\|A_0(E)\|_{\mathcal{T}_1} = \|v\|_1 \|(H_0 - E)^{-\frac{1}{2}}(0,\cdot)\|_2^2 < 1$ so that we can use the Neumann series. Since $A_0(E)$ is an orthogonal projection times a positive constant we get

$$(I - A_0(E))^{-1} = I + \frac{1}{1 - \|A_0(E)\|_{\mathcal{T}_1}} A_0(E).$$

By inserting the latter in (2.1) we get

$$(H-E)^{-1} = (H_0 - E)^{-1} + \frac{(H_0 - E)^{-\frac{1}{2}} A_0(E)(H_0 - E)^{-\frac{1}{2}}}{1 - \|A_0(E)\|_{\mathcal{T}_1}}.$$

In order to find an expression of the resolvent defined on the whole resolvent set $\rho(H)$ we must use the following facts:

(i) $\|(H_0 - E)^{-\frac{1}{2}}(0,\cdot)\|_2^2 = \sum_{l=0}^\infty \frac{\Phi_l^2(0)}{(l+\frac{1}{2}-E)} = (H_0 - E)^{-1}(0,0)$ for $E < \frac{1}{2}$;

(ii) $(H_0 - E)^{-\frac{1}{2}}|(H_0 - E)^{-\frac{1}{2}}(\cdot,0)\rangle = (H_0 - E)^{-\frac{1}{2}} \sum_{l=0}^\infty \frac{\Phi_l(0)\Phi_l}{(l+\frac{1}{2}-E)^{\frac{1}{2}}} = |(H_0 - E)^{-1}(\cdot,0)\rangle$.

Since the r.h. sides of (i) and (ii) are well defined over $\mathbb{C}\setminus\{E_n\}$ we can extend the resolvent by means of the formula

$$(H-E)^{-1} = (H_0 - E)^{-1} + \frac{|(H_0 - E)^{-1}(\cdot,0)\rangle\langle(H_0 - E)^{-1}(0,\cdot)|}{\frac{1}{\|v\|_1} - (H_0 - E)^{-1}(0,0)}.$$

□

3 The bound state equation

As is well known, the eigenvalues of the operator H are given by the poles of its resolvent. The odd eigenfunctions of H_0 and its eigenvalues are not affected by the perturbation since the second summand on the r.h.s. of (2.1) vanishes on the antisymmetric functions due to the fact that $(H_0 - E)^{-1}(0,\cdot\,)$ is even.

By computing $(\Phi_{2k}, (H-E)^{-1}\Phi_{2k})$ near $2k+\frac{1}{2}$ one can see that, because of a cancellation, such an eigenvalue is no longer in the spectrum of H. Therefore, the spectrum of H is given by the odd eigenvalues of H_0 and the solutions of the following equation:

$$\frac{1}{\|v\|_1} = \sum_{l=0}^{\infty} \frac{\Phi_{2l}^2(0)}{(2l+\frac{1}{2}-E)}. \tag{3.1}$$

Let us now try to determine the even eigenvalues. We can sketch the graph of the series as a function of the real variable E. The following plot has been obtained using the first 200 terms of the series. Regardless of the slow convergence of the series, our truncation is sufficient to enable us to observe that the series vanishes at the odd eigenvalues of H_0.

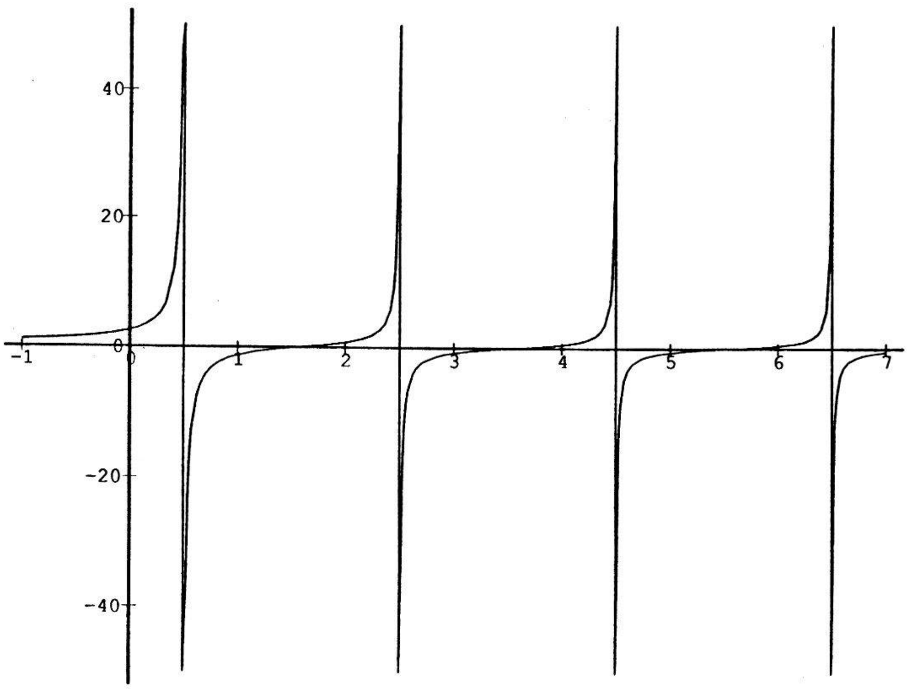

Figure 1. Graph of the r.h.s. of (3.1) as a function of E.

In order to study equation (3.1), we can use the explicit expression for $\Phi_{2l}(0)$ given in the introduction to get

$$\frac{1}{\|v\|_1} = \sum_{l=0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l+\frac{1}{2}-E)}.$$

If we are interested in the first eigenvalue E_0 we can set $E = \frac{1}{2} - \epsilon \quad (\epsilon > 0)$ to get

$$\frac{1}{\|v\|_1} = (H_0 - E)^{-1}(0,0) \sum_{l=1}^{\infty} \frac{1}{\sqrt{\pi}} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l+\epsilon)} + \frac{1}{\epsilon\sqrt{\pi}}. \tag{3.2}$$

We can determine the explicit expression of $(H_0 - E)^{-1}(0,0)$ for any $E < \frac{1}{2}$ by means of the explicit formula for $e^{-iH_0\tau}(x,y)$ given in [3,(3.59)]. By inserting an imaginary time into the latter (i.e. $t = -i\tau$) we can get the explicit expression of $e^{-H_0t}(x,y)$, which is the integral kernel of the semigroup generated by H_0. Then

$$e^{-H_0t}(0,0) = [\frac{1}{2\pi \sinh t}]^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}(H_0 - E)^{-1}(0,0) &= \sum_{l=0}^{\infty} \frac{\Phi_{2l}^2(0)}{(2l + \frac{1}{2} - E)} \\ &= \int_0^{\infty} \sum_{l=0}^{\infty} \Phi_{2l}^2(0) e^{-(2l+\frac{1}{2}-E)t} dt \\ &= \int_0^{\infty} e^{-(H_0-E)t}(0,0) dt \\ &= \int_0^{\infty} \frac{e^{Et}}{\sqrt{2\pi \sinh t}} dt \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{E+\frac{1}{2}t}}{(e^{2t}-1)^{\frac{1}{2}}} dt.\end{aligned}$$

Therefore, (3.2) can be rewritten as follows:

$$\frac{1}{\|v\|_1} = (H_0 - \frac{1}{2} + \epsilon)^{-1}(0,0) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{(1-\epsilon)t}}{(e^{2t}-1)^{\frac{1}{2}}} dt. \tag{3.3}$$

The integral on the right hand side of (3.3) can be transformed into

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{(\cosh t)^{\epsilon}} dt = \frac{1}{\sqrt{\pi}} \frac{1}{\epsilon} \int_0^1 (\frac{2}{1+\xi^{\frac{2}{\epsilon}}})^{\epsilon} d\xi.$$

Therefore the first eigenvalue $\tilde{E}_0 = \frac{1}{2} - \epsilon_0$ is determined by solving the equation

$$\frac{\sqrt{\pi}}{\|v\|_1} = \frac{1}{\epsilon} \int_0^1 (\frac{2}{1+\xi^{\frac{2}{\epsilon}}})^{\epsilon} d\xi. \tag{3.4}$$

Remark. It is easy to show that $\exists a, b > 0$ such that the integral in (3.4) is in [a,b] so that for any ϵ solving (3.4) we have

$$\frac{a}{\epsilon} \leq \frac{\sqrt{\pi}}{\|v\|_1} \leq \frac{b}{\epsilon}.$$

It follows that when $\|v\|_1$ goes to $+\infty$, ϵ diverges as well and consequently the *ground state* $\tilde{E}_0 = \frac{1}{2} - \epsilon$ goes to $-\infty$.

To find an analogous equation determining the second even eigenvalue $\tilde{E}_2 = \frac{5}{2} - \epsilon_2$ we must write the series in (3.1) as

$$\sum_{l=0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l-2+\epsilon)} = \frac{1}{\sqrt{\pi}} (\sum_{l=1}^{\infty} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l-2+\epsilon)} - \frac{1}{2-\epsilon}). \quad (3.5)$$

We can write the series as an integral also in this case. First we consider

$$\begin{aligned}\sum_{l=1}^{\infty} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l+z)} &= \int_0^{\infty} \frac{e^{(1-z)t}}{(e^{2t}-1)^{\frac{1}{2}}} dt - \frac{1}{z} \\ &= \int_0^{\infty} (\frac{e^{(1-z)t}}{(e^{2t}-1)^{\frac{1}{2}}} - e^{-zt}) dt \\ &= \int_0^{\infty} e^{-zt} \frac{1}{(e^{2t}-1)^{\frac{1}{2}}(e^t + (e^{2t}-1)^{\frac{1}{2}})} dt, \quad z > 0.\end{aligned}$$

Since the integral is still convergent for $z > -2$ we can write

$$\sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l}(l!)^2} \frac{1}{(2l-2+\epsilon)} = \int_0^{\infty} \frac{e^{(2-\epsilon)t}}{(e^{2t}-1)^{\frac{1}{2}}(e^t + (e^{2t}-1)^{\frac{1}{2}})} dt.$$

Substituting in (3.5) and handling the integral as in (3.3), the equation determining the second eigenvalue is

$$\frac{\sqrt{\pi}}{\|v\|_1} = -\frac{1}{2-\epsilon} + \frac{1}{\epsilon} \int_0^1 (\frac{1+\xi^{\frac{2}{\epsilon}}}{2})^{1-\epsilon} d\xi. \quad (3.6)$$

Equations (3.4) and (3.6) are in a form extremely suitable to compute their numerical solutions (see figures 2 and 3).

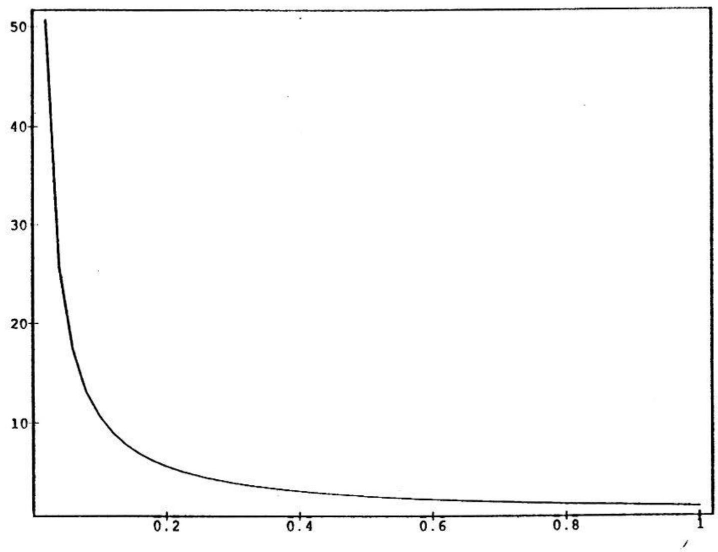

Figure 2.The r.h.s. of (3.4) as a function of ϵ.

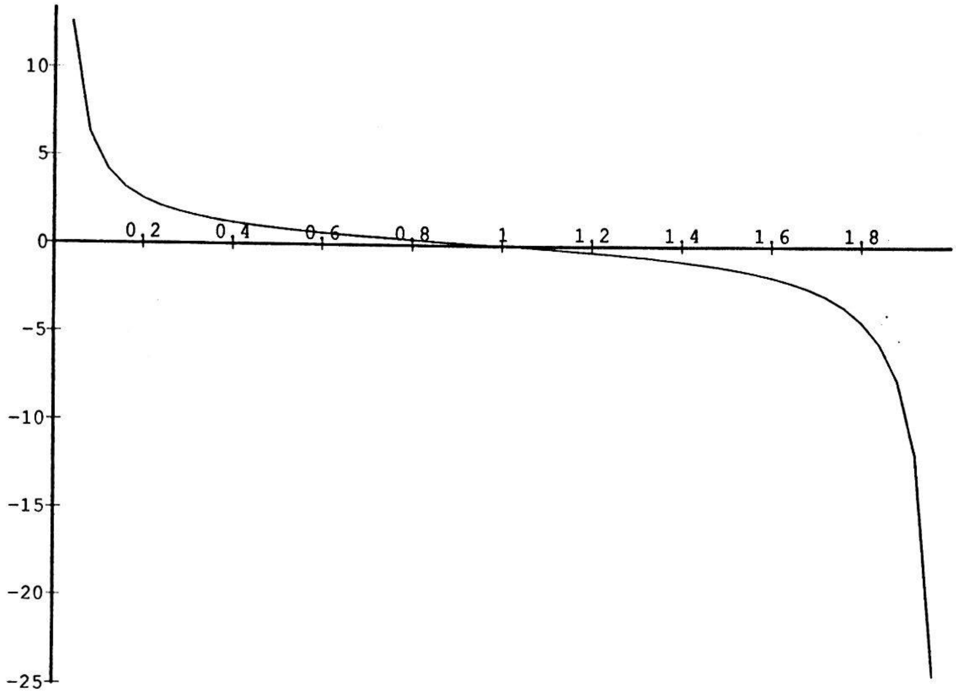

Figure 3. The r.h.s. of (3.6) as a function of ϵ.

At this stage we would like to add that even though it seems possible to iterate the above procedure to determine the higher order eigenvalues, this would lead to much more complicated expressions. Of course, one could approximate such eigenvalues by truncating as far as possible the series in (3.1). However, this procedure is made not very precise by the slow convergence of the series itself.

Remark. It is important to point out that the integral expression of $(H_0 - E)^{-1}(0,0)$ on the left of the ground state $E_0 = \frac{1}{2}$ can be written as a ratio of Γ functions by using the formula:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{1}{\Gamma(b-a)} \int_0^\infty e^{-(z+a)t}(1-e^{-t})^{b-a-1}dt.$$

In fact

$$(H_0 - E)^{-1}(0,0) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{1}{2}(\frac{1}{2}-E)t}}{(1-e^{-t})^{\frac{1}{2}}}dt.$$

By setting $\lambda = E - \frac{1}{2}$ (and $z = -\frac{1}{2}\lambda$, $a = 0$, $b = \frac{1}{2}$) we get

$$(H_0 - \lambda - \frac{1}{2})^{-1}(0,0) = \frac{\Gamma(-\frac{1}{2}\lambda)}{2\Gamma(\frac{1}{2}(1-\lambda))}.$$

Therefore, the equation determining the bound states can be written as

$$\frac{2}{\|v\|_1} = \frac{\Gamma(-\frac{1}{2}\lambda)}{\Gamma(\frac{1}{2}(1-\lambda))}$$

i.e. equation (12) in Avakian's paper [2] (the $\frac{1}{2}$ comes from our using the operator $H_0 = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2))$.

Our way of arriving at the result has the advantage of not relying too heavily on the theory of special functions since it uses the spectral representation of H_0. Furthermore, it provides nice formulae for the numerical calculation of the perturbed ground state and of the first excited symmetric eigenstate.

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